CHAPTER II.3. INTERACTION OF QCoh AND IndCoh

Contents

Introduction

One of the first things one notices about the category $\mathrm{IndCoh}(X)$ (for a scheme X) is that it is equipped with an action of the (symmetric) monoidal category $\mathrm{QCoh}(X)$, see [Chapter II.1, Sect. 1.2.9].

In this chapter we will study how (or, rather, in what sense) this action extends, when we want to consider IndCoh as a functor out of the category of correspondences.

We should say that the contents of this chapter are rather technical (and are largely included for completeness), and thus can be skipped on the first pass.

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0.1. **Why does this chapter exist?**

0.1.1. The first question to ask is, indeed, why bother? The true answer is that if we really care about IndCoh as a functor out of category of correspondences and about the action of $\mathrm{QCoh}(X)$ on $\mathrm{IndCoh}(X)$, then we must understand how they interact.

However, in addition to that, the material of this chapter will have some practical consequences.

0.1.2. We recall that, say, for an individual scheme X, both categories $\mathrm{QCoh}(X)$ and $\mathrm{IndCoh}(X)$ have a canonical symmetric monoidal structure. We will show that the action of $\mathrm{QCoh}(X)$ on $\mathrm{IndCoh}(X)$ comes from a symmetric monoidal functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X),$$

which as a plain functor looks like

$$\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X.$$

The functoriality properties established in this chapter will allow us to extend the assignment $X \rightsquigarrow \Upsilon_X$ from schemes to prestacks.

0.1.3. We will see that the functor *dual* to Υ_X with respect to the Serre auto-duality of $\mathrm{IndCoh}(X)$ and the naive auto-duality of $\mathrm{QCoh}(X)$, is the natural transformation

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X),$$

again in a way functorial with respect to X.

0.2. **The action of** $\mathrm{QCoh}(X)$ **on** $\mathrm{IndCoh}(X)$ **and correspondences.** Let us explain how we encode the action of $\mathrm{QCoh}(X)$ on $\mathrm{IndCoh}(X)$ in the framework of the $(\infty, 2)$-categories of correspondences.

0.2.1. Recall that in [Chapter II.2] we extended the assignment $X \rightsquigarrow \mathrm{IndCoh}(X)$ into a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

We now consider the assignment

$$X \rightsquigarrow (\mathrm{QCoh}(X), \mathrm{IndCoh}(X)),$$

where we regard $\mathrm{QCoh}(X)$ as a monoidal DG category (i.e., an algebra object in $\mathrm{DGCat}_{\mathrm{cont}}$), and $\mathrm{IndCoh}(X)$ as a $\mathrm{QCoh}(X)$-module category, i.e., an object of $\mathrm{QCoh}(X)\text{-}\mathbf{mod}$.

We want to extend this assignment to a functor out of $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}$. The challenge is to identify the target $(\infty, 2)$-category, so that it will account for the pieces of structure that we need, also one for which such a construction will be possible.

0.2.2. The sought-for $(\infty, 2)$-category, denoted $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$, is introduced in Sect. 1. As expected, its objects are pairs $(\mathbf{O}, \mathbf{C})$, where $\mathbf{O}$ is a monoidal DG category and $\mathbf{C}$ is a module $\mathbf{O}$-category.

But 1-morphisms are less obvious. We refer the reader to Sect. 1.1 for the definition. It is designed so that there is a natural forgetful functor

$$\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}} \tag{0.1}$$

that at the level of objects sends $(\mathbf{O}, \mathbf{C})$ to $\mathbf{C}$.

Here is how the desired functor

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}} \tag{0.2}$$

is constructed[1].

0.2.3. Consider the $(\infty,1)$-category $\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}$, whose objects are pairs $(\mathbf{O},\mathbf{C})$, but where the space of morphisms from $(\mathbf{O}_1,\mathbf{C}_1)$ to $(\mathbf{O}_2,\mathbf{C}_2)$ is that of pairs

$$(F_{\mathbf{O}}:\mathbf{O}_2\to\mathbf{O}_1, F_{\mathbf{C}}:\mathbf{C}_1\to\mathbf{C}_2),$$

where $F_{\mathbf{O}}$ is a monoidal functor (note the direction of the arrow), and $F_{\mathbf{C}}$ is a map of $\mathbf{O}_2$-module categories.

As our initial input we start with the functor

$$(\mathrm{QCoh}^*,\mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}:\mathrm{Sch}_{\mathrm{aft}}\to\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}} \tag{0.3}$$

that sends a scheme X to $(\mathrm{QCoh}(X),\mathrm{IndCoh}(X))$ and a morphism $X\overset{f}{\to}Y$ to the pair $(f^*,f_*^{\mathrm{IndCoh}})$.

0.2.4. Next, from the definition of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ it follows that there exists a canonically defined functor

$$\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}\to\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}.$$

Precomposing with (0.3) we obtain a functor

$$(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Sch}_{\mathrm{aft}}}:\mathrm{Sch}_{\mathrm{aft}}\to\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}. \tag{0.4}$$

Now, to get from (0.4) to (0.2) we repeat the procedure of [Chapter II.2, Sect. 2.1].

0.3. **The natural transformation Υ.** We shall now explain how the existence of the functor $(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ in (0.2) leads to the natural transformation from Sect. 0.1.1.

0.3.1. First, we note that $(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ comes equipped with a canonically defined symmetric monoidal structure, where $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}$ acquires a symmetric monoidal structure from the operation of Cartesian product on $\mathrm{Sch}_{\mathrm{aft}}$, and the symmetric monoidal structure on $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ is given by component-wise tensor product.

0.3.2. Let $\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$ denote the $(\infty,1)$-category, whose objects are pairs $(\mathbf{O},\mathbf{C})$, but where the space of morphisms from $(\mathbf{O}_1,\mathbf{C}_1)$ to $(\mathbf{O}_2,\mathbf{C}_2)$ is that of pairs

$$(F_{\mathbf{O}}:\mathbf{O}_1\to\mathbf{O}_2, F_{\mathbf{C}}:\mathbf{C}_1\to\mathbf{C}_2),$$

where $F_{\mathbf{O}}$ is a monoidal functor and $F_{\mathbf{C}}$ is a map of $\mathbf{O}_1$-module categories. (Note that the difference between $\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$ and $\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}$ is in the direction of the arrow $F_{\mathbf{O}}$.)

From the definition of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ it follows that there exists a canonically defined (symmetric monoidal) functor

$$\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\to\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}.$$

[1]This essentially mimics the construction of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ in [Chapter II.2, Sect. 2.1].

0.3.3. Restricting $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ to $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}$ we obtain a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}},$$

and one shows that it factors through a canonically defined (symmetric monoidal) functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}. \tag{0.5}$$

Since every object in $\mathrm{Sch}_{\mathrm{aft}}$ has a canonical structure of co-commutative co-algebra (via the diagonal map), the functor (0.5) gives rise to a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{ComAlg}(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}). \tag{0.6}$$

0.3.4. We note that the category $\mathrm{ComAlg}(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})$ identifies with that of triples

$$(\mathbf{O}, \mathbf{O}', \alpha : \mathbf{O} \to \mathbf{O}'),$$

where $\mathbf{O}$ and $\mathbf{O}'$ are symmetric monoidal categories, and α is a symmetric monoidal functor.

Hence, the data of the functor (0.6) is equivalent to that of a natural transformation

$$\Upsilon : \mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}, \tag{0.7}$$

where $\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}}$ and $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ are viewed as functors

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{cont}}) = \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}.$$

0.3.5. For an individual scheme X we thus obtain a functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X),$$

which is obtained by the action of $\mathrm{QCoh}(X)$ on the monoidal unit in $\mathrm{IndCoh}(X)$, i.e., $\omega_X \in \mathrm{IndCoh}(X)$.

For a morphism $X \xrightarrow{f} Y$, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{\Upsilon_X} & \mathrm{IndCoh}(X) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^!} \\ \mathrm{QCoh}(Y) & \xrightarrow{\Upsilon_Y} & \mathrm{IndCoh}(Y). \end{array}$$

The above observation allows to extend the assignment $X \rightsquigarrow \Upsilon_X$ from schemes to prestacks. I.e., for every $\mathcal{Y} \in \mathrm{PreStk}$ we also have a canonically defined (symmetric monoidal) functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}).$$

0.4. **Relationship with Ψ.** Let us finally explain the relationship between the functors

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X) \text{ and } \Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X).$$

Remark 0.4.1. Recall that the functor Ψ_X played a fundamental role in the initial steps of setting up the theory of ind-coherent sheaves. Specifically, it was used in the definition of the direct image functor

$$X \xrightarrow{f} Y \quad \rightsquigarrow \quad \mathrm{IndCoh}(X) \xrightarrow{f^{\mathrm{IndCoh}}_*} \mathrm{IndCoh}(Y).$$

However, Ψ is really a feature of schemes. In particular, it does not have an intrinsic meaning for prestacks.

0.4.2. Recall the categories $\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}}$ and $\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}$, and note that they contain full subcategories

$$(\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}})^{\mathrm{dualizable}} \subset \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}}$$

and

$$(\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}})^{\mathrm{dualizable}} \subset \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}},$$

respectively that consist of pairs $(\mathbf{O}, \mathbf{C})$ where $\mathbf{C}$ is dualizable as a plain DG category.

The operation of dualization $(\mathbf{O}, \mathbf{C}) \mapsto (\mathbf{O}, \mathbf{C}^\vee)$ defines an equivalence

$$((\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}})^{\mathrm{dualizable}})^{\mathrm{op}} \to (\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}})^{\mathrm{dualizable}}. \tag{0.8}$$

0.4.3. Recall (see [Chapter II.2, Sect. 4.2]) that Serre duality for IndCoh was a formal consequence of the existence of the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$, equipped with its symmetric monoidal structure.

