THE LOCAL AND GLOBAL VERSIONS OF THE WHITTAKER CATEGORY

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To Professor Kyoji Saito, with admiration

Contents

Introduction 2
0.1. Whittaker categories 2
0.2. The global Whittaker category 4
0.3. What is actually done in this paper? 7
0.4. Structure of the paper 8
0.5. Conventions 9
0.6. Acknowledgements 11
1. Preliminaries 11
1.1. The geometric objects 11
1.2. Categories of sheaves 12
1.3. Ind-schemes 13
1.4. Categories acted on by groups 15
1.5. The twisted case 17
1.6. Characteristic 0 situation 17
2. The local Whittaker category 18
2.1. Definition of the local Whittaker category 18
2.2. Structure of the local Whittaker category 20
2.3. A stratification 21
2.4. Adapted objects 23
2.5. Proof of Theorem 2.4.2 24
3. A dual definition of the local Whittaker category 26
3.1. Digression: invariant functors and categorical *coinvariants* 26
3.2. The dual local Whittaker category 27
3.3. Properties of the dual Whittaker category 28
3.4. The pseudo-identity functor 30
4. The global Whittaker category 32
4.1. Drinfeld's compactification 32
4.2. Stratifications of $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$ 33
4.3. Adding a level structure 34
4.4. Action of the loop groupoid away from the level 35
4.5. Definition of the global Whittaker category (with an auxiliary point) 36
4.6. Properties of the global Whittaker category (with an auxiliary point) 37
4.7. Definition of the global Whittaker category 39
4.8. Duality for the global Whittaker category 41
5. The local vs global comparison 42
5.1. The local-to-global map 42
5.2. A strata-wise equivalence 42
5.3. Local-to-global functor and duality 45

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5.4. Comparing the averaging procedures 46
5.5. Statement of the local-to-global equivalence 47
6. Ran version and the proof of the main theorem 48
6.1. Ran geometry 48
6.2. The Ran version of the Whittaker category 49
6.3. Proof of the main theorem 50
6.4. Unital structure 51
6.5. Proof of Theorem 6.4.2 53
6.6. Proof of Theorem 6.4.8 55
7. Generalizations 57
7.1. Full level structure 57
7.2. Multi-point version 58
7.3. "Abstract" Whittaker categories 59
Appendix A. Proof of Theorem 2.5.5 62
A.1. Compactification of the action map 62
A.2. Proof of the cleanness statement 64
A.3. Proof of Proposition A.2.4 65
A.4. Proof of Lemma A.3.3 66
Appendix B. Invariants vs coinvariants for group actions 67
B.1. The statement 67
B.2. Locally constant actions 68
B.3. Rigidity 70
B.4. Recovering the category category from invariants 70
B.5. The maximal subcategory with a locally constant action 72
Appendix C. Sheaf theory in infinite type 72
C.1. Placid (ind-)schemes 72
C.2. The category of sheaves on a placid (ind-)scheme 73
C.3. Sheaves on the loop group 75
Appendix D. Invariants and coinvariants for loop group actions 76
D.1. Categories acted on by the loop group 76
D.2. Proof of Theorem D.1.4 76
D.3. Proof of Proposition D.2.9 78
D.4. Proof of Proposition D.2.7 79
References 83

Introduction

0.1. **Whittaker categories.**

0.1.1. Passage to the Whittaker model is an important tool in representation theory of reductive groups over local fields, and in the theory of automorphic functions.

In the local situation, given a representation V of $G(\mathbf{K})$ (let us say, for $\mathbf{K}$ non-Archimedian), the Whittaker model of V is defined as the space of coinvariants

$$\mathrm{Whit}(V) := V_{N(\mathbf{K}),\chi},$$

where $N \subset G$ is the maximal unipotent, and $\chi : N(\mathbf{K}) \to \mathbb{C}^*$ is a non-degenerate character.

In the global situation (let us say over a function field K), the global Whittaker space is

$$\mathrm{Whit}_{\mathrm{glob}} := \mathrm{Funct}((N(\mathbb{A}), \chi)\backslash G(\mathbb{A})),$$

where $\chi : N(\mathbb{A}) \to \mathbb{C}^*$ is chosen so that it is trivial on $N(K)$.

A key feature of the global Whittaker space, which makes it particularly useful for *local-to-global* constructions is that, unlike the space $\mathrm{Funct}(G(K)\backslash G(\mathbb{A}))$ of automorphic functions, the space $\mathrm{Whit}_{\mathrm{glob}}$ is local in nature in that it splits as the (restricted) tensor product

$$\underset{x}{\otimes}\ \mathrm{Funct}((N(\mathbf{K}_x),\chi)\backslash G(\mathbf{K}_x))$$

(here x runs through the set of places of K, and for a given place we denote by $\mathbf{K}_x$ the corresponding local field).

0.1.2. In this paper we work in the geometric context, which by its nature forces us to go one level up in the hierarchy

$$\text{Elements} \to \text{Sets} \to \text{Categories} \to \text{2-Categories} \to \text{etc.}$$

Locally, our object of study is *categories equipped with an action of the loop group* $\mathfrak{L}(G)$ (Sect. D.1 for what this means). In practice (and for the most part in this paper), we will consider the action of $\mathfrak{L}(G)$ on the category of sheaves on the quotient $\mathfrak{L}(G)/K_n$, where $K_n \subset \mathfrak{L}(G)$ is a congruence subgroup.

Given a category $\mathbf{C}$ with an action of $\mathfrak{L}(G)$, we wish to attach to it its Whittaker model. However, geometry allows more flexibility than the classical theory, and as a result there are two ways in which one can proceed[1]:

One can consider the category $\mathrm{Whit}(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}(N),\chi}$ by imposing $\mathfrak{L}(N)$-equivariance against χ. Or one can consider the corresponding category of coinvariants $\mathrm{Whit}(\mathbf{C})_{\mathrm{co}} := \mathbf{C}_{\mathfrak{L}(N),\chi}$.

Now, if instead of $\mathfrak{L}(N)$ we had a finite-dimensional algebraic group (or a pro-finite dimensional algebraic group), we would know that the two definitions agree (see Theorem B.1.2). However, $\mathfrak{L}(N)$ is a group ind-scheme, and there is a priori no reason for such an invariants/coinvariants equivalence to hold[2]. Yet, one can define a functor (by a non-tautological procedure)

$$\text{Ps-Id} : \mathrm{Whit}(\mathbf{C})_{\mathrm{co}} \to \mathrm{Whit}(\mathbf{C}). \tag{0.1}$$

One of the key results of the paper [Ras] is that the functor (0.1) is an equivalence. In the present paper, we give an alternative proof of this result (by methods that use global geometry).

Let us emphasize that the fact that (0.1) is an equivalence is a specialty of the Whittaker situation. For example, an analogously defined functor would *not* be an equivalence if instead of the non-degenerate character χ we considered the trivial character (i.e., just invariants/coinvariants for $\mathfrak{L}(N)$, instead of the χ-twisted version).

0.1.3. The fact that the equivalence (0.1) holds is really good news in that it says that the operation of passage to the Whittaker model in the local geometric situation is a well-behaved operation. For example, it implies that the assignment

$$\mathbf{C} \mapsto \mathrm{Whit}(\mathbf{C}) \tag{0.2}$$

commutes with limits and colimits, and with the operation of passage to the dual category.

0.1.4. *Why do we care about the local Whittaker model?* The assignment (0.2), viewed as a (2)-functor from the (2)-category of DG categories equipped with an action of $\mathfrak{L}(G)$ to that of plain DG categories, plays a key role in the *local geometric Langlands correspondence.*

To explain this, we need to place ourselves in the context of D-modules. In this case for every choice of *level* (which is a W-invariant symmetric bilinear form $\Lambda \otimes \Lambda \to k$, where Λ is the coweight lattice of G) there corresponds the notion of *category acted on by* $\mathfrak{L}(G)$ *at level* κ. Denote the (2)-category of such by $\mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa$.

Assume that κ is non-degenerate, i.e., it defines an *isomorphism*

$$\mathfrak{t} \simeq k \underset{\mathbb{Z}}{\otimes} \Lambda \to k \underset{\mathbb{Z}}{\otimes} \check{\Lambda} \simeq \check{\mathfrak{t}}.$$

[1] We emphasize that the dichotomy explained below does not seem to have an immediate analog in the classical theory.

[2] Such an equivalence does, however, hold for $\mathfrak{L}(G)$ with G reductive, see Theorem D.1.4.

Transferring κ to $\check{\mathfrak{t}}$, we obtain a form $\check{\kappa}$ on $\check{\Lambda}$.

The local geometric Langlands conjecture says that there exists a canonical (2)-equivalence of (2)-categories

$$\mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa \simeq \mathfrak{L}(\check{G})\text{-}\mathbf{mod}_{-\check{\kappa}}. \tag{0.3}$$

Now, the Whittaker model plays a crucial role in characterizing the (2)-equivalence (0.3). Namely, if

$$\mathbf{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa \text{ and } \check{\mathbf{C}} \in \mathfrak{L}(\check{G})\text{-}\mathbf{mod}_{-\check{\kappa}}$$

are two objects that *correspond to each other* under the (2)-equivalence (0.3), then the Whittaker model of $\mathbf{C}$, i.e., $\mathrm{Whit}(\mathbf{C})$, and the *Kac-Moody model* of $\check{\mathbf{C}}$ are equivalent *as DG categories*. And vice versa, i.e., when the roles of G and $\check{G}$ are swapped.

Here the Kac-Moody model of a category $\mathbf{C}$ acted on by the loop group $\mathfrak{L}(G)$ at level κ, denoted $\mathrm{KM}(\mathbf{C})$, is the DG category of *weak invariants* on $\mathbf{C}$ with respect to the loop group. Equivalently,

$$\mathrm{KM}(\mathbf{C}) = \mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa}(\widehat{\mathfrak{g}}\text{-mod}_\kappa, \mathbf{C}),$$

where $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ is the category of modules for the Kac-Moody algebra at level κ, viewed as an object of $\mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa$.

0.2. **The global Whittaker category.**

0.2.1. We now come to the main point of focus of this paper. Let us take $\mathbf{C}$ to be the category of sheaves on $\mathfrak{L}(G)/K_n$, so that

$$\mathrm{Whit}(\mathbf{C}) = \mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}.$$

There are two issues one needs to address for practical applications:

(a) The category $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ is inherently infinite-dimensional in nature in that all of its objects have infinite-dimensional support[3]. So it would be desirable to find another description of $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ that would involve sheaves on finite-dimensional algebro-geometric objects (such as algebraic stacks).

(b) In the global Geometric Langlands theory, one studies the functor that relates the category $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ to the category of sheaves on $\mathrm{Shv}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}})$, where $\mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}}$ is the moduli stack of G-bundles (on a given curve X) with structure of level n at x. The functor in question is

$$\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi} \to \mathrm{Shv}(\mathfrak{L}(G)/K_n) \to \mathrm{Shv}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}}),$$

where the first arrow is the forgetful functor, and the second arrow is the functor of !-direct image along the uniformization map $\mathfrak{L}(G)/K_n \to \mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}}$. However, in order to control various properties of this functor (e.g., behavior with respect to Verdier duality), it would again be desirable a finite-dimensional model for $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$, as well as the above functor itself.

The goal of this paper is to describe such a finite-dimensional model for $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$, addressing points (a) and (b) above.

[3]That said, Raskin's results in [Ras] show that $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ is a union of full subcategories, such that compact objects in each of them can be expressed through sheaves with finite-dimensional support.

0.2.2. In order to explain what this finite-dimensional model is, let us return to the classical situation. Consider the space of Whittaker functions that are non-ramified away from a particular place x, i.e.,

$$\mathrm{Funct}((N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/K_n \times G(\mathbb{O}^x)),$$

where

$$\mathbb{O}^x = \prod_{x'\neq x} \mathbf{O}_{x'},$$

and K_n is the n-th congruence subgroup at x. We normalize χ so that its conductor is $N(\mathbb{O}) \subset N(\mathbb{A})$.

Let

$$\mathrm{Funct}((N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/K_n \times G(\mathbb{O}^x))_0 \subset \mathrm{Funct}((N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/K_n \times G(\mathbb{O}^x))$$

be the subspace of functions that are supported on

$$G(\mathbf{K}_x) \times N(\mathbb{A}^x) \cdot G(\mathbb{O}^x) \subset G(\mathbf{K}_x) \times G(\mathbb{A}^x) = G(\mathbb{A}),$$

where

$$\mathbb{A}^x = \prod_{x'\neq x} \mathbf{K}_{x'},$$

Then we have an isomorphism

$$\mathrm{Funct}((N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/K_n \times G(\mathbb{O}^x))_0 \simeq \mathrm{Funct}((N(\mathbf{K}_x),\chi)\backslash G(\mathbf{K}_x)/K_n). \tag{0.4}$$

Thus,

$$\mathrm{Funct}((N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/K_n \times G(\mathbb{O}^x))_0 \tag{0.5}$$

is isomorphic to $\mathrm{Whit}(\mathrm{Funct}(G(\mathbf{K}_x)/K_n))$.

0.2.3. The next observation is that the space (0.5) can be realized as the subspace of functions on the quotient

$$N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^x). \tag{0.6}$$

Moreover, this subspace can be characterized by a certain equivariance property, as follows. Choose a point $x' \neq x$, and consider the space

$$N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^{x\cup x'}), \tag{0.7}$$

where $\mathbb{O}^{x\cup x'}$ is defined in the same was as $\mathbb{O}^x$ above with two places instead of one.

This space is acted on by $N(\mathbf{K}_{x'})$ on the right, so that

$$N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^x) \simeq \Big(N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^{x\cup x'})\Big)/N(\mathbf{O}_{x'}).$$

Now

$$\mathrm{Funct}((N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/K_n \times G(\mathbb{O}^x))_0 \subset N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^x) \tag{0.8}$$

consists of those elements that after pullback along

$$N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^{x\cup x'}) \to N(K)\backslash G(\mathbf{K}_x) \times N(\mathbb{A}^x)/K_n \times N(\mathbb{O}^x)$$

transform along the character $\chi|_{\mathbf{N}(K_{x'})}$ with respect to the above action of $N(\mathbf{K}_{x'})$.

0.2.4. The point is that the space (0.6) (as well as (0.7)) has a direct analog in geometry, and the resulting geometric object is an (ind)-algebraic stack, with one caveat.

The (ind)-algebraic stack whose k-points (almost) match (0.6) is a version of Drinfeld's compactification, denoted $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$; it is introduced in Sect. 4.1. (The (ind)-algebraic stack corresponding to (0.7) is introduced in Sect. 4.4.3).

We introduce the global version of the Whittaker category to be a full subcategory

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}), \tag{0.9}$$

by mimicking the recipe in Sect. 0.2.3.

Remark 0.2.5. The caveat, alluded to above, is that k-points of $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$ do not really match (0.6). In fact, the former is a proper subset of the latter. Geometrically, $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$ has a stratification, and (0.6) corresponds to the union of k-points of some of the strata (let us call these strata *relevant*, and the other strata *irrelevant*).

That said, a feature of the category $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ is that for all of its objects, their restrictions (both !- and $*$-versions) to the irrelevant strata vanish. So, the subcategory (0.9) is a geometric analog of the subspace (0.8).

0.2.6. We have a tautologically defined map

$$\pi : \mathfrak{L}(G)/K_n \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x},$$

and one shows that the pullback functor

$$\pi^! : ((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathfrak{L}(G)/K_n)$$

sends

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \mapsto \mathrm{Whit}(\mathfrak{L}(G)/K_n) := \mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}.$$

The main theorem of the present paper, Theorem 5.5.2, says that the resulting functor

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathfrak{L}(G)/K_n) \tag{0.10}$$

is an equivalence.

Theorem 5.5.2 is a geometric analog of the (more or less tautological) function-theoretic isomorphism (0.4). However, Theorem 5.5.2 is not tautological. It is easy to show that the functor (0.10) induces a *strata-wise* equivalence (and at the level of functions this is all one needs to show). But the fact that the subquotients on both sides corresponding to different strata glue in the same way requires a non-trivial argument.

0.2.7. The left-hand side of the equivalence (0.10) provides the sought-for fininite-dimensional model for $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$, see Sect. 0.2.1(a). It also provides the answer to Sect. 0.2.1(b). Namely, the corresponding functor

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}})$$

is the composite

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}}),$$

where the first arrow is the tautological inclusion, and the second arrow is !-direct image with respect to the natural morphism of algebraic stacks:

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x} \to \mathrm{Bun}_G^{\mathrm{level}_{n\cdot x}}.$$

Remark 0.2.8. Historically, one has been using $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ as a surrogate for the local Whittaker category long before the appearance of the direct local definition of $\mathrm{Whit}(\mathfrak{L}(G)/K_n)$ as $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$. So this paper provides a justification of why this surrogate is valid.

