

QUANTUM LANGLANDS CORRESPONDENCE

This note summarizes some conjectures on the theme of quantum geometric Langlands correspondence, which arose in the course of discussions around October-November 2006 between A. Beilinson, R. Bezrukavnikov, A. Braverman, M. Finkelberg, D. Gaitsgory and J. Lurie. We also thank E. Frenkel and E. Witten for stimulating conversations. ¹

Let G be a reductive group over a field of characteristic 0, and let \check{G} be its Langlands dual. By a level c we will mean a choice of a symmetric invariant form on the Lie algebra \mathfrak{g} . We will absorb the critical shift into the notation, i.e., $c = 0$ means the critical level. Given c , we will denote by $\frac{1}{c}$ the dual level for \check{G} , obtained by identifying the corresponds Cartan subalgebras as duals one one another.

1. CATEGORIES ACTED ON BY THE LOOP GROUP

We will assume having the following notions at our disposal:

1.1. The notion of category \mathcal{C} acted on by the loop group $G((t))$ at level c (for an abelian category this notion is developed, e.g., in [FG2]).

1.2. Example. The category $\mathfrak{D}^c(G((t))\text{-mod}$ of (c -twisted) D-modules on the loop group itself, carries an action of $G((t))$ at level c and a commuting action at level $-c$. From now on we will denote it by $\mathfrak{D}^{c,-c}(G((t))\text{-mod}$. We have a canonical equivalence

$$\mathfrak{D}^{c,-c}(G((t))\text{-mod} \simeq \mathfrak{D}^{-c,c}(G((t))\text{-mod},$$

that interchanges the two actions.

1.3. To \mathcal{C}_1 acted on by $G((t))$ at level c , and \mathcal{C}_2 acted on by $G((t))$ at level $-c$, we should be able to associate their tensor product over $G((t))$, denoted $\mathcal{C}_1 \otimes_{G((t))} \mathcal{C}_2$.

1.4. Example. let K be a subgroup of $G[[t]]$, and let $\mathcal{C}_1 = \mathfrak{D}^c(G((t))/K)\text{-mod}$. Then for $\mathcal{C}_2 := \mathcal{C}$ as above, $\mathfrak{D}^c(G((t))/K)\text{-mod} \otimes_{G((t))} \mathcal{C}$ should be equivalent to the category \mathcal{C}^K of K -equivariant objects in \mathcal{C} . In particular, we have:

$$(1) \quad \mathfrak{D}^{c,-c}(G((t))\text{-mod} \otimes_{G((t))} \mathcal{C} \simeq \mathcal{C},$$

as categories acted on by $G((t))$ at level $-c$.

Date: Nov 13, 2007.

¹The sole bearer of responsibility for the speculations that follow is D.G., who took it upon himself to write them down. Other people mentioned are responsible only insofar as they personally choose to do so. Accordingly, we shall not specify individual credits, except for Jacob's initial guess of how to get the quantum group from twisted Whittaker sheaves.

1.5. Example. Let $\mathcal{C}_1 = \widehat{\mathfrak{g}}^c\text{-mod}$ —the category of smooth Kac-Moody modules at level c . For $\mathcal{C}_2 := \mathcal{C}$ consider the category $\widehat{\mathfrak{g}}^c\text{-mod} \otimes_{G((t))} \mathcal{C}$.

If instead of $G((t))$ we had a group-scheme H , the corresponding category $\mathfrak{h}\text{-mod} \otimes_H \mathcal{C}$ would identify with the category $\mathcal{C}^{H,w}$ of weakly H -equivariant objects in \mathcal{C} . In this case, the tensor category $\text{Rep}(H)$ would act on $\mathcal{C}^{H,w}$. In the sequel, we will see what replaces this structure when instead of H we have the loop group $G((t))$.

1.6. To \mathcal{C} acted on by $G((t))$ at level c , we should be able to assign the category $\text{Whit}(\mathcal{C})$ that corresponds to $(N((t)), \chi)$ -equivariant objects, where $\chi : N((t)) \rightarrow \mathbb{G}_a$ is a non-degenerate character.