In the same way, we use the functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$, equipped with its symmetric monoidal structure, to show that the composition of the functor $(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}$ of (0.3) with (0.8) identifies with the functor $(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}$ of (0.5).

0.4.4. Next, by construction, the functor $(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}$ comes equipped with the natural transformation

$$(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}} \to (\mathrm{QCoh}^*, \mathrm{QCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}$$

as functors $\mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}}$. Applying (0.8), we obtain a natural transformation

$$(\mathrm{Id}, \Psi^\vee)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}} \to (\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}} \tag{0.9}$$

as functors $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}$.

0.4.5. What we show is that that the above natural transformation (0.9) is canonically isomorphic to the natural transformation

$$(\mathrm{Id}, \Upsilon)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}} \to (\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}, \tag{0.10}$$

the latter being part of the data of the functor (0.7).

0.4.6. For an individual scheme X this means that the functors

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X) \text{ and } \Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$$

are canonically duals of each other.

Here $\mathrm{IndCoh}(X)$ is identified with its own dual via the Serre duality functor $\mathbf{D}_X^{\mathrm{Serre}}$ (see [Chapter II.2, Sect. 4.2.6]). The category $\mathrm{QCoh}(X)$ is identified with its own dual via the "naive" duality functor

$$\mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X),$$

whose evaluation map $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{Vect}$ is

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \overset{\otimes}{\to} \mathrm{QCoh}(X) \overset{\Gamma(X,-)}{\longrightarrow} \mathrm{Vect}\,.$$

0.5. **What is done in this chapter?**

0.5.1. In Sect. 1 we define the $(\infty,2)$-category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ that will be the recipient of the functor

$$(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}:\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}\to\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}. \tag{0.11}$$

In doing so we allow ourselves a certain sloppiness: we say what the space of objects of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ is, and what is the $(\infty,1)$-category of morphisms between any two objects.

We leave it to the reader to complete this to an actual definition of a $(\infty,2)$-category (as those are defined in [Chapter A.1, Sect. 2.1]).

0.5.2. In Sect. 2 we carry out the construction of the functor (0.11) along the lines indicated in Sect. 0.2.

0.5.3. In Sect. 3 we discuss the symmetric monoidal structure on the functor (0.11), and how it gives rise to the natural transformation Υ, as described in Sect. 0.3.

0.5.4. In Sect. 4 we discuss the self-duality feature of the assignment

$$X\quad\rightsquigarrow\quad\mathrm{QCoh}(X)\in\mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}},\mathrm{IndCoh}(X)\in\mathrm{QCoh}(X)\text{-}\mathbf{mod},$$

and the relationship between the natural transformations

$$\Upsilon_X:\mathrm{QCoh}(X)\to\mathrm{IndCoh}(X)$$

and

$$\Psi_X:\mathrm{IndCoh}(X)\to\mathrm{QCoh}(X).$$

1. The $(\infty,2)$-category of pairs

In this section we introduce a general framework that encodes the $(\infty,2)$-category of pairs $(\mathbf{O},\mathbf{C})$, where $\mathbf{O}$ us a monoidal DG category, and $\mathbf{C}$ is an $\mathbf{O}$-module category.

This $(\infty,2)$-category will be the recipient of the functor from the $(\infty,2)$-category of correspondences that assigns to a scheme X the pair $(\mathrm{QCoh}(X),\mathrm{IndCoh}(X)$.

The reason that we need this rather involved $(\infty,2)$-category instead of the more easily defined underlying 1-category is that $(\infty,2)$-category are necessary for the construction of the assignment

$$X\rightsquigarrow(\mathrm{QCoh}(X),\mathrm{IndCoh}(X)$$

as a functor, see [Chapter II.2, Sect. 2.1].

1.1. The category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$.

In this subsection we introduce our two category of pairs.

1.1.1. We introduce the $(\infty,2)$-category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ as follows. Its objects are pairs $(\mathbf{O},\mathbf{C})$, where $\mathbf{O}\in\mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}}$, and $\mathbf{C}\in\mathbf{O}\text{-}\mathbf{mod}$.

Given two objects $(\mathbf{O}_1,\mathbf{C}_1)$ and $(\mathbf{O}_2,\mathbf{C}_2)$ of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$, the objects of the $(\infty,1)$-category of 1-morphisms $(\mathbf{O}_1,\mathbf{C}_1)\to(\mathbf{O}_2,\mathbf{C}_2)$ are the data of:

- An $(\mathbf{O}_2,\mathbf{O}_1)$-bimodule category $\mathbf{M}$;
- A map $F:\mathbf{M}\underset{\mathbf{O}_1}{\otimes}\mathbf{C}_1\to\mathbf{C}_2$ in $\mathbf{O}_2\text{-}\mathbf{mod}$;
- A distinguished object $\mathbf{1}_{\mathbf{M}}\in\mathbf{M}$.

1.1.2. Given two objects $(\mathbf{M}^s, F^s, \mathbf{1}_{\mathbf{M}^s})$ and $(\mathbf{M}^t, F^t, \mathbf{1}_{\mathbf{M}^t})$ as above, the space of 2-morphisms

$$(\mathbf{M}^s, F^s, \mathbf{1}_{\mathbf{M}^s}) \to (\mathbf{M}^t, F^t, \mathbf{1}_{\mathbf{M}^t})$$

is that of the following data:

- A map of bimodules $\Phi : \mathbf{M}^t \to \mathbf{M}^s$ (note the direction of the arrow!);
- A natural transformation of maps of $\mathbf{O}_2$-bimodules
$$T : F^s \circ (\Phi \otimes \mathrm{Id}_{\mathbf{C}_1}) \Rightarrow F^t;$$
- A map in $\mathbf{M}^s$ as a plain DG category
$$\psi : \mathbf{1}_{\mathbf{M}^s} \to \Phi(\mathbf{1}_{\mathbf{M}^t}).$$

1.1.3. Compositions of 1-morphisms are defined naturally: for

$$(\mathbf{M}_{1,2}, F_{1,2}, \mathbf{1}_{\mathbf{M}_{1,2}}) : (\mathbf{O}_1, \mathbf{C}_1) \to (\mathbf{O}_2, \mathbf{C}_2)$$

and

$$(\mathbf{M}_{2,3}, F_{2,3}, \mathbf{1}_{\mathbf{M}_{2,3}}) : (\mathbf{O}_2, \mathbf{C}_2) \to (\mathbf{O}_3, \mathbf{C}_3),$$

their composition is defined by means of

$$\mathbf{M}_{2,3} := \mathbf{M}_{2,3} \underset{\mathbf{O}_2}{\otimes} \mathbf{M}_{1,2},$$

the data of $F_{1,3}$ equal to the composition

$$\mathbf{M}_{2,3} \underset{\mathbf{O}_2}{\otimes} \mathbf{M}_{1,2} \underset{\mathbf{O}_1}{\otimes} \mathbf{C}_1 \overset{F_{1,2}}{\longrightarrow} \mathbf{M}_{2,3} \underset{\mathbf{O}_2}{\otimes} \mathbf{C}_2 \overset{F_{2,3}}{\longrightarrow} \mathbf{C}_3,$$

and the data of $\mathbf{1}_{\mathbf{M}_{1,3}}$ being

$$\mathbf{1}_{\mathbf{M}_{2,3}} \otimes \mathbf{1}_{\mathbf{M}_{1,2}} \in \mathbf{M}_{2,3} \underset{\mathbf{O}_2}{\otimes} \mathbf{M}_{1,2}.$$

Compositions of 2-morphisms are also defined naturally.

The higher-categorical structure on $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ is defined in a standard fashion.

1.2. **Some forgetful functors.** In this subsection we discuss two (obvious) forgetful functors from $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ to some more familiar 2-categories.

1.2.1. First, we observe that $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ comes equipped with a forgetful functor to $\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$.

At the level of objects we send $(\mathbf{O}, \mathbf{C})$ to $\mathbf{C}$. At the level of 1-morphisms, given

$$(\mathbf{M}, F, \mathbf{1}_{\mathbf{M}}) : (\mathbf{O}_1, \mathbf{C}_1) \to (\mathbf{O}_2, \mathbf{C}_2)$$

we define the corresponding functor between plain DG categories $\overline{F} : \mathbf{C}_1 \to \mathbf{C}_2$ as the composition

$$\mathbf{C}_1 \overset{\mathbf{1}_{\mathbf{M}} \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathbf{M} \underset{\mathbf{O}_1}{\otimes} \mathbf{C}_1 \overset{F}{\longrightarrow} \mathbf{C}_2.$$

Given a 2-morphism

$$(\Phi, T, \psi) : (\mathbf{M}^s, F^s, \mathbf{1}_{\mathbf{M}^s}) \to (\mathbf{M}^t, F^t, \mathbf{1}_{\mathbf{M}^t}),$$

the corresponding natural transformation $\overline{F}^s \to \overline{F}^t$ is the composition

$$\overline{F}^s := F^s \circ (\mathbf{1}_{\mathbf{M}^s} \otimes \mathrm{Id}_{\mathbf{C}_1}) \overset{\psi}{\Rightarrow} F^s \circ (\Phi \otimes \mathrm{Id}_{\mathbf{C}_1}) \circ (\mathbf{1}_{\mathbf{M}^t} \otimes \mathrm{Id}_{\mathbf{C}_1}) \overset{T}{\Rightarrow} F^t \circ (\mathbf{1}_{\mathbf{M}^t} \otimes \mathrm{Id}_{\mathbf{C}_1}) = \overline{F}^t.$$

1.2.2. Let $\mathrm{DGCat}^{\mathrm{Mon,ext}}$ denote the $(\infty,2)$-category, where:

- The objects are $\mathbf{O} \in \mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}}$;
- Given $\mathbf{O}_1, \mathbf{O}_2 \in \mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}}$, the objects of $(\infty,1)$-category of 1-morphisms from $\mathbf{O}_1$ to $\mathbf{O}_2$ are $(\mathbf{O}_2, \mathbf{O}_1)$-bimodule categories;
- For a pair of 1-morphims from $\mathbf{O}_1$ to $\mathbf{O}_2$, given by bimodule categories $\mathbf{M}^s$ and $\mathbf{M}^t$ respectively, the space of 2-morphisms from $\mathbf{M}^s$ to $\mathbf{M}^t$ is that of maps of bimodules $\mathbf{M}^t \to \mathbf{M}^s$ (note the direction of the arrow)[2].