The model for $\mathrm{Whit}(\mathfrak{L}(G)/K_n)$ as $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ had been used for both local considerations (see, e.g., [FGV], where it is used to prove the geometric Casselman-Shalika formula, or [Ga3]), and for global ones (see, e.g., [FGKV, Ga4, Ga5]).

The reason for this was that in order to define $\mathrm{Whit}(\mathfrak{L}(G)/K_n)$ as $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$, one needed to overcome several (mostly psychological) obstructions:

For one thing, when definining $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ one needs to work with the *large* category $\mathrm{Shv}(\mathfrak{L}(G)/K_n)$ (i.e., the ind-completion of the more conventional category of sheaves with finite-dimensional support).

Second, as we shall see in Proposition 2.2.8, the objects of $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ are *invisible* from the point of view of the t-structure on $\mathrm{Shv}(\mathfrak{L}(G)/K_n)$ (technically, all these objects are *infinitely connective*). Thus, one had to really leave the world of abelian categories to define $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$. (That said, we should mention that the category $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$ carries its own t-structure with a non-trivial heart.)

And third, prior to Raskin's paper or our Theorem 5.5.2, even if one defined $\mathrm{Whit}(\mathfrak{L}(G)/K_n)$ as $\mathrm{Shv}(\mathfrak{L}(G)/K_n)^{\mathfrak{L}(N),\chi}$, it would be totally unclear how to compute anything in it: we need a finite-dimensional model to perform actual computations.

0.3. **What is actually done in this paper?** Here is a brief synopsis of the mathematical contents of this paper.

0.3.1. *Definition of the local Whittaker category.*

–We define the Whittaker category $\mathrm{Whit}(\mathcal{Y})$ for $\mathcal{Y} = \mathfrak{L}(G)/K_n$ as $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$.

–We state (and subsequently prove) the non-obvious fact that $\mathrm{Whit}(\mathcal{Y})$ is compactly generared.

–We define a stratification on $\mathrm{Whit}(\mathcal{Y})$ that corresponds to the stratification of $\mathrm{Gr}_G = \mathfrak{L}(G)/\mathfrak{L}^+(G)$ by $\mathfrak{L}(N)$-orbits. We show that the category on each stratum can be expressed in terms of finite-dimensional algebro-geometric objects.

0.3.2. *Dual definition.*

–We define the dual version of the Whittaker category, denoted $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$.

–We study the strata-wise behavior of $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$, and we show that it reproduces that of $\mathrm{Whit}(\mathcal{Y})$.

–We define the functor $\mathrm{Ps\text{-}Id} : \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Whit}(\mathcal{Y})$ and state (and subsequently prove) the theorem that says that it is an equivalence.

0.3.3. *Global definition.*

–We define the global Whittaker category $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$.

–We construct a functor $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathcal{Y})$ and state (and subsequently prove) our main result, Theorem 5.5.2, which says that the above functor is an equivalence. We explain that the non-trivial part is the fully-faithfulness assertion (something that does not have a counterpart in the classical theory).

–We introduce a *Ran version* of $\mathrm{Whit}(\mathfrak{L}(G)/K_n)$, denoted $\mathrm{Whit}((\mathfrak{L}(G)/K_n)_{\mathrm{Ran}})$. We prove that the pullback functor $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}((\mathfrak{L}(G)/K_n)_{\mathrm{Ran}})$ is fully faithful.

–We prove the equivalence $\mathrm{Whit}((\mathfrak{L}(G)/K_n) \times \mathrm{Ran}(X)_x) \simeq \mathrm{Whit}((\mathfrak{L}(G)/K_n)_{\mathrm{Ran}})$. This implies the fully-faithfulness of the functor in Theorem 5.5.2 by an easy retraction argument.

0.3.4. *Generalizations.*

–We define the Whittaker models $\mathrm{Whit}(\mathbf{C})$ and $\mathrm{Whit}(\mathbf{C})_{\mathrm{co}}$ for an abstract DG category $\mathbf{C}$ with an action of $\mathfrak{L}(G)$, and use Theorem 5.5.2 to deduce an equivalence $\mathrm{Whit}(\mathbf{C}) \simeq \mathrm{Whit}(\mathbf{C})_{\mathrm{co}}$.

–We consider the "factorizable" situation, when instead of a fixed formal disc (that the loop group $\mathfrak{L}(G)$ is attached to), we consider the *multi-disc* parameterized by points of X^n for some integer n. The results of this paper hold in this more general situation with no major modifications.

0.3.5. *Groups acting on categories.*

–We review the theory of actions of a (finite-dimensional) algebraic group on a DG category. We prove that in this case, the categories of invariants and coinvariants are canonically equivalent.

–We review the theory of *placid (ind)-schemes* and sheaves on these objects.

–We review the notion of action of a loop group $\mathfrak{L}(G)$ on a DG category. We show that for G reductive, the resulting categories of invariants and coinvariants are canonically equivalent.

0.4. **Structure of the paper.** We will now describe the contents of the paper, section-by-section.

0.4.1. In Sect. 1 we collect some preliminaries: loop and arc spaces, categories of sheaves, group actions on categories and equivariance.

The reader familiar with this material is advised to skip this section and return to it when necessary.

0.4.2. In Sect. 2 we introduce our basic object of study: the local Whittaker category.

We define the local Whittaker category $\mathrm{Whit}(\mathcal{Y})$ as $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$, where $\mathcal{Y} = \mathfrak{L}(G)/K_n$. We show how the stratification of the affine Grassmannian $\mathfrak{L}(G)/\mathfrak{L}^+(G)$ by $\mathfrak{L}(N)$-orbits defines a stratification of $\mathrm{Whit}(\mathcal{Y})$ with explicit subquotients.

We discuss the question of compact generation of $\mathrm{Whit}(\mathcal{Y})$. This is not obvious, as the definition of $\mathrm{Whit}(\mathcal{Y})$ involves an infinite limit. We introduce the notion of *adapted object* of $\mathrm{Shv}(\mathcal{Y})$; these are objects for which the functor of !-averaging along $\mathfrak{L}(N)$ against the character χ is defined and well-behaved. We show that if $\mathrm{Shv}(\mathcal{Y})$ has "enough" adapted objects, then $\mathrm{Whit}(\mathcal{Y})$ is compactly generated. We then exhibit a supply of adapted objects, using a recipe from [Ras].

0.4.3. In Sect. 3 we define the other version of the local Whittaker category, denoted $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$, as $(\mathfrak{L}(N),\chi)$-coinvariants of $\mathrm{Shv}(\mathcal{Y})$. We show that the stratification of $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$ that arises from the stratification of the affine Grassmannian has subquotients isomorphic to those of $\mathrm{Whit}(\mathcal{Y})$.

We show that the supply of adapted objects in $\mathrm{Shv}(\mathcal{Y})$ makes $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$ compactly generated as well, and that it is the category dual of $\mathrm{Whit}(\mathcal{Y})$ (up to replacing χ by its inverse).

We introduce the "non-standard" averaging functor

$$\mathrm{Ps\text{-}Id} : \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Whit}(\mathcal{Y}), \tag{0.11}$$

which would automatically be an equivalence if instead of $\mathfrak{L}(N)$ we had a (pro)finite-dimensional algebraic group. We state the theorem that Ps-Id is an equivalence in our case as well. We emphasize that for the validity of this assertion it is crucial that we are working with a non-degenerate character χ.

0.4.4. In Sect. 4 we introduce the global Whittaker category, by mimicking the procedure in Sect. 0.2.3. The underlying geometric object is a version of Drinfeld's compactification, denoted $(\overline{\mathrm{Bun}}_N)^{G\text{-}\mathrm{level}_{n\cdot x}}_{\infty\cdot x}$. To spell out the definition, we first choose a collection of auxiliary points $\underline{y}$ on our curve, and then we show that the definition is independent of this choice.

We show that a natural stratification (by the order of degeneration) on $(\overline{\mathrm{Bun}}_N)^{G\text{-}\mathrm{level}_{n\cdot x}}_{\infty\cdot x}$ defines a stratification on the Whittaker category $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-}\mathrm{level}_{n\cdot x}}_{\infty\cdot x})$, with only the "relevant" strata carrying non-zero objects.

0.4.5. In Sect. 5 we show that pullback along the map

$$\pi : \mathcal{Y} = \mathfrak{L}(G)/K_n \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$$

defines a functor

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathcal{Y}). \tag{0.12}$$

We first show that this functor is a strata-wise equivalence. We then proceed to state our main result, Theorem 5.5.2, which says that the functor (0.12) is an equivalence. Given the strata-wise equivalence, we see that Theorem 5.5.2 is equivalent to the assertion that the functor (0.12) is fully faithful.

We show that the equivalence (0.11) follows formally from Theorem 5.5.2.

0.4.6. In Sect. 6 we prove Theorem 5.5.2. The idea of the proof is to consider two more versions of the category $\mathrm{Whit}(\mathcal{Y})$ that involve the Ran space, denoted $\mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}})$ and $\mathrm{Whit}(\mathcal{Y} \times \mathrm{Ran}(X)_x)$. We will have a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \arrow[r] \arrow[d, "\pi^!"'] & \mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}}) \arrow[d] \\
\mathrm{Whit}(\mathcal{Y}) \arrow[r] & \mathrm{Whit}(\mathcal{Y} \times \mathrm{Ran}(X)_x).
\end{tikzcd}$$

We will see that the two horizontal functors are fully faithful: this is a general contractibility-type assertion. Finally, we will show that the right vertical arrow is an equivalence. This would involve showing that there are "enough" adapted objects, so we will essentially use Raskin's recipe again. As a result, we will see that $\pi^!$ is fully faithful, as required.

0.4.7. In Sect. 7 we discuss several generalizations of Theorem 5.5.2.

We show that instead of considering a fixed punctured disc, we can consider the multi-disc parameterized by points of X^n. In this way we obtain a *factorizable version* of the results obtained in the preceding sections.

We consider the abstract setting of a DG category $\mathbf{C}$ acted on by the loop group $\mathfrak{L}(G)$, and study its associated Whittaker categories $\mathrm{Whit}(\mathbf{C})$ and $\mathrm{Whit}(\mathbf{C})_{\mathrm{co}}$. We show that the fact that the functor (0.11) is an equivalence implies that the corresponding functor $\mathrm{Whit}(\mathbf{C})_{\mathrm{co}} \to \mathrm{Whit}(\mathbf{C})$ is also an equivalence.

0.4.8. In Sect. A we review the input we need from Raskin's work [Ras] for the present paper; we provide a detailed proof of the relevant geometric results.

0.4.9. In Sect. B we show that for a category $\mathbf{C}$ acted on by a finite-dimensional group H, the functor $\mathbf{C}_H \to \mathbf{C}^H$, given by *-averaging with respect to H, is an equivalence.

0.4.10. In Sect. C we review the notion of *placid* scheme (resp., ind-scheme). These are algebro-geometric objects of *inifinite type*, but ones for which one can easily bootstrap the theory of sheaves from the finite type situation[4].

We should emphasize, however, that for a placid (ind)-scheme $\mathcal{Y}$, the resulting category $\mathrm{Shv}(\mathcal{Y})$ does *not* come equipped with a t-structure. Choosing a t-structure on $\mathrm{Shv}(\mathcal{Y})$ involves a trivialization of a certain $\mathbb{Z}$-gerbe (the dimension gerbe). We will not pursue this in the present paper.

We show that the loop group $\mathfrak{L}(G)$ is a placid ind-scheme, so the category $\mathrm{Shv}(\mathfrak{L}(G))$ is something manageable.

0.4.11. Finally, in Sect. D we show that if G is reductive, for a category $\mathbf{C}$ acted on by $\mathfrak{L}(G)$, there exists a canonical equivalence $\mathbf{C}^{\mathfrak{L}(G)} \simeq \mathbf{C}_{\mathfrak{L}(G)}$. This extends the result from Sect. B from the case of a finite-dimensional group to the case of a loop group of a reductive group G.

0.5. **Conventions.**

[4]In the present paper we are not (yet) trying to attack sheaf theory in infinite type directly, e.g., à la [BS].

0.5.1. We will be working over an algebraically closed ground field, denoted k. In this paper we will *not* need derived algebraic geometry (this is because we will work with sheaf theories of topological nature, see Sect. 1.2.1).

We let $\mathrm{Sch}^{\mathrm{aff}}$ denote the category of affine schemes over k, and by $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset \mathrm{Sch}^{\mathrm{aff}}$ its full subcategory consisting of affine schemes of finite type over k.

All other algebro-geometric objects that we will encounter are classical *prestacks*, i.e., (accessible) functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Grpds}, \tag{0.13}$$

where Grpds is the category of classical groupoids.

We let $\mathrm{PrStk}_{\mathrm{lft}} \subset \mathrm{PrStk}$ be the full subcategory of prestacks locally of finite type. By definition, an object $\mathcal{Y} \in \mathrm{PrStk}$ belongs to $\mathrm{PrStk}_{\mathrm{lft}}$, if when viewed as a functor (0.13), it takes filtered limits in $\mathrm{Sch}^{\mathrm{aff}}$ to colimits in Grpds. Equivalently, $\mathcal{Y} \in \mathrm{PrStk}_{\mathrm{lft}}$ if it is isomorphic to the left Kan extension of its own restriction to

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}.$$

Thus, we can identify $\mathrm{PrStk}_{\mathrm{lft}}$ with the category of functors

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Grpds}\,.$$

0.5.2. We let e denote the field of coefficients. We will be working with DG categories over e; we refer the reader to [GR1, Chapter 1, Sect. 10] for a detailed exposition of the theory of DG categories.

Unless specified otherwise, we will assume our DG categories to be cocomplete (i.e., contain infinite direct sums, equivalently colimits).

We let $\mathrm{DGCat}_{\mathrm{cont}}$ denote the ∞-category, whose objects are *cocomplete* DG categories and whose 1-morphisms are *continuous* (i.e., colimit preserving) functors.

The category $\mathrm{DGCat}_{\mathrm{cont}}$ carries a symmetric monoidal structure, given by the Lurie tensor product. Thus, for $\mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}$ it makes sense to talk about its dualizability as an object of $\mathrm{DGCat}_{\mathrm{cont}}$.

0.5.3. Given a DG category $\mathbf{C}$, we let $\mathbf{C}_c$ denote its full (but not cocomplete) subcategory consisting of compact objects.

We remind the reader that if $\mathbf{C}$ is a compactly generated category, then it is dualizable, and we have a canonical equivalence

$$(\mathbf{C}^\vee)_c \simeq (\mathbf{C}_c)^{\mathrm{op}}.$$

0.5.4. We will fix a sheaf theory, see Sect. 1.2.1, whose field of coefficients is e. So for every $\mathcal{Y} \in \mathrm{PrStk}_{\mathrm{lft}}$, we have a well-defined object $\mathrm{Shv}(\mathcal{Y}) \in \mathrm{DGCat}_{\mathrm{cont}}$.

This category is compactly generated for $\mathcal{Y} = S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, more or less by definition. From here one can deduce that it is compactly generated also for *schemes* (resp., *ind-schemes*) that are of finite type (resp., of ind-finite type). In general, the question of compact generation of $\mathrm{Shv}(\mathcal{Y})$ for a given prestack is a non-trivial one.

0.5.5. We let G denote a reductive group over k. We fix a Bore subgroup $B \subset G$. Let $N \subset B$ denote its unipotent radical, and let T denote the Cartan quotient of B.

We let Λ denote the coweight lattice of T, and $\check{\Lambda}$ its dual, i.e., the weight lattice. Let $\Lambda^+ \subset \Lambda$ denote the sub-monoid of dominant coweights, and similarly for $\check{\Lambda}$.

We let $\Lambda^{\mathrm{pos}} \subset \Lambda$ be sub-monoid equal to the non-negative integral span of simple coroots.

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1. Preliminaries

In this section we collect some miscellanea: loop spaces, sheaf theory, ind–schemes and group actions on categories.

The reader in encouraged to skip this section and return to it when necessary.

1.1. **The geometric objects.**

1.1.1. Let Z be a scheme of finite type. We define the scheme of arcs $\mathfrak{L}^+(Z)$ to represent the functor on k-algebras

$$R \mapsto \mathrm{Hom}(\mathrm{Spec}(R[\![t]\!]), Z).$$

We have:

$$\mathfrak{L}^+(Z) \simeq \lim_{n\in\mathbb{Z}^{\geq 0}} \mathfrak{L}^+(Z)_n,$$

where each $\mathfrak{L}^+(Z)_n$ is a scheme (of finite type) given by the functor

$$R \mapsto \mathrm{Hom}(\mathrm{Spec}(R[t]/t^n), Z).$$

Note that if Z is smooth, then the transition maps

$$\mathfrak{L}^+(Z)_{n''} \to \mathfrak{L}^+(Z)_{n'}, \quad n'' \geq n'$$

are smooth.