1.7. Example. One of the principal players for us will be the category

$$(2) \quad \text{Whit}^c(\text{Gr}_G) := \text{Whit}(\mathfrak{D}^c(\text{Gr}_G)\text{-mod}).$$

Another important example is $\text{Whit}^c(G((t))) := \text{Whit}(\mathfrak{D}^{c,-c}(G((t)))\text{-mod})$. By transport of structure, the latter carries an action of $G((t))$ at level $-c$. We have:

$$(3) \quad \text{Whit}(\mathfrak{D}^{c,-c}(G((t)))\text{-mod}) \otimes_{G((t))} \mathcal{C} \simeq \text{Whit}(\mathcal{C}).$$

1.8. Example. Consider $\mathcal{C} = \widehat{\mathfrak{g}}^c\text{-mod}$. We are supposed to have

$$(4) \quad \text{Whit}(\widehat{\mathfrak{g}}^c\text{-mod}) \simeq \mathcal{W}_{\mathfrak{g}}^c\text{-mod},$$

where $\mathcal{W}_{\mathfrak{g}}^c$ is the W-algebra corresponding to \mathfrak{g} at level c .

2. CHIRAL CATEGORIES

2.1. Let X be an algebraic curve. Another notion that we assume having at our disposal is that of chiral category over X .

The data of a chiral category \mathcal{A} assigns to each integer n an \mathcal{O} -module of categories \mathcal{A}^n over X^n , equipped with factorization isomorphisms, which we will spell out for $n = 2$:

The restriction $\mathcal{A}^2|_{X \times X - \Delta(X)}$ should be identified with $\mathcal{A}^1 \boxtimes \mathcal{A}^1|_{X \times X - \Delta(X)}$, and the restriction $\mathcal{A}^2|_{\Delta(X)}$ should be identified with \mathcal{A}^1 .

In addition, \mathcal{A}^n is supposed to have a unit object, analogously to the case of chiral algebras. The latter endows the sheaf of categories \mathcal{A}^n with a connection along X .

2.2. We will usually think of a chiral category as an \mathcal{O} -module of categories \mathcal{A}^1 over X itself, endowed with an extra structure. When a chiral category is obtained by a universal procedure (i.e., is a vertex operator category), we will think of it as a plain category \mathcal{A} equal to the fiber of \mathcal{A}^1 at a point $x \in X$, endowed with an extra structure.

The notion of chiral category should be regarded as a D-module version of the notion of E_2 -category.

2.3. Example. A sheaf of symmetric monoidal categories over X , endowed with a connection along X gives rise to a chiral category.

2.4. When working over \mathbb{C} , and we choose a coordinate on our curve, there is a transcendental procedure that assigns to a ribbon category a chiral category. (We are not going to use it.) This procedure is fully faithful: a chiral category comes in this way from a ribbon category when a certain representability condition is satisfied. This is how a monoidal structure arises in [KL].

2.5. Example. Let A be a chiral algebra. Then the category $A\text{-mod}$ of chiral A -modules is naturally a chiral category.

2.6. Example. A construction of Beilinson and Drinfeld endows the category $\mathfrak{D}^c(\text{Gr}_G\text{-mod})$ with a structure of chiral category.

2.7. Example. The category $\text{Whit}^c(\text{Gr}_G)$ is a chiral category.

2.8. By analogy with the theory of chiral algebras, given a chiral category, it makes sense to consider module categories over it. We will consider module categories supported at a fixed point x of the curve, with t being a local coordinate.

2.9. Generalizing the construction of Beilinson and Drinfeld, we obtain that the category $\mathfrak{D}^c(G((t))\text{-mod})$ is naturally a module category with respect to $\mathfrak{D}^c(\text{Gr}_G\text{-mod})$. In addition, it carries a commuting action of $G((t))$ at level $-c$. Hence, for any category \mathcal{C} acted on by $G((t))$ at level c , the category $\mathfrak{D}^{c-c}(G((t))\text{-mod}) \otimes_{G((t))} \mathcal{C}$ is a module category with respect to $\mathfrak{D}^c(\text{Gr}_G\text{-mod})$. Taking into account (1), we obtain that on a given category \mathcal{C} an action of $G((t))$ at level c gives rise to a structure of module category with respect to $\mathfrak{D}^c(\text{Gr}_G\text{-mod})$.

Conjecture 2.10. *For a category \mathcal{C} , the data of an action of $G((t))$ at level c is equivalent to a structure of module category with respect to $\mathfrak{D}^c(\text{Gr}_G\text{-mod})$.*

2.11. Example. Independent of the above conjecture, the category $\text{Whit}^c(G((t)))$ has a structure of module category with respect to $\text{Whit}^c(\text{Gr}_G)$, and carries a commuting action of $G((t))$ at level $-c$.

Hence, by (3), for any category \mathcal{C} acted on by $G((t))$ at level c , the category $\text{Whit}(\mathcal{C})$ is a module category with respect to $\text{Whit}^c(\text{Gr}_G)$.