We have a naturally defined forgetful functor

$$\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}} \to \mathrm{DGCat}^{\mathrm{Mon,ext}}$$

that sends $(\mathbf{O}, \mathbf{C})$ to $\mathbf{O}$.

1.3. **Two 2-full subcategories.** In this subsection we single out two 1-subcategories of the $(\infty,2)$-category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ that we will ultimately be interested in.

1.3.1. Let $\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$ be the $(\infty,1)$-category, where

- The objects are the same as those of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$;
- 1-morphisms between $(\mathbf{O}_1, \mathbf{C}_1) \to (\mathbf{O}_2, \mathbf{C}_2)$ are pairs $(R_{\mathbf{O}}, R_{\mathbf{C}})$, where
  - $R_{\mathbf{O}} : \mathbf{O}_1 \to \mathbf{O}_2$ is a 1-morphism in $\mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}}$;
  - $R_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_2$ is a map of $\mathbf{O}_1$-module categories.

1.3.2. We claim that there is a canonically defined 1-fully faithful functor [3]

$$\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}} \to (\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}})^{1\text{-}\mathrm{Cat}}.$$

Indeed, the functor in question is the identity on objects. At the level of 1-morphisms its essential image corresponds to those pairs $(\mathbf{M}, \mathbf{1}_{\mathbf{M}})$, for which the functor

$$\mathbf{O}_2 \to \mathbf{M},$$

defined by $\mathbf{1}_{\mathbf{M}}$, is an equivalence.

1.3.3. Let $\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}$ be the $(\infty,1)$-category, where

- The objects are the same as those of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$;
- 1-morphisms between $(\mathbf{O}_1, \mathbf{C}_1) \to (\mathbf{O}_2, \mathbf{C}_2)$ are pairs $(R_{\mathbf{O}}, R_{\mathbf{C}})$, where
  - $R_{\mathbf{O}} : \mathbf{O}_2 \to \mathbf{O}_1$ (note the direction of the arrow!) is a 1-morphism in $\mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}}$;
  - $R_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_2$ is a map of $\mathbf{O}_2$-module categories.

1.3.4. We claim that there is a canonically defined 2-fully faithful functor

$$\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}} \to (\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}})^{1\text{-}\mathrm{Cat}}.$$

Indeed, the functor in question is the identity on objects. At the level of 1-morphisms its essential image corresponds to those pairs $(\mathbf{M}, \mathbf{1}_{\mathbf{M}})$, for which the functor

$$\mathbf{O}_1 \to \mathbf{M},$$

defined by $\mathbf{1}_{\mathbf{M}}$, is an equivalence.

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[2]We emphasize that the latter is considered as a space and not as an $(\infty,1)$-category.

[3]We remind that a functor between $(\infty,2)$-categories is said to be 1-fully faithful if it defines a fully faithful functor on $(\infty,1)$-categories of 1-morphisms.

2. The functor of IndCoh, equipped with the action of QCoh

In this section we will construct the assignment

$$X \rightsquigarrow (\mathrm{QCoh}(X), \mathrm{IndCoh}(X))$$

as a functor from $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ to the $(\infty,2)$-category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ defined in the previous section.

2.1. **The goal.** In this subsection we explain the idea behind the assignment

$$X \rightsquigarrow (\mathrm{QCoh}(X), \mathrm{IndCoh}(X)).$$

2.1.1. In [Chapter II.2, Sect. 2], we constructed the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}.$$

In this section we will extend this functor to a functor

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}.$$

The original functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ is recovered from $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ by composing with the forgetful functor of Sect. 1.2.1.

2.1.2. The meaning of the functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ is that it 'remembers' the category $\mathrm{IndCoh}(-)$ together with the action og $\mathrm{QCoh}(-)$.

Namely, we recall that for $X \in \mathrm{Sch}_{\mathrm{aft}}$, the category $\mathrm{IndCoh}(X)$ is naturally a module over the monoidal category $\mathrm{QCoh}(X)$, see [Chapter II.1, Sect. 1.2.9].

Now, the functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ encodes the fact that for $f : X \to Y$, the functors

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X) \text{ and } f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

each has a natural structure of morphim in $\mathrm{QCoh}(Y)\text{-}\mathbf{mod}$, where $\mathrm{QCoh}(Y)$ acts on $\mathrm{IndCoh}(X)$ via the monoidal functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$.

Moreover, if f is proper, the $(f^{\mathrm{IndCoh}}_*, f^!)$-adjunction also upgrades to one in the $(\infty,2)$-category $\mathrm{QCoh}(Y)\text{-}\mathbf{mod}^{2\text{-Cat}}$.

2.2. **The input.** As in the case of IndCoh, the only input for the construction of the functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ is the ability to take direct images. However, this time we need to do it for IndCoh and QCoh simultaneously, in a compatible way.

In this subsection we construct the required direct image procedure.

2.2.1. Recall the categories $\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$ and $\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}$.

Let

$$(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}} \subset \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$$

and

$$(\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}} \subset \mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}$$

be 1-full subcategories, where we restrict 1-morphisms to those pairs $(R_{\mathbf{O}}, R_{\mathbf{C}})$, where we require that $R_{\mathbf{C}}$ admit a right (resp., left) adjoint in the $(\infty,2)$-category of $\mathbf{O}_1\text{-}\mathbf{mod}^{2\text{-Cat}}$ (resp., $\mathbf{O}_2\text{-}\mathbf{mod}^{2\text{-Cat}}$).

The operation of passing to the right/left adjoint (see [Chapter A.3, Corollary 1.3.4]) defines an equivalence

$$\left((\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}}\right)^{\mathrm{op}} \simeq (\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}}. \tag{2.1}$$

2.2.2. Note that we have a tautologically defined functor of $(\infty,1)$-categories

$$\mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}, \quad \mathbf{O} \mapsto (\mathbf{O},\mathbf{O}). \tag{2.2}$$

We start with the functor $\mathrm{QCoh}^*_{\mathrm{Sch_{aft}}}$, considered as a functor

$$(\mathrm{Sch_{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{Mon}}_{\mathrm{cont}},$$

and consider its composition with (2.2). We obtain a functor

$$(\mathrm{QCoh}^*,\mathrm{QCoh}^*)_{\mathrm{Sch_{aft}}} : (\mathrm{Sch_{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}. \tag{2.3}$$

2.2.3. It is easy to see that the functor (2.3) factors (automatically, canonically) via the subcategory

$$(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}} \subset \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}.$$

Thus, we obtain a functor

$$(\mathrm{Sch_{aft}})^{\mathrm{op}} \to (\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}}. \tag{2.4}$$

Composing (2.5) with the equivalence (2.1) we obtain a functor

$$\mathrm{Sch_{aft}} \to (\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}}. \tag{2.5}$$

We follow (2.5) by the forgetful functor

$$(\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}})_{\mathrm{adjtble}} \subset \mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}$$

and obtain a functor

$$(\mathrm{QCoh}^*,\mathrm{QCoh}_*)_{\mathrm{Sch_{aft}}} : \mathrm{Sch_{aft}} \to \mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}. \tag{2.6}$$

Explicitly, the functor (2.6) sends $X \in \mathrm{Sch_{aft}}$ to the pair $(\mathrm{QCoh}(X),\mathrm{QCoh}(X))$, and a morphism $f : X \to Y$ to the pair

$$f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X), \quad f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y).$$

2.2.4. Recall that for an individual object $X \in \mathrm{Sch_{aft}}$, the DG category $\mathrm{IndCoh}(X)$ carries a canonical action of the monoidal category $\mathrm{QCoh}(X)$, see [Chapter II.1, Sect. 1.2.9].

By construction, the functor

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$

carries a unique structure of map of $\mathrm{QCoh}(X)$-module categories.

Furthermore, the following results from the construction in [Chapter II.1, Proposition 2.2.1]:

Lemma 2.2.5. *For a map $f : X \to Y$, the functor*

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

can be equipped with a unique structure of map of $\mathrm{QCoh}(Y)$-module categories, in such a way that

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\Psi_X} & \mathrm{QCoh}(X) \\ {\scriptstyle f^{\mathrm{IndCoh}}_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \mathrm{QCoh}(Y) \end{array}$$

is a commutative diagram in $\mathrm{QCoh}(Y)\text{-}\mathbf{mod}$.