We define the prestack of loops $\mathfrak{L}(Z)$ to represent the functor on k-algebras

$$R \mapsto \mathrm{Hom}(\mathrm{Spec}(R(\!(t)\!)), Z).$$

One easily shows (by reducing to the case of the affine space) that when Z is affine, the prestack $\mathfrak{L}(Z)$ is an ind-scheme, which contains $\mathfrak{L}^+(Z)$ as a closed subscheme.

1.1.2. Let G be a an affine algebraic group. Then by functoriality $\mathfrak{L}^+(G)$ and all $\mathfrak{L}^+(G)_n$ are group-schemes, and $\mathfrak{L}(G)$ is a group ind-scheme. In what follows we will denote by $K_n \subset \mathfrak{L}^+(G)$ the k-th congruence subgroup, i.e., the kernel of the projection $\mathfrak{L}^+(G) \to \mathfrak{L}^+(G)_n$.

For each n we can consider the stack-quotient $\mathcal{Y} := \mathfrak{L}(G)/K_n$ (i.e., we take the prestack quotient, and sheafify it in the étale (or, which would produce the same result, fpqc) topology).

The prestack $\mathcal{Y}$ is known to be an ind-scheme of ind-finite type, and it represents the functor that sends a k-algebra R to the set of triples $(\mathcal{P}_G, \gamma, \epsilon)$, where $\mathcal{P}_G$ is a G-bundle on $\mathrm{Spec}(R[\![t]\!])$, γ is a trivialization of the restriction to $\mathcal{P}_G$ to $\mathrm{Spec}(R(\!(t)\!))$, and ϵ is a trivialization of the restriction to $\mathcal{P}_G$ to $\mathrm{Spec}(R[t]/t^n)$.

We can write $\mathcal{Y}$ as a (filtered) union of its closed $\mathfrak{L}^+(G)$-stable subschemes Y_i. The action of $\mathfrak{L}^+(G)$ on each Y_i factors through the quotient $\mathfrak{L}^+(G) \to \mathfrak{L}^+(G)_n$ for all n sufficiently large (depending on Y_i).

1.1.3. Let T be a torus and let $\lambda : \mathbb{G}_m \to T$ be a co-character. We will denote by $\lambda(t)$ the point of $\mathfrak{L}(T)$ given by the map

$$\operatorname{Spec}(k(\!(t)\!)) \to \operatorname{Spec}(k[t, t^{-1}]) = \mathbb{G}_m \xrightarrow{\lambda} T.$$

1.1.4. Let N be the unipotent radical of a Borel inside a reductive group G. Consider the corresponding group ind-scheme $\mathfrak{L}(N)$. We observe that in addition to being a group ind-scheme (i.e., a group-object in the category of ind-schemes), it is naturally an ind group-scheme (i.e., an ind-object in the category of group-schemes). In other words, $\mathfrak{L}(N)$ can be written as a filtered union of its closed group sub-schemes:

$$\mathfrak{L}(N) \simeq \underset{\alpha \in A}{\operatorname{colim}}\, N^\alpha, \tag{1.1}$$

where A is a filtered category.

Indeed, we can take the index category to be the set of dominant coweights in the adjoint quotient of G; for each such coweight (denoted λ), we let the corresponding subgroup be

$$\operatorname{Ad}_{-\lambda(t)}(\mathfrak{L}^+(N)).$$

Each group-scheme N^α, can be written as

$$\lim_{\beta \in B_{\alpha,i}} N^\alpha_\beta, \tag{1.2}$$

where:

(i) $B_{\alpha,i}$ is a filtered category;

(ii) Each N^α_β is a unipotent algebraic group (of finite type);

(iii) For every $(\beta \to \beta') \in B_{\alpha,i}$ the corresponding map $N^\alpha_{\beta'} \to N^\alpha_\beta$ is surjective.

Moreover, if N^α acts on a scheme Y of finite type, this action comes from a compatible family of actions of N^α_β's on Y.

1.1.5. The choice of the uniformizer $t \in k[\![t]\!]$ gives rise to a homomorphism

$$\mathfrak{L}(\mathbb{G}_a) \to \mathbb{G}_a, \quad \underset{n}{\Sigma}, a_n \cdot t^n \mapsto a_{-1}.$$

From here we obtain a homomorphism $\chi : \mathfrak{L}(N) \to \mathbb{G}_a$ equal to

$$\mathfrak{L}(N) \to \mathfrak{L}(N/[N,N]) \to \mathfrak{L}(\mathbb{G}_a) \to \mathbb{G}_a,$$

where the second arrows comes from the map

$$N/[N,N] \simeq \underset{i \in \mathcal{I}}{\Pi}\, \mathbb{G}_a \xrightarrow{\text{sum}} \mathbb{G}_a,$$

where $\mathcal{I}$ is the set of vertices of the Dynkin diagram of G.

1.2. Categories of sheaves.

1.2.1. We adopt the conventions regarding sheaf theory from [Ga6]. We will denote by

$$\operatorname{Shv} : (\operatorname{Sch}^{\operatorname{aff}}_{\operatorname{ft}})^{\operatorname{op}} \to \operatorname{DGCat}_{\operatorname{cont}} \tag{1.3}$$

the functor that attaches to an affine scheme of finite type S the (DG) category $\operatorname{Shv}(S)$ and to a morphism $S_1 \xrightarrow{f} S_2$ the pullback functor $f^! : \operatorname{Shv}(S_2) \to \operatorname{Shv}(S_1)$.

Examples of such theories are:

(i) If the ground field k has characteristic 0, we can take $\operatorname{Shv}(S) = \operatorname{D-mod}(S)$ (see [GR2, Chapter 4, Sect. 1.2] with the caveat that *loc.cit.* the notation is $\operatorname{Crys}(S)$ rather than $\operatorname{D-mod}(S)$). In this case the field of coefficients e equals k.

(ii) For any ground field, one can take $\operatorname{Shv}(S)$ to be the ind-completion of the DG category of constructible $\mathbb{Q}_\ell$-adic sheaves (for ℓ invertible in k), as defined in [GL, Sect. 2.3.2]. In this case, the field e is coefficients is $\mathbb{Q}_\ell$ or a finite extension thereof.

(iii) If the ground field is the field of complex numbers $\mathbb{C}$, one can take $\operatorname{Shv}(S)$ to be the ind-completion of the DG category of constructible e-sheaves, for any field characteristic zero field e.

1.2.2. We apply to (1.3) the procedure of right Kan extension and obtain a functor

$$\mathrm{Shv} : (\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}} . \tag{1.4}$$

In particular, for every $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ we have a well-defined DG category $\mathrm{Shv}(\mathcal{Y})$, and for a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ we have a continuous functor

$$f^! : \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1).$$

1.2.3. The functor (1.4) has a remarkable feature that it encodes not only the !-pullback functor, but also the *-pushforward functor for schematic morphisms.

Namely, for a quasi-compact schematic map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, we have a well-defined functor

$$f_* : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2)$$

that satisfies base change against !-pullbacks, and which is the left adjoint of $f^!$ is f is proper.

Following [GR1, Chapter 7], one can combine the !-pullback and *-pushforward functoriality in saying that the functor Shv uniquely extends to a functor

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{proper}}_{\mathrm{sch,all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}, \tag{1.5}$$

where $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{proper}}_{\mathrm{qc\text{-}sch,all}}$ is the 2-category of correspondences, whose objects are prestacks $\mathcal{Y}$ locally of finite type, 1-morphisms are diagrams

$$\begin{tikzcd} \mathcal{Y}_{0,1} \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{Y}_0 \\ \mathcal{Y}_1 & \end{tikzcd}$$

with f schematic and quasi-compact, and where 2-morphisms are given by maps $h : \mathcal{Y}'_{0,1} \to \mathcal{Y}''_{0,1}$ that are schematic and proper.

1.2.4. The functor

$$\mathrm{Shv} : \mathrm{PreStk}_{\mathrm{lft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is *lax* symmetric monoidal, i.e., for $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}_{\mathrm{lft}}$ we have a natural map

$$\mathrm{Shv}(\mathcal{Y}_1) \otimes \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2), \tag{1.6}$$

given by the external product $\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \otimes \mathcal{F}_2$.

1.2.5. The functor (1.6) is in general not an equivalence. However, we have the following two observations:

(a) If $\mathrm{Shv}(-) = \mathrm{D\text{-}mod}(-)$, then (1.6) is an equivalence for $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. This formally implies that (1.6) is an equivalence any time either $\mathrm{Shv}(\mathcal{Y}_1)$ or $\mathrm{Shv}(\mathcal{Y}_2)$ is dualizable as a DG category, see [GR1, Chapter 3, Proposition 3.1.7]. In particular, this is the case when $\mathcal{Y}_i$ is a scheme of finite type (or more generally, an ind-scheme, see Sect. 1.3.1 below).

(b) For sheaf theories (i) and (ii) in Sect. 1.2.1, for $\mathcal{Y}_i = S_i \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, the functor (1.6) sends compacts to compacts and is fully faithful. This formally implies that the same remains true for the functor (1.6) for prestacks any time either $\mathrm{Shv}(\mathcal{Y}_1)$ or $\mathrm{Shv}(\mathcal{Y}_2)$ is dualizable as a DG category.

1.3. **Ind-schemes.**

1.3.1. We will be particularly interested in evaluating the functor (1.4) on

$$\mathrm{IndSch}_{\mathrm{lft}} \subset \mathrm{PreStk}_{\mathrm{lft}} .$$

Recall that an object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ is an ind-scheme if it can be written as a *filtered* colimit

$$\mathcal{Y} \simeq \underset{i}{\mathrm{colim}}\, Y_i, \tag{1.7}$$

where Y_i are schemes of finite type, and for every arrow $i \xrightarrow{\alpha} j$ in the category of indices, the corresponding map $Y_i \xrightarrow{f_\alpha} Y_j$ is a closed embedding.

1.3.2. Given an ind-scheme $\mathcal{Y}$, we can consider *category* of its presentations as (1.7). This category is contractible. In fact, it has a final object, where I is the category of all closed subfunctors of $\mathcal{Y}$.

So, any two presentations of $\mathcal{Y}$ as (1.7) are essentially equivalent.

1.3.3. Recall the following general phenomenon. Let

$$I \to \mathrm{DGCat}_{\mathrm{cont}}, \quad i \mapsto \mathbf{C}_i$$

be a functor, where I is some index category. Suppose that for every arrow $(i \to j) \in I$, the corresponding functor $\mathbf{C}_i \to \mathbf{C}_j$ admits a continuous right adjoint. By passing to the right adjoints we obtain another functor

$$I^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Then for every $i_0 \in I$, the tautological functor

$$\mathrm{ins}_{i_0} : \mathbf{C}_{i_0} \to \underset{i\in I}{\mathrm{colim}}\, \mathbf{C}_i$$

admits a continuous right adjoint. Furthermore, the resulting functor

$$\underset{i\in I}{\mathrm{colim}}\, \mathbf{C}_i \to \underset{i\in I^{\mathrm{op}}}{\mathrm{colim}}\, \mathbf{C}_i$$

is an equivalence.

1.3.4. We apply the situation of Sect. 1.3.3, by setting

$$\mathbf{C}_i := \mathrm{Shv}(Y_i),$$

where for $(i \overset{\alpha}{\to} j) \in I$ we have the functor

$$\mathrm{Shv}(Y_i) \overset{(f_\alpha)_*}{\longrightarrow} \mathrm{Shv}(Y_j)$$

and its right adjoint

$$\mathrm{Shv}(Y_i) \overset{(f_\alpha)^!}{\longleftarrow} \mathrm{Shv}(Y_j).$$

Hence, we obtain that $\mathrm{Shv}(\mathcal{Y})$, which is, by definition, given as

$$\underset{i\in I^{\mathrm{op}}}{\lim}\, \mathrm{Shv}(Y_i),$$

with respect to !-pullbacks, can also be written as

$$\underset{i\in I}{\mathrm{colim}}\, \mathrm{Shv}(Y_i), \tag{1.8}$$

with respect to *-pushforwards.

In particular, $\mathrm{Shv}(\mathcal{Y})$ is compactly generated by the essential images of $\mathrm{Shv}(Y_{i_0})_c$ under the tautological functors

$$\mathrm{ins}_{i_0} : \mathrm{Shv}(Y_{i_0}) \to \mathrm{Shv}(\mathcal{Y}).$$

1.3.5. If $\mathcal{Y}$ is an ind-scheme, the category $\mathrm{Shv}(\mathcal{Y})$, being compactly generated, is also dualizable. However, Sect. 1.3.4 implies that $\mathrm{Shv}(\mathcal{Y})$ is canonically self-dual. This self-duality can be described in the following equivalent ways:

(i) Under the identifications

$$\mathrm{Shv}(\mathcal{Y})^\vee \simeq \mathrm{Shv}(\mathcal{Y}) \text{ and } \mathrm{Shv}(Y_i)^\vee \simeq \mathrm{Shv}(Y_i),$$

the functor dual to $\mathrm{ins}_i : \mathrm{Shv}(Y_i) \to \mathrm{Shv}(\mathcal{Y})$ is the evaluation functor $\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(Y_i)$.

(ii) The functors ins_i are compatible with the contravariant equivalences

$$\mathbb{D}_{\mathcal{Y}} : \mathrm{Shv}(\mathcal{Y})_c \to \mathrm{Shv}(\mathcal{Y})_c \text{ and } \mathbb{D}_{Y_i} : \mathrm{Shv}(Y_i)_c \to \mathrm{Shv}(Y_i)_c.$$

(iii) The pairing $\mathrm{Shv}(\mathcal{Y})\otimes\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$ is given, in terms of (1.8), by the compatible family of pairings $\mathrm{Shv}(Y_i) \otimes \mathrm{Shv}(Y_i) \to \mathrm{Vect}$, corresponding to the usual self-duality of each $\mathrm{Shv}(Y_i)$.

1.3.6. Let $\mathcal{Y}$ be an ind-scheme. The presentation of $\mathrm{Shv}(\mathcal{Y})$ as in (1.8) shows that it carries a unique t-structure compatible with colimits, characterized by the property that the functors

$$\mathrm{ins}_i : \mathrm{Shv}(Y_i) \to \mathrm{Shv}(\mathcal{Y})$$

are t-exact.

Equivalently, an object of $\mathrm{Shv}(\mathcal{Y})$ is coconnective if and only if its restriction to every Y_i is coconnective as an object of $\mathrm{Shv}(Y_i)$.

1.4. **Categories acted on by groups.**

1.4.1. Let H be an algebraic group (of finite type). We consider $\mathrm{Shv}(H)$ as a monoidal category, where the monoidal operation is *convolution*, i.e.,

$$\mathrm{Shv}(H)^{\otimes n} \xrightarrow{\boxtimes} \mathrm{Shv}(H^n) \xrightarrow{\mathrm{mult}^n_*} \mathrm{Shv}(H).$$

Verdier duality on H defines an equivalence $\mathrm{Shv}(H)^\vee \simeq \mathrm{Shv}(H)$. This gives $\mathrm{Shv}(H)$ a comonoidal structure. This comonoidal structure can be expressed in terms of the pullback

$$\mathrm{Shv}(H) \xrightarrow{(\mathrm{mult}^n)^!} \mathrm{Shv}(H^n) \tag{1.9}$$

as follows.

The co-tensor product

$$\mathrm{Shv}(H) \to \mathrm{Shv}(H)^{\otimes n}$$

is the composite of (1.9) and the functor

$$\mathrm{Shv}(H^n) \to \mathrm{Shv}(H)^{\otimes n}$$

right adjoint to the fully faithful functor $\mathrm{Shv}(H)^{\otimes n} \to \mathrm{Shv}(H^n)$, see Sect. 1.2.5.

1.4.2. By an action of H on a DG category $\mathbf{C}$ we shall mean an action on $\mathbf{C}$ of the monoidal category $\mathrm{Shv}(H)$. We denote the ∞-category of DG categories acted on by H by H-**mod**.

1.4.3. *An example.* Let $\mathcal{Y}$ be a prestack acted on by H. Then the operation of pushforward defines an action of H on $\mathrm{Shv}(\mathcal{Y})$.

