NOTES ON GEMETRIC LANGLANDS: QUASI-COHERENT SHEAVES ON STACKS

DENNIS GAITSGORY

Like [GL:Stacks], this paper isn't really a paper either. I will try to record the basic facts and definitions regarding quasi-coherent sheaves on stacks in the DG setting. The vast majority of the material here is what I've learned from Jacob Lurie.

The conventions regarding stacks in the DG setting follow [GL:Stacks]. The conventions regarding DG categories follow [GL:DG].

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1. Quasi-coherent sheaves

1.1. **Definition.**

1.1.1. In [GL:Stacks] we considered the notion of *prestack*, which was a functor

$$\mathrm{DGSch}^{\mathrm{aff}} \to \infty\text{-}\mathrm{Grpd};$$

the category of prestacks was denoted PreStk.

However, this notion makes sense for functors out of $\mathrm{DGSch}^{\mathrm{aff}}$ with values in any $(\infty, 1)$-category as a target. Among the targets that we will consider in this paper are ∞-Cat, DGCat and $\mathrm{DGCat}^{\mathrm{SymMon}}$, all considered as $(\infty, 1)$-categories.

For a target $\mathbf{C}$, we shall denote the resulting category of prestacks by $\mathrm{PreStk}_{\mathbf{C}}$.

1.1.2. For an affine DG scheme S we consider the symmetric monoidal DG category $\mathrm{QCoh}(S)$. The assignment $S \mapsto \mathrm{QCoh}(S)$ is naturally an object

$$\mathrm{QCoh}(-) \in \mathrm{PreStk}_{\mathrm{DGCat}^{\mathrm{SymMon}}}\,. \tag{1}$$

1.1.3. For $\mathcal{Y} \in \mathrm{PreStk}$, consider the ∞-category, denoted $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$, consisting of pairs

$$(S, y_S \in \mathcal{Y}(S))$$

for $S \in \mathrm{DGSch}^{\mathrm{aff}}$. The assignment

$$(S, y_S \in \mathcal{Y}(S)) \mapsto \mathrm{QCoh}(S)$$

defines a functor $(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op} \to \mathrm{DGCat}^{\mathrm{SymMon}}$.

By definition, the symmetric monoidal DG category $\mathrm{QCoh}(\mathcal{Y})$ is

$$\underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op}}{lim} \mathrm{QCoh}(S),$$

where the limit is taken in $\mathrm{DGCat}^{\mathrm{SymMon}}$.

1.1.4. Explicitly, this means that an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is an assignment

$$\Big(S \in \mathrm{DGSch}^{\mathrm{aff}}, y_S \in \mathcal{Y}(S)\Big) \mapsto \mathcal{F}_{y_S} \in \mathrm{QCoh}(S),$$

compatible under pull-backs.

1.1.5. The above definition can be rephrased as follows: we have a functor

$$\mathrm{PreStk}^{op} \to \mathrm{DGCat}^{\mathrm{SymMon}} : \mathcal{Y} \mapsto \mathrm{QCoh}(\mathcal{Y})$$

given by the right Kan extension of the functor (1) along the tautological embedding

$$\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}\,.$$

1.1.6. For a prestack $\mathcal{Y}$, the DG category underlying $\mathrm{QCoh}(\mathcal{Y})$ can be also recovered as follows:

$$\mathrm{QCoh}(\mathcal{Y}) = \mathrm{Hom}_{\mathrm{PreStk}_{\infty\text{-}\mathrm{Cat}}}(\mathcal{Y}, \mathrm{QCoh}(-)),$$

where $\mathrm{QCoh}(-)$ is viewed as an object of $\mathrm{PreStk}_{\mathrm{DGCat}}$ via the forgetful functor

$$\mathrm{PreStk}_{\mathrm{DGCat}^{\mathrm{SymMon}}} \to \mathrm{PreStk}_{\mathrm{DGCat}}\,.$$

1.2. **Basic properties.**

1.2.1. *Non-convergence.* We begin with a negative observation. Recall the notion of *convergence* for a prestack, see [GL:Stacks], Sect. 1.2.

We observe that the prestack $\mathrm{QCoh}(-)$ is *not* convergent.

Remark 1.2.2. As we shall see in Sect. 4.1.3, the prestack $\mathrm{QCoh}(-)$ contains a canonical full sub-prestack $\mathrm{QCoh}(-)^{\mathrm{perf}}$, which is convergent.

1.2.3. *t-structure.* The category $\mathrm{QCoh}(\mathcal{Y})$ carries a canonical t-structure: an object $\mathcal{F}$ is declared to belong to $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ if for any $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and $y_S : S \to \mathcal{Y}$, the corresponding $\mathcal{F}_{y_S}$ belongs to $\mathrm{QCoh}(S)^{\leq 0}$.

1.2.4. *Quasi-coherent sheaves and n-coconnective prestacks.* Assume that $\mathcal{Y}$ is n-coconnective (see [GL:Stacks], Sect. 1.1.5), i.e., as a functor $(\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-Grpd}$, $\mathcal{Y}$ is a left Kan extension along the embedding

$$^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}.$$

From Sect. 1.1.6, we obtain:

Lemma 1.2.5. *Under the above circumstances, the natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{(S,y_S)\in({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op}}{lim} \mathrm{QCoh}(S)$$

is an equivalence.

I.e., for $\mathcal{Y}$ being n-coconnective, in the definition of quasi-coherent sheaves, it is enough to consider only those affine DG schemes mapping to $\mathcal{Y}$ that are themselves n-coconnective.

In particular, if $\mathcal{Y}$ is a classical prestack, it is sufficient to consider only classical affine schemes mapping to $\mathcal{Y}$.

1.2.6. *Quasi-coherent sheaves on stacks locally of finite type.*

Let $\mathcal{Y} \in \mathrm{PreStk}$ be n-coconnective as above, and assume, moreover, that it is locally of finite type (see [GL:Stacks], Sect. 1.3.2). I.e., $\mathcal{Y}|_{^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ is the left Kan extension along the embedding

$$^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}.$$

Lemma 1.2.7. *Under the above circumstances, the natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{(S,y_S)\in({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}/\mathcal{Y}})^{op}}{lim} \mathrm{QCoh}(S)$$

is an equivalence.

I.e., for n-coconnective prestacks locally of finite type, in the definition of quasi-coherent sheaves, it is enough to consider only those affine DG schemes mapping to $\mathcal{Y}$ that are themselves n-coconnective *and* are of finite type.

1.3. **Descent.**

1.3.1. The notion of stack, i.e., a prestack satisfying descent, was introduced in [GL:Stacks] for the category PreStk. Again, this notion makes sense for an arbitrary target category $\mathbf{C}$. We shall denote the resulting full subcategory of $\mathrm{PreStk}_{\mathbf{C}}$ by $\mathrm{Stk}_{\mathbf{C}}$.

Note, however, that the condition of descent for an object of $\mathrm{PreStk}_{\mathbf{C}}$ can be reduced to that in PreStk. Namely, for $\mathcal{F} \in \mathrm{PreStk}_{\mathbf{C}}$ and $\mathbf{c} \in \mathbf{C}$ we can consider $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}, \mathcal{F})$ as an object of PreStk, and by definition $\mathcal{F}$ satisfies descent as an object of $\mathrm{PreStk}_{\mathbf{C}}$ if and only if $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}, \mathcal{F})$ does as an object of PreStk for every $\mathbf{c} \in \mathbf{C}$.