2.12. By Sect 2.5, the category $\widehat{\mathfrak{g}}^c\text{-mod}$ is a chiral category. Let $\text{KL}_G^c \subset \widehat{\mathfrak{g}}^c\text{-mod}$ be the subcategory consisting of $G[[t]]$ -integrable representations. (The symbol KL stands for Kazhdan-Lusztig, who studied this category in [KL].) I.e.,

$$\text{KL}_G^c := (\widehat{\mathfrak{g}}^c\text{-mod})^{G[[t]]}.$$

This is also a chiral category. We will regard the fiber of $\widehat{\mathfrak{g}}^c\text{-mod}$ at $x \in X$ as a module category with respect to KL_G^c .

The following is established in [FG1]:

Theorem 2.13. *The following two pieces of structure defined on $\widehat{\mathfrak{g}}^c\text{-mod}$ commute: the action of $G((t))$ at level c , and the structure of module category with respect to KL_G^c .*

2.14. Let \mathcal{C} be again a category acted on by $G((t))$ at level $-c$, and consider the category $\text{KL}(\mathcal{C}) := \widehat{\mathfrak{g}}^c\text{-mod} \otimes_{G((t))} \mathcal{C}$ of Sect 1.5. From Theorem 2.13 we obtain that this category is naturally a module category with respect to KL_G^c .

It is this structure that we regard as a substitute for the action of $\text{Rep}(H)$, alluded to in Sect 1.5.

3. LOCAL QUANTUM LANGLANDS

3.1. The following is a version of a conjecture proposed by J. Lurie:

Conjecture 3.2. *For c not rational negative, the chiral categories $\text{Whit}^c(\text{Gr}_G)$ and $\text{KL}_{\check{G}}^{\frac{1}{c}}$ are equivalent.*

This conjecture has been essentially proven in [Ga] for c irrational, by identifying both sides with a third chiral category, namely, that of factorizable sheaves of [BFS].

3.3. We can now formulate the Local Quantum Geometric Langlands conjecture, which we literally believe to hold only for irrational values of c :

Conjecture 3.4. *There exists a 2-equivalence $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$:*

$$\{ \text{Categories acted on by } G((t)) \text{ at level } c \} \rightarrow \{ \text{Categories acted on by } \check{G}((t)) \text{ at level } -\frac{1}{c} \},$$

characterized by **either** of the following two properties: for \mathcal{C} acted on by $G((t))$ and $\check{\mathcal{C}} := \Psi^{G \rightarrow \check{G}}(\mathcal{C})$, we need that:

- The category $\text{Whit}(\mathcal{C})$, regarded as a module category with respect to $\text{Whit}^c(\text{Gr}_G)$, is equivalent to $\text{KL}(\check{\mathcal{C}})$, regarded as a module category with respect to $\text{KL}_{\check{G}}^{\frac{1}{c}}$, when we identify $\text{Whit}^c(\text{Gr}_G) \simeq \text{KL}_{\check{G}}^{\frac{1}{c}}$ via Conjecture 3.2.
- The category $\text{KL}(\mathcal{C})$, regarded as a module category with respect to KL_G^{-c} , is equivalent to $\text{Whit}(\check{\mathcal{C}})$, regarded as a module category with respect to $\text{Whit}^{-\frac{1}{c}}(\text{Gr}_{\check{G}})$, when we identify $\text{KL}_G^{-c} \simeq \text{Whit}^{-\frac{1}{c}}(\text{Gr}_{\check{G}})$ via Conjecture 3.2.

3.5. We can divide Conjecture 3.4 into three steps:

Conjecture 3.6. *For c irrational, the assignment $\mathcal{C} \mapsto \text{Whit}(\mathcal{C})$ establishes a 2-equivalence $\{ \text{Categories acted on by } G((t)) \text{ at level } c \} \rightarrow \{ \text{Module categories with respect to } \text{Whit}^c(\text{Gr}_G) \}$.*

Conjecture 3.7. *The assignment $\mathcal{C} \mapsto \text{KL}(\mathcal{C})$ establishes a 2-equivalence*

$$\{ \text{Categories acted on by } G((t)) \text{ at level } c \} \rightarrow \{ \text{Module categories with respect to } \text{KL}_G^{-c} \}.$$

Conjecture 3.8. *Assuming the above two conjectures, the two composed 2-functors*

$$(\text{KL for } \check{G})^{-1} \circ (\text{Whit for } G)$$

and

$$(\text{Whit for } \check{G})^{-1} \circ (\text{KL for } G)$$

are isomorphic.

$$\{ \text{Categories acted on by } G((t)) \text{ at level } c \} \rightarrow \{ \text{Categories acted on by } \check{G}((t)) \text{ at level } -\frac{1}{c} \},$$