1.4.4. We shall say that an action is trivial if it factors through the augmentation

$$\mathrm{Shv}(H) \to \mathrm{Vect}, \quad \mathcal{F} \mapsto \mathrm{C}^\cdot(H, \mathcal{F}).$$

Unless specified otherwise, we will regard Vect as equipped with the trivial action of H.

For $\mathbf{C} \in H\text{-}\mathbf{mod}$ we let

$$\mathbf{C}^H = \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C}).$$

Equivalently, using the self-duality of $\mathrm{Shv}(H)$, we can rewrite $\mathbf{C}^H$ as the totalization of the cosimplicial category $\mathbf{C}^\bullet$ with terms

$$\mathbf{C}^n := \mathrm{Shv}(H)^{\otimes n} \otimes \mathbf{C}.$$

1.4.5. Let us return to the example of Sect. 1.4.3. Note that one can give an a priori different definition of the category $\mathrm{Shv}(\mathcal{Y})^H$, namely by setting it be equal to

$$\mathrm{Shv}(\mathcal{Y}/H) := \mathrm{Tot}(\mathcal{Y}_\bullet), \quad \mathcal{Y}_n = H^n \times \mathcal{Y}.$$

We claim, however, that the two definitions agree. Indeed, the right adjoints to the fully faithful functors

$$\mathrm{Shv}(H)^{\otimes n} \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(H^n \times \mathcal{Y})$$

define a map of cosimplicial categories

$$\mathrm{Shv}(\mathcal{Y}_\bullet) \to \mathrm{Shv}(\mathcal{Y})^\bullet, \tag{1.10}$$

and we claim that the functor (1.10) induces an equivalence on totalizations.

Proof. This follows from the fact that each of the the individual functors

$$\mathrm{Shv}(H^n \times \mathcal{Y}) = \mathrm{Shv}(\mathcal{Y}_n) \to \mathrm{Shv}(\mathcal{Y})^n = \mathrm{Shv}(H)^{\otimes n} \otimes \mathrm{Shv}(\mathcal{Y})$$

is fully faithful when restricted to the full subcategory

$$\mathrm{Shv}(H)^{\otimes n} \otimes \mathrm{Shv}(\mathcal{Y}) \subset \mathrm{Shv}(H^n \times \mathcal{Y}),$$

which in turn contains all the objects of the form

$$\mathsf{e}_H^{\boxtimes n} \boxtimes \mathcal{F}, \quad \mathcal{F} \in \mathrm{Shv}(\mathcal{Y}).$$

□

1.4.6. The cosimplicial DG category $\mathbf{C}^\bullet$ of Sect. 1.4.4 satisfies the *comonadic Beck-Chevalley condition*; see [Ga7, Defn. C.1.3] for what this means. In particular, this implies that the forgetful functor

$$\mathbf{oblv}_H : \mathbf{C}^H \to \mathbf{C}$$

admits a right adjoint, to be denoted Av_*^H, and $\mathbf{C}^H$ identifies with comodules in $\mathbf{C}$ for the comonad $\mathrm{Av}_*^H \circ \mathbf{oblv}_H$.

It follows formally that the endo-functor $\mathrm{Av}_*^H \circ \mathbf{oblv}_H$ of $\mathbf{C}$ is given by

$$c \mapsto \mathsf{e}_H \star c.$$

1.4.7. Note that in the situation of Sect. 1.4.5, the functor Av_*^H identifies with direct image along the map $\mathcal{Y} \to \mathcal{Y}/H$. Hence, the functor $\mathbf{oblv}_H$ identifies with its left adjoint, which is the functor of *-pullback along the above map.

The endo-functor $\mathrm{Av}_*^H \circ \mathbf{oblv}_H$ of $\mathrm{Shv}(\mathcal{Y})$ is given by

$$\mathcal{F} \mapsto \mathrm{act}_* \circ \mathrm{pr}^*(\mathcal{F}),$$

where

$$\mathrm{pr}, \mathrm{act} : H \times \mathcal{Y} \to \mathcal{Y}$$

are the projection and the action maps, respectively.

1.4.8. Although we will not need this in the main body of the paper, we remark that the functor

$$\mathbf{C} \mapsto \mathbf{C}^H, \quad H\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

has the following (non-tautological property): it commutes with colimits, see Corollary B.1.5 for a proof.

This formally implies that if $\mathbf{C}$ is dualizable as a plain category, then so is $\mathbf{C}^H$. Moreover, we have a canonical identification

$$(\mathbf{C}^H)^\vee \simeq (\mathbf{C}^\vee)^H,$$

so that the dual of the functor $\mathbf{oblv}_H : \mathbf{C}^H \to \mathbf{C}$ is the functor $\mathrm{Av}_*^H : \mathbf{C}^\vee \to (\mathbf{C}^\vee)^H$, and the dual of the functor $\mathrm{Av}_*^H : \mathbf{C} \to \mathbf{C}^H$ is the functor $\mathbf{oblv}_H : (\mathbf{C}^\vee)^H \to \mathbf{C}^\vee$.

1.4.9. The category H-**mod** carries a natural symmetric monoidal structure that commutes with the forgetful functor to $\mathrm{DGCat}_{\mathrm{cont}}$.

Namely, for $\mathbf{C}_1, \mathbf{C}_2 \in H$-**mod** we define the action of $\mathrm{Shv}(H)$ on $\mathbf{C}_1 \otimes \mathbf{C}_2$ by precomposing the natural action of $\mathrm{Shv}(H) \otimes \mathrm{Shv}(H)$ on $\mathbf{C}_1 \otimes \mathbf{C}_2$ with the monoidal functor

$$\mathrm{Shv}(H) \xrightarrow{\Delta_*} \mathrm{Shv}(H \times H) \to \mathrm{Shv}(H) \otimes \mathrm{Shv}(H),$$

where the second arrow is the right adjoint to $\mathrm{Shv}(H) \otimes \mathrm{Shv}(H) \to \mathrm{Shv}(H \times H)$.

1.5. **The twisted case.**

1.5.1. *Character sheaves.* Let $\mathcal{L}$ be a 1-dimensional local system on H, which is character sheaf, i.e., we have an isomorphism

$$\mathrm{mult}^*(\mathcal{L}) \simeq \mathcal{L} \boxtimes \mathcal{L},$$

that behave associatively.

1.5.2. Given a character sheaf $\mathcal{L}$, the functor

$$\mathrm{Shv}(H) \to \mathrm{Vect}, \quad \mathcal{F} \mapsto \mathrm{C}^{\cdot}(H, \mathcal{F} \otimes \mathcal{L})$$

has a natural monoidal structure.

This defines on Vect a different structure of category acted on by H; we denote the resulting object of H-**mod** by $\mathrm{Vect}_{\mathcal{L}}$.

1.5.3. For $\mathbf{C} \in H$-**mod**, we can twist the initial action of H on $\mathbf{C}$ by considering the object

$$\mathbf{C}_{\mathcal{L}} := \mathbf{C} \otimes \mathrm{Vect}_{\mathcal{L}} \in H\text{-}\mathbf{mod}.$$

Explicitly, $\mathbf{C}_{\mathcal{L}}$ identifies with $\mathbf{C}$ as a plain DG category, but the new action is given by the formula

$$\mathcal{F} \overset{\mathrm{new}}{\star} c := \mathrm{C}^{\cdot}(H, \mathcal{F} \otimes \mathcal{L}) \otimes \mathcal{F} \overset{\mathrm{old}}{\star} c.$$

1.5.4. We denote

$$\mathbf{C}^{H,\mathcal{L}} := \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathrm{Vect}_{\mathcal{L}}, \mathbf{C}) \simeq (\mathbf{C}_{\mathcal{L}^{-1}})^H.$$

We let

$$\mathbf{oblv}_{H,\mathcal{L}} : \mathbf{C} \rightleftarrows \mathbf{C}^{H,\mathcal{L}} : \mathrm{Av}_*^{H,\mathcal{L}}$$

denote the resulting adjoint pair.

Note that the endo-functor $\mathbf{oblv}_{H,\mathcal{L}} \circ \mathrm{Av}_*^{H,\mathcal{L}}$ of $\mathbf{C}$ is given by

$$c \mapsto \mathcal{L}^{-1} \star c.$$

1.5.5. The basic examples of character sheaves are the Kummer sheaf on $\mathbb{G}_m$ and the Artin-Schreier sheaf on $\mathbb{G}_m$, denoted A-Sch, the latter being more relevant for this paper.

A priori, A-Sch is defined either for $\mathrm{Shv}(-) = \text{D-mod}$ in the guise of the exponential D-module, or in the context of ℓ-adic sheaves when the ground field k has positive characteristic, in which case it depends on the choice of a non-trivial character $\mathbb{F}_p \to \mathsf{e}^{\times}$.

For a group H and a homomorphism $\chi : H \to \mathbb{G}_a$, we will often write

$$\mathbf{C}^{H,\chi}$$

instead of

$$\mathbf{C}^{H,\chi^*(\text{A-Sch})}.$$

1.6. **Characteristic** 0 **situation.**

1.6.1. The Artin-Schreier sheaf does not exist as a constructible sheaf if the ground field k has characteristic 0. E.g., it does not exist for the sheaf theory (iii) of Sect. 1.2.1. So for $\mathbf{C} \in H$-**mod**, and a homomorphism $\chi : H \to \mathbb{G}_m$, the notation $\mathbf{C}^{H,\chi}$ does not make sense.

However, one *can* define a category equivalent to $\mathbf{C}^{H,\chi}$, given some additional data.

1.6.2. First, let us replace $\mathbf{C}$ by $\mathbf{C}^{H'}$, where $H' := \ker(\chi)$, so we can assume that we are dealing with a category acted on by $\mathbb{G}_a$ itself.

Assume now that the $\mathbb{G}_a$-action on $\mathbf{C}$ extends to an action of the semi-direct product $\mathbb{G}_m \ltimes \mathbb{G}_a$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ my dilations.

Consider the full subcategory

$$\ker(\mathrm{Av}_*^{\mathbb{G}_a}) =: \mathbf{C}' \subset \mathbf{C},$$

and set

$$\mathrm{Kir}(\mathbf{C}) := (\mathbf{C}')^{\mathbb{G}_m}.$$

(Here "Kir" is a short-hand for the "Kirillov model".)

We have a tautological forgetful functor

$$\mathrm{Kir}(\mathbf{C}) \to \mathbf{C}^{\mathbb{G}_m},$$

which admits a left adjoint, given by

$$c \mapsto \mathrm{Cone}(\mathrm{Av}_*^{\mathbb{G}_a}(c) \to c).$$

1.6.3. We claim that $\mathrm{Kir}(\mathbf{C})$ is a substitute of

$$\mathrm{Whit}(\mathbf{C}) := \mathbf{C}^{\mathbb{G}_a,\text{A-Sch}}$$

for all practical purposes.

For instance, in the situation when A-Sch is defined, we claim that there exists a canonical equivalence

$$\mathrm{Whit}(\mathbf{C}) \simeq \mathrm{Kir}(\mathbf{C}). \tag{1.11}$$

Namely, the functor $\to$ in (1.11) is defined by

$$c \mapsto \mathrm{Av}_*^{\mathbb{G}_m} \circ \mathbf{oblv}_{\mathbb{G}_a,\text{A-Sch}}(c).$$

The functor $\leftarrow$ in (1.11) is defined by

$$c' \mapsto \mathrm{Av}_!^{\mathbb{G}_a,\text{A-Sch}} \circ \mathbf{oblv}_{\mathbb{G}_m}(c'),$$

where one can show that $\mathrm{Av}_!^{\mathbb{G}_a,\text{A-Sch}}$ is defined and isomorphic to $\mathrm{Av}_*^{\mathbb{G}_a,\text{A-Sch}}[2]$ on the essential image of $\mathbf{oblv}_{\mathbb{G}_m}$.

2. The local Whittaker category

In this section we define the local Whittaker category and study its basic properties.

2.1. **Definition of the local Whittaker category.** In this subsection we introduce the main object in this paper–the local Whittaker category. We do this by imposing equivariance with respect to the group indscheme $\mathfrak{L}(N)$. The "ind" direction in $\mathfrak{L}(N)$ will cause objects of this category to be of substantially infinite-dimensional nature.

2.1.1. Consider the ind-scheme $\mathcal{Y} := \mathfrak{L}(G)/K_n$ as acted on from the left by $\mathfrak{L}(G)$. We define the Whittaker category

$$\mathrm{Whit}(\mathcal{Y}) := \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \subset \mathrm{Shv}(\mathcal{Y})$$

to be the full subcategory that consists of $(\mathfrak{L}(N), \chi)$-equivariant objects.

Let us decipher what this means (we will essentially copy the definition from [Ga1, Sect. 1.2]).

2.1.2. Recall the presentation (1.1). We set

$$\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} := \lim_{\alpha} \mathrm{Shv}(\mathcal{Y})^{N^\alpha,\chi}, \tag{2.1}$$

where each $\mathrm{Shv}(\mathcal{Y})^{N^\alpha,\chi}$ is a full subcategory of $\mathrm{Shv}(\mathcal{Y})$, and for $(\alpha' \to \alpha'') \in A$, we have

$$\mathrm{Shv}(\mathcal{Y})^{N^{\alpha''},\chi} \subset \mathrm{Shv}(\mathcal{Y})^{N^{\alpha'},\chi}$$

as full subcategories in $\mathrm{Shv}(\mathcal{Y})$. Note that the limit in (2.1) amounts to the intersection

$$\bigcap_{\alpha} \mathrm{Shv}(\mathcal{Y})^{N^\alpha,\chi}$$

as full subcategories in $\mathrm{Shv}(\mathcal{Y})$.

Let us now explain what the subcategories

$$\mathrm{Shv}(\mathcal{Y})^{N^\alpha,\chi} \subset \mathrm{Shv}(\mathcal{Y})$$

are.

2.1.3. For a fixed index α, the ind-scheme $\mathcal{Y}$, when viewed as equipped with an action of N^α, is naturally an ind-object in the category of schemes equipped with an action of N^α.

I.e., we can represent $\mathcal{Y}$ as

$$\underset{i\in I}{\mathrm{colim}}\, Y_i, \quad Y_i \xrightarrow{f_{i,i'}} Y_{i'} \tag{2.2}$$

where each Y_i is stable under the N^α-action and $f_{i,i'}$ are closed embeddings (automatically compatible with the N^α-actions).

We set

$$\mathrm{Shv}(\mathcal{Y})^{N^\alpha,\chi} := \lim_{i\in I} \mathrm{Shv}(Y_i)^{N^\alpha,\chi},$$

viewed as a full subcategory of

$$\mathrm{Shv}(\mathcal{Y}) \simeq \lim_{i\in I} \mathrm{Shv}(Y_i).$$

Thus, it remains to explain what we mean by

$$\mathrm{Shv}(Y_i)^{N^\alpha,\chi} \subset \mathrm{Shv}(Y_i)$$

for each α and i, so that for $(i \to i')$, the corresponding functor

$$\mathrm{Shv}(Y_{i'}) \xrightarrow{f^!_{i,i'}} \mathrm{Shv}(Y_i)$$

sends $\mathrm{Shv}(Y_{i'})^{N^\alpha,\chi}$ to $\mathrm{Shv}(Y_i)^{N^\alpha,\chi}$.

2.1.4. Recall the presentation (1.2). For any $\beta \in B_{\alpha,i}$, we can consider the corresponding equivariant category $\mathrm{Shv}(Y_i)^{N^\alpha_\beta,\chi}$. Since N^α_β is unipotent, the forgetful functor

$$\mathrm{Shv}(Y_i)^{N^\alpha_\beta,\chi} \to \mathrm{Shv}(Y_i)$$

is fully faithful, and for every $(\beta' \to \beta'') \in B_{\alpha,i}$, we have

$$\mathrm{Shv}(Y_i)^{N^\alpha_{\beta'},\chi} = \mathrm{Shv}(Y_i)^{N^\alpha_{\beta''},\chi}$$

as subcategories of $\mathrm{Shv}(Y_i)$.

We set $\mathrm{Shv}(Y_i)^{N^\alpha,\chi} \subset \mathrm{Shv}(Y_i)$ to be $\mathrm{Shv}(Y_i)^{N^\alpha_\beta,\chi}$ for some/any $\beta \in B_{\alpha,i}$.

2.1.5. Going back, it is clear that for a map $(i \to i') \in I$, the corresponding functor

$$\mathrm{Shv}(Y_{i'}) \xrightarrow{f^!_{i,i'}} \mathrm{Shv}(Y_i)$$

indeed sends $\mathrm{Shv}(Y_{i'})^{N^\alpha,\chi}$ to $\mathrm{Shv}(Y_i)^{N^\alpha,\chi}$.