1.3.2. In [GL:Stacks] the definition of prestack was given as a functor

$$\mathrm{DGSch}^{\mathrm{aff}} \to \infty\text{-Grpd}\,.$$

1.3.3. The following assertion is a version of Grothendieck's faithfully flat descent:

Theorem 1.3.4. *The prestack* $\mathrm{QCoh}(-) : S \mapsto \mathrm{QCoh}(S)$*, viewed as a presheaf of DG categories on* $\mathrm{DGSch}^{\mathrm{aff}}$*, is a stack, i.e., it satisfies faithfully flat descent descent.*

Since the forgetful functor $\mathrm{DGCat}^{\mathrm{SymMon}} \to \mathrm{DGCat}$ is conservative and commutes with limits, we obtain that $\mathrm{QCoh}(-)$ satisfies descent also as an object of $\mathrm{PreStk}_{\mathrm{DGCat}^{\mathrm{SymMon}}}$.

1.3.5. Since any fpppf morphism (see [GL:Stacks], Sect. 2.1.2) is by definition faithfully flat, from the above theorem and Sect. 1.1.6 we obtain:

Corollary 1.3.6. *Let* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *be a map in* PreStk*, which is an fpppf equivalence. Then it induces an equivalence:*

$$\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1).$$

From this, we obtain, tautologically:

Corollary 1.3.7. *For* $\mathcal{Y} \in \mathrm{PreStk}$*, the canonical map* $\mathcal{Y} \to L(\mathcal{Y})$ *induces an equivalence:*

$$\mathrm{QCoh}(L(\mathcal{Y})) \to \mathrm{QCoh}(\mathcal{Y}).$$

1.3.8. The last corollary has a two-fold significance:

First, to specify a stack we may often have to start from a prestack given explicitly, and then apply the functor L. Corollary 1.3.7 implies that in order to calculate the category QCoh of the resulting stack we can work with the initial prestack.

Secondly, we obtain that for the purposes of QCoh, we will lose no information if we work with the subcategory Stk rather than all of PreStk.

1.3.9. From [GL:Stacks], Corollary 2.3.3, we obtain:

Corollary 1.3.10. *Let* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *be an fpppf surjection. Then the natural map*

$$\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{Tot}\left(\mathrm{QCoh}(\mathcal{Y}_1^\bullet/\mathcal{Y}_2)\right)$$

is an equivalence.

1.3.11. *Quasi-coherent sheaves on n-coconnective stacks.*

Recall the notion of n-coconnective stack, see [GL:Stacks], Sect. 2.4.7. (Note that if $\mathcal{Y}$ is n-coconnective as a stack, then this does *not* mean that its is n-coconnective as a prestack.)

However, combining Corollary 1.3.7 and Lemma 1.2.5, we obtain:

Corollary 1.3.12. *Let* $\mathcal{Y}$ *be an* n*-coconnective stack. Then the natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{(S,y_S)\in({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op}}{lim} \mathrm{QCoh}(S)$$

is an equivalence.

1.4. **Behavior with respect to products.** Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be prestacks. Consider their product $\mathcal{Y}_1 \times \mathcal{Y}_2$.

1.4.1. We claim that there is a canonical morphism in $\mathrm{DGCat}^{\mathrm{SymMon}}$:

$$\mathrm{QCoh}(\mathcal{Y}_1)\otimes \mathrm{QCoh}(\mathcal{Y}_2)\to \mathrm{QCoh}(\mathcal{Y}_1\times\mathcal{Y}_2). \tag{2}$$

Defining such map amounts to specifying for each system of assignments

$$(S\in \mathrm{DGSch}^{\mathrm{aff}}, y_{1,S}\in \mathcal{Y}_1(S))\mapsto \mathcal{F}_{y_{1,S}}\in \mathrm{QCoh}(S),$$
$$(S\in \mathrm{DGSch}^{\mathrm{aff}}, y_{2,S}\in \mathcal{Y}_2(S))\mapsto \mathcal{F}_{y_{2,S}}\in \mathrm{QCoh}(S)$$

an assignment

$$(S\in \mathrm{DGSch}^{\mathrm{aff}}, y_{1,S}\in \mathcal{Y}_1(S), y_{2,S}\in \mathcal{Y}_2(S))\mapsto \mathcal{F}_{y_{12,S}}\in \mathrm{QCoh}(S).$$

We set $\mathcal{F}_{y_{12,S}}:=\mathcal{F}_{y_{1,S}}\underset{\mathcal{O}_S}{\otimes}\mathcal{F}_{y_{2,S}}$.

1.4.2. We can rephrase the above functor as follows. We have:

$$\mathrm{QCoh}(\mathcal{Y}_1)\otimes \mathrm{QCoh}(\mathcal{Y}_2)\simeq \tag{3}$$
$$\simeq\left(\underset{(S_1,y_{S_1})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S_1)\right)\otimes\left(\underset{(S_2,y_{S_2})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S_2)\right)\to$$
$$\to \underset{(S_1,y_{S_1})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1})^{op},(S_2,y_{S_2})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S_1)\otimes\mathrm{QCoh}(S_2).$$

However, the category $(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1})^{op}\times(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}$ is cofinal in $(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1\times\mathcal{Y}_2})^{op}$, so the last limit is isomorphic to

$$\underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1\times\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S)\simeq\mathrm{QCoh}(\mathcal{Y}_1\times\mathcal{Y}_2).$$

1.4.3. The following assertion (plagiarized from J. Lurie) will be useful in the sequel:

Proposition 1.4.4. *Assume that $\mathcal{Y}_1$ is such that the category $\mathrm{QCoh}(\mathcal{Y}_1)$, viewed as an object of DGCat, is dualizable. Then for any $\mathcal{Y}_2$, the functor (2) is an equivalence.*

Proof. We need to show that the second arrow in (3) is an isomorphism. We can write it as a composition

$$\mathrm{QCoh}(\mathcal{Y}_1)\otimes\left(\underset{(S_2,y_{S_2})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S_2)\right)\to$$
$$\underset{(S_2,y_{S_2})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}(\mathrm{QCoh}(\mathcal{Y}_1)\otimes\mathrm{QCoh}(S_2))\simeq$$
$$\simeq\underset{(S_2,y_{S_2})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}\left(\left(\underset{(S_1,y_{S_1})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S_1)\right)\otimes\mathrm{QCoh}(S_2)\right)\to$$
$$\to\underset{(S_2,y_{S_2})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2})^{op}}{\underset{\longleftarrow}{lim}}\left(\underset{(S_1,y_{S_1})\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_1})^{op}}{\underset{\longleftarrow}{lim}}\mathrm{QCoh}(S_1)\otimes\mathrm{QCoh}(S_2)\right).$$

The first arrow is an isomorphism since tensoring with a dualizable category commutes with limits (see [GL:DG], Lemma 2.1.6). The third arrow is an isomorphism for the same reason, as $\mathrm{QCoh}(S)$ for an affine scheme S is dualizable.

$\square$

1.5. **Inverse and direct images.**

1.5.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk. Interpreting QCoh as in Sect. 1.1.5, we obtain a canonical functor

$$f^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1).$$

Explicitly, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_2)$, the object $f^*(\mathcal{F}) \in \mathrm{QCoh}(\mathcal{Y}_2)$ is described as follows: For $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and $y_{1,S} : S \to \mathcal{Y}_1$, let $y_{2,S} := f \circ y_{1,S}$, and we set

$$(f^*(\mathcal{F}))_{y_{1,S}} := \mathcal{F}_{y_{2,S}}.$$

We shall denote the resulting functor

$$\mathrm{PreStk}^{op} \to \mathrm{DGCat}_{\mathrm{cont}}$$

by $\mathrm{QCoh}^*_{\mathrm{PreStk}}$.

1.5.2. By the adjoint functor theorem ([Lu0], Cor. 5.5.2.9), the functor f^* admits a right adjoint, that we denote f_*. Thus, passing to right adjoints, we obtain another functor

$$\mathrm{QCoh}_{\mathrm{PreStk},*} : \mathrm{PreStk} \to \mathrm{DGCat}\,.$$

We emphasize that the target of the above functor is DGCat, and not $\mathrm{DGCat}_{\mathrm{cont}}$: indeed the functor f_* is *not* in general continuous (i.e., does not commute with colimits).