From here we obtain:

Corollary 2.2.6. *There exists a uniquely defined functor*

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}},$$

equipped with a natural transformation

$$(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}} \Rightarrow (\mathrm{QCoh}^*, \mathrm{QCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}},$$

such that

- *The composition of* $(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}$ *and* $(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}}$ *with the forgetful functor*
$$\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}} \to (\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}})^{\mathrm{op}}$$
is the identity on $\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}}$*;*
- *At the level of objects and 1-morphisms,* $(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}}$ *is given by the structure specified in Lemma 2.2.5.*
- *The composition of* $(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}$ *and* $(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}}$ *with the forgetful functor*
$$\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}} \to \mathrm{DGCat}_{\mathrm{cont}}$$
is the pair $(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}, \Psi_{\mathrm{Sch}_{\mathrm{aft}}})$ *of* [Chapter II.1, Proposition 2.2.3].

2.3. **The construction.** In this subsection we will finally construct the sought-for functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$. The method will be analogous to that by which we constructed the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ in [Chapter II.2, Sect. 2.1].

2.3.1. As in the case of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$, the point of departure for the sought-for functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ is a functor

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}. \tag{2.7}$$

To construct the functor (2.7) we proceed as follows. We start with the functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}, \tag{2.8}$$

of Corollary 2.2.6, and compose it with the functor of Sect. 1.3.4 to obtain the desired functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Sch}_{\mathrm{aft}}}$ in (2.7).

2.3.2. We shall now extend the functor (2.7) to a functor

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}. \tag{2.9}$$

As in [Chapter II.2, Sect. 2.1.2], in order to do so, it suffices to prove:

Proposition 2.3.3. *The functor* $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Sch}_{\mathrm{aft}}}$*, viewed as a functor*

$$\mathrm{Sch}_{\mathrm{aft}} \to \left(\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}\right)^{2\text{-op}},$$

satisfies the left Beck-Chevalley condition *with respect to open embeddings.*

This proposition will be proved in Sect. 2.4. We proceed with the construction of the functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$.

2.3.4. We will now show that the functor (2.9) admits a unique extension to a functor

$$(\text{QCoh}, \text{IndCoh})_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{all;all}}} : \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{all;all}} \to \text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}}. \tag{2.10}$$

As in [Chapter II.2, Sect. 2.1.5], we need to verify the following two statements. One is the next proposition, proved in Sect. 2.5:

Proposition 2.3.5. *The functor* $(\text{QCoh}, \text{IndCoh})_{\text{Sch}_{\text{aft}}}$ *satisfies the* left Beck-Chevalley condition *with respect to the class of proper maps.*

Another is that the condition of [Chapter V.1, Sect. 5.2.2] is satisfied. This will be proved in Sect. 2.6.

2.4. **Open adjunction.** In this subsection we will prove Proposition 2.3.3.

2.4.1. Recall that the value of $(\text{QCoh}, \text{IndCoh})_{\text{Sch}_{\text{aft}}}$ on $X \in \text{Sch}_{\text{aft}}$ is $(\text{QCoh}(X), \text{IndCoh}(X))$, where $\text{QCoh}(X)$ acts on $\text{IndCoh}(X)$ as in [Chapter II.1, Sect. 1.1.5].

For a morphism $f : X \to Y$, the 1-morphism

$$(\text{QCoh}(X), \text{IndCoh}(X)) \to (\text{QCoh}(Y), \text{IndCoh}(Y)) \in \text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}}. \tag{2.11}$$

is given by the pair $(\mathbf{M}, F, \mathbf{1_M})$, where

- $$\mathbf{M} := \text{QCoh}(X),$$
  regarded as an $\text{QCoh}(X)$-module tautologically and as a $\text{QCoh}(Y)$-module via the functor $f^* : \text{QCoh}(Y) \to \text{QCoh}(X)$.
- $$F : \text{IndCoh}(X) \simeq \text{QCoh}(X) \underset{\text{QCoh}(X)}{\otimes} \text{IndCoh}(X) \to \text{IndCoh}(Y)$$
  is the functor f^{IndCoh}_*;
- $\mathbf{1_M} = \mathcal{O}_X$.

2.4.2. Let f be an open embedding. We need to show that in this case the corresponding 1-morphism (2.11) admits a *left* adjoint, and that the corresponding base change property holds.

We construct the left adjoint

$$(\text{QCoh}(Y), \text{IndCoh}(Y)) \to (\text{QCoh}(X), \text{IndCoh}(X)) \in \text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}}$$

as follows. It is given by the pair $(\mathbf{N}, G, \mathbf{1_N})$, where

- $$\mathbf{N} := \text{QCoh}(X),$$
  regarded as an $\text{QCoh}(X)$-module tautologically and as a $\text{QCoh}(Y)$-module via the functor $f^* : \text{QCoh}(Y) \to \text{QCoh}(X)$.
- $$G : \text{QCoh}(X) \underset{\text{QCoh}(Y)}{\otimes} \text{IndCoh}(Y) \to \text{IndCoh}(X)$$
  is obtained by tensoring up from the functor $f^{\text{IndCoh},*} : \text{QCoh}(Y) \to \text{QCoh}(Y)$;
- $\mathbf{1_N} = \mathcal{O}_X$.

Let us construct the adjunction data.

2.4.3. The composition

$$(\mathbf{M}, F, \mathbf{1}_{\mathbf{M}}) \circ (\mathbf{N}, G, \mathbf{1}_{\mathbf{N}}) : \\ (\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y)) \to (\mathrm{QCoh}(X), \mathrm{IndCoh}(X)) \to (\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y))$$

is given by:

- The $(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y))$-bimodule $\mathrm{QCoh}(X)$;
- The functor
$$\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y),$$
which under the identification
$$\mathrm{IndCoh}(X) \simeq \mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y)$$
of [Chapter II.1, Proposition 4.1.6] goes over to
$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y);$$
- The object $\mathcal{O}_X \in \mathrm{QCoh}(X)$.

The unit of the adjunction is a 2-morphism (Φ, T, ψ)

$$\mathrm{Id}_{(\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y))} \to (\mathbf{M}, F, \mathbf{1}_{\mathbf{M}}) \circ (\mathbf{N}, G, \mathbf{1}_{\mathbf{N}}),$$

where

- Φ is the functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$;
- T is the identity natural transformation on f_*^{IndCoh};
- ψ is the canonical map from $\mathcal{O}_X$ to $f_*(\mathcal{O}_Y)$.

2.4.4. The composition

$$(\mathbf{N}, G, \mathbf{1}_{\mathbf{N}}) \circ (\mathbf{M}, F, \mathbf{1}_{\mathbf{M}}) : \\ (\mathrm{QCoh}(X), \mathrm{IndCoh}(X)) \to (\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y)) \to (\mathrm{QCoh}(X), \mathrm{IndCoh}(X))$$

is given by:

- The bimodule $\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X)$;
- The identity functor on $\mathrm{IndCoh}(X)$;
- The object $\mathcal{O}_X \in \mathrm{QCoh}(X)$.

The co-unit for the adjunction is a 2-morphism (Φ, T, ψ)

$$(\mathbf{N}, G, \mathbf{1}_{\mathbf{N}}) \circ (\mathbf{M}, F, \mathbf{1}_{\mathbf{M}}) \to \mathrm{Id}_{(\mathrm{QCoh}(X), \mathrm{IndCoh}(X))},$$

where

- Φ is the identity functor;
- T is the identity functor;
- ψ is the identity map.

The fact that the unit and co-unit specified above indeed satisfy the adjunction axioms is a straightforward verification.

2.4.5. We will now verify the base change property for the open adjunction. Let

$$\begin{array}{ccc} U_1 & \xrightarrow{j_1} & X_1 \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f_X} \\ U_2 & \xrightarrow{j_2} & X_2 \end{array}$$

be a Cartesian diagram in $\mathrm{Sch}_{\mathrm{aft}}$, in which the horizontal arrows are open. Consider the commutative diagram of 1-morphisms in $\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}$

$$\begin{array}{ccc} (\mathrm{QCoh}(U_1), \mathrm{IndCoh}(U_1)) & \longrightarrow & (\mathrm{QCoh}(X_1), \mathrm{IndCoh}(X_1)) \\ \downarrow & & \downarrow \\ (\mathrm{QCoh}(U_2), \mathrm{IndCoh}(U_2)) & \longrightarrow & (\mathrm{QCoh}(X_2), \mathrm{IndCoh}(X_2)). \end{array}$$

By passing to left adjoints along the horizontal arrows, we obtain a diagram that commutes up to a 2-morphism as indicated:

$$\begin{array}{ccc} (\mathrm{QCoh}(U_1), \mathrm{IndCoh}(U_1)) & \longleftarrow & (\mathrm{QCoh}(X_1), \mathrm{IndCoh}(X_1)) \\ \downarrow & \Uparrow & \downarrow \\ (\mathrm{QCoh}(U_2), \mathrm{IndCoh}(U_2)) & \longleftarrow & (\mathrm{QCoh}(X_2), \mathrm{IndCoh}(X_2)). \end{array} \tag{2.12}$$

We need to show that the 2-morphism in question is an isomorphism.

For the clockwise circuit in (2.12), the corresponding $(\mathrm{QCoh}(X_1), \mathrm{QCoh}(U_2))$-bimodule is $\mathrm{QCoh}(U_1)$, equipped with the distinuished object $\mathcal{O}_{U_1} \in \mathrm{QCoh}(U_1)$.

For the anti-clockwise circuit, the corresponding $(\mathrm{QCoh}(X_1), \mathrm{QCoh}(U_2))$-bimodule is

$$\mathrm{QCoh}(U_2) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{QCoh}(X_1),$$

equippped with the distinguished object $\mathcal{O}_{U_2} \underset{X_2}{\boxtimes} \mathcal{O}_{X_1}$.

The datum Φ of the 2-morphism in (2.12) is the canonical equivalence

$$\mathrm{QCoh}(U_1) \simeq \mathrm{QCoh}(U_2) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{QCoh}(X_1),$$

and the datum ψ is the identity map on $\mathrm{QCoh}(U_1)$.