It is also clear that for a map $(\alpha' \to \alpha'') \in A$, we have

$$\mathrm{Shv}(\mathrm{Gr})^{N^{\alpha''},\chi} \subset \mathrm{Shv}(\mathrm{Gr})^{N^{\alpha'},\chi}$$

as full subcategories of $\mathrm{Shv}(\mathrm{Gr})$.

This completes the definition of $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ as a full subcategory of $\mathrm{Shv}(\mathcal{Y})$.

2.2. **Structure of the local Whittaker category.** In this subsection we will discuss the very first general properties of the local Whittaker category $\mathrm{Whit}(\mathcal{Y})$.

2.2.1. For what follows, let us note that for every fixed α, the forgetful functor

$$\mathbf{oblv}_{N^\alpha,\chi} : \mathrm{Shv}(\mathcal{Y})^{N^\alpha,\chi} \to \mathrm{Shv}(\mathcal{Y})$$

admits a *continuous* right adjoint $\mathrm{Av}_*^{N^\alpha,\chi}$.

Let us describe the functor $\mathrm{Av}_*^{N^\alpha,\chi}$ explicitly. Writing $\mathcal{Y}$ as (2.2), the functor $\mathrm{Av}_*^{N^\alpha,\chi}$ corresponds to a compatible family of functors

$$\mathrm{Av}^{N^\alpha,\chi} : \mathrm{Shv}(Y_i) \to \mathrm{Shv}(Y_i)^{N^\alpha,\chi}. \tag{2.3}$$

For every individual i, writing N^α as (1.2), the corresponding functor (2.3) equals the functor

$$\mathrm{Av}_*^{N^\alpha_\beta,\chi} : \mathrm{Shv}(Y_i) \to \mathrm{Shv}(Y_i)^{N^\alpha_\beta,\chi} = \mathrm{Shv}(Y_i)^{N^\alpha,\chi}. \tag{2.4}$$

2.2.2. The functor $\mathbf{oblv}_{\mathfrak{L}(N),\chi}$, being continuous, admits a right adjoint, to be denoted $\mathrm{Av}_*^{\mathfrak{L}(N),\chi}$. But the functor $\mathrm{Av}_*^{\mathfrak{L}(N),\chi}$ is *discontinuous.* Explicitly, we have

$$\mathrm{Av}_*^{\mathfrak{L}(N),\chi}(\mathcal{F}) \simeq \lim_{\alpha} \mathrm{Av}_*^{N^\alpha,\chi}(\mathcal{F}),$$

where the limit is taken in $\mathrm{Shv}(\mathcal{Y})$.

Even though each individual functor $\mathrm{Av}_*^{N^\alpha,\chi}$ is continuous, the inverse limit destroys this property.

2.2.3. Consider again the forgetful functor

$$\mathbf{oblv}_{\mathfrak{L}(N),\chi} : \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \to \mathrm{Shv}(\mathcal{Y}).$$

As any functor, it admits a partially defined left adjoint[5], to be denoted $\mathrm{Av}_!^{\mathfrak{L}(N),\chi}$.

We *do not* claim that $\mathrm{Av}_!^{\mathfrak{L}(N),\chi}$ is defined on all of $\mathrm{Shv}(\mathcal{Y})$ (however, this is the case when $n = 0$).

Nevertheless, it will turn out that the functor $\mathrm{Av}_!^{\mathfrak{L}(N),\chi}$ *is* defined on a sufficiently large class of objects of $\mathrm{Shv}(\mathcal{Y})$ to ensure that the category $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ is well-behaved, see Theorem 2.4.2 below. In particular, in Sect. 2.4.3 we will prove:

Theorem 2.2.4. *The category* $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ *is compactly generated.*

---

[5]For a functor $F : \mathbf{C} \to \mathbf{D}$, we shall say that its partially defined adjoint F^L is defined on $d \in \mathbf{D}$, if the functor $c \mapsto \mathcal{H}om_{\mathbf{D}}(d, F(c))$ is co-representable. In this case we set $F^L(d) \in \mathbf{C}$ to be the co-representing object.

2.2.5. Note that the definition of the Whittaker category $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ has a variant

$$(\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi},$$

where $\mathbf{C}$ is an arbitrary DG category.

We have the forgetful functor

$$(\mathbf{oblv}_{\mathfrak{L}(N),\chi} \otimes \mathrm{Id}_{\mathbf{C}}) : \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \otimes \mathbf{C} \to \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C},$$

whose essential image is easily seen to belong to $(\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi}$. Hence, we obtain a functor:

$$F_{\mathbf{C}} : \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \otimes \mathbf{C} \to (\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi}. \tag{2.5}$$

In Sect. 2.4 we will prove:

Theorem 2.2.6. *The functor* $F_{\mathbf{C}}$ *of* (2.5) *is an equivalence for any* $\mathbf{C}$.

Warning: the assertion of Theorem 2.2.6 is not at all tautological.

2.2.7. Recall (see Sect. 1.3.6) that the category $\mathrm{Shv}(\mathcal{Y})$ is equipped with a t-structure. A feature that makes $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ "very non-classical" is that the objects of this subcategory are "invisible" from the point of view of this t-structure. Namely, we will prove:

Proposition 2.2.8. *Every* $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ *is* infinitely connective*, i.e., lies in* $(\mathrm{Shv}(\mathcal{Y}))^{\leq -n}$ *for every* n.

2.3. **A stratification.** The stratification of the affine Grassmannian $\mathrm{Gr}_{G,x}$ by $\mathfrak{L}(N)$-orbits gives rise to a stratification of $\mathcal{Y}$. This will define a stratification of $\mathrm{Whit}(\mathcal{Y})$ but some more easily understood categories.

2.3.1. Consider the projection

$$\mathcal{Y} \to \mathfrak{L}(G)/K_0 = \mathrm{Gr}_{G,x}. \tag{2.6}$$

Recall that $\mathfrak{L}(N)$-orbits on $\mathrm{Gr}_{G,x}$ are in bijection with elements of the coweight lattice Λ; for each $\mu \in \Lambda$, let us denote by S^μ the corresponding orbit, i.e.,

$$S^\mu = \mathfrak{L}(N) \cdot t^\mu.$$

Let $\mathcal{Y}^\mu$ denote the preimage of S^μ under (2.6). Let

$$\iota^\mu : \mathcal{Y}^\mu \hookrightarrow \mathcal{Y}$$

denote the corresponding locally closed embedding.

2.3.2. Consider the corresponding full subcategory

$$\mathrm{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi} \subset \mathrm{Shv}(\mathcal{Y}^\mu),$$

defined in the same way as

$$\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \subset \mathrm{Shv}(\mathcal{Y}).$$

The functors

$$(\iota^\mu)_* : \mathrm{Shv}(\mathcal{Y}^\mu) \to \mathrm{Shv}(\mathcal{Y}) \text{ and } (\iota^\mu)^! : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}^\mu)$$

restrict to functors on the corresponding $(\mathfrak{L}(N),\chi)$-equivariant subcategories.

We will prove:

Proposition 2.3.3.

(a) *The category* $\mathrm{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi}$ *is non-zero only when* $\mu + n \cdot \rho \in \Lambda^+_{\mathbb{Q}}$.

(b) *The category* $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ *is generated by the essential images of the functors* $(\iota^\mu)_*$ *for* $\mu + n\cdot\rho \in \Lambda^+_{\mathbb{Q}}$.

(c) *An object of* $\mathrm{Shv}(\mathcal{Y})$ *belongs to* $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$ *if and only if its !-restriction to each* $\mathcal{Y}^\mu$ *belongs to* $\mathrm{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi} \subset \mathrm{Shv}(\mathcal{Y}^\mu)$.

2.3.4. *Example.* For $n=0$, we obtain that the category $\operatorname{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi}$ is non-zero if and only if μ is dominant, and in the latter case it is equivalent to Vect.

When we go from n to $n+1$ more and more strata $\operatorname{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi}$ become non-zero. The geometric reason for that is that the stabilizers become smaller.

2.3.5. For what follows we will need some more notation:

For a fixed $\mu \in \Lambda$, let $Y^\mu \subset \mathcal{Y}^\mu$ denote the preimage of the point $t^\mu \in \operatorname{Gr}_{G,x}$ under (2.6). Denote

$$N^\mu := \operatorname{Ad}_{t^\mu}(\mathfrak{L}^+(N)) \subset \mathfrak{L}(N).$$

The action of N^μ on $\mathcal{Y}^\mu$ preserves Y^μ. Moreover, it is easy to see that we can find a group ind-scheme $N' \subset \mathfrak{L}(N)$ so that

$$\mathfrak{L}(N) = N^\mu \cdot N', \quad N^\mu \cap N' = \{1\}.$$

Hence, we can identify

$$\mathcal{Y}^\mu \simeq Y^\mu \times N'.$$

Lemma 2.3.6. *Restriction along $Y^\mu \hookrightarrow \mathcal{Y}^\mu$ defines an equivalence*

$$\operatorname{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi} \simeq \operatorname{Shv}(Y^\mu)^{N^\mu,\chi},$$

so that the forgetful functor

$$\mathbf{oblv}_{\mathfrak{L}(N),\chi} : \operatorname{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi} \to \operatorname{Shv}(\mathcal{Y}^\mu)$$

identifies with

$$\operatorname{Shv}(Y^\mu)^{N_\mu,\chi} \overset{\mathbf{oblv}_{N^\mu,\chi}}{\longrightarrow} \operatorname{Shv}(Y^\mu) \overset{-\boxtimes\chi^!(\text{A-Sch})}{\longrightarrow} \operatorname{Shv}(Y^\mu \times N') = \operatorname{Shv}(\mathcal{Y}^\mu).$$

Proof. We can choose the family of subgroups N^α to be of the form $N^\mu \cdot N'_\alpha$ for $N'_\alpha \subset N'$. We have:

$$\operatorname{Shv}(\mathcal{Y}^\mu)^{N^\mu\cdot N',\chi} = \operatorname{Shv}(Y^\mu \times N')^{N^\mu\cdot N',\chi} \simeq \lim_{\alpha} \operatorname{Shv}(Y^\mu \times N')^{N^\mu\cdot N'_\alpha,\chi} \simeq$$

$$\simeq \lim_{\alpha} \lim_{\alpha_1\geq\alpha} \operatorname{Shv}(Y^\mu \times N'_{\alpha_1})^{N^\mu\cdot N'_\alpha,\chi} \simeq \lim_{\alpha_1\geq\alpha} \operatorname{Shv}(Y^\mu \times N'_{\alpha_1})^{N^\mu\cdot N'_\alpha,\chi}$$

Since the diagonal $\{\alpha_1 = \alpha\}$ is cofinal in the poset of $\{\alpha_1 \leq \alpha\}$, the above limit identifies with

$$\lim_{\alpha} \operatorname{Shv}(Y^\mu \times N'_\alpha)^{N^\mu\cdot N'_\alpha,\chi}.$$

Now, it is clear that for each α, the restriction functor

$$\operatorname{Shv}(Y^\mu \times N'_\alpha)^{N^\mu\cdot N'_\alpha,\chi} \to \operatorname{Shv}(Y^\mu)^{N^\mu,\chi}$$

is an equivalence and the forgetful functor

$$\mathbf{oblv}_{N^\mu\cdot N'_\alpha,\chi} : \operatorname{Shv}(Y^\mu \times N'_\alpha)^{N^\mu\cdot N'_\alpha,\chi} \to \operatorname{Shv}(Y^\mu \times N'_\alpha)$$

identifies with

$$\operatorname{Shv}(Y^\mu)^{N_\mu,\chi} \overset{\mathbf{oblv}_{N^\mu,\chi}}{\longrightarrow} \operatorname{Shv}(Y^\mu) \overset{-\boxtimes\chi^!(\text{A-Sch})}{\longrightarrow} \operatorname{Shv}(Y^\mu \times N'_\alpha).$$

Hence,

$$\lim_{\alpha} \operatorname{Shv}(Y^\mu \cdot N'_\alpha)^{N^\mu\times N'_\alpha,\chi} \to \operatorname{Shv}(Y^\mu)^{N^\mu,\chi}$$

is also an equivalence, and the statement concerning $\mathbf{oblv}_{\mathfrak{L}(N),\chi}$ follows as well. $\square$

We can now prove Proposition 2.2.8:

Proof of Proposition 2.2.8. Since each (finite-dimensional) closed subscheme $Y \subset \mathcal{Y}$ intersects finitely many of the strata $\mathcal{Y}^\mu$, it is enough to show that the !-restriction of $\mathcal{F}$ to any $\mathcal{Y}^\mu$ is infinitely connective.

However, this follows immediately from Lemma 2.3.6, since

$$\chi^!(\text{A-Sch}) \in \text{Shv}(N')$$

is infinitely connective. Indeed, its further restriction to every $N^\alpha \cap N'$ lives in the (perverse) cohomological degree equal to $-\dim(N^\alpha \cap N')$.

□

Finally, let us prove Proposition 2.3.3:

Proof. Suppose $\mu + n \cdot \rho \notin \Lambda^+_{\mathbb{Q}}$. Then there exists a simple root $\check{\alpha}_i$ such that

$$\mu(\check{\alpha}_i) < n.$$

Then the subgroup

$$\mathbb{G}_a \overset{t^{-1}}{\to} \mathfrak{L}(\mathbb{G}_a) \overset{\check{\alpha}_i}{\to} \mathfrak{L}(N)$$

belongs to N^μ, and acts trivially on Y^μ. Since the restriction of χ to the above subgroup is non-trivial, this implies that

$$\text{Shv}(Y^\mu)^{\mathbb{G}_a} = 0 \Rightarrow \text{Shv}(Y^\mu)^{N^\mu,\chi} = 0.$$

This implies point (a) of the proposition using Lemma 2.3.6.

Point (a) formally implies point (b). Indeed, for every connected component of $\mathcal{Y}$, every subset of μ's for which stratum $\mathcal{Y}^\mu$ intersects this component and for which $\mu + n \cdot \rho \in \Lambda^+_{\mathbb{Q}}$ has a minimal element with respect to the standard order relation

$$\mu_1 \leq \mu_2 \Leftrightarrow \mu_2 - \mu_1 \in \Lambda^{\text{pos}}.$$

Point (c) follows similarly.

□

2.4. **Adapted objects.** In this subsection we will describe a procedure to construct a particularly manageable family of compact objects inside $\text{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$.

2.4.1. We will say that an object $\mathcal{F} \in \text{Shv}(\mathcal{Y})$ is "$(\mathfrak{L}(N),\chi)$-adapted" if:

- For any DG category $\mathbf{C}$ and $\mathbf{c} \in \mathbf{C}$, the partially defined functor $\text{Av}_!^{\mathfrak{L}(N),\chi}$, left adjoint to
$$\mathbf{oblv}_{\mathfrak{L}(N),\chi} : (\text{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi} \to \text{Shv}(\mathcal{Y}) \otimes \mathbf{C},$$
is defined on $\mathcal{F} \otimes \mathbf{c} \in \text{Shv}(\mathcal{Y}) \otimes \mathbf{C}$;
- The tautological map
$$\text{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F} \otimes \mathbf{c}) \to F_{\mathbf{C}}(\text{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}) \otimes \mathbf{c})$$
(where $F_{\mathbf{C}}$ is as in (2.5)), is an isomorphism.

Since $\mathbf{oblv}_{\mathfrak{L}(N),\chi}$ is continuous, if $\mathcal{F} \in \text{Shv}(\mathcal{Y})$ is compact and $(\mathfrak{L}(N),\chi)$-adapted, then

$$\text{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}) \in \text{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$$

is compact.

We will prove:

Theorem 2.4.2. *For any DG category* $\mathbf{C}$*, the category* $(\text{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi}$ *is generated by objects of the form* $\text{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F} \otimes \mathbf{c})$ *with* $\mathcal{F} \in \text{Shv}(\mathcal{Y})$ *compact and* $(\mathfrak{L}(N),\chi)$*-adapted.*

We will prove this theorem in Sect. 2.5. We will deduce it from the simplest part of S. Raskin's paper [Ras], namely, Sect. 2.11 of *loc. cit.* (i.e., the case of Theorem 2.7.1(1) of *loc.cit.* for $m = \infty$).

2.4.3. Note that Theorem 2.4.2 (for $\mathbf{C} = \text{Vect}$) immediately implies Theorem 2.2.4.

2.4.4. The rest of this subsection is devoted to the proof of Theorem 2.2.6.

First off, Theorem 2.4.2 readily implies that the essential image of the functor

$$F_{\mathbf{C}} : \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \otimes \mathbf{C} \to (\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi}$$

generates the target category. Hence, it remains to show that $F_{\mathbf{C}}$ is fully faithful.