We should also add that the functor f_* does not satisfy the base change formula (even for open embeddings) and does not have the base change property.

2. Direct images under "nice" morphisms

2.1. **Schematic morphisms.**

The next assertion is thoroughly plagiarized; we don't refer to any particular source since it is not entirely clear which one is the original.

Proposition 2.1.1. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. Assume that f is schematic, quasi-compact and quasi-separated.*

(1) *The functor f_* is continuous.*

(2) *The base change property is satisfied:*

$$\begin{CD} \mathcal{Y}'_1 @>{g'}>> \mathcal{Y}_1 \\ @V{f'}VV @VV{f}V \\ \mathcal{Y}'_2 @>{g}>> \mathcal{Y}_2 \end{CD}$$

be a Cartesian square in PreStk. *Then the natural transformation*

$$g^* \circ f_* \to f'_* \circ g'^*$$

is an isomorphism.

(3) *Projection formula holds: for $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)$ and $\mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y}_2)$, the natural map*

$$\mathcal{F}_2 \otimes f_*(\mathcal{F}_1) \to f_*(f^*(\mathcal{F}_2) \otimes \mathcal{F}_1)$$

is an isomorphism.

Remark 2.1.2. In Sect. 2.2 we will generalize this statement to the case when f is no longer required to be schematic, but just representable (but still quasi-compact and quasi-separated).

Proof. Define *a* functor $f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$ as follows. We need to assign to a compatible collection

$$\mathcal{F}_1 := \Big(\Big(S_1 \in \mathrm{DGSch}^{\mathrm{aff}}, y_{S_1} \in \mathcal{Y}_1(S_1)\Big) \mapsto \mathcal{F}_{y_{S_1}} \in \mathrm{QCoh}(S_1)\Big). \tag{4}$$

a compatible collection

$$\mathcal{F}_2 := \Big(\Big(S_2 \in \mathrm{DGSch}^{\mathrm{aff}}, y_{S_2} \in \mathcal{Y}_1(S_2)\Big) \mapsto \mathcal{F}_{y_{S_2}} \in \mathrm{QCoh}(S_2)\Big). \tag{5}$$

For (S_2, y_{S_2}) set

$$X_1 := S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1.$$

The assumption on f implies that X_1 is a quasi-compact and quasi-separated DG scheme.

Pullback under $X_1 \to \mathcal{Y}_1$ gives rise to an object $\mathcal{F}_{X_1} \in \mathrm{QCoh}(X_1)$. We set $\mathcal{F}_{y_{S_2}} \in \mathrm{QCoh}(S_2)$ to be the direct image of $\mathcal{F}_{X_1}$ under $f_{y_{S_2}} : X_1 \to S_2$. Since $f_{y_{S_2}}$ is quasi-compact and quasi-separated scheme, the assignment

$$\mathcal{F}_1 \mapsto \mathcal{F}_{y_{S_2}}$$

commutes with colimits. For the same reason, it commutes with any further base change $S_2' \to S_2$. Hence, the collection

$$(S_2, y_{S_2}) \mapsto \mathcal{F}_{y_{S_2}}$$

is compatible, i.e., gives rise to a well-defined object as in (5).

Moreover, it is easy to see that the functor f_* thus constructed is a right adjoint to f^*. This establishes point (1) of the Proposition. Points (2) and (3) follow formally by base change $S_2 \to \mathcal{Y}_2'$.

□

2.1.3. Let us denote by $\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}$ the non-full subcategory of PreStk where we restrict 1-morphisms to be schematic, quasi-compact and quasi-separated.

Consider the functor

$$\mathrm{QCoh}_{\mathrm{PreStk},*}|_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}} : \mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}} \to \mathrm{DGCat},$$

where $\mathrm{QCoh}_{\mathrm{PreStk},*}$ is an in Sect. 1.5.2.

Proposition 2.1.1 implies that the above functor canonically factors through a functor

$$\mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}},*} : \mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

2.2. **Representable morphisms.**

2.2.1. We are now going to show how to generalize Proposition 2.1.1 to allow the morphism f to be representable. Tracing through the proof, all we need to establish is the following fact:

Proposition 2.2.2. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic space. Then the functor*

$$\Gamma(\mathcal{X}, -) : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Vect}$$

is continuous, i.e., the object $\mathcal{O}_{\mathcal{X}} \in \mathrm{QCoh}(\mathcal{X})$ is compact.

2.2.3. The proof of Proposition 2.2.2 that we are going to present (due to Drinfeld[1]) will use the notion of Nisnevich cover of an algebraic space.

Namely, let $\mathcal{X}$ be a quasi-compact and quasi-separated (but not necessarily locally Noetherian) algebraic space. We claim:

Lemma 2.2.4. *There exists an affine DG scheme Z equipped with an étale map $f : Z \to \mathcal{X}$ and a finite sequence of closed algebraic subspaces*

$$\mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \mathcal{X}_2 \supset ... \supset \mathcal{X}_n \supset \emptyset$$

such that the ideal of each $\mathcal{X}_i$ in $\mathcal{X}$ is finitely generated, and such that for for every i, the morphism

$$Z \underset{\mathcal{X}}{\times} (\mathcal{X}_i - \mathcal{X}_{i+1}) \to (\mathcal{X}_i - \mathcal{X}_{i+1})$$

admits a section.

2.2.5. Another ingredient in the proof of Proposition 2.2.2 will be the following observation. Let $\mathcal{Y}$ be a prestack. Let $i : \mathcal{V} \hookrightarrow \mathcal{Y}$ be a schematic closed embedding of prestacks, and let $j : \mathcal{U} \hookrightarrow \mathcal{Y}$ be the complementary open.

Let $f : \mathcal{Y}' \to \mathcal{Y}$ be a schematic étale map, and let us assume that

$$\mathcal{Y}' \underset{\mathcal{Y}}{\times} \mathcal{V} =: \mathcal{V}' \overset{f_{\mathcal{V}}}{\to} \mathcal{V}$$

is an isomorphism. Denote $\mathcal{U}' := \mathcal{Y}' \underset{\mathcal{Y}}{\times} \mathcal{U}$, and its map to $\mathcal{U}$ by $f_{\mathcal{U}}$.

Lemma 2.2.6. *Assume that the maps f and j are quasi-compact. Then the diagram*

$$\begin{array}{ccc} \text{QCoh}(\mathcal{Y}) & \xrightarrow{f^*} & \text{QCoh}(\mathcal{Y}') \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j'^*} \\ \text{QCoh}(\mathcal{U}) & \xrightarrow{f_{\mathcal{U}}^*} & \text{QCoh}(\mathcal{U}'). \end{array} \tag{6}$$

is a pull-back square of DG categories.

Proof. The pull-back category in (6) is by definition the category of quintuples:

$$(\mathcal{F}_{\mathcal{U}} \in \text{QCoh}(\mathcal{U}), \mathcal{F}_{\mathcal{Y}'} \in \text{QCoh}(\mathcal{Y}'), \mathcal{F}_{\mathcal{U}'}, \alpha : j'^*(\mathcal{F}_{\mathcal{Y}'}) \simeq \mathcal{F}_{\mathcal{U}'}, \beta : f_{\mathcal{U}}^*(\mathcal{F}_{\mathcal{U}}) \simeq \mathcal{F}_{\mathcal{U}'}).$$

We have a natural functor $\text{QCoh}(\mathcal{Y}) \to \text{QCoh}(\mathcal{Y}') \underset{\text{QCoh}(\mathcal{U}')}{\times} \text{QCoh}(\mathcal{U})$ given by restriction, and its right adjoint given by

$$(\mathcal{F}_{\mathcal{U}}, \mathcal{F}_{\mathcal{Y}'}, \mathcal{F}_{\mathcal{U}'}, \alpha, \beta) \mapsto \text{Cone}\left(f_*(\mathcal{F}_{\mathcal{Y}'}) \oplus j_*(\mathcal{F}_{\mathcal{U}}) \to (j \circ f_{\mathcal{U}})_*(\mathcal{F}_{\mathcal{U}'})\right).$$

Let us show that the unit of the adjunction is an isomorphism. Note that the morphisms j and f are automatically quasi-separated, so by Proposition 2.1.1(2), the base-change property holds. This implies that the unit of the adjunction induces an isomorphism between the composition

$$\text{QCoh}(\mathcal{U}) \overset{j_*}{\longrightarrow} \text{QCoh}(\mathcal{Y}) \to \text{QCoh}(\mathcal{Y}') \underset{\text{QCoh}(\mathcal{U}')}{\times} \text{QCoh}(\mathcal{U}) \to \text{QCoh}(\mathcal{Y}) \tag{7}$$

and the functor $j_* : \text{QCoh}(\mathcal{U}) \to \text{QCoh}(\mathcal{Y})$.