Under the above identification

$$\mathrm{QCoh}(U_2) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{QCoh}(X_1) \simeq \mathrm{QCoh}(U_1),$$

the datum of F of the 2-morphism in (2.12) is the identity map on the functor

$$(f_U)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(U_1) \simeq \mathrm{QCoh}(U_1) \underset{\mathrm{QCoh}(X_1)}{\otimes} \mathrm{IndCoh}(X_1) \to \mathrm{IndCoh}(U_2).$$

2.5. **Proper adjunction.** In this subsection we will prove Proposition 2.3.5.

2.5.1. Let now f be proper. We need to show that in this case the corresponding 1-morphism (2.11) admits a *right* adjoint.

We construct the right adjoint

$$(\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y)) \to (\mathrm{QCoh}(X), \mathrm{IndCoh}(X)) \in \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$$

as follows. It is given by the pair $(\mathbf{N}, G, \mathbf{1_N})$, where

- $\mathbf{N} := \mathrm{QCoh}(X)$;
- The functor

$$G : \mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$$

  is obtained by tensoring up from the functor $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ (here we are using [Chapter II.1, Sect. 5.1.7]).
- The object $\mathbf{1_N}$ is $\mathcal{O}_X \in \mathrm{QCoh}(X)$.

Let us construct the adjunction data.

2.5.2. The composition

$$(\mathbf{M}, F, \mathbf{1_M}) \circ (\mathbf{N}, G, \mathbf{1_N}) :$$
$$(\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y)) \to (\mathrm{QCoh}(X), \mathrm{IndCoh}(X)) \to (\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y))$$

is given by:

- The $(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y))$-bimodule $\mathrm{QCoh}(X)$;
- The functor $\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y)$ is

$$(\mathcal{E}, \mathcal{F}) \mapsto f^{\mathrm{IndCoh}}_*(\mathcal{E} \otimes f^!(\mathcal{F}));$$

- The object $\mathcal{O}_X \in \mathrm{QCoh}(X)$.

The co-unit of the adjunction is a 2-morphism (Φ, T, ψ)

$$(\mathbf{M}, F, \mathbf{1_M}) \circ (\mathbf{N}, G, \mathbf{1_N}) \to \mathrm{Id}_{(\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y))},$$

where

- Φ is the functor f^*;
- T is the natural transformation between $f^{\mathrm{IndCoh}}_* \circ f^!$ and $\mathrm{Id}_{\mathrm{IndCoh}(Y)}$ equal to the co-unit of the $(f^{\mathrm{IndCoh}}_*, f^!)$-adjunction
- ψ is the natural isomorphism.

2.5.3. The composition

$$(\mathbf{N}, G, \mathbf{1_N}) \circ (\mathbf{M}, F, \mathbf{1_M}) :$$
$$(\mathrm{QCoh}(X), \mathrm{IndCoh}(X)) \to (\mathrm{QCoh}(Y), \mathrm{IndCoh}(Y)) \to (\mathrm{QCoh}(X), \mathrm{IndCoh}(X))$$

is given by:

- The $(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$-bimodule $\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \underset{Y}{\times} X)$;
- The functor $\mathrm{QCoh}(X \underset{Y}{\times} X) \underset{\mathrm{QCoh}(X)}{\otimes} \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$ is

$$(\mathcal{E}, \mathcal{F}) \mapsto (p_2)^{\mathrm{IndCoh}}_*(\mathcal{E} \otimes p_1^!(\mathcal{F}));$$

- The object $\mathcal{O}_{X \underset{Y}{\times} X} \in \mathrm{QCoh}(X \underset{Y}{\times} X)$;

The unit of the adjunction is a 2-morphism (Φ, T, ψ)

$$\mathrm{Id}_{(\mathrm{QCoh}(X),\mathrm{IndCoh}(X))} \to (\mathbf{N}, G, \mathbf{1_N}) \circ (\mathbf{M}, F, \mathbf{1_M}),$$

where

- Φ is the functor $\Delta^*_{X/Y}$, where $\Delta_{X/Y}$ is the diagonal map
$$X \to X \underset{Y}{\times} X;$$
- T is the natural transformation $\Delta^*_{X/Y}(\mathcal{E}) \otimes \mathcal{F} \to (p_2)^{\mathrm{IndCoh}}_*(\mathcal{E} \otimes p_1^!(\mathcal{F}))$ equal to

$$\Delta^*_{X/Y}(\mathcal{E}) \otimes \mathcal{F} \simeq (p_2)^{\mathrm{IndCoh}}_* \circ (\Delta_{X/Y})^{\mathrm{IndCoh}}_* \left(\Delta^*_{X/Y}(\mathcal{E}) \otimes \mathcal{F}\right) \simeq$$
$$\simeq (p_2)^{\mathrm{IndCoh}}_* \left(\mathcal{E} \otimes (\Delta_{X/Y})^{\mathrm{IndCoh}}_*(\mathcal{F})\right) \simeq (p_2)^{\mathrm{IndCoh}}_* \left(\mathcal{E} \otimes (\Delta_{X/Y})^{\mathrm{IndCoh}}_* \circ \Delta^!_{X/Y} \circ p_1^!(\mathcal{F})\right) \to$$
$$\to (p_2)^{\mathrm{IndCoh}}_* \left(\mathcal{E} \otimes p_1^!(\mathcal{F})\right),$$

  where the last arrow coves from the co-unit for the $((\Delta_{X/Y})^{\mathrm{IndCoh}}_*, \Delta^!_{X/Y})$-adjunction.
- ψ is the natural isomorphism.

Again, the fact that the unit and co-unit specified above indeed satisfy the adjunction axioms is a straightforward verification.

2.5.4. We will now verify the base change property for the proper adjunction. Let

$$\begin{array}{ccc} Y_1 & \xrightarrow{g_1} & X_1 \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f_X} \\ Y_2 & \xrightarrow{g_2} & X_2 \end{array}$$

be a Cartesian diagram in $\mathrm{Sch}_{\mathrm{aft}}$, in which the vertical arrows are proper. Consider the commutative diagram of 1-morphisms in $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$

$$\begin{array}{ccc} (\mathrm{QCoh}(Y_1), \mathrm{IndCoh}(Y_1)) & \longrightarrow & (\mathrm{QCoh}(X_1), \mathrm{IndCoh}(X_1)) \\ \downarrow & & \downarrow \\ (\mathrm{QCoh}(Y_2), \mathrm{IndCoh}(Y_2)) & \longrightarrow & (\mathrm{QCoh}(X_2), \mathrm{IndCoh}(X_2)). \end{array}$$

By passing to right adjoints along the vertical arrows, we obtain a diagram that commutes up to a 2-morphism as indicated:

(2.13)
$$\begin{array}{ccc} (\mathrm{QCoh}(Y_1), \mathrm{IndCoh}(Y_1)) & \longrightarrow & (\mathrm{QCoh}(X_1), \mathrm{IndCoh}(X_1)) \\ \uparrow & \searrow\!\!\!\!\searrow & \uparrow \\ (\mathrm{QCoh}(Y_2), \mathrm{IndCoh}(Y_2)) & \longrightarrow & (\mathrm{QCoh}(X_2), \mathrm{IndCoh}(X_2)). \end{array}$$

We need to show that the 2-morphism in question is an isomorphism.

For the clockwise circuit in (2.13), the corresponding $(\mathrm{QCoh}(X_1), \mathrm{QCoh}(Y_2))$-module is by definition $\mathrm{QCoh}(Y_1)$, equipped with the distinguished object $\mathcal{O}_{Y_1} \in \mathrm{QCoh}(Y_1)$.

For the anti-clockwise circuit in (2.13), the corresponding $(\mathrm{QCoh}(X_1), \mathrm{QCoh}(Y_2))$-module is

$$\mathrm{QCoh}(X_1) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{QCoh}(Y_2),$$

equipped with the distinguished object $\mathcal{O}_{X_1} \underset{X_1}{\boxtimes} \mathcal{O}_{Y_2}$.

The datum of Φ of the 2-morphism in (2.13) is the canonical functor

$$\mathrm{QCoh}(X_1) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{QCoh}(Y_2) \to \mathrm{QCoh}(X_1),$$

which is known to be an equivalence (see, e.g., [Chapter I.3, Proposition 3.5.3]). Under this identification, the datum of ψ of the 2-morphism in (2.13) is the identity isomorphism on $\mathcal{O}_{Y_1}$.

It remains to show that the natural transformation T is an isomorphism.

For a triple

$$\mathcal{E}_1 \in \mathrm{QCoh}(X_1), \quad \mathcal{E}_2 \in \mathrm{QCoh}(Y_2), \quad \mathcal{F} \in \mathrm{IndCoh}(Y_2)$$

and the corresponding object

$$(\mathcal{E}_1 \otimes \mathcal{E}_2) \otimes \mathcal{F} \in \left(\mathrm{QCoh}(X_1) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{QCoh}(Y_2)\right) \underset{\mathrm{QCoh}(Y_2)}{\otimes} \mathrm{IndCoh}(Y_2),$$

the functor F^t sends it to

$$\mathcal{E}_1 \otimes \left(f_X^! \circ (g_2)_*^{\mathrm{IndCoh}}(\mathcal{E}_2 \otimes \mathcal{F})\right),$$

and the functor $F^s \circ (\Phi \otimes \mathrm{Id}_{\mathbf{C}_1})$ sends it to

$$(g_1)_*^{\mathrm{IndCoh}}\left((g_1^*(\mathcal{E}_1) \otimes f_Y^*(\mathcal{E}_2)) \otimes f_Y^!(\mathcal{F})\right).$$

Under the above identifications, the natutal transformation T acts as follows:

$$(g_1)_*^{\mathrm{IndCoh}}\left((g_1^*(\mathcal{E}_1) \otimes f_Y^*(\mathcal{E}_2)) \otimes f_Y^!(\mathcal{F})\right) \simeq (g_1)_*^{\mathrm{IndCoh}}\left((g_1^*(\mathcal{E}_1) \otimes f_Y^!(\mathcal{E}_2 \otimes \mathcal{F})\right) \simeq$$
$$\simeq \mathcal{E}_1 \otimes (g_1)_*^{\mathrm{IndCoh}}\left(f_Y^!(\mathcal{E}_2 \otimes \mathcal{F})\right) \simeq \mathcal{E}_1 \otimes f_X^!\left((g_2)_*^{\mathrm{IndCoh}}(\mathcal{E}_2 \otimes \mathcal{F})\right),$$

which is an isomorphism, as required.