3.9. Let us consider some examples of how the 2-functor $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ is supposed to act. We claim that we have:

$$\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\mathfrak{D}^c(\text{Gr}_G)\text{-mod}) \simeq \mathfrak{D}^{-\frac{1}{c}}(\text{Gr}_{\check{G}})\text{-mod}.$$

This follows from either of the characterizing properties of Ψ , since

$$\text{Whit}(\mathfrak{D}^c(\text{Gr}_G)\text{-mod}) := \text{Whit}^c(\text{Gr}_G)$$

and

$$\text{KL}(\mathfrak{D}^c(\text{Gr}_G)\text{-mod}) := \widehat{\mathfrak{g}}^{-c}\text{-mod}_{G((t))} \otimes \mathfrak{D}^c(\text{Gr}_G)\text{-mod} \simeq \widehat{\mathfrak{g}}^{-c}\text{-mod}^{G[[t]]} =: \text{KL}_G^{-c}.$$

Conjecture 3.10. $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\mathfrak{D}^c(\mathrm{Fl}_G)\text{-mod}) \simeq \mathfrak{D}^{-\frac{1}{c}}(\mathrm{Fl}_{\check{G}})\text{-mod}$.

This is an interesting conjecture in its own right, as it translates as:

Conjecture 3.11.

$$\mathrm{Whit}^c(\mathrm{Fl}_G) \simeq \widehat{\mathfrak{g}}^{\frac{1}{c}}\text{-mod}^{\check{I}}.$$

3.12. Duality of \mathcal{W} -algebras. Let us recall the assertion of [FF] that there exists an isomorphism

$$(5) \quad \mathcal{W}_{\mathfrak{g}}^c \simeq \mathcal{W}_{\check{\mathfrak{g}}}^{\frac{1}{c}}.$$

In particular, the corresponding categories of modules are equivalent.

Note, however, that the identification (4) defines on $\mathcal{W}_{\mathfrak{g}}^c\text{-mod}$ the commuting structures of module over the chiral categories $\mathrm{Whit}^c(\mathrm{Gr}_G)$ and KL_G^c .

We propose the following strengthening of (5):

Conjecture 3.13. *The equivalence of categories*

$$\mathcal{W}_{\mathfrak{g}}^c\text{-mod} \simeq \mathcal{W}_{\check{\mathfrak{g}}}^{\frac{1}{c}}\text{-mod},$$

induced by (5), respects the module structures with respect to the chiral categories

$$\mathrm{Whit}^c(\mathrm{Gr}_G) \simeq \mathrm{KL}_{\check{G}}^{\frac{1}{c}} \quad \text{and} \quad \mathrm{KL}_G^c \simeq \mathrm{Whit}^{\frac{1}{c}}(\mathrm{Gr}_{\check{G}}).$$

This conjecture formally implies that

$$(6) \quad \Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\widehat{\mathfrak{g}}^c\text{-mod}) \simeq \mathrm{Whit}^{\frac{1}{c}}(\check{G}((t))) \quad \text{and} \quad \Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\mathrm{Whit}^c(G((t)))) \simeq \widehat{\mathfrak{g}}\text{-mod}^{\frac{1}{c}}.$$

3.14. Let us denote by $\mathbf{M}_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ the category $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\mathfrak{D}^{c, -c}(G((t)))\text{-mod})$. It carries an action of \check{G} at level $-\frac{1}{c}$ and a commuting action of $G((t))$ at level $-c$, and the functor $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ can be realized as

$$\mathcal{C} \mapsto \mathbf{M}_{G \rightarrow \check{G}}^{c, -\frac{1}{c}} \otimes_{G((t))} \mathcal{C}.$$

Isomorphism (6) implies that

$$(7) \quad \mathbf{M}_{G \rightarrow \check{G}}^{c, -\frac{1}{c}} \simeq \mathbf{M}_{\check{G} \rightarrow G}^{\frac{1}{c}, -c}.$$

The category $\mathbf{M}_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ has the following properties

$$(8) \quad (\mathrm{Whit} \text{ for } G)(\mathbf{M}_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}) \simeq \widehat{\mathfrak{g}}\text{-mod}^{-\frac{1}{c}} \quad \text{and} \quad (\mathrm{Whit} \text{ for } \check{G})(\mathbf{M}_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}) \simeq \widehat{\mathfrak{g}}\text{-mod}^{-c}.$$

Remark. In [Sto] it is suggested that there should exist a chiral algebra $M_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$, which receives maps with commuting images from the Kac-Moody chiral algebras $A_{\mathfrak{g}, -c}$ and $A_{\check{\mathfrak{g}}, -\frac{1}{c}}$, corresponding to \mathfrak{g} and $\check{\mathfrak{g}}$ at levels $-c$ and $-\frac{1}{c}$, respectively, such that the Drinfeld-Sokolov reduction of $M_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ with respect to \mathfrak{g} is isomorphic to $A_{\check{\mathfrak{g}}, -\frac{1}{c}}$, and with respect to $\check{\mathfrak{g}}$ is isomorphic to $A_{\mathfrak{g}, -c}$. The above discussion does not produce such a chiral algebra, but rather a chiral category with the corresponding properties.