[1]This fact was independently established by Lurie.

Let $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}}$ (resp., $\mathrm{QCoh}(\mathcal{Y}')_{\mathcal{V}'}$) denote the full subcategory $\ker(j^*)$ (resp., $\ker(j'^*)$). By (7), in order to prove that the unit of the adjunction is an isomorphism, it suffices to show that the unit of the adjunction induces an isomorphism between

$$\mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}} \hookrightarrow \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{U}')}{\times} \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\mathcal{Y}) \tag{8}$$

and the tautological embedding $\mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}} \hookrightarrow \mathrm{QCoh}(\mathcal{Y})$.

We observe that the functor in (8) is canonically isomorphic to $f_* \circ f^*|_{\mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}}}$, such that the unit of the adjunction comes from the unit of the adjunction $\mathrm{Id} \to f_* \circ f^*$. However, the assumption of the lemma implies that the functors (f^*, f_*) define mutually inverse equivalences:

$$f^* : \mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}} \rightleftarrows \mathrm{QCoh}(\mathcal{Y}')_{\mathcal{V}'} : f_*,$$

implying that $\mathrm{Id}\,|_{\mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}}} \to f_* \circ f^*|_{\mathrm{QCoh}(\mathcal{Y})_{\mathcal{V}}}$ is an isomorphism.

It remains to show that the right adjoint functor is conservative. By the base change formula, we obtain that the composition

$$\mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{U}')}{\times} \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\mathcal{Y}) \to$$
$$\to \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{U}')}{\times} \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\mathcal{U})$$

is canonically isomorphic to the forgetful functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{U}')}{\times} \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\mathcal{U}).$$

Hence, it remains to show that the right adjoint is conservative on the full subcategory

$$\ker\left(\mathrm{QCoh}(\mathcal{Y}') \underset{\mathrm{QCoh}(\mathcal{U}')}{\times} \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\mathcal{U})\right) \simeq \mathrm{QCoh}(\mathcal{Y}')_{\mathcal{V}'}.$$

The restriction of the right adjoint to this category is canonically isomorphic to f_*, and the assertion follows from the fact that the latter is conservative.

□

2.2.7. *Proof of Proposition 2.2.2* (Drinfeld). For a quasi-compact and quasi-separated algebraic stack $\mathcal{X}$, choose a Nisnevich cover $f : Z \to \mathcal{X}$ given by Lemma 2.2.4. Set $\mathcal{X}^k := \mathcal{X} - \mathcal{X}_{k+1}$. We will argue by induction on k.

For $k = 0$, we obtain that the map

$$Z \underset{\mathcal{X}}{\times} \mathcal{X}^0 := Z^0 \to \mathcal{X}^0$$

admits a section. Hence $\mathcal{X}^0$ maps isomorphically to a locally closed DG subscheme of Z^0. So, $\mathcal{X}^0$ is a quasi-compact and quasi-separated (and, in fact, quasi-affine) DG scheme, and both assertions of the proposition are known in this case.

To execute the induction step, we can assume being in the situation of Lemma 2.2.6, where the assertion of the proposition holds for $\mathrm{QCoh}(\mathcal{Y}')$, $\mathrm{QCoh}(\mathcal{U})$ and $\mathrm{QCoh}(\mathcal{U}')$. The proposition follows now from (6) and the next lemma:

Lemma 2.2.8. *Let $i \mapsto \mathbf{C}_i$ be a finite diagram of DG categories, and let $\mathbf{C} = \underset{i}{\lim}\, \mathbf{C}_i$. Let $\mathbf{c} \in \mathbf{C}$ be an object such that its image in each $\mathbf{C}_i$ is compact. Then $\mathbf{c}$ is compact.*

2.3. Behavior with respect to products, revisited.

2.3.1. The following Proposition provides a partial converse to Proposition 1.4.4.

Recall the notion of a rigid monoidal category, [GL:DG], Sect. 6.

Proposition 2.3.2. *Let $\mathcal{Y}$ be a prestack, such that the diagonal map $\Delta_\mathcal{Y}$ is representable, quasi-compact and quasi-separated, and such that the object $\mathcal{O}_\mathcal{Y} \in \mathrm{QCoh}(\mathcal{Y})$ is compact. Then the following conditions are equivalent:*

- *(i) The functor $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}')$ is an equivalence for any $\mathcal{Y}'$.*
- *(ii) The functor $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y})$ is an equivalence.*
- *(iii) The category $\mathcal{Y}$ is rigid as a monoidal category.*
- *(iv) The category $\mathrm{QCoh}(\mathcal{Y})$ is dualizable.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are tautological. The implication (iv) $\Rightarrow$ (i) is the content of Proposition 1.4.4. It remains to show (ii) $\Rightarrow$ (iii).

Given (ii), we can identify the map $mult^*_{\mathrm{QCoh}(\mathcal{Y})}$ (we are using the notation of [GL:DG], Sect. 6.1) with

$$\Delta^*_\mathcal{Y} : \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$

The fact that it satisfies the assumptions of *loc. cit.* follows from Proposition 2.1.1 combined with Remark 2.1.2.

□

2.3.3. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between prestacks, such that both $\mathrm{QCoh}(\mathcal{Y}_1)$ and $\mathrm{QCoh}(\mathcal{Y}_2)$ are rigid. From [GL:DG], Lemma 6.2.1(2) we obtain:

Lemma 2.3.4. *The functor $f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$ is continuous, and under the identifications*

$$\mathrm{QCoh}(\mathcal{Y}_i)^\vee \simeq \mathrm{QCoh}(\mathcal{Y}_i),$$

we have $f_* \simeq (f^*)^\vee$.

2.3.5. *The case of algebraic spaces.* Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic space. By Proposition 2.2.2, $\mathcal{O}_\mathcal{X} \in \mathrm{QCoh}(\mathcal{X})$ is compact.

Proposition 2.3.6. *Any quasi-compact and quasi-separated algebraic space satisfies the equivalent conditions of Proposition 2.3.2.*

2.3.7. *Proof of Proposition 2.3.6.* By Proposition 2.3.2, we need to show that $\mathrm{QCoh}(\mathcal{X})$ is dualizable. The proof follows the same strategy as that of Proposition 2.2.2. Thus, we need to show the following:

Let us be in the situation of Lemma 2.2.6, where the categories $\mathrm{QCoh}(\mathcal{Y}')$, $\mathrm{QCoh}(\mathcal{U})$ and $\mathrm{QCoh}(\mathcal{U}')$ are rigid. We claim that in this case $\mathrm{QCoh}(\mathcal{Y})$ is dualizable.