For the latter, it suffices to show that for $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ compact and $(\mathfrak{L}(N),\chi)$-adapted, any $\mathbf{c} \in \mathbf{C}$ and any $\widetilde{\mathcal{F}} \in \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \otimes \mathbf{C}$, the map

$$\begin{aligned}(2.7) \quad & \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}\otimes\mathbf{C}}(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}) \otimes \mathbf{c}, \widetilde{\mathcal{F}}) \to \\ & \to \mathcal{H}om_{(\mathrm{Shv}(\mathcal{Y})\otimes\mathbf{C})^{\mathfrak{L}(N),\chi}}(F_{\mathbf{C}}(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}) \otimes \mathbf{c}), \widetilde{\mathcal{F}}) \simeq \\ & \simeq \mathcal{H}om_{(\mathrm{Shv}(\mathcal{Y})\otimes\mathbf{C})^{\mathfrak{L}(N),\chi}}(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F} \otimes \mathbf{c}), \widetilde{\mathcal{F}}) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})\otimes\mathbf{C}}(\mathcal{F} \otimes \mathbf{c}, \mathbf{oblv}_{\mathfrak{L}(N),\chi}(\widetilde{\mathcal{F}}))\end{aligned}$$

is an isomorphism.

2.4.5. Consider the following general paradigm:

Let $\mathbf{D}$ be a compactly generated DG category, and $\mathbf{d} \in \mathbf{D}^c$. For another DG category $\mathbf{C}$, consider the (continuous) functor

$$\mathcal{H}om_{\mathbf{D}}(\mathbf{d}, -) : \mathbf{D} \otimes \mathbf{C} \to \mathbf{C}, \quad \mathbf{d}_1 \otimes \mathbf{c}_1 \mapsto \mathcal{H}om_{\mathbf{D}}(\mathbf{d}, \mathbf{d}_1) \otimes \mathbf{c}_1.$$

We have:

Lemma 2.4.6. *For any $\mathbf{c} \in \mathbf{C}$ and $\widetilde{\mathbf{d}} \in \mathbf{D} \otimes \mathbf{C}$, we have a canonical isomorphism*

$$\mathcal{H}om_{\mathbf{D}\otimes\mathbf{C}}(\mathbf{d} \otimes \mathbf{c}, \widetilde{\mathbf{d}}) \simeq \mathcal{H}om_{\mathbf{C}}(\mathbf{c}, \mathcal{H}om_{\mathbf{D}}(\mathbf{d}, \widetilde{\mathbf{d}})).$$

Proof. Follows by interpreting $\mathbf{D} \otimes \mathbf{C}$ as

$$\mathrm{Funct}((\mathbf{D}^c)^{\mathrm{op}}, \mathbf{C}).$$

□

2.4.7. We apply Lemma 2.4.6 to the two sides in (2.7). We obtain that the left-hand side identifies with

$$\mathcal{H}om_{\mathbf{C}}(\mathbf{c}, \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}}(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}), \widetilde{\mathcal{F}})),$$

and the right-hand side with

$$\mathcal{H}om_{\mathbf{C}}(\mathbf{c}, \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}, \mathbf{oblv}_{\mathfrak{L}(N),\chi}(\widetilde{\mathcal{F}}))).$$

Hence, it remains to show that the two functors $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \otimes \mathbf{C} \to \mathbf{C}$, given by

$$\widetilde{\mathcal{F}} \mapsto \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}}(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}), \widetilde{\mathcal{F}}) \text{ and } \widetilde{\mathcal{F}} \mapsto \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}, \mathbf{oblv}_{\mathfrak{L}(N),\chi}(\widetilde{\mathcal{F}}))$$

are isomorphic.

For that it suffices to identify the corresponding functors $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \times \mathbf{C} \rightrightarrows \mathbf{C}$ that send $\mathcal{F}_1 \times \mathbf{c}_1$ to

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}}(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}(\mathcal{F}), \mathcal{F}_1) \otimes \mathbf{c}_1 \text{ and } \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}, \mathbf{oblv}_{\mathfrak{L}(N),\chi}(\mathcal{F}_1)) \otimes \mathbf{c}_1,$$

respectively.

Now the assertion follows from the $(\mathrm{Av}_!^{\mathfrak{L}(N),\chi}, \mathbf{oblv}_{\mathfrak{L}(N),\chi})$-adjunction.

2.5. **Proof of Theorem 2.4.2.** The proof of Theorem 2.4.2, given below, is based on a geometric construction due to S. Raskin.

2.5.1. For $j \geq 1$ let $\overset{\circ}{I}{}^j$ denote the subgroup of $\mathfrak{L}^+(G)$ consisting of points that belong to the preimage of

$$\mathfrak{L}^+(N)_j \subset \mathfrak{L}^+(G)_j$$

under the projection

$$\mathfrak{L}^+(G) \to \mathfrak{L}^+(G)_j := \mathfrak{L}^+(G)/K_j.$$

(I.e., this is the subgroup consisting of points that belong to N modulo t^j.)

Note that for $j = 1$, the subgroup $\overset{\circ}{I}{}^1$ is the unipotent radical of the Iwahori subgroup. By convention, for $j = 0$ we set $\overset{\circ}{I}{}^0 = \mathfrak{L}^+(G)$.

Denote

$$I^j := \mathrm{Ad}_{-j\check{\rho}(t)}(\overset{\circ}{I}{}^j) \subset \mathfrak{L}(G).$$

2.5.2. Consider the intersection

$$I^j \cap \mathfrak{L}(N).$$

We claim that the character $\chi|_{I^j \cap \mathfrak{L}(N)}$ can be canonically extended to all of I^j. Namely, $\chi|_{I^j \cap \mathfrak{L}(N)}$ factors through the projection

$$I^j \cap \mathfrak{L}(N) \overset{\mathrm{Ad}_{j\check{\rho}(t)}}{\longrightarrow} \overset{\circ}{I}{}^j \cap \mathfrak{L}(N) = \mathfrak{L}^+(N) \to \mathfrak{L}^+(N)_j,$$

and we define the sought-for extension (to be denoted also by χ) to be the restriction of the resulting character on $\mathfrak{L}^+(N)_j$ along

$$I^j \overset{\mathrm{Ad}_{j\check{\rho}(t)}}{\longrightarrow} \overset{\circ}{I}{}^j \to \mathfrak{L}^+(N)_j.$$

2.5.3. For any j we can consider the category $\mathrm{Shv}(\mathcal{Y})^{I^j,\chi}$, or more generally

$$(\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{I^j,\chi}$$

for a test DG category $\mathbf{C}$.

Note that for $j \geq 1$, the group-scheme I^j is pro-unipotent, and so $(\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{I^j,\chi}$ is a full subcategory of $\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C}$.

We have the functor

$$\mathrm{Av}_*^{I^j,\chi} \circ \mathbf{oblv}_{\mathfrak{L}(N),\chi} : (\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi} \to (\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{I^j,\chi}. \tag{2.8}$$

Note that for $j \geq 1$ we have $I^j = (I^j \cap \mathfrak{L}^+(B^-)) \cdot (I^j \cap \mathfrak{L}(N))$, and so the above functor identifies with

$$\mathrm{Av}_*^{I^j \cap \mathfrak{L}^+(B^-)} \circ \mathbf{oblv}_{\mathfrak{L}(N),\chi}.$$

The functor $\mathrm{Av}_*^{I^j,\chi} \circ \mathbf{oblv}_{\mathfrak{L}(N),\chi}$ considered above has a partially defined left adjoint given by

$$\mathrm{Av}_!^{\mathfrak{L}(N),\chi} \circ \mathbf{oblv}_{I^j,\chi}. \tag{2.9}$$

2.5.4. We have the following key result due to S. Raskin (this is the case of $m = \infty$ in Theorem 2.7.1 in [Ras], which is the most elementary part of that paper):

Theorem 2.5.5. *Any object in the essential image of* $\mathbf{oblv}_{I^j,\chi}$ *is* $(\mathfrak{L}(N), \chi)$*-adapted.*

As an immediate corollary, we obtain:

Corollary 2.5.6. *The left adjoint* (2.9) *of* (2.8) *is defined.*

For completeness, we will sketch the proof of Theorem 2.5.5 in Sect. A.

2.5.7. Let us now use Theorem 2.5.5 to prove Theorem 2.4.2.

First off, the category $\mathrm{Shv}(\mathcal{Y})^{I^j,\chi}$ is compactly generated (e.g., by [DrGa]). Moreover, the functor

$$\mathbf{oblv}_{I^j,\chi} : \mathrm{Shv}(\mathcal{Y})^{I^j,\chi} \to \mathrm{Shv}(\mathcal{Y})$$

sends compacts to compacts (being a left adjoint of the continuous functor $\mathrm{Av}_*^{I^j,\chi}$).

Thus, it remains to see that the essential images of the functors (2.9) (for all j) generate the category $(\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),\chi}$. This is equivalent to saying that the intersection of the kernels of the functors (2.8) is zero.

We will take $j \geq 1$. We will show that the intersection of the kernels of the functors $\mathrm{Av}_*^{I^j\cap\mathfrak{L}^+(B^-)}$ is zero on all of $\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C}$. The latter assertion is equivalent to the fact that the essential images of the functors

$$\mathbf{oblv}_{I^j\cap\mathfrak{L}^+(B^-)} : (\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{I^j\cap\mathfrak{L}^+(B^-)} \to \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C}$$

generates $\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C}$.

However, the latter is obvious, as $I^j \cap \mathfrak{L}^+(B^-)$ shrink as $j \to \infty$.

$\square$[Theorem 2.4.2]

3. A dual definition of the local Whittaker category

In this section we will define another version of the local Whittaker category, by following a procedure dual to that used in the definition of $\mathrm{Whit}(\mathcal{Y})$: instead of invariants we will use coinvariants.

We will eventually see that the new category, denoted $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$, is equivalent to the original $\mathrm{Whit}(\mathcal{Y})$. But the functor establishing this equivalence will be something non-tautological.

3.1. **Digression: invariant functors and categorical *coinvariants*.** In order to prepare for the dual definition of the local Whittaker category, we will first consider the finite-dimensional situation.

3.1.1. First, let N' be a unipotent group equipped with a character $\chi : N' \to \mathbb{G}_a$, and an action on a scheme Y.

For a DG category $\mathbf{C}$, we let

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C})^{N',\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C}) \tag{3.1}$$

be the full subcategory that consists of continuous functors $F : \mathrm{Shv}(Y) \to \mathbf{C}$, for which the natural transformation

$$F \circ \mathrm{Av}_*^{N',\chi} \to F$$

is an isomorphism.

The inclusion (3.1) admits a right adjoint, given by

$$F \mapsto \mathrm{Av}_*^{N',\chi} \circ F.$$

3.1.2. Note that using the Verdier self-duality

$$\mathrm{Shv}(Y) \simeq \mathrm{Shv}(Y)^\vee, \quad \langle \mathcal{F}, \mathcal{F}' \rangle := \Gamma(Y, \mathcal{F} \overset{!}{\otimes} \mathcal{F}')$$

we can identify

$$\mathrm{Shv}(Y) \otimes \mathbf{C} \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C}), \quad \mathcal{F} \otimes \mathbf{c} \mapsto (\mathcal{F}' \mapsto \langle \mathcal{F}, \mathcal{F}' \rangle \otimes \mathbf{c}).$$

In terms of this identification, we have

$$(\mathrm{Shv}(Y) \otimes \mathbf{C})^{N',-\chi} \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C})^{N',\chi},$$

where the LHS is understood in the sense of Sect. 1.5.4.

3.1.3. We define the category $\mathrm{Shv}(Y)_{N',\chi}$ to be universal among DG categories $\mathbf{C}$ equipped with a functor

$$F : \mathrm{Shv}(Y) \to \mathbf{C}, \quad F \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C})^{N',\chi}.$$

Denote the resulting universal functor

$$\mathrm{Shv}(Y) \to \mathrm{Shv}(Y)_{N',\chi}$$

by $p^{N',\chi}$.

We claim, however:

Proposition 3.1.4. *There exists a canonical identification of pairs*

$$(\mathrm{Shv}(Y)_{N',\chi}, p^{N',\chi}) \simeq (\mathrm{Shv}(Y)^{N',\chi}, \mathrm{Av}_*^{N',\chi}).$$

Proof. We need to establish an equivalence

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C})^{N',\chi} \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y)^{N',\chi}, \mathbf{C}),$$

such that the forgetful functor (3.1) corresponds to

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y)^{N',\chi}, \mathbf{C}) \xrightarrow{-\circ \mathrm{Av}_*^{N',\chi}} \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C}), \tag{3.2}$$

in a way functorial in $\mathbf{C}$.

Note that (3.2) is fully faithful and admits a right adjoint given by restriction along $\mathbf{oblv}_{N',\chi}$. Hence, it is enough to show that the corresponding two idempotents on $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y), \mathbf{C})$ match up. However, they are both given by pre-composition with $\mathrm{Av}_*^{N',\chi}$.

□

Corollary 3.1.5.

(a) *The composite functor*

$$\mathrm{Shv}(Y)^{N',\chi} \xrightarrow{\mathbf{oblv}_{N',\chi}} \mathrm{Shv}(Y) \xrightarrow{p^{N',\chi}} \mathrm{Shv}(Y)_{N',\chi}$$

is an equivalence.

(b) *The inverse equivalence, precomposed with $p^{N',\chi}$, identifies with $\mathrm{Av}_*^{N',\chi}$.*

Proof. In terms of the identification of Proposition 3.1.4, the functor in question corresponds to the endofunctor

$$\mathrm{Av}_*^{N',\chi} \circ \mathbf{oblv}_{N',\chi}$$

of $\mathrm{Shv}(Y)^{N',\chi}$, which is isomorphic to the identity. □

3.1.6. The pair $(\mathrm{Shv}(Y)_{N',\chi}, p^{N',\chi})$ can be also described as a Verdier quotient.

Namely, it is obtained by taking the quotient of $\mathrm{Shv}(Y)$ by the full DG subcategory consisting of annihilated by the functor $\mathrm{Av}_*^{N',\chi}$.

3.2. **The dual local Whittaker category.** We will now define a dual version of the Whittaker category, to be denoted

$$\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} := \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}.$$

3.2.1. For given a DG category $\mathbf{C}$, we can consider the DG category

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C}).$$

We define the full subcategory

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{\mathfrak{L}(N),\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C}),$$

essentially by mimicking the procedure in Sect. 2.1:

Namely, for $\mathfrak{L}(N)$ written as in (1.1), we set

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{\mathfrak{L}(N),\chi} := \lim_{\alpha} \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{N^\alpha,\chi} \simeq$$

$$\simeq \bigcap_{\alpha} \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{N^\alpha,\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C}),$$

so we have to make sense of $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{N^\alpha,\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})$.

Using Sect. 1.3.3, we have

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C}) = \mathrm{Funct}_{\mathrm{cont}}(\operatorname*{colim}_{i} \mathrm{Shv}(Y_i), \mathbf{C}) \simeq \lim_{i} \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C}),$$

and in terms of this equivalence, we set

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{N^\alpha,\chi} = \lim_{i} \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C})^{N^\alpha,\chi} \subset \lim_{i} \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C}).$$

Thus, it remains to define

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C})^{N^\alpha,\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C}).$$

We set

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C})^{N^\alpha,\chi} := \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C})^{N^\alpha_\beta,\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(Y_i), \mathbf{C}),$$

for N^α presented as in (1.2).

This completes the definition of the full subcategry

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{\mathfrak{L}(N),\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C}).$$

3.2.2. We are now ready to define $\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$. Namely, we let it be the universal among DG categories $\mathbf{C}$ equipped with a functor

$$F : \mathrm{Shv}(\mathcal{Y}) \to \mathbf{C}, \quad F \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})^{\mathfrak{L}(N),\chi}.$$

Denote the resulting universal functor

$$\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$$

by $p^{\mathfrak{L}(N),\chi}$.

3.2.3. It follows from the definitions that $\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$ identifies tautologically with the colimit in $\mathrm{DGCat}_{\mathrm{cont}}$

$$\operatorname*{colim}_{\alpha} \mathrm{Shv}(\mathcal{Y})_{N^\alpha,\chi},$$

where the colimit is taken in $\mathrm{DGCat}_{\mathrm{cont}}$.