3.15. Equation (7) implies that for \mathcal{C}_1 , acted on by $G((t))$ at level c , \mathcal{C}_2 , acted on by $G((t))$ at level $-c$, and

$$\check{\mathcal{C}}_1 := \Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\mathcal{C}_1), \quad \check{\mathcal{C}}_2 := \Psi_{G \rightarrow \check{G}}^{-c, \frac{1}{c}}(\mathcal{C}_2),$$

we have:

$$(9) \quad \mathcal{C}_1 \otimes_{G((t))} \mathcal{C}_2 \simeq \check{\mathcal{C}}_1 \otimes_{\check{G}((t))} \check{\mathcal{C}}_2.$$

In particular, for \mathcal{C} acted on by $G((t))$ at level c and $\check{\mathcal{C}} := \Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}(\mathcal{C})$ we obtain:

$$(10) \quad \mathcal{C}^{G[[t]]} \simeq \mathcal{C} \otimes_{G((t))} \mathfrak{D}^{-c}(\mathrm{Gr}_G)\text{-mod} \simeq \check{\mathcal{C}} \otimes_{\check{G}((t))} \mathfrak{D}^{\frac{1}{c}}(\mathrm{Gr}_{\check{G}})\text{-mod} \simeq \check{\mathcal{C}}^{\check{G}[[t]]}.$$

Assuming (3.10), we also obtain

$$(11) \quad \mathcal{C}^I \simeq \check{\mathcal{C}}^I.$$

3.16. Harish-Chandra bimodules. Let us denote by $\mathrm{HCh}_G^{c, -c}$ the category

$$\widehat{\mathfrak{g}}^c\text{-mod} \otimes_{G((t))} \widehat{\mathfrak{g}}^{-c}\text{-mod}.$$

We remark that for a group scheme H , the corresponding category $\mathfrak{h}\text{-mod} \otimes_H \mathfrak{h}\text{-mod}$ is indeed tautologically equivalent to the category of Harish-Chandra modules for the pair $(\mathfrak{h} \oplus \mathfrak{h}, H)$.

Equation (8) implies that we have the equivalence

Conjecture 3.17.

$$\mathrm{HCh}_G^{c, -c} \simeq (\mathrm{Whit} \times \mathrm{Whit})(\mathfrak{D}^{\frac{1}{c}, -\frac{1}{c}}(\check{G}((t)))\text{-mod}.$$

4. GLOBAL QUANTUM LANGLANDS

4.1. Assume now that X is a complete curve. Let Bun_G denote the moduli stack of G -bundles on X .

The following conjecture was proposed in [Sto]:

Conjecture 4.2. *There exists an equivalence of categories*

$$\mathfrak{D}^c(\mathrm{Bun}_G)\text{-mod} \simeq \mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}_{\check{G}})\text{-mod}.$$

We will now couple this with Conjecture 3.2, which would, conjecturally, fix the equivalence of Conjecture 4.2 uniquely.

4.3. Let x_1, \dots, x_n be a finite collection of points on X . On the one hand, we have a localization functor

$$\mathrm{Loc} : \mathrm{KL}_{G, x_1}^c \times \dots \times \mathrm{KL}_{G, x_n}^c \rightarrow \mathfrak{D}^c(\mathrm{Bun}_G),$$

obtained by considering conformal blocks of $\widehat{\mathfrak{g}}^c$ -modules.

On the other hand, we the Poincare series functor

$$\mathrm{Poinc} : \mathrm{Whit}^{-c}(\mathrm{Gr}_{G, x_1}) \times \dots \times \mathrm{Whit}^{-c}(\mathrm{Gr}_{G, x_n}) \rightarrow \mathfrak{D}^c(\mathrm{Bun}_G),$$

corresponding to the diagram

$$\mathrm{Gr}_{G, x_1} / N((t)) \times \dots \times \mathrm{Gr}_{G, x_n} / N((t)) \leftarrow \mathrm{Gr}_{G, x_1} \times \dots \times \mathrm{Gr}_{G, x_n} / N_{out} \rightarrow \mathrm{Gr}_{G, x_1} \times \dots \times \mathrm{Gr}_{G, x_n} / G_{out},$$

where

$$(\mathrm{Gr}_{G, x_1} \times \dots \times (\mathrm{Gr}_{G, x_n})) / G_{out} \simeq \mathrm{Bun}_G.$$