Note that the proof of Lemma 2.2.6 shows that for any $\mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}$, the diagram

$$\begin{array}{ccc} \mathbf{C} \otimes \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{f^*} & \mathbf{C} \otimes \mathrm{QCoh}(\mathcal{Y}') \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j'^*} \\ \mathbf{C} \otimes \mathrm{QCoh}(\mathcal{U}) & \xrightarrow{f^*_\mathcal{U}} & \mathbf{C} \otimes \mathrm{QCoh}(\mathcal{U}') \end{array}$$

is a pull-back square. The assertion of the proposition follows now from the following general lemma:

Lemma 2.3.8. *Let* $(i \in I) \mapsto \mathbf{C}_i$ *be a diagram in* $\mathrm{DGCat}_{\mathrm{cont}}$*, and let* $\mathbf{C} := \underset{I}{lim}\, \mathbf{C}_i$*, where each* $\mathbf{C}_i$ *is dualizable. Suppose that for any* $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$*, the functor*

$$\mathbf{C} \otimes \mathbf{D} \to \underset{I}{lim}\, \mathbf{C}_i \otimes \mathbf{D}$$

is an equivalence. Then $\mathbf{C}$ *is dualizable.*

Proof. Consider the diagram $(i \in I^{op}) \mapsto \mathbf{C}_i^\vee$ and set $\mathbf{C}' := \underset{I^{op}}{colim}\, \mathbf{C}_i^\vee$. We have $\mathbf{C} \simeq \mathrm{Hom}(\mathbf{C}', \mathrm{Vect})$, and for any $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$ we obtain a functor

$$\mathbf{C} \otimes \mathbf{D} \to \mathrm{Hom}(\mathbf{C}', \mathbf{D}).$$

However, we claim that the latter functor is an equivalence: indeed, it can be rewritten as the composition

$$\mathbf{C} \otimes \mathbf{D} \to \underset{I}{lim}\, \mathbf{C}_i \otimes \mathbf{D} \simeq \underset{I}{lim}\, \mathrm{Hom}(\mathbf{C}_i^\vee, \mathbf{D}) \simeq \mathrm{Hom}(\mathbf{C}', \mathbf{D}),$$

which is an equivalence, by assumption.

This shows that $\mathbf{C}'$ is dualizable and that $\mathbf{C}$ is its dual. Hence, $\mathbf{C}$ is dualizable as well.

□

□(Proposition)

2.3.9. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between prestacks, which is quasi-compact and quasi-affine.

Consider $f_*(\mathcal{O}_{\mathcal{Y}_1})$ as an algebra in the monoidal category $\mathrm{QCoh}(\mathcal{Y}_2)$.

The next assertion is plagiarized from J. Lurie:

Proposition 2.3.10. *Under these circumstances, the natural functor*

$$f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to f_*(\mathcal{O}_{\mathcal{Y}_1})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2))$$

is an equivalence.

Proof. Consider the functor

$$f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2).$$

By Proposition 2.1.1(1) it commutes with colimits, and admits a left adjoint. Moreover, the quasi-affiness hypothesis implies that it is conservative. Hence, we find ourselves in the (easy case of) the situation of the Barr-Beck-Lurie theorem, [GL:DG], Sect. 3.1.2. We obtain that f_* upgrades to an equivalence between $\mathrm{QCoh}(\mathcal{Y}_1)$ and the category of modules in $\mathrm{QCoh}(\mathcal{Y}_2)$ with respect to the monad $f_* \circ f^*$. However, from Proposition 2.1.1(2), it is easy to see that the monad in question identifies with

$$\mathcal{F} \mapsto f_*(\mathcal{O}_{\mathcal{Y}_1}) \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \mathcal{F}.$$

□

3. Tensoring over QCoh

3.1. **The setting.**

3.1.1. Let

$$\begin{array}{ccc} \mathcal{Y}'_1 & \longrightarrow & \mathcal{Y}_1 \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \mathcal{Y}'_2 & \xrightarrow{g} & \mathcal{Y}_2 \end{array}$$

be a Cartesian square of prestacks.

Note that we have a canonical map:

$$\text{QCoh}(\mathcal{Y}_1) \underset{\text{QCoh}(\mathcal{Y}_2)}{\otimes} \text{QCoh}(\mathcal{Y}'_2) \to \text{QCoh}(\mathcal{Y}'_1). \tag{9}$$

The general question we would like to ask is when the map in (9) is an equivalence.

3.1.2. The map (9) is not an isomorphism for general prestacks. For example, let us take $\mathcal{Y}_1 = \mathcal{Y}_2 = \text{pt}$, and $\mathcal{Y} = \text{pt}/A$, where A is an abelian variety. Then $\mathcal{Y}_1 \underset{\mathcal{Y}}{\times} \mathcal{Y}_2 \simeq A$, while

$$\text{QCoh}(\text{pt}/A) \simeq H\text{-mod},$$

where $H = (\Gamma(A, \mathcal{O}_A))^\vee$ is an algebra with respect to convolution, and is isomorphic to $\text{Sym}(H^1(X, \mathcal{O}_A)^\vee[1])$. So

$$\text{Vect} \underset{H\text{-mod}}{\otimes} \text{Vect} \simeq \text{Sym}(H^1(X, \mathcal{O}_A)^\vee[2])\text{-mod}.$$

3.2. **Quasi-affine case.** We are going to show:

Proposition 3.2.1. *Assume that the map f (and hence f') is quasi-compact and quasi-affine. Then then map* (9) *is an equivalence.*

Proof. By Proposition 2.3.10

$$\text{QCoh}(\mathcal{Y}'_1) \simeq f'_*(\mathcal{O}_{\mathcal{Y}'_1})\text{-mod}(\text{QCoh}(\mathcal{Y}'_2)).$$

Now, again by Proposition 2.3.10 and [GL:DG], Prop. 4.8.1,

$$\text{QCoh}(\mathcal{Y}_1) \underset{\text{QCoh}(\mathcal{Y}_2)}{\otimes} \text{QCoh}(\mathcal{Y}'_2) \simeq g^*(f_*(\mathcal{O}_{\mathcal{Y}_1}))\text{-mod}(\text{QCoh}(\mathcal{Y}'_2)).$$

Finally, by Proposition 2.1.1(2),

$$f'_*(\mathcal{O}_{\mathcal{Y}'_1}) \simeq g^*(f_*(\mathcal{O}_{\mathcal{Y}_1}))$$

as algebras in $\text{QCoh}(\mathcal{Y}'_2)$.

□

3.3. **Passable prestacks.**

3.3.1. We shall say that a prestack $\mathcal{Y}$ is *passable* if

(1) The diagonal morphism of $\mathcal{Y}$ is quasi-affine and quasi-compact,
(2) $\mathcal{O}_\mathcal{Y} \in \text{QCoh}(\mathcal{Y})$ is compact,
(3) The category $\text{QCoh}(\mathcal{Y})$ is dualizable.

By Proposition 2.3.2, we obtain that if $\mathcal{Y}$ is passable, then $\text{QCoh}(\mathcal{Y})$ is rigid as a monoidal category.

3.3.2. We are going to show that passable prestacks are adapted to having the map in (9) an equivalence:

Let $\mathcal{Y}$ be a passable prestack. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be prestacks mapping to $\mathcal{Y}$.

Proposition 3.3.3. *If under the above circumstances* $\mathrm{QCoh}(\mathcal{Y}_1)$ *is dualizable as a category, the natural functor*

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \underset{\mathcal{Y}}{\times} \mathcal{Y}_2)$$

is an equivalence.

Proof. By [GL:DG], Corollaries 4.3.2 and 6.4.2, the rigidity of $\mathrm{QCoh}(\mathcal{Y})$ and the fact that $\mathrm{QCoh}(\mathcal{Y}_1)$ is dualizable imply that the operation

$$\mathbf{C} \mapsto \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} : \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

commutes with limits.

This allows to replace $\mathcal{Y}_2$ by $S \in \mathrm{DGSch}^{\mathrm{aff}}$. But then the map $S \to \mathcal{Y}$ is quasi-compact and quasi-affine, and we find ourselves in the situation of Proposition 3.2.1.