2.6. **Verification of compatibility.** In this subsection we will show that the condition of [Chapter V.1, Sect. 5.2.2] is satisfied for the functor $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Sch}_{\mathrm{aft}}}$.

2.6.1. Let

$$\begin{array}{ccc} U_1 & \xrightarrow{j_1} & X_1 \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f_X} \\ U_2 & \xrightarrow{j_2} & X_2 \end{array}$$

be a Cartesian diagram in $\mathrm{Sch}_{\mathrm{aft}}$, with the vertical arrows being proper and the horizontal arrows being open.

According to [Chapter V.1, Sect. 5.2.2], from the base change isomorphism of Sect. 2.4.5, we obtain a 2-morphism

$$(2.14)\qquad \begin{array}{ccc} (\mathrm{QCoh}(U_1), \mathrm{IndCoh}(U_1)) & \longleftarrow & (\mathrm{QCoh}(X_1), \mathrm{IndCoh}(X_1)) \\ \uparrow & \swArrow & \uparrow \\ (\mathrm{QCoh}(U_2), \mathrm{IndCoh}(U_2)) & \longleftarrow & (\mathrm{QCoh}(X_2), \mathrm{IndCoh}(X_2)). \end{array}$$

We need to show that this morphism is an isomorphism.

2.6.2. By the description of the left and right adjoint functors in Sects. 2.4 and 2.5, the $(\mathrm{QCoh}(U_1), \mathrm{QCoh}(X_2))$-bimodule corresponding to both circuits in the diagram (2.14) is the category $\mathrm{QCoh}(U_1)$, equipped with the distinguished object $\mathcal{O}_{U_1} \in \mathrm{QCoh}(U_1)$.

Under this identification, the data of Φ and γ in the 2-morphism in (2.14) are the identity maps. Hence, it remains to show that the natural transformation T is an isomorphism.

2.6.3. The natural transformation T is a 2-morphism in $\mathrm{QCoh}(U_1)\text{-}\mathbf{mod}^{2\text{-Cat}}$ between two functors

$$\mathrm{QCoh}(U_1) \underset{\mathrm{QCoh}(X_2)}{\otimes} \mathrm{IndCoh}(X_2) \rightrightarrows \mathrm{IndCoh}(U_1).$$

Such functors and natural transformations are in bijection with those in $\mathrm{QCoh}(X_2)\text{-}\mathbf{mod}^{2\text{-Cat}}$

$$\mathrm{IndCoh}(X_2) \rightrightarrows \mathrm{IndCoh}(U_1).$$

Now, the assertion follows from the fact that in the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(U_1) & \xleftarrow{j_1^{\mathrm{IndCoh},*}} & \mathrm{IndCoh}(X_1) \\ {\scriptstyle f_U^!}\uparrow & \swArrow & \uparrow{\scriptstyle f_X^!} \\ \mathrm{IndCoh}(U_2) & \xleftarrow[j_2^{\mathrm{IndCoh},*}]{} & \mathrm{IndCoh}(X_2) \end{array}$$

the 2-morphism is an isomorphism (which is [Chapter II.1, Proposition 5.3.4]).

3. The multiplicative structure

In this section we will further amplify the functor

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}.$$

Namely, we will show that it has a natural symmetric monoidal structure.

This will imply certain expected properties of IndCoh, regarded as equipped with an action of QCoh.

3.1. **Upgrading to a symmetric monoidal functor.** In this subsection we state the existence of $(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ as a symmetric monoidal functor.

3.1.1. Recall that the $(\infty,2)$-category $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}$ carries a natural symmetric monoidal functor, which at the level of objects is given by Cartesian product.

Note that the $(\infty,2)$-category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$ also carries a symmetric monoidal structure, given by term-wise tensor product.

3.1.2. As in [Chapter II.2, Theorem 4.1.2], we have:

Theorem 3.1.3. *The functor*

$$(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}:\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}\to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$$

carries a canonical symmetric monoidal structure.

3.2. **Consequences for the !-pullback.** In this subsection we will show that the action of QCoh on IndCoh comes from a symmetric monoidal functor

$$\mathrm{QCoh}(-)\to \mathrm{IndCoh}(-),$$

whose formation is compatible with pullbacks (the *-pullback for QCoh and the !-pullback for IndCoh).

3.2.1. Let us restrict the functor $(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$ to

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}\subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}.$$

We obtain a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}\to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}. \tag{3.1}$$

However, the explicit description of the !-pullback functors in Sects. 2.4 and 2.5 imply that the functor (3.1) factors (automatically canonically) through the 1-fullly faithful functor

$$\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}.$$

We denote the resulting functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}\to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$$

by $(\mathrm{QCoh}^*,\mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}$.

Remark 3.2.2. The meaning of the functor $(\mathrm{QCoh}^*,\mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}$ is that it encodes that for a map $f:X\to Y$, the functor $f^!:\mathrm{IndCoh}(Y)\to\mathrm{IndCoh}(X)$ has a natural structure of map of $\mathrm{QCoh}(Y)$-module categories. By contrast with $(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$, we discard the information pertaining to the functor f^{IndCoh}_*.

3.2.3. Taking into account Theorem 3.1.3, we obtain that the functor $(\mathrm{QCoh}^*,\mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}$ has a natural symmetric monoidal structure with respect to the symmetric monoidal structure on $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}$, induced by the Cartesian product on $\mathrm{Sch}_{\mathrm{aft}}$ and the symmetric monoidal structure on $\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$, given by term-wise tensor product.

Recall now (see [Chapter II.2, Sect. 4.1.3]) that any symmetric monoidal functor from $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}$ with values in a symmetric monoidal category naturally upgrades to a functor from $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}$ with values in the category of commutative algebras in that symmetric monoidal category.

In particular, we obtain that the functor $(\mathrm{QCoh}^*,\mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}$ naturally extends to a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}\to \mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right). \tag{3.2}$$

The composition of the functor (3.2) with the forgetful functor

$$\mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right) \to \mathrm{ComAlg}\left(\mathrm{DGCat}_{\mathrm{cont}}\right) = \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}$$

is the functor of [Chapter II.2, Formula (4.1)].

3.2.4. Note that the category $\mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right)$ can be identified with the category

$$\mathrm{Funct}([1], \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}),$$

i.e., the category of triples

$$(\mathbf{O}, \mathbf{O}', \alpha : \mathbf{O} \to \mathbf{O}'),$$

where $\mathbf{O}$ and $\mathbf{O}'$ are symmetric monoidal categories, and α is a symmetric monoidal functor.

Hence, the content of the functor (3.2) is that the assignment

$$X \rightsquigarrow (\mathrm{QCoh}(X), \mathrm{IndCoh}(X))$$

is the functor from the category (opposite to that) of schemes to the category of pairs of symmetric monoidal DG categories, where:

- $\mathrm{QCoh}(X)$ is a symmetric monoidal DG category via the usual $*$-tensor product operation;
- $\mathrm{IndCoh}(X)$ is a symmetric monoidal DG category via the usual !-tensor product operation (see [Chapter II.2, Sect. 4.1.3])
- The symmetric monoidal functor $\mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$ is given by the action of $\mathrm{QCoh}(X)$ on the unit object in $\mathrm{IndCoh}(X)$, when $\mathrm{IndCoh}(X)$ is regarded as a $\mathrm{QCoh}(X)$-module category.

3.2.5. As a consequence, we obtain a natural transformation between the functors

$$(\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}} \Rightarrow (\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}, \quad (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}.$$

In particular, we obtain a natural transformation between the functors

$$\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}} \Rightarrow \mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}, \quad (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

We denote the latter transformation by $\Upsilon_{\mathrm{Sch}_{\mathrm{aft}}}$ and the former by

$$(\mathrm{Id}, \Upsilon)_{\mathrm{Sch}_{\mathrm{aft}}}.$$

For an individual scheme X, we will denote the corresponding fonctor

$$\mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$$

by Υ_X.

By construction, this functor is given by

$$\mathcal{E} \mapsto \mathcal{E} \otimes \omega_X, \quad \mathcal{E} \in \mathrm{QCoh}(X),$$

where $\otimes$ is the action of $\mathrm{QCoh}(X)$ on IndCoh_X, and ω_X is the dualizing object of $\mathrm{IndCoh}(X)$ (the !-pullback of k under $X \to \mathrm{pt}$, or equivalently, the unit for the !-symmetric monoidal structure on $\mathrm{IndCoh}(X)$).

3.3. **Extension to prestacks.** We will now use the theory developed above, to show that for a prestack $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ acquires a natural action of the monoidal category $\mathrm{QCoh}(\mathcal{X})$.