4.4. Global unramified quantum Langlands. We propose:

Conjecture 4.5. *There exists an equivalence as in Conjecture 4.2, which for every collection of points x_1, \dots, x_n makes the diagram*

$$\begin{array}{ccc} \mathrm{KL}_{G,x_1}^c \times \dots \times \mathrm{KL}_{G,x_n}^c & \xrightarrow{\text{Conjecture 3.2}} & \mathrm{Whit}^{\frac{1}{c}}(\mathrm{Gr}_{\check{G},x_1}) \times \dots \times \mathrm{Whit}^{\frac{1}{c}}(\mathrm{Gr}_{\check{G},x_n}) \\ \mathrm{Loc} \downarrow & & \mathrm{Poinc} \downarrow \\ \mathfrak{D}^c(\mathrm{Bun}_G)\text{-mod} & \xrightarrow{\text{Conjecture 4.2}} & \mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}_{\check{G}})\text{-mod} \end{array}$$

commute.

4.6. The ramified case. For a point $x \in X$ let $\mathrm{Bun}_{G,x}$ be the moduli space of G -bundles with a full level structure at x . The category $\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod}$ carries a natural action of $G((t))$ at level c .

We propose the following:

Conjecture 4.7. *The 2-functor $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ sends the category $\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod}$ to $\mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}_{\check{G},x})$.*

We emphasize that this conjecture *does not* say that the categories $\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod}$ and $\mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}_{\check{G},x})$ are equivalent. Rather, it says that they correspond to each other under $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$.

4.8. Compatibility with the unramified picture. Let us couple Conjecture 4.7 with (10) and (11). We obtain

$$(12) \quad (\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod})^{G[[t]]} \simeq \left(\mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}_{\check{G},x}) \right)^{\check{G}[[t]]}$$

and

$$(13) \quad (\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod})^I \simeq \left(\mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}_{\check{G},x}) \right)^{\check{I}},$$

respectively.

However, $(\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod})^{G[[t]]} \simeq \mathfrak{D}^c(\mathrm{Bun}_G)\text{-mod}$, so (12) recovers Conjecture 4.2.

Note that $(\mathfrak{D}^c(\mathrm{Bun}_{G,x})\text{-mod})^I \simeq \mathfrak{D}^c(\mathrm{Bun}'_G)\text{-mod}$, where Bun'_G denotes the moduli space of G -bundles with a parabolic structure at x . So, (13) leads to an equivalence

$$\mathfrak{D}^c(\mathrm{Bun}'_G)\text{-mod} \simeq \mathfrak{D}^{-\frac{1}{c}}(\mathrm{Bun}'_{\check{G}})\text{-mod},$$

generalizing (4.2).

4.9. One can give a version of Conjecture 4.7 along the lines of Conjecture 4.5. Let us instead of one point x have a collection x_1, \dots, x_n . We can consider the functors

$$\mathrm{Loc}_{ram} : \widehat{\mathfrak{g}}_{x_1}^c\text{-mod} \times \dots \times \widehat{\mathfrak{g}}_{x_n}^c\text{-mod} \rightarrow \mathfrak{D}^c(\mathrm{Bun}_{G,x_1,\dots,x_n})\text{-mod}$$

and

$$\mathrm{Poinc}_{ram} : \mathrm{Whit}^{-c}(G((t))_{x_1}) \times \dots \times \mathrm{Whit}^{-c}(G((t))_{x_n}) \rightarrow \mathfrak{D}^c(\mathrm{Bun}_{G,x_1,\dots,x_n})\text{-mod}.$$

Conjecture 4.10. *The 2-functor $\Psi_{G \rightarrow \check{G}}^{c, -\frac{1}{c}}$ takes the functor Loc_{ram} for G at level c to the functor Poinc_{ram} for \check{G} at level $-\frac{1}{c}$.*

5. LOCAL CORRESPONDENCE AT $c = 0/\infty$

Let us now consider the limiting cases, that correspond to the "classical", i.e., non-quantum, geometric Langlands correspondence. We remind that $c = 0$ means the critical level.

5.1. First, some comments are due as to how the corresponding objects look at $c = \infty$.

Let $\text{Conn}_G(\mathcal{D}^\times)$ denote the ind-scheme of G -connections over the formal punctured disc \mathcal{D}^\times . We have an action of $G((t))$ on $\text{Conn}(\mathcal{D}^\times)$ by gauge transformations.