□

3.4. Examples and stability properties of passable prestacks.

3.4.1. *Initial examples.* Any stack which is perfect (see Sect. 4.3) is passable. In particular, any quasi-compact and quasi-separated DG scheme with a quasi-affine diagonal is passable when viewed as a prestack.

More generally, Proposition 2.3.6 implies that any quasi-compact and quasi-separated algebraic space with a quasi-affine diagonal is passable.

3.4.2. Let $\mathcal{Y}$ be a passable prestack. We claim that its sheafification $L(\mathcal{Y})$ is also passable. Indeed, by Corollary 1.3.7, conditions (2) and (3) do not distinguish between $\mathcal{Y}$ and $L(\mathcal{Y})$. The fact that the diagonal of $L(\mathcal{Y})$ is quasi-affine follows from the fact that an fpppf descent of a quasi-affine scheme is a quasi-affine scheme.

3.4.3. Let $\mathcal{Y}$ be a prestack covered by two opens U_1 and U_2, each of which is passable.

Lemma 3.4.4. *Under the above circumstances, the stack* $\mathcal{Y}$ *is passable.*

Proof. The fact that the diagonal of $\mathcal{Y}$ is quasi-affine is immediate. The fact that $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact follows from Lemma 2.2.8. The fact that $\mathrm{QCoh}(\mathcal{Y})$ is dualizable was established in the course of the proof of Proposition 2.3.6.

□

3.4.5. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map of prestacks. Assume that f is representable, separated and quasi-compact, and that $\mathcal{Y}_2$ is passable.

Proposition 3.4.6. *Under the above circumstances,* $\mathcal{Y}_1$ *is also passable.*

Proof. The fact that the diagonal of $\mathcal{Y}_1$ is quasi-affine is immediate from the definitions.

The fact that $\mathcal{O}_{\mathcal{Y}_1} \in \mathrm{QCoh}(\mathcal{Y}_1)$ is compact follows from the corresponding fact for $\mathcal{Y}_2$ and the fact that f^* sends compacts to compacts (as its right adjoint f_* commutes with colimits).

It remains to show that $\mathrm{QCoh}(\mathcal{Y}_1)$ is dualizable. By [GL:DG], Corollary 6.4.2, it is enough to show that $\mathrm{QCoh}(\mathcal{Y}_1)$ is dualizable as a module category over $\mathrm{QCoh}(\mathcal{Y}_2)$. We claim that it is in fact self-dual as a module over $\mathrm{QCoh}(\mathcal{Y}_2)$. Indeed, by Proposition 3.3.3, we have:

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}_1) \simeq \mathrm{QCoh}(\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1),$$

and the duality datum is supplied by the functors

$$\Delta^* : \mathrm{QCoh}(\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1) \rightleftarrows \mathrm{QCoh}(\mathcal{Y}_1) : \Delta_*,$$

where Δ_* commutes with colimits by Proposition 2.1.1, combined with Remark 2.1.2.

□

3.4.7. Finally, let us add the following to our list of examples of passable prestacks:

Let $\mathcal{Z}$ be a passable prestack acted on by an algebraic group G, and let $\mathcal{Y} := \mathcal{Z}/G$ (by which we mean the geometric realization of the corresponding simplicial prestack). We will show in [GL:WA] that such $\mathcal{Y}$ is passable. (In *loc. cit.* it will be shown that in char. 0, the procedure of passing to the G-equivariant category preserves dualizability.)

4. Perfectness

4.1. **The perfect subcategory.**

4.1.1. Let S be an affine DG scheme. We recall the following:

Lemma 4.1.2. *For* $M \in \mathrm{QCoh}(S)$ *the following conditions are equivalent:*

(1) *M is compact.*
(2) *M is dualizable as an object of the monoidal category* $\mathrm{QCoh}(S)$*.*

In what follows, objects of $\mathrm{QCoh}(S)$ with the above properties will be called perfect, and denote the corresponding full (symmetric monoidal) DG subcategory

$$\mathrm{QCoh}(S)^{\mathrm{perf}} \subset \mathrm{QCoh}(S).$$

Note that the category $\mathrm{QCoh}(S)^{\mathrm{perf}}$ is, of course, *not* cocomplete.

4.1.3. We can regard the assignment

$$\mathrm{QCoh}(-)^{\mathrm{perf}} : S \mapsto \mathrm{QCoh}(S)^{\mathrm{perf}}$$

as a prestack on $\mathrm{DGSch}^{\mathrm{aff}}$.

The next result follows from Theorem 1.3.4:

Lemma 4.1.4. $\mathrm{QCoh}(-)^{\mathrm{perf}} \in \mathrm{Stk}_{\mathrm{DGCat}}$*, i.e., it satisfies fpppf descent.*

Moreover, we have:

Lemma 4.1.5. *As a prestack of DG categories,* $\mathrm{QCoh}(-)^{\mathrm{perf}}$ *is* convergent *(see* [GL:Stacks]*, Sect. 1.2).*

4.1.6. For a prestack $\mathcal{Y}$, we define a non-complete symmetric monoidal DG category $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ by

$$\underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op}}{\underleftarrow{lim}} \mathrm{QCoh}(S)^{\mathrm{perf}}.$$

By construction, we have a natural (symmetric monoidal) functor

$$\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}} \to \mathrm{QCoh}(\mathcal{Y}).$$

It is fully faithful embedding, because a limit of fully faithful functors is fully faithful. The essential image of the above functor can be tautologically characterized as follows: $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is perfect if $\mathcal{F}_{y_S} \in \mathrm{QCoh}(S)$ is perfect for every $(S, y_S) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$.

4.2. **Some properties of** $\mathrm{QCoh}(-)^{\mathrm{perf}}$**.**

4.2.1. *Perfectness and dualizability.* We have:

Lemma 4.2.2. *An object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is perfect if and only if it is dualizable as an object of the monoidal category $\mathrm{QCoh}(\mathcal{Y})$.*

Proof. This follows from the following general assertion, due to J. Lurie:

Let $i \mapsto \mathbf{C}_i$ be a functor from an ∞-category I to $\infty\text{-}\mathrm{Cat}^{\mathrm{Mon}}$, and let $\mathbf{C} = \underset{I}{lim}\,(\mathbf{C}_i)$. Then $\mathbf{c} \in \mathbf{C}$ is dualizable if and only its image in every $\mathbf{C}_i$ is dualizable. □

4.2.3. *Perfectness and compactness.* We shall study to what extent Lemma 4.1.2 generalizes to stacks:

Proposition 4.2.4.

(1) *Suppose that the diagonal morphism of $\mathcal{Y}$ is representable, quasi-compact and quasi-separated. Then any compact object of $\mathrm{QCoh}(\mathcal{Y})$ is perfect.*

(2) *Suppose that $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact. Then any perfect object of $\mathrm{QCoh}(\mathcal{Y})$ is compact.*

Proof. Point (2) follows from Lemma 4.2.2: for any monoidal category in which the unit object is compact, dualizability implies compactness, see [GL:DG], Lemma 5.2.1.

To prove point (1) we have to show that the functor $\mathcal{F} \mapsto f^*(\mathcal{F})$ for $f : S \to \mathcal{Y}$ with S affine, sends compact objects to compact ones. However, this is true, since the functor in question admits a right adjoint, namely, f_*, which commutes with colimits, by Proposition 2.1.1, combined with Remark 2.1.2.

□

4.3. **Perfect stacks.** The discussion of perfect stacks is entirely plagiarized from [BFN]. Following *loc. cit.*, we shall say that a stack $\mathcal{Y}$ is perfect if

(1) The diagonal morphism of $\mathcal{Y}$ is affine,
(2) $\mathrm{QCoh}(\mathcal{Y}) \simeq \mathrm{Ind}(\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}})$.

By Proposition 4.2.4, the above conditions can be reformulated as follows:

(1) The diagonal morphism of $\mathcal{Y}$ is affine,
(2) $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact,
(3) $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

Since every compactly generated category is dualizable, we obtain that every perfect stack is passable.