3.3.1. We consider again the functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right)$$

of (3.2) and apply the right Kan extension along

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

Denote the resulting functor by

$$(\widetilde{\mathrm{QCoh}}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right).$$

The value of this funcor on a given prestack $\mathcal{Y}$ is $(\widetilde{\mathrm{QCoh}}(\mathcal{Y}), \mathrm{IndCoh}(\mathcal{Y}))$, where

$$\widetilde{\mathrm{QCoh}}^* : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}$$

is right Kan extension of $\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}}$ along $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}$, i.e.,

$$\widetilde{\mathrm{QCoh}}(\mathcal{Y}) := \lim_{X \in ((\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(X).$$

Remark 3.3.2. The difference between $\widetilde{\mathrm{QCoh}}(\mathcal{Y})$ and $\mathrm{QCoh}(\mathcal{Y})$ is that in the latter we take the limit of $\mathrm{QCoh}(X)$ over all schemes X mapping to $\mathcal{Y}$, and in the former only $X \in \mathrm{Sch}_{\mathrm{aft}}$. Under some (mild) conditions on $\mathcal{Y}$, the restriction functor $\mathrm{QCoh}(\mathcal{Y}) \to \widetilde{\mathrm{QCoh}}(\mathcal{Y})$ is an equivalence. For example, this happens if $\mathcal{Y}$ admits *deformation theory*, see [Chapter III.1, Proposition 9.1.4].

3.3.3. Note that there is a canonically defined natural transformation of functors

$$\mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{laft}}} \Rightarrow \widetilde{\mathrm{QCoh}}_{\mathrm{PreStk}_{\mathrm{laft}}}, (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}.$$

Composing with the functor $(\widetilde{\mathrm{QCoh}}^*, \mathrm{IndCoh}^!)_{\mathrm{PreStk}_{\mathrm{laft}}}$, we obtain a functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right).$$

The value of the latter functor on $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ is now

$$(\mathrm{QCoh}(\mathcal{Y}), \mathrm{IndCoh}(\mathcal{Y})).$$

3.3.4. The content of the functor $(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{PreStk}_{\mathrm{laft}}}$ is the natural transformation

$$\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}} : \mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{laft}}} \Rightarrow \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}},$$

as functors

$$\mathrm{PreStk}_{\mathrm{laft}} \rightrightarrows \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}.$$

For an individual $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ we shall denote the resulting functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$$

by $\Upsilon_{\mathcal{Y}}$.

3.3.5. Applying the forgetful functor $\mathrm{ComAlg}\left(\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}\right) \to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}$, we can view $(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{PreStk}_{\mathrm{laft}}}$ as a functor

$$(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}}.$$

I.e., for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, the DG category $\mathrm{IndCoh}(\mathcal{Y})$ acquires a structure of $\mathrm{QCoh}(\mathcal{Y})$-module, functorially with respect to the !-pullback on IndCoh and *-pullback on QCoh.

The functor $\Upsilon_{\mathcal{Y}}$ is given by the monoidal action of $\mathrm{QCoh}(\mathcal{Y})$ on the object $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$.

3.3.6. The following assertion is often useful:

Lemma 3.3.7. *For any* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$*, the functor*

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}} \to \mathrm{IndCoh}(\mathcal{Y})$$

is fully faithful and the essential image consists of objects dualizable with respect to the $\overset{!}{\otimes}$ *symmetric monoidal structure.*

Proof. Since both functors $\mathrm{QCoh}(-)^{\mathrm{perf}}$ and $\mathrm{IndCoh}(-)$ are convergent (by [Chapter I.3, Proposition 3.6.10] and [Chapter II.1, Proposition 6.4.3], respectively), the assertion reduces to the case when $\mathcal{Z} = S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

In this case, the functor Υ_S is fully faithful on all of $\mathrm{QCoh}(S)$.

Let $\mathcal{F} \in \mathrm{IndCoh}(S)$ be a dualizable object. Since the unit object $\omega_S \in \mathrm{IndCoh}(S)$ is compact, we obtain that $\mathcal{F}$ is compact, i.e., it belongs to $\mathrm{Coh}(S)$.

Consider $\mathcal{E} := \mathbb{D}^{\mathrm{Serre}}_S(\mathcal{F}) \in \mathrm{Coh}(S)$. It suffices to show that $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}} \subset \mathrm{Coh}(S)$. For that it suffices to show that all the *-fibers of $\mathcal{E}$ are finite-dimensional.

But the *-fibers of $\mathcal{E}$ are the duals of the !-fibers of $\mathcal{F}$, and the latter are finite-dimensional by the dualizability hypothesis: indeed, taking the !-fiber is a symmetric monoidal functor from $(\mathrm{IndCoh}(Z), \overset{!}{\otimes})$ to Vect.

□

4. Duality

In this section we will study the interaction of the Serre duality on IndCoh with the naive (i.e., usual) duality on QCoh.

4.1. Duality on the category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$.

In this subsection we will explicitly describe dualizable objects in the symmetric monoidal category $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$, and how the duality involution on dualizable objects looks like.

4.1.1. Let $(\mathbf{O}, \mathbf{C})$ be an object of $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$.

The forgetful symmetric monoidal functor

$$\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

implies that if $(\mathbf{O}, \mathbf{C})$ is dualizable with respect to the symmetric monoidal structure on $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$, then $\mathbf{C}$ is dualizable as a plain DG category.

4.1.2. Vice versa, we claim that if $\mathbf{C}$ is dualizable as a DG category, then $(\mathbf{O}, \mathbf{C})$ is dualizable in $\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$, with the dual being $(\mathbf{O}^{\text{rev-mult}}, \mathbf{C}^{\vee})$, where $\mathbf{O}^{\text{rev-mult}}$ is the monoidal category, obtained from $\mathbf{O}$ by reveresing the multiplication.

Namely, the unit

$$(\mathrm{Vect}, \mathrm{Vect}) \to (\mathbf{O}, \mathbf{C}) \otimes (\mathbf{O}^{\text{rev-mult}}, \mathbf{C}^{\vee}) = (\mathbf{O} \otimes \mathbf{O}^{\text{rev-mult}}, \mathbf{C} \otimes \mathbf{C}^{\vee})$$

is given by the data $(\mathbf{M}, F, \mathbf{1}_{\mathbf{M}})$, where:

- $\mathbf{M} = \mathbf{O}$, regarded as a $\mathbf{O} \otimes \mathbf{O}^{\text{rev-mult}}$-module category;
- F is the functor of $\mathbf{O} \to \mathbf{C} \otimes \mathbf{C}^{\vee}$, given by the action of $\mathbf{O}$ on $\mathbf{C}$;
- $\mathbf{1}_{\mathbf{M}}$ is the unit object of $\mathbf{O}$.

The co-unit

$$(\mathbf{O}^{\text{rev-mult}} \otimes \mathbf{O}, \mathbf{C}^\vee \otimes \mathbf{C}) = (\mathbf{O}^{\text{rev-mult}}, \mathbf{C}^\vee) \otimes (\mathbf{O}, \mathbf{C}) \to (\text{Vect}, \text{Vect})$$

is given by the data $(\mathbf{M}, F, \mathbf{1}_\mathbf{M})$, where:

- $\mathbf{M} = \mathbf{O}$, regarded as a *right* $\mathbf{O}^{\text{rev-mult}} \otimes \mathbf{O}$-module category;
- F is the functor

$$\mathbf{O} \underset{\mathbf{O}^{\text{rev-mult}} \otimes \mathbf{O}}{\otimes} (\mathbf{C}^\vee \otimes \mathbf{C}) \simeq \mathbf{C}^\vee \underset{\mathbf{O}}{\otimes} \mathbf{C} \to \text{Vect},$$

  where the last arrow is the canonical pairing;
- $\mathbf{1}_\mathbf{M}$ is the unit object of $\mathbf{O}$.

4.1.3. Let

$$(\text{DGCat}^{\text{Mon+Mod}}_{\text{cont}})^{\text{dualizable}} \subset \text{DGCat}^{\text{Mon+Mod}}_{\text{cont}}$$

and

$$(\text{DGCat}^{\text{Mon}^{\text{op}}+\text{Mod}}_{\text{cont}})^{\text{dualizable}} \subset \text{DGCat}^{\text{Mon}^{\text{op}}+\text{Mod}}_{\text{cont}}$$

be the full subcategories corresponding to the pairs $(\mathbf{O}, \mathbf{C})$ in which $\mathbf{C}$ is dualizable as a plain DG category.

Note that we have a canonically defined functor

$$\Big((\text{DGCat}^{\text{Mon+Mod}}_{\text{cont}})^{\text{dualizable}}\Big)^{\text{op}} \to (\text{DGCat}^{\text{Mon}^{\text{op}}+\text{Mod}}_{\text{cont}})^{\text{dualizable}}, \quad (\mathbf{O}, \mathbf{C}) \mapsto (\mathbf{O}^{\text{rev-mult}}, \mathbf{C}^\vee). \tag{4.1}$$

By Sect. 4.1.2, the functors

$$\text{DGCat}^{\text{Mon+Mod}}_{\text{cont}} \to \text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}} \leftarrow \text{DGCat}^{\text{Mon}^{\text{op}}+\text{Mod}}_{\text{cont}}$$

send the subcategories

$$(\text{DGCat}^{\text{Mon+Mod}}_{\text{cont}})^{\text{dualizable}} \text{ and } (\text{DGCat}^{\text{Mon}^{\text{op}}+\text{Mod}}_{\text{cont}})^{\text{dualizable}}$$

to $(\text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}})^{\text{dualizable}}$.