By definition, a category \mathcal{C} acted on by $G((t))$ at level ∞ is a category over $\text{Conn}_G(\mathcal{D}^\times)$, equipped with a compatible *weak* action of $G((t))$.

The category $\widehat{\mathfrak{g}}^\infty\text{-mod}$ is by definition $\text{QCoh}(\text{Conn}_G(\mathcal{D}^\times))$.

The category KL_G^∞ is the category of quasi-coherent sheaves on $\text{Conn}_G(\mathcal{D}^\times)$ that are supported on subscheme $\text{Conn}_G^{\text{reg}}(\mathcal{D}^\times)$ of regular connections (=without poles) and that are equivariant with respect to $G[[t]]$. I.e., this is the category $\text{QCoh}(\text{Conn}_G^{\text{reg}}(\mathcal{D}^\times)/G[[t]])$. However, $\text{Conn}_G^{\text{reg}}(\mathcal{D}^\times)/G[[t]] \simeq \text{pt}/G$, so

$$\text{KL}_G^\infty \simeq \text{Rep}(G).$$

Similarly,

$$\widehat{\mathfrak{g}}^\infty\text{-mod}^I \simeq \text{QCoh}(\mathfrak{n}/B).$$

5.2. Conjecture 3.7 translates into the following:

Conjecture 5.3. *The assignment $\mathcal{C} \mapsto \mathcal{C}^{G((t)),w}$ establishes a 2-equivalence between the 2-category of categories over $\text{Conn}_G(\mathcal{D}^\times)$ equipped with a weak $G((t))$ action, and the 2-category of modules with respect to the chiral category $\text{Rep}(G)$.*

Let $\text{LocSys}_G(\mathcal{D}^\times)$ denote the quotient stack $\text{Conn}_G(\mathcal{D}^\times)/G((t))$.

Note that the assignment $\mathcal{C} \mapsto \mathcal{C}^{G((t)),w}$ can be alternatively interpreted as a bijection between the 2-category of categories over $\text{Conn}_G(\mathcal{D}^\times)$ equipped with a weak $G((t))$ action, and the 2-category of categories over the stack $\text{LocSys}_G(\mathcal{D}^\times)$.

Hence, Conjecture 5.3 implies:

Conjecture 5.4. *For a category \mathcal{C} the following two pieces of structure are equivalent: a structure of category over the stack $\text{LocSys}_G(\mathcal{D}^\times)$, and a structure of module category with respect to $\text{Rep}(G)$.*

5.5. The category $\mathcal{W}_G^\infty\text{-mod}$ identifies with $\text{QCoh}(\text{Op}_G(\mathcal{D}^\times))$, where $\text{Op}_G(\mathcal{D}^\times)$ is the ind-scheme of G -opers on \mathcal{D}^\times .

The category $\text{Whit}^\infty(\text{Gr}_G)$ identifies with $\text{QCoh}(\text{Op}_G^{\text{unr}}(\mathcal{D}^\times))$, where $\text{Op}_G^{\text{unr}}(\mathcal{D}^\times)$ is the ind-scheme of unramified local systems.

Similarly, the category $\text{Whit}^\infty(\text{Fl}_G)$ identifies with $\text{QCoh}(\text{Op}_G^{\text{nilp.ram.}}(\mathcal{D}^\times))$, where we denote by $\text{Op}_G^{\text{nilp.ram.}}(\mathcal{D}^\times)$ is the ind-scheme of opers that have a nilpotent ramification as local systems.

Remark. The ind-schemes $\text{Op}_G^{\text{unr}}(\mathcal{D}^\times)$ and $\text{Op}_G^{\text{nilp.ram.}}(\mathcal{D}^\times)$ should not be confused with their sub-schemes Op_G^{reg} and $\text{Op}_G^{\text{nilp}}$ that correspond to opers with a regular and nilpotent singularity, respectively.

Finally, the category $(\text{Whit} \times \text{Whit})(\mathcal{D}^{\infty,-\infty}(G((t)))\text{-mod})$ identifies with $\text{QCoh}(\text{Isom}_G(\mathcal{D}^\times))$, where $\text{Isom}_G(\mathcal{D}^\times)$ is the isomonodromy groupoid over $\text{Op}_G(\mathcal{D}^\times)$.

5.6. Let us now specialize some of the conjectures mentioned above to $c = 0$ and ∞ .

Conjecture 3.2 for $c = 0$ reads

$$\text{Whit}^0(\text{Gr}_G) \simeq \text{Rep}(\check{G}),$$

as chiral categories. This is a valid assertion.