4.3.1. *Examples.* In [BFN], following the arguments of [TT] and [Ne], it is shown that any quasi-compact and quasi-separated DG scheme, considered as a prestack, is perfect.

Moreover, in [BFN] it is shown that if $\mathcal{Y}$ is of the form X/G, where G is an algebraic group and X is a DG scheme endowed with a G-equivariant ample line bundle, then $\mathcal{Y}$ is perfect (under our assumption that the ground field k is of char. 0).

4.3.2. *Stability properties of perfect stacks.* Here is a simple procedure to produce perfect stacks from existing ones:

Proposition 4.3.3. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map of stacks. Assume that f is schematic, quasi-compact and quasi-affine, and that $\mathcal{Y}_2$ is perfect. Then $\mathcal{Y}_1$ is also perfect.*

Proof. The fact that the diagonal of $\mathcal{Y}_1$ is affine, follows tautologically. The other two conditions follow from the fact that f^* sends compact objects to compact ones (as its right adjoint f_* commutes with colimits, see Proposition 2.1.1), and that its essential image generates $\mathrm{QCoh}(\mathcal{Y}_1)$ (since f_* is conservative).

□

In particular, we obtain that if $\mathcal{Y}$ is perfect and $\mathcal{Y}' \hookrightarrow \mathcal{Y}$ is a quasi-compact locally closed embedding, then $\mathcal{Y}'$ is perfect as well.

4.3.4. Let now $\mathcal{Y}$ be a stack, covered by two opens $U_1, U_2 \subset \mathcal{Y}$. The following question was asked by Drinfeld:

Question 4.3.5. *Assume that U_1 and U_2 are perfect. Is it true that $\mathcal{Y}$ is also perfect?*

5. Quasi-coherent sheaves an Artin stacks

5.1. QCoh **on Artin stacks.**

5.1.1. Let $\mathcal{Y}$ be an k-Artin stack. We claim that in this case, there is a more concise expression for $\mathrm{QCoh}(\mathcal{Y})$.

Let $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth}$ and $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr}$ denote the following subcategories of $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$: the former is the full subcategory of consisting of those pairs $(S, y : S \to \mathcal{Y})$ for which y_S is smooth, whereas, $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr}$ is obtained from $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth}$ by only allowing those 1-morphisms that are themselves smooth.

Proposition 5.1.2.

(a) *The natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth})^{op}}{lim} \mathrm{QCoh}(S)$$

is an equivalence.

(b) *The natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr})^{op}}{lim} \mathrm{QCoh}(S)$$

is an equivalence.

Proof. Assume by induction that both statements are true for $k' < k$. Let $f : Z \to \mathcal{Y}$ be a smooth atlas. By Corollary 1.3.10, the map

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Tot}\left(\mathrm{QCoh}(Z^\bullet/\mathcal{Y})\right)$$

is an equivalence. We are going to construct a map

$$\underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr})^{op}}{lim} \mathrm{QCoh}(S) \to \mathrm{Tot}\left(\mathrm{QCoh}(Z^\bullet/\mathcal{Y})\right), \tag{10}$$

inverse to the composition

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth})^{op}}{lim} \mathrm{QCoh}(S) \to \underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr})^{op}}{lim} \mathrm{QCoh}(S).$$

Namely, for every i, composition defines a map

$$\underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr})^{op}}{lim} \mathrm{QCoh}(S) \to \underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/(Z^i/\mathcal{Y}),smooth,restr})^{op}}{lim} \mathrm{QCoh}(S),$$

where the RHS, by the induction hypothesis, identifies with $\mathrm{QCoh}(Z^i/\mathcal{Y})$.

It is clear that the maps

$$\underset{(S,y_S)\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},smooth,restr})^{op}}{lim} \mathrm{QCoh}(S) \to \mathrm{QCoh}(Z^i/\mathcal{Y})$$

thus constructed are compatible, i.e., we obtain a map as in (10). It is also clear from the construction that the resulting map (10) is inverse to the one in the proposition.

□

5.2. **t-structure in the Artin case.**

5.2.1. The next proposition gives an explicit description of the t-structure on $\mathrm{QCoh}(\mathcal{Y})$ in terms of an atlas:

Proposition 5.2.2.

(a) *Let $\mathcal{Y}$ be a k-Artin stack and let $f : S \to \mathcal{Y}$ be an atlas, where $S \in \mathrm{DGSch}$. Then an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ (resp., $\mathrm{QCoh}(\mathcal{Y})^{>0}$) if and only if $f^*(\mathcal{F})$ belongs to $\mathrm{QCoh}(S)^{\leq 0}$ (resp., $\mathrm{QCoh}(S)^{>0}$).*

(b) *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a flat map between k-Artin stacks. Then the functor $\pi^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$ is t-exact.*

Clearly, point (a) is a particular case of point (b).

Proof. We will argue by induction, assuming that both statements are true for $k' < k$. Let us first prove point (a). It is enough to show that the functor f^* is compatible with the truncation functors.

Thus, let $\mathcal{F}$ be an object of $\mathrm{QCoh}(\mathcal{Y})$, and let

$$\mathcal{F}|_{S^\bullet/\mathcal{Y}} \in \mathrm{QCoh}(S^\bullet/\mathcal{Y})$$

be the corresponding object. We claim that

$$i \mapsto \tau^{\leq 0}(\mathcal{F}|_{S^i/\mathcal{Y}}) \text{ and } i \mapsto \tau^{>0}(\mathcal{F}|_{S^i/\mathcal{Y}})$$

both belong to $\mathrm{QCoh}(S^\bullet/\mathcal{Y})$. This follows by the induction hypothesis from the fact that the face maps in the simplicial stack $S^\bullet/\mathcal{Y}$ are flat. It is clear that the object $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y})$ that corresponds to $\tau^{\leq 0}(\mathcal{F}|_{S^\bullet/\mathcal{Y}})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$. We claim that the object $\mathcal{F}'' \in \mathrm{QCoh}(\mathcal{Y})$ that corresponds to $\tau^{>0}(\mathcal{F}|_{S^\bullet/\mathcal{Y}})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{>0}$. Indeed, for $\mathcal{F}''' \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$, we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}''', \mathcal{F}') \simeq \mathrm{Tot}\left(\mathrm{Hom}_{\mathrm{QCoh}(S^\bullet/\mathcal{Y})}(\mathcal{F}'''|_{S^\bullet/\mathcal{Y}}, \tau^{>0}(\mathcal{F}|_{S^i/\mathcal{Y}}))\right),$$

and the right-hand side vanishes, since $\mathcal{F}'''|_{S^i/\mathcal{Y}} \in \mathrm{QCoh}(S^i/\mathcal{Y})^{\leq 0}, \forall i$.

Let us now prove point (b). By point (a), we can assume that $\mathcal{Y}_1$ is a scheme X (replace the initial $\mathcal{Y}_1$ by its atlas). So, we are dealing with a flat map π from a scheme X to a k-Artin stack $\mathcal{Y} = \mathcal{Y}_2$. Let $f : S \to \mathcal{Y}$ be an atlas. Consider the Cartesian square:

$$\begin{array}{ccc} X \underset{\mathcal{Y}}{\times} S & \xrightarrow{\pi'} & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\pi} & \mathcal{Y}. \end{array}$$

Again, by point (a), it is sufficient to show that the functor

$$f'^* \circ \pi^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(X \underset{\mathcal{Y}}{\times} S)$$

is exact. However, $f'^* \circ \pi^* \simeq \pi'^* \circ f^*$, and f^* is exact by point (a), and π'^* is exact by the induction hypothesis.