Moreover the construction of Sect. 4.1.2 can be upgraded to the following statement:

Lemma 4.1.4. *The following square*

$$\begin{array}{ccc} \Big((\text{DGCat}^{\text{Mon+Mod}}_{\text{cont}})^{\text{dualizable}}\Big)^{\text{op}} & \longrightarrow & \Big((\text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}})^{\text{dualizable}}\Big)^{\text{op}} \\ {\scriptstyle (4.1)}\downarrow & & \downarrow{\scriptstyle \text{dualization in } \text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}}} \\ (\text{DGCat}^{\text{Mon}^{\text{op}}+\text{Mod}}_{\text{cont}})^{\text{dualizable}} & \longrightarrow & (\text{DGCat}^{\text{Mon+Mod,ext}}_{\text{cont}})^{\text{dualizable}} \end{array}$$

canonically commutes.

4.2. **The linearity structure on Serre duality.** In this subsection we study how Serre duality on IndCoh is compatible with the action of QCoh.

4.2.1. Consider again the symmetric monoidal functor

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}$$

of Theorem 3.1.3. Consider its restriction

$$(\mathrm{QCoh}, \mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}}.$$

As in [Chapter II.2, Theorem 4.2.3], we obtain:

Corollary 4.2.2. *The following diagram of functors canonically commutes:*

$$\begin{array}{ccc} (\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}})^{\mathrm{op}} & \xrightarrow{((\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}})^{\mathrm{op}}} & \left((\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}})^{\mathrm{dualizable}}\right)^{\mathrm{op}} \\ \varpi \downarrow & & \downarrow \text{dualization in } \mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}} \\ \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} & \xrightarrow{(\mathrm{QCoh},\mathrm{IndCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}}} & (\mathrm{DGCat}^{\mathrm{Mon+Mod,ext}}_{\mathrm{cont}})^{\mathrm{dualizable}}, \end{array}$$

where ϖ *is as* [Chapter II.2, Sect. 4.2.2].

4.2.3. Combining Corollary 4.2.2 with Lemma 4.1.4, we obtain:

Corollary 4.2.4. *The following diagram canonically commutes*

$$\begin{array}{ccc} \mathrm{Sch}_{\mathrm{aft}} & \xrightarrow{(\mathrm{QCoh}^*,\mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}} & (\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}})^{\mathrm{dualizable}} \\ \mathrm{Id} \downarrow & & \downarrow (4.1) \\ \mathrm{Sch}_{\mathrm{aft}} & \xrightarrow{((\mathrm{QCoh}^*,\mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}})^{\mathrm{op}}} & \left((\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})^{\mathrm{dualizable}}\right)^{\mathrm{op}}. \end{array}$$

The content of Corollary 4.2.4 is that for an individual $X \in \mathrm{Sch}_{\mathrm{aft}}$, the Serre duality equivalence

$$\mathbf{D}^{\mathrm{Serre}}_X : \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X)$$

is compatible with the action of $\mathrm{QCoh}(X)$.

Furthermore, for a morphism $f : X \to Y$, the identification

$$f^! \simeq (f^{\mathrm{IndCoh}}_*)^\vee$$

is also compatible with the action of $\mathrm{QCoh}(Y)$.

4.3. **Digression:** QCoh **as a functor out of the category of correspondences.** The contents of this subsection are more or less tautological: they encode that the operation of direct image on QCoh is compatible with the action of QCoh on itself by tensor products.

4.3.1. The goal of this subsection is to prove the following analog of Corollary 4.2.4 for the pair of functors $(\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}}$ and $(\mathrm{QCoh}^*, \mathrm{QCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}$ of (2.3) and (2.6), respectively:

Proposition 4.3.2. *The following diagram canonically commutes*

$$\begin{array}{ccc} \mathrm{Sch}_{\mathrm{aft}} & \xrightarrow{(\mathrm{QCoh}^*,\mathrm{QCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}} & (\mathrm{DGCat}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}})^{\mathrm{dualizable}} \\ \mathrm{Id} \downarrow & & \downarrow (4.1) \\ \mathrm{Sch}_{\mathrm{aft}} & \xrightarrow{((\mathrm{QCoh}^*,\mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}})^{\mathrm{op}}} & \left((\mathrm{DGCat}^{\mathrm{Mon+Mod}}_{\mathrm{cont}})^{\mathrm{dualizable}}\right)^{\mathrm{op}}. \end{array}$$

The content of Proposition 4.3.2 is that for an individual $X \in \mathrm{Sch}_{\mathrm{aft}}$, there is a canonical duality equivalence

$$\mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X), \tag{4.2}$$

which is compatible with the action of $\mathrm{QCoh}(X)$.

Furthermore, for a morphism $f : X \to Y$, we have an identification

$$f^* \simeq (f_*)^\vee,$$

which is also compatible with the action of $\mathrm{QCoh}(Y)$.

4.3.3. It will follow from the construction below that the duality (4.2) is that given by the unit

$$\mathrm{Vect} \overset{k \to \mathcal{O}_X}{\longrightarrow} \mathrm{QCoh}(X) \overset{(\Delta_X)_*}{\longrightarrow} \mathrm{QCoh}(X \times X) \simeq \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X),$$

and the co-unit

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \overset{\Delta_X^*}{\longrightarrow} \mathrm{QCoh}(X) \overset{\Gamma(X,-)}{\longrightarrow} k.$$

Equivalently, the duality (4.2) is induced by the anti-self equivalence of

$$\mathrm{QCoh}(X)^c \simeq \mathrm{QCoh}(X)^{\mathrm{perf}},$$

given by

$$\mathcal{E} \mapsto \mathcal{E}^\vee := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X).$$

4.3.4. As in the case of Corollary 4.2.4, in order to prove Proposition 4.3.2, it is sufficient to construct a symmetric monoidal functor

$$(\mathrm{QCoh}, \mathrm{QCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}. \tag{4.3}$$

We will obtain $(\mathrm{QCoh}, \mathrm{QCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}}$ by restriction from a symmetric monoidal functor

$$(\mathrm{QCoh}, \mathrm{QCoh})_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{all}} \to (\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}})^{2\text{-op}}. \tag{4.4}$$

4.3.5. To construct (4.4), we start with the functor

$$(\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}},$$

of (2.3) and follow it by the functor

$$\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}$$

of Sect. 1.3.4.

We obtain a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}}. \tag{4.5}$$

4.3.6. We now claim that the functor (4.5), viewed as a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon+Mod,ext}})^{2\text{-op}}$$

satisfies the right Beck-Chevalley condition.

This is proved in the same was as in Sect. 2.4.

Now, applying [Chapter V.1, Theorem 3.2.2(b)], we obtain that the functor (4.5) uniquely gives rise to the sought-for functor (4.4).

4.4. **Compatibility with the functor Ψ.** In this subsection we will use the theory developed above to show that the dual of the functor Υ identifies with the functor Ψ of [Chapter II.1, Sect. 1.1.2].

4.4.1. Recall (see Corollary 2.2.6) that the functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}^{\mathrm{op}}+\mathrm{Mod}}$$

was constructed in such a way that it was equipped with a natural transformation

$$(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{QCoh}^*, \mathrm{IndCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}} \Rightarrow (\mathrm{QCoh}^*, \mathrm{QCoh}_*)_{\mathrm{Sch}_{\mathrm{aft}}}.$$

Applying the functor (4.1), and taking into account Corollary 4.2.4 and Proposition 4.3.2, from the natural transformation $(\mathrm{Id}, \Psi)_{\mathrm{Sch}_{\mathrm{aft}}}$, we obtain a natural transformation

$$(\mathrm{Id}, \Psi^\vee)_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}} \to (\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{Sch}_{\mathrm{aft}}}$$

as functors

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}+\mathrm{Mod}}.$$

For an individual $X \in \mathrm{Sch}_{\mathrm{aft}}$, we let the resulting functor

$$\mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$$

be denoted by $\Psi_X^\vee$.

4.4.2. The goal of this subsection is to prove the following:

Theorem 4.4.3. *There is a canonical isomomorphism of natural transformations*

$$(\mathrm{Id}, \Psi^\vee)_{\mathrm{Sch}_{\mathrm{aft}}} \simeq (\mathrm{Id}, \Upsilon)_{\mathrm{Sch}_{\mathrm{aft}}},$$

where $(\mathrm{Id}, \Upsilon)_{\mathrm{Sch}_{\mathrm{aft}}}$ *is as in Sect. 3.2.5.*

The content of this theorem is that for an individual $X \in \mathrm{Sch}_{\mathrm{aft}}$, with respect to the identifications

$$\mathbf{D}_X^{\mathrm{Serre}} : \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X) \text{ and } \mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X),$$

the dual of the functor

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$

is the functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \omega_X.$$

Furthermore, these identifications of functors are compatible with respect to maps $f : X \to Y$.

4.4.4. *Proof of Theorem 4.4.3.* Since the functor $(\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}}$ corresponds to the 'free module on one generator', a datum of a natural transformation out of it to some other functor

$$\mathsf{F} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}+\mathrm{Mod}},$$

whose composition with $\mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}+\mathrm{Mod}} \to \mathrm{DGCat}_{\mathrm{cont}}^{\mathrm{Mon}}$ is the functor $\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}}$, is equivalent to the datum of a 1-morphism

$$(\mathrm{QCoh}^*, \mathrm{QCoh}^*)_{\mathrm{Sch}_{\mathrm{aft}}}(\mathrm{pt}) \to \mathsf{F}(\mathrm{pt}).$$

I.e., in order to prove the theorem, it is sufficient to perform the identification

$$\Psi_X^\vee \simeq \Upsilon_X$$

for $X = \mathrm{pt}$. However, the latter is evident.

$\square$