Conjecture 3.2 for $c = \infty$ reads

$$\text{KL}_G^0 \simeq \text{QCoh}(\text{Op}_{\check{G}}^{\text{unr}}(\mathcal{D}^\times)).$$

This is also a theorem, established in [FG3].

Conjecture 3.11 for $c = 0$ reads as

$$\text{Whit}^0(\text{Fl}_G) \simeq \text{QCoh}(\check{\mathfrak{n}}/\check{B}),$$

which is the theorem of [AB].

Conjecture 3.11 for $c = \infty$ reads as

Conjecture 5.7.

$$\widehat{\mathfrak{g}}^0\text{-mod}^I \simeq \text{QCoh}(\text{Op}_{\check{G}}^{\text{nlp.ram.}}(\mathcal{D}^\times)).$$

The latter can be viewed as a generalization of the main conjecture from [FG2].

Conjecture 3.17 for $c = 0$ reads

Conjecture 5.8.

$$\widehat{\mathfrak{g}}^0\text{-mod} \otimes_{G((t))} \widehat{\mathfrak{g}}^0\text{-mod} \simeq \text{QCoh}(\text{Isom}_G(\mathcal{D}^\times)).$$

5.9. Consider now the 2-functor $\Psi_{G \rightarrow \check{G}}^{0,\infty}$. We will compose it with the 2-equivalence of the (plausible) Conjecture 5.3 and Conjecture 5.4, and thus consider the 2-functor, denoted $\Phi_{G \rightarrow \check{G}}$ from the 2-category of categories acted on by $G((t))$ at level 0 to that of categories over the stack $\text{LocSys}_G(\mathcal{D}^\times)$.

By construction, the 2-functor in question is

$$\mathcal{C} \mapsto \text{Whit}(\mathcal{C}),$$

when the latter is regarded as a module category with respect to the chiral category $\text{Rep}(\check{G})$.

Remark. At level 0, the 2-functor Whit is not a 2-equivalence for obvious reasons: it kills the category $\mathcal{C} = \text{Vect}$, equipped with the trivial action of $G((t))$. Thus, in order to have a Langlands-type equivalence in this case, one has to enhance the RHS, presumably by adding an Arthur SL_2 .

6. GLOBAL CORRESPONDENCE AT $c = 0/\infty$

6.1. The unramified case. First, we note that the equivalence (4.2) specializes at $c = 0$ to the usual geometric Langlands:

Conjecture 6.2. *There exists an equivalence*

$$\mathfrak{D}^0(\text{Bun}_G)\text{-mod} \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(X)).$$

The commutativity of the diagram in Conjecture 4.5 for $c = 0$ amounts to the Beilinson-Drinfeld construction of Hecke eigensheaves via opers.

When we exchange the roles of G and \check{G} and replace 0 by ∞ , the commutative diagram of Conjecture 4.5 amounts to the expectation that the equivalence of (6.2) takes the "Whittaker coefficient" D-modules on the LHS to the tautological coherent sheaves associated to points of X and representations of \check{G} on the RHS.

6.3. The ramified case. Consider the 2-functor $\Phi_{G \rightarrow \check{G}}$ applied to the category $\mathfrak{D}^0(\text{Bun}_{G,x})$. This is the category $\text{Whit}(\mathfrak{D}^0(\text{Bun}_{G,x}))$ over $\text{LocSys}_G(\mathcal{D}^\times)$. By Conjecture 4.7 this category is equivalent to

$$\text{QCoh}(\text{LocSys}_{\check{G},x}(X))^{\check{G}((t)),w}.$$

Note that the stack $\text{LocSys}_{\check{G},x}(X)$ classifies \check{G} -local systems on X with an arbitrary ramification at x , and a full level structure at x on the underlying G -bundle. Hence, the stack $\text{LocSys}_{\check{G},x}(X)/\check{G}((t))$ identifies with the stack $\text{LocSys}_{\check{G}}(X-x)$ of \check{G} -local systems defined on the punctured curve. The category

$$\text{QCoh}(\text{LocSys}_{\check{G},x}(X))^{\check{G}((t)),w} \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(X-x))$$

is naturally a category over $\text{LocSys}_G(\mathcal{D}^\times)$ via the map of stacks

$$\text{LocSys}_{\check{G}}(X-x) \rightarrow \text{LocSys}_G(\mathcal{D}^\times).$$

Summarizing, we obtain:

Conjecture 6.4. *The category $\text{Whit}(\mathfrak{D}^0(\text{Bun}_{G,x}))$ is equivalent to $\text{QCoh}(\text{LocSys}_{\check{G}}(X-x))$, as categories over $\text{LocSys}_G(\mathcal{D}^\times)$.*

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