3.2.4. Using Sect. 3.1.6, we can also describe $\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$ as the quotient of $\mathrm{Shv}(\mathcal{Y})$ by the full DG subcategory *generated* by objects

$$\{\mathcal{F} \,|\, \exists \alpha \text{ such that } \mathrm{Av}_*^{N^\alpha,\chi}(\mathcal{F}) = 0\}.$$

3.3. **Properties of the dual Whittaker category.** We will now discuss some basic properties of $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$. We will see that it is really the dual category of $\mathrm{Whit}(\mathcal{Y})$ (up to replacing χ by its inverse). An essential feature of our infinite-dimensional setting is that the tautological composite functor

$$\mathrm{Whit}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$$

is identically equal to zero, by stark contrast with the finite-dimensional situation.

3.3.1. By Sect. 3.1.2, we obtain:

Lemma 3.3.2. *For any* $\mathbf{C}$*, under the Verdier duality identification*

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C}) \simeq \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C},$$

the full subcategory

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}, \mathbf{C}) \hookrightarrow \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathbf{C})$$

corresponds to

$$(\mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C})^{\mathfrak{L}(N),-\chi} \subset \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C}.$$

Combined with Theorem 2.2.6:

Corollary 3.3.3. *The category* $\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$ *identifies with the dual of* $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),-\chi}$*, so that the functor*

$$p^{\mathfrak{L}(N),\chi} : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$$

is the dual of

$$\mathbf{oblv}_{\mathfrak{L}(N),\chi} : \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),-\chi} \to \mathrm{Shv}(\mathcal{Y}).$$

Since $\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),-\chi}$ is is compactly generated, we further obtain:

Corollary 3.3.4.

(a) *The category* $\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$ *is compactly generated.*

(b) *Let* $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c$ *be such that the functor* $\mathrm{Av}_!^{\mathfrak{L}(N),-\chi}$ *is defined on* $\mathcal{F}$*. Then*

$$p^{\mathfrak{L}(N),\chi}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \in \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$$

is compact, and

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}}(p^{\mathfrak{L}(N),\chi}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})), -) \simeq \langle \mathrm{Av}_!^{\mathfrak{L}(N),-\chi}, - \rangle,$$

where

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}(\mathcal{Y})^c)^{\mathrm{op}} \to \mathrm{Shv}(\mathcal{Y})^c$$

denotes the Verdier duality functor and $\langle -, - \rangle$ *denotes the canonical pairing*

$$\mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),-\chi} \otimes \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi} \to \mathrm{Vect}\,.$$

3.3.5. In order to develop a "feel" for what $\mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$ is like, let us describe the corresponding category $\mathrm{Shv}(\mathcal{Y}^\mu)_{\mathfrak{L}(N),\chi}$, where $\mathcal{Y}^\mu$ is, as in Sect. 2.3.

We have the following counterpart of Lemma 2.3.6 (with the same proof):

Lemma 3.3.6. *We have a canonical identification*

$$\mathrm{Shv}(\mathcal{Y}^\mu)_{\mathfrak{L}(N),\chi} \simeq \mathrm{Shv}(Y^\mu)_{N^\mu,\chi}$$

so that the projection functor

$$\mathrm{Shv}(\mathcal{Y}^\mu) \overset{p^{\mathfrak{L}(N),\chi}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y}^\mu)_{\mathfrak{L}(N),\chi}$$

goes over to

$$\mathrm{Shv}(\mathcal{Y}^\mu) \simeq \mathrm{Shv}(Y^\mu \times N') \overset{\overset{!}{\otimes} - \chi^!(\text{A-Sch})}{\longrightarrow} \mathrm{Shv}(Y^\mu \times N') \to \mathrm{Shv}(Y^\mu) \overset{p^{N^\mu,\chi}}{\longrightarrow} \mathrm{Shv}(Y^\mu)_{N^\mu,\chi},$$

*where the third arrow is the functor of *-direct image.*

3.3.7. Consider the composite functor

$$\operatorname{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} \overset{\mathbf{oblv}_{\mathfrak{L}(N),\chi}}{\longrightarrow} \operatorname{Shv}(\mathcal{Y}) \overset{p^{\mathfrak{L}(N),\chi}}{\longrightarrow} \operatorname{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}. \tag{3.3}$$

We note that *contrary* to the finite-dimensional situation described by Corollary 3.1.5(a), we have:

Proposition 3.3.8. *The functor* (3.3) *is identically equal to zero.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi} & \xrightarrow{\mathbf{oblv}_{\mathfrak{L}(N),\chi}} & \operatorname{Shv}(\mathcal{Y}^\mu) & \xrightarrow{p^{\mathfrak{L}(N),\chi}} & \operatorname{Shv}(\mathcal{Y}^\mu)_{\mathfrak{L}(N),\chi} \\
{\scriptstyle (\iota^\mu)_*}\downarrow & & {\scriptstyle (\iota^\mu)_*}\downarrow & & \downarrow{\scriptstyle (\iota^\mu)_*} \\
\operatorname{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi} & \xrightarrow{\mathbf{oblv}_{\mathfrak{L}(N),\chi}} & \operatorname{Shv}(\mathcal{Y}) & \xrightarrow{p^{\mathfrak{L}(N),\chi}} & \operatorname{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}.
\end{array}$$

By Proposition 2.3.3(b), it suffices to show that the corresponding functor

$$\operatorname{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}(N),\chi} \overset{\mathbf{oblv}_{\mathfrak{L}(N),\chi}}{\longrightarrow} \operatorname{Shv}(\mathcal{Y}^\mu) \overset{p^{\mathfrak{L}(N),\chi}}{\longrightarrow} \operatorname{Shv}(\mathcal{Y}^\mu)_{\mathfrak{L}(N),\chi} \tag{3.4}$$

is zero.

Using Lemmas 2.3.6 and 3.3.6, it suffices to show that the functor

$$\operatorname{Shv}(Y^\mu) \overset{-\boxtimes\chi^!(\text{A-Sch})}{\longrightarrow} \operatorname{Shv}(Y^\mu \times N') \overset{\overset{!}{\otimes}-\chi^!(\text{A-Sch})}{\longrightarrow} \operatorname{Shv}(Y^\mu \times N') \to \operatorname{Shv}(Y^\mu)$$

is zero.

However, the latter functor is given by tensoring with

$$\Gamma(N', \omega_{N'}) \simeq \underset{\alpha}{\operatorname{colim}}\, \Gamma(N'_\alpha, \omega_{N'_\alpha}) \in \operatorname{Vect},$$

and the latter is zero, as it is infinitely connective.

□

3.4. **The pseudo-identity functor.** As we have just seen, an analog of Corollary 3.1.5(a) completely fails in our situation: the corresponding composite functor is identically equal to 0.

However, we will be able to salvage Corollary 3.1.5(b). Namely, we will define a (renormalized) analog of the functor of *-averaging with respect to $(\mathfrak{L}(N), \chi)$ that would factor through $\operatorname{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi}$ and give rise to an equivalence $\operatorname{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi} \to \operatorname{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}$.

The definition of this functor depends on the choice of a lattice $N_0 \subset \mathfrak{L}(N)$; a natural such choice is $N_0 = \mathfrak{L}^+(N)$.

3.4.1. Choose a presentation of $\mathfrak{L}(N)$ as in (1.1). With no restriction of generality, we can assume that $N_0 \subset N^\alpha$ for all α.

For each α we consider the endofunctor

$$\mathbf{oblv}_{N^\alpha,\chi} \circ \operatorname{Av}_*^{N^\alpha,\chi}[2\dim(N^\alpha/N_0)]$$

of $\operatorname{Shv}(\mathcal{Y})$. We claim that the assignment

$$\alpha \rightsquigarrow \mathbf{oblv}_{N^\alpha,\chi} \circ \operatorname{Av}_*^{N^\alpha,\chi}[2\dim(N^\alpha/N_0)]$$

lifts to a functor

$$A \to \operatorname{Funct}_{\text{cont}}(\operatorname{Shv}(\mathcal{Y}), \operatorname{Shv}(\mathcal{Y})),$$

i.e., we have a homotopy-coherent system of natural transformations

$$\mathbf{oblv}_{N^{\alpha'},\chi} \circ \operatorname{Av}_*^{N^{\alpha'},\chi}[2\dim(N^{\alpha'}/N_0)] \to \mathbf{oblv}_{N^{\alpha''},\chi} \circ \operatorname{Av}_*^{N^{\alpha''},\chi}[2\dim(N^{\alpha''}/N_0)]$$

for $N^{\alpha'} \subset N^{\alpha''}$.

3.4.2. Namely, in terms of the action of $\mathrm{Shv}(\mathfrak{L}(N))$ on $\mathrm{Shv}(\mathcal{Y})$ (see Sect. C.3.4), the functor

$$\mathbf{oblv}_{N^\alpha,\chi} \circ \mathrm{Av}_*^{N^\alpha,\chi}$$

is given by convolution with the object

$$\mathsf{e}_{N^\alpha} \overset{!}{\otimes} \chi^!(\text{A-Sch}) \in \mathrm{Shv}(\mathfrak{L}(N^\alpha)) \subset \mathrm{Shv}(\mathfrak{L}(N)).$$

Now, we claim that the assignment

$$\alpha \rightsquigarrow \mathsf{e}_{N^\alpha}[2\dim(N^\alpha/N_0)]$$

extends to a functor

$$A \to \mathrm{Shv}(\mathfrak{L}(N)).$$

Indeed, the object e_{N^α} is the *-pullback of $\mathsf{e}_{N^\alpha/N_0} \in \mathrm{Shv}(\mathfrak{L}(N)/N_0)$ along $\mathfrak{L}(N) \to \mathfrak{L}(N)/N_0$, while since N^α/N_0 is smooth, we have

$$\mathsf{e}_{N^\alpha/N_0} \simeq \omega_{N^\alpha/N_0}[-2\dim(N^\alpha/N_0)].$$

Now, the desired functor comes from the functor

$$A \to \mathrm{Shv}(\mathfrak{L}(N)/N_0), \quad \alpha \mapsto \omega_{N^\alpha/N_0}, \quad N^{\alpha'} \subset N^{\alpha''} \mapsto (\omega_{N^{\alpha'}/N_0} \to \omega_{N^{\alpha''}/N_0}).$$

3.4.3. We define

$$\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi} := \underset{\alpha\in A}{\mathrm{colim}}\, \mathbf{oblv}_{N^\alpha,\chi} \circ \mathrm{Av}^{N^\alpha,\chi}[2\dim(N^\alpha/N_0)].$$

We claim that the essential image of $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi}$ is contained in the essential image of $\mathbf{oblv}_{\mathfrak{L}(N),\chi}$.

Indeed, by definition, we need to show that the essential image of $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi}$ is contained in the essential image of $\mathbf{oblv}_{N_{\alpha'},\chi}$ for every $\alpha' \in A$. However, for every $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ and $\alpha' \in A$, the objects $\mathbf{oblv}_{N^\alpha,\chi} \circ \mathrm{Av}^{N^\alpha,\chi}[2\dim(N^\alpha/N_0)](\mathcal{F})$ belong to the essential image of $\mathbf{oblv}_{N_{\alpha'},\chi}$ for $\alpha \geq \alpha'$.

Remark 3.4.4. One can view $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi}$ as a renormalized version of *-averaging with respect to $(\mathfrak{L}(N),\chi)$ in the following sense:

In the situation of Sect. 3.1 (say, for the trivial character), the functor $\mathrm{Av}_*^{N'}$ is given by

$$\mathrm{act}_* \circ p^*,$$

where

$$\mathrm{act}, p : N' \times Y \rightrightarrows Y$$

are the action and the projection maps. Set

$$\mathrm{Av}_{*,\mathrm{ren}}^{N'} := \mathrm{act}_* \circ p^!.$$

We have:

$$\mathrm{Av}_{*,\mathrm{ren}}^{N'} \simeq \mathrm{Av}_*^{N'}[2\dim(N')].$$

Now, in the situation when N' is a group ind-scheme *of ind-finite type*, the functor p^* makes no sense (or, rather, defines a pro-object). So we have to use $p^!$, and we get a well-defined functor $\mathrm{Av}_{*,\mathrm{ren}}^{N'}$.

When N' is a group ind-scheme not of ind-finite type, such as $\mathfrak{L}(N)$, in order to have a well-defined $p^!$, we need a choose a lattice $N_0 \subset N'$. This leads to the definition of $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi}$ given above.

3.4.5. For the same reason as in Sect. 3.4.3, we have:

$$\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi} \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathcal{Y}), \mathrm{Shv}(\mathcal{Y}))^{\mathfrak{L}(N),\chi}.$$

Hence, we obtain that the functor $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N),\chi}$ factors as

$$\mathrm{Shv}(\mathcal{Y}) \overset{p^{\mathfrak{L}(N),\chi}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi} \overset{\text{Ps-Id}_{\mathrm{Whit}}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}.$$

for a uniquely defined functor

$$\text{Ps-Id}_{\mathrm{Whit}} : \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}(N),\chi} \to \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N),\chi}.$$

3.4.6. *An example.* Consider the functor $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)}$ applied to the category $\mathrm{Shv}(\mathfrak{L}(N)/N')$, where $N' \subset N_0$ is a group-subscheme of finite codimension. We have the canonical identifications

$$\mathrm{Vect} \simeq \mathrm{Shv}(\mathfrak{L}(N)/N_0)^{\mathfrak{L}(N)}, \quad \mathsf{e} \mapsto \omega_{\mathfrak{L}(N)/N_0}$$

and

$$\mathrm{Shv}(\mathfrak{L}(N)/N_0)_{\mathfrak{L}(N)} \simeq \mathrm{Vect}, \quad \mathcal{F} \mapsto \Gamma(\mathfrak{L}(N)/N_0, \mathcal{F}).$$

With respect to the above identifications, the functor $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)}$, viewed as an endo-functor of Vect is the cohomological shift by $[-2(\dim(N_0/N'))]$.

By contrast, if we apply the functor $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)}$ to $\mathrm{Shv}(\mathrm{pt}) \simeq \mathrm{Vect}$, we obtain the zero functor.

3.4.7. We have the following key statement that replaces Corollary 3.1.5(b) in our infinite-dimensional setting:

Theorem 3.4.8. *The functor* $\text{Ps-Id}_{\mathrm{Whit}}(\mathcal{F})$ *is an equivalence.*

Theorem 3.4.8, as stated above, is due to S. Raskin. It had been conjectured by the author in 2008 and established by him for $n = 0$ (unpublished). The case of an arbitrary n and $G = GL_r$ was done by D. Beraldo in [Be]. The general case was established by S. Raskin using a new geometric insight.

In this paper we will give an alternative proof of Theorem 3.4.8, see Corollary 5.5.6. However, our proof is not altogether disjoint from that of Raskin: we will use the main geometric tool of [Ras], namely the subgroups I^j introduced in Sect. 2.5. Yet, we will use only the simplest part of [Ras], incarnated by Theorem 2.5.5 (or rather its Ran space version).

4. The global Whittaker category

In this section we fix a smooth and complete curve X and a point $x \in X$. We will define a *global* version of the Whittaker category, using various enhancements of the moduli stack Bun_G of G-bundles on X. The idea is to mimic the definition of the global Whittaker space in the classical theory of automorphic functions.

We will ultimately prove that the global Whittaker category is equivalent to the local one. The corresponding phenomenon in the classical theory is that the global Whittaker space splits as the tensor product of local Whittaker spaces.

4.1. **Drinfeld's compactification.** In this subsection we recall the definition of the *Drinfeld compactification*, which is an (ind)-algebraic stack used in the definition of the global Whittaker category .

4.1.1. Let $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$ be the version of Drinfeld's compactification introduced in [Ga3]. Namely, $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$ is the prestack that classifies the data of a G-bundle $\mathcal{P}_G$ on X equipped with injective maps of coherent sheaves

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\infty \cdot x), \quad \check{\lambda} \in \check{\Lambda}^+ \tag{4.1}$$

(here $\mathcal{V}^{\check{\lambda}}$ denotes the Weyl module of highest weight $\check{\lambda}$), such that the maps $\kappa^{\check{\lambda}}$ satisfy the Plücker relations, i.e., they define a reduction of $\mathcal{P}_G$ to B at the generic point of X.

Remark 4.1.2. When the derived group of G is not simply connected, in addition to the Plücker relations one imposes another closed condition, restricting the possible defect of the maps (4.1), see [Sch, Sect. 7]. However, for the purposes of defining the global Whittaker category, the difference is material, as the objects satisfying the Whittaker condition will be supported on the closed substack in question.

4.1.3. For $\mu \in \Lambda$, let

$$(\overline{\mathrm{Bun}}_N)_{\leq \mu\cdot x} \subset (\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$$

be the closed subfunctor where we require that for every $\check{\lambda} \in \check{\Lambda}^+$, the corresponding map (4.1) has a pole of order $\leq \langle \mu, \check{\lambda}\rangle$, i.e., corresponds to a regular map

$$(\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\langle \mu, \check{\lambda}\rangle \cdot x). \tag{4.2}$$

For

$$\mu_2 \leq \mu_1 \Leftrightarrow \mu_2 - \mu_1 \in \Lambda^{\mathrm{pos}} \tag{4.3}$$

we have an inclusion

$$(\overline{\mathrm{Bun}}_N)_{\leq \mu_1\cdot x} \subset (\overline{\mathrm{Bun}}_N)_{\leq \mu_2\cdot x},$$

and

$$(\overline{\mathrm{Bun}}_N)_{\infty\cdot x} \simeq \underset{\mu\in\Lambda}{\mathrm{colim}}\, (\overline{\mathrm{Bun}}_N)_{\leq \mu\cdot x}, \tag{4.4}$$

where Λ^{pos} is understood as a poset with the standard order relation, i.e., (4.3).

For each μ, the prestack $(\overline{\mathrm{Bun}}_N)_{\leq \mu\cdot x}$ is an algebraic stack; thus (4.4) shows that $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$ is an ind-algebraic stack.

Remark 4.1.4. Although the poset Λ is *not* filtered, its subset corresponding to those μ, for which $(\overline{\mathrm{Bun}}_N)_{\leq \mu\cdot x}$ intersects a given connected component of $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$, is filtered.

4.2. **Stratifications of** $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$**.** In this subsection we review various stratifications of Drinfeld's compactification, which will be used in the analysis of the structure of the global Whittaker category.

4.2.1. We denote by

$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x} \subset (\overline{\mathrm{Bun}}_N)_{\leq \mu\cdot x}$$

the open substack, where we require that for for every $\check{\lambda} \in \check{\Lambda}^+$, the corresponding map (4.1) has a pole of order *equal* to $\langle \mu, \check{\lambda}\rangle$ at x. I.e., the map (4.2) is a bundle map *on a neighborhood* of x.

4.2.2. One can further subdivide each $(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}$ into locally closed substacks, according to the order of vanishing of the maps (4.1) away from x.

Namely, let

$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{good\,elswhr}} \subset (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}$$

be the open substack where we require that the maps (4.1) do not vanish away from x, i.e., (4.2) is a bundle map.

For each $\lambda \in \Lambda^{\mathrm{pos}}$, let

$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{def}=\lambda} \subset (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}$$

be the locally closed substack where each of the maps (4.1) factors as

$$(\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho\rangle} \to (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho\rangle}(D) \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\langle \mu, \check{\lambda}\rangle \cdot x),$$

where D is a divisor of degree $\langle \lambda, \check{\lambda}\rangle$ on $X - x$, and the second map is a bundle map.

We have a well-defined map

$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{def}=\lambda} \to (X - x)^{\lambda},$$

where for $\lambda = \Sigma\, n_i \cdot \alpha_i$ (here α_i's are the positive coroots) we have

$$(X - x)^{\lambda} := \prod_i (X - x)^{(n_i)}.$$

We have

$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{good\,elswhr}} = (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{def}=0}$$

and

$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x} = \underset{\lambda\in\Lambda^{\mathrm{pos}}}{\cup} (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{def}=\lambda}.$$

Remark 4.2.3. In addition to the locally closed substacks
$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{good elswhr}} \subset (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}$$
for an individual μ, can define $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good elswhr}}$ as a subfunctor of $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$.

The caveat here is that
$$(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good elswhr}} \hookrightarrow (\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$$
is *not* a locally closed embedding, and $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good elswhr}}$ is not even an algebraic stack.

4.2.4. Note that $(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}$ is *not* quasi-compact.

Let
$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{def}\leq\lambda} \subset (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}$$
be the open substack equal to
$$\underset{0\leq\lambda'\leq\lambda}{\cup} (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{def}=\lambda'}.$$

Then each $(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{def}\leq\lambda}$ is quasi-compact.

4.2.5. Let $\underline{y}=\{y_1,...,y_m\}$ be a finite collection of points on $X-x$. We define an open subfunctor
$$(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}} \subset (\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$$
by requiring that the maps (4.1) do not vanish at the points $y_1,...,y_m$.

We will use the notation
$$(\overline{\mathrm{Bun}}_N)_{\leq\mu\cdot x,\text{good at }\underline{y}} := (\overline{\mathrm{Bun}}_N)_{\leq\mu\cdot x} \cap (\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}},$$
$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{good at }\underline{y}} := (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x} \cap (\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}},$$
$$(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{def}=\lambda,\text{good at }\underline{y}} := (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{def}=\lambda} \cap (\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}},$$
etc.

Note that the open subfunctors $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}}$ for $\underline{y}$ being singletons $\underline{y}=\{y\}$ cover $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$. We have
$$(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}} = \underset{i=1,...,m}{\cap} (\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }y_i}.$$

4.3. **Adding a level structure.** In order to find an analog of the Whittaker category on $\mathfrak{Y}=\mathfrak{L}(G)/K$ where $K\subsetneq\mathfrak{L}^+(G)$, we will need to introduce a variant of Drinfeld's compactification that has to do with G-level structures at x.

4.3.1. Let $\mathrm{Bun}_G^{G\text{-level}_{n\cdot x}}$ is the moduli stack of G-bundles with structure of level n at x.

The forgetful map
$$\mathrm{Bun}_G^{G\text{-level}_{n\cdot x}} \to \mathrm{Bun}_G$$
is a $\mathfrak{L}^+_x(G)_n$-torsor.

4.3.2. Consider the forgetful map
$$(\overline{\mathrm{Bun}}_N)_{\infty\cdot x} \to \mathrm{Bun}_G,$$
and denote
$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x} := (\overline{\mathrm{Bun}}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G^{G\text{-level}_{n\cdot x}}.$$

We will denote by
$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x} \subset (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$$
the corresponding locally closed substack, and similarly for
$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x},\ (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}},\ (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{def}=\lambda},\ (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{def}\leq\lambda},\ (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}},$$
etc.

4.3.3. Note that for a fixed $\mu \in \Lambda$, we have a well-defined map

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x} \to (\mathfrak{L}^+_x(G)_n/\mathfrak{L}^+_x(N)_n) \overset{\mathfrak{L}^+_x(T)}{\times} \mathcal{P}^{\omega^\rho}_T(\mu\cdot x). \tag{4.5}$$

4.4. **Action of the loop groupoid away from the level.** We will now introduce a key tool needed for the definition of the global Whittaker category: the action of the loop group $\mathfrak{L}(N)$ by "regluing". A feature if this construction is that it takes place at points of the curve *different* from x, which is our point of interest.

4.4.1. Given $\underline{y}$ as above, we can consider the usual loop (resp., arc) groups $\mathfrak{L}_{\underline{y}}(N)$, $\mathfrak{L}_{\underline{y}}(B)$ and $\mathfrak{L}_{\underline{y}}(G)$ (resp., $\mathfrak{L}^+_{\underline{y}}(N)$, $\mathfrak{L}^+_{\underline{y}}(B)$ and $\mathfrak{L}^+_{\underline{y}}(G)$). However, we will change the notation slightly and will use the above symbols to denote certain twists of these objects.

Namely, consider the T-torsor induced from the line bundle $\omega^{\frac{1}{2}}$ by means of the homomorphism $2\rho : \mathbb{G}_m \to T$; denote it $\mathcal{P}^{\omega^\rho}_T$. Using a (chosen) splitting $T \to B$, we can consider the B- and G-torsors

$$\mathcal{P}^{\omega^\rho}_B := B \overset{T}{\times} \omega^\rho \text{ and } \mathcal{P}^{\omega^\rho}_G := G \overset{T}{\times} \omega^\rho$$

over X. Let B^{ω^ρ} (resp., G^{ω^ρ}) be the group-scheme of automorphisms of $\mathcal{P}^{\omega^\rho}_B$ (resp., $\mathcal{P}^{\omega^\rho}_G$). In other words, B^{ω^ρ} (resp., G^{ω^ρ}) is the inner twist of the constant group-scheme with fiber B (resp., G) over X by means of $\mathcal{P}^{\omega^\rho}_B$ (resp., $\mathcal{P}^{\omega^\rho}_G$).

Let N^{ω^ρ} be the group-scheme of automorphisms of $\mathcal{P}^{\omega^\rho}_B$ that project to the identity automorphism of $\mathcal{P}^{\omega^\rho}_T$ (in other words, N^{ω^ρ} is the twist of the constant group-scheme with fiber N over X by means of the T-torsor ω^ρ using the adjoint action of T on N).

From now on, we will use the symbol $\mathfrak{L}^+_{\underline{y}}(N)$ (resp., $\mathfrak{L}_{\underline{y}}(N)$) to denote the group-scheme (resp., group ind-scheme) of sections of N over the formal (resp., formal punctured) disc around $\underline{y}$. And similarly for $\mathfrak{L}^+_{\underline{y}}(B)$ and $\mathfrak{L}^+_{\underline{y}}(G)$ (resp., $\mathfrak{L}_{\underline{y}}(B)$ and $\mathfrak{L}_{\underline{y}}(G)$).

The above twist is made in order to have a canonical character

$$\chi_{\underline{y}} : \mathfrak{L}_{\underline{y}}(N) \to \mathbb{G}_a,$$

which is trivial on $\mathfrak{L}^+_{\underline{y}}(N)$.

4.4.2. Note that a point of $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}}$ defines a B-torsor on the formal disc around $\underline{y}$, with the induced T-torsor identified with $\mathcal{P}^{\omega^\rho}_T$.

Let

$$(\overline{\mathrm{Bun}}_N)^{N\text{-level}_{\infty\cdot\underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}}$$

denote the moduli space that classifies the data of a point of $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}}$ plus the data of isomorphism of the above B-torsor with $\mathcal{P}^{\omega^\rho}_B$ that induces the identity automorphism on $\mathcal{P}^{\omega^\rho}_T$.

The forgetful map

$$(\overline{\mathrm{Bun}}_N)^{N\text{-level}_{\infty\cdot\underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}} \to (\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\text{good at }\underline{y}}$$

is a $\mathfrak{L}^+_{\underline{y}}(N)$-torsor.

4.4.3. Denote

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}} := (\overline{\mathrm{Bun}}_N)^{N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G^{G\text{-level}_{n\cdot x}}.$$

A crucial piece of structure is that the $\mathfrak{L}^+_{\underline{y}}(N)$-action on $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}}$ extends to an action of the group ind-scheme $\mathfrak{L}_{\underline{y}}(N)$.

In particular,

$$\mathfrak{L}^+_{\underline{y}}(N)\backslash \mathfrak{L}_{\underline{y}}(N) \overset{\mathfrak{L}^+_{\underline{y}}(N)}{\times} (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}} \tag{4.6}$$

has a natural structure of groupoid acting on $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}}$.

4.4.4. Recall the map (4.5). For future use we note the following:

Lemma 4.4.5. *For a fixed $\mu \in \Lambda$ and $\lambda \in \Lambda^{\mathrm{pos}}$, the group ind-scheme $\mathfrak{L}_{\underline{y}}(N)$ acts transitively along the fibers of the map*

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{=\mu\cdot x,\text{def}=\lambda,\text{good at }\underline{y}} \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{def}=\lambda,\text{good at }\underline{y}} \overset{(4.5)}{\longrightarrow}$$
$$\to \left((\mathfrak{L}^+_x(G)_n/\mathfrak{L}^+_x(N)_n) \overset{\mathfrak{L}^+_x(T)}{\times} \mathcal{P}^{\omega^\rho}_T(\mu\cdot x)\right) \times (X - \{x\cup \underline{y}\})^\lambda.$$

Furthermore, by Riemann-Roch, we have:

Lemma 4.4.6. *For every integer k there exists a large enough group sub-scheme of $\mathfrak{L}_{\underline{y}}(N)$ such that for $\mu \in \Lambda$ and $\lambda \in \Lambda^{\mathrm{pos}}$ satisfying $\langle \mu-\lambda,\check{\rho}\rangle \leq k$, this subgroup acts transitively along the orbits of $\mathfrak{L}_{\underline{y}}(N)$ on $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{=\mu\cdot x,\text{def}=\lambda,\text{good at }\underline{y}}$ (i.e., along the fibers of the map in Lemma 4.4.5).*

4.5. **Definition of the global Whittaker category (with an auxiliary point).** Our goal is to define a certain full subcategory

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}).$$

We will first do so on the locus $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}$, i.e., we will define

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}).$$

This will be done by imposing an equivariance condition with respect to the action of the groupoid (4.6).

4.5.1. Note that the operation of *-pullback defines an equivalence

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}) \simeq \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}^+_{\underline{y}}(N)} \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}}).$$

We define $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}})$ to be the full subcategory of $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}})$ that maps under the above equivalence to

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}_{\underline{y}}(N),\chi_{\underline{y}}} \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}^+_{\underline{y}}(N)}.$$

Let us rewrite this definition in terms that only involve algebro-geometric objects locally of finite type.

4.5.2. Let us write

$$\mathfrak{L}_{\underline{y}}(N) \simeq \underset{\alpha \in A}{\operatorname{colim}} N_{\underline{y}}^{\alpha}$$

as in (1.1). With no restriction of generality we can assume that

$$\mathfrak{L}^+_{\underline{y}}(N) \subset N_{\underline{y}}^{\alpha},\ \forall \alpha.$$

First off, we have:

$$\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}_{\underline{y}}(N),\chi_{\underline{y}}} = \underset{\alpha}{\lim}\ \operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{N^{\alpha}_{\underline{y}},\chi_{\underline{y}}},$$

where each

$$\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{N^{\alpha}_{\underline{y}},\chi_{\underline{y}}}$$

is a full subcategory of

$$\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}^+_{\underline{y}}(N)} \subset \operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}}).$$

In particular, the above $\underset{\alpha}{\lim}$ amounts to the intersection of these subcategories. We will now describe $\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{N^{\alpha}_{\underline{y}},\chi_{\underline{y}}}$ as a full subcategory of $\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}})$.

4.5.3. For each α let $N^{-\alpha}_{\underline{y}} \subset N^{\alpha}_{\underline{y}}$ be a normal subgroup of finite codimension contained in $\mathfrak{L}^+_{\underline{y}}(N)$. Then the character $\chi_{\underline{y}}|_{N^{\alpha}_{\underline{y}}}$ factors through

$$N^{\alpha}_{\underline{y}} \twoheadrightarrow N^{\alpha}_{\underline{y}}/N^{-\alpha}_{\underline{y}}.$$

Consider the ind-algebraic stack

$$N^{-\alpha}_{\underline{y}}\backslash(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}};$$

the action of $\mathfrak{L}^+_{\underline{y}}(N)/N^{-\alpha}_{\underline{y}}$ on it extends to an action of $N^{\alpha}_{\underline{y}}/N^{-\alpha}_{\underline{y}}$.

Then we have:

$$\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{N^{\alpha}_{\underline{y}},\chi_{\underline{y}}} \simeq \operatorname{Shv}(N^{-\alpha}_{\underline{y}}\backslash(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{N^{\alpha}_{\underline{y}}/N^{-\alpha}_{\underline{y}},\chi_{\underline{y}}},$$

where we identify the RHS with a full subcategory of $\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}})$ as follows:

We have:

$$\operatorname{Shv}(N^{-\alpha}_{\underline{y}}\backslash(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{N^{\alpha}_{\underline{y}}/N^{-\alpha}_{\underline{y}},\chi_{\underline{y}}} \subset$$
$$\subset \operatorname{Shv}(N^{-\alpha}_{\underline{y}}\backslash(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}^+_{\underline{y}}(N)/N^{-\alpha}_{\underline{y}}},$$

whereas *-pullback along

$$N^{-\alpha}_{\underline{y}}\backslash(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}} \to (\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}$$

identifies

$$\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}) \simeq \operatorname{Shv}(N^{-\alpha}_{\underline{y}}\backslash(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at }\underline{y}})^{\mathfrak{L}^+_{\underline{y}}(N)/N^{-\alpha}_{\underline{y}}}.$$

4.6. **Properties of the global Whittaker category (with an auxiliary point).** In this subsection we will study some basic properties of the category $(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at }\underline{y}}$; in particular, its (in)dependence of $\underline{y}$.

4.6.1. Replacing $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}}$ by its (locally) closed substacks

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda,\mathrm{good\ at}\ \underline{y}} \subset (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}} \subset (\overline{\mathrm{Bun}}_N)_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$$

we can similarly define the corresponding full categories

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda,\mathrm{good\ at}\ \underline{y}}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda,\mathrm{good\ at}\ \underline{y}});$$

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}});$$

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}).$$

The corresponding !-pullback and *-pushforward functors maps these full subcategories to one another. Furthermore, since we are dealing with unipotent groups, we have:

Lemma 4