□

5.2.3. The above proposition has the following corollary:

Corollary 5.2.4. *Let $\mathcal{Y}$ be a k-Artin stack for some k.*

(a) *The t-stucture on* $\mathrm{QCoh}(\mathcal{Y})$ *is compatible with filtered colimits, i.e., the truncation functors on* $\mathrm{QCoh}(\mathcal{Y})$ *are compatible with filtered colimits (or, equivalently, the subcategory* $\mathrm{QCoh}(\mathcal{Y})^{>0}$ *is closed under filtered colimits).*

(b) *The t-structure on* $\mathrm{QCoh}(\mathcal{Y})$ *is* left-complete *and* right-complete *, i.e., for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, the natural maps*

$$\mathcal{F} \to \underset{n\in\mathbb{N}}{lim}\, \tau^{\geq -n}(\mathcal{F})$$

$$\underset{n\in\mathbb{N}}{colim}\, \tau^{\leq n}(\mathcal{F}) \to \mathcal{F}$$

are isomorphisms, where τ denotes the truncation functor.

The proof follows from Proposition 5.1.2(b) and the fact that both assertions are true for affine schemes: the corresponding properties are inherited under taking limits in DGCat, under continuous and t-exact functors.

5.3. **Direct images.**

5.3.1. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between Artin stacks. As was mentioned in Sect. 1.5.2 The functor

$$\pi_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$$

is not necessarily continuous. However, the following does hold:

Proposition 5.3.2. *For every $n \in \mathbb{N}$, the restriction $\pi_*|_{\mathrm{QCoh}(\mathcal{Y}_1)^{\geq 0}}$ maps to* $\mathrm{QCoh}(\mathcal{Y}_2)^{\geq 0}$*, and commutes with filtered colimits.*

Proof. Again, we argue by induction on k, assuming that the assertion is true for $k' < k$. Let $f : Z \to \mathcal{Y}_1$ be a smooth (or even flat) atlas. Let π^i denote the composition

$$Z^i/\mathcal{Y}_1 \to \mathcal{Y}_1 \overset{\pi}{\to} \mathcal{Y}_2.$$

By Corollary 1.3.10, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)$, we have:

$$\pi_*(\mathcal{F}) \simeq \mathrm{Tot}\left(\pi^\bullet_*(\mathcal{F}|_{Z^\bullet/\mathcal{Y}_1})\right).$$

By the induction hypothesis, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)^{\geq 0}$, each term of the co-simplicial object

$$i \mapsto \pi^i_*(\mathcal{F}|_{Z^i/\mathcal{Y}_1}) \tag{11}$$

is in $\mathrm{QCoh}(\mathcal{Y}_2)^{\geq 0}$, and, when viewed as a functors of $\mathcal{F}$, the assignment (11) commutes with filtered colimits. This implies the assertion of the proposition, since we are dealing with a spectral sequence concentrated in the positive quadrant.

□

Corollary 5.3.3. *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ a map between Artin stacks, and let*

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g'} & \mathcal{Y}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}'_2 & \xrightarrow{g} & \mathcal{Y}_2 \end{array}$$

be a Cartesian square, where the morphism g is flat. Then the diagram of functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}'_1)^+ & \xleftarrow{g'^*} & \mathrm{QCoh}(\mathcal{Y}_1)^+ \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(\mathcal{Y}'_2)^+ & \xleftarrow{g^*} & \mathrm{QCoh}(\mathcal{Y}_2)^+ \end{array}$$

is commutative.

5.4. **Classical algebraic stacks.**

5.4.1. For any Artin stack $\mathcal{Y}$ there is a canonical t-exact functor

$$D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit) \to \mathrm{QCoh}(\mathcal{Y}),$$

but in general this is not an equivalence.

5.4.2. Assume now that $\mathcal{Y}$ is an algebraic stack (i.e., a 1-Artin stack), and assume that it is classical. Moreover, assume that $\mathcal{Y}$ is quasi-compact and that the diagonal morphism of $\mathcal{Y}$ is affine. For the sake of completeness, we will give a proof of the following (well-known) assertion:

Proposition 5.4.3. *Under the above circumstances, the functor $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)^+ \to \mathrm{QCoh}(\mathcal{Y})^+$ is an equivalence.*

Remark 5.4.4. The above proposition implies that, under the specified assumptions, the category $\mathrm{QCoh}(\mathcal{Y})$ identifies with the left-completion of $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$. We do not know what are the general conditions that guarantee that $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$ itself is left-complete. For example, this is true for quasi-compact and quasi-separated DG schemes. More generally, the proof of Proposition 2.2.2 shows that this is also true for quasi-compact and quasi-separated algebraic spaces. It is also easy to see that this is true for algebraic stacks of the form Z/G, where Z is a quasi-projective DG scheme and G an algebraic group acting linearly on Z (we remind that we are working over a field of characteristic 0).

Proof. The proof will follow from the following general lemma:

Lemma 5.4.5. *Let* $\mathbf{C}$ *be a DG category equipped with a t-structure that commutes with filtered colimits, and which is right-complete. Assume that for every object* $\mathbf{c} \in \mathbf{C}^{\heartsuit}$ *there exists an injection* $\mathbf{c} \to \mathbf{c}_0$*, where* $\mathbf{c}_0 \in \mathbf{C}^{\heartsuit}$ *is such that* $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}', \mathbf{c}_0[i]) = 0$ *for* $i > 0$ *and all* $\mathbf{c}' \in \mathbf{C}^{\heartsuit}$*. Then the natural functor*

$$D(\mathbf{C}^{\heartsuit})^{+} \to \mathbf{C}^{+}$$

is an equivalence.

We apply this lemma to $\mathbf{C} = \mathrm{QCoh}(\mathcal{Y})$. Let $f : S \to \mathcal{Y}$ be a map, where S is an affine scheme. Since the diagonal morphism of $\mathcal{Y}$ is affine, the map f itself is affine. Hence, if $\mathcal{F}_S \in \mathrm{QCoh}(S)^{\heartsuit}$, then $f_*(\mathcal{F}_S) \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$. Moreover, if f is flat and $\mathcal{F}_S \in \mathrm{QCoh}(S)^{\heartsuit}$ is injective, we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}', f_*(\mathcal{F}_S)[i]) = 0, \ \forall \mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}.$$

If $\mathcal{F}$ is an object of $\mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$, choose an injective $f^*(\mathcal{F}) \hookrightarrow \mathcal{F}_S$, and $\mathcal{F}$ embed into $f_*(\mathcal{F}_S)$.

□

5.5. **QCA stacks.**

5.5.1. In this subsection we will announce some results which would be proved in [DrGa]. (We remind again that we are working over a field of char. 0, which is crucial for what follows.)

Let $\mathcal{Y}$ be a classical algebraic stack, i.e., a 1-Artin stack.

Definition 5.5.2. *We shall say that* $\mathcal{Y}$ *is QCA if*

(1) *It is quasi-compact;*
(2) *The diagonal morphism* $\mathcal{Y}' \to \mathcal{Y}' \times \mathcal{Y}'$ *is quasi-compact and quasi-separated;*
(3) *The automorphism groups of its field-valued points are affine;*
(4) *The classical inertia stack, i.e.,* ${}^{cl}(\mathcal{Y} \underset{\mathcal{Y}\times\mathcal{Y}}{\times} \mathcal{Y})$ *is of finite presentation over* ${}^{cl}\mathcal{Y}$*.*

Definition 5.5.3. *An algebraic stack* $\mathcal{Y}$ *is QCA if its underlying classical stack is QCA in the above sense.*

Under these circumstances the following will be established in [DrGa]:

Theorem 5.5.4. *The functor* $\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$ *is continuous, i.e., the object* $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ *is compact.*

5.5.5. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. We shall say that it is QCA if for every affine DG scheme S mapping to $\mathcal{Y}_2$, the base change $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is a QCA algebraic stack.

From Theorem 5.5.4 we obtain:

Corollary 5.5.6. *The assertions of Proposition 2.1.1 remain valid if* f *is assumed QCA.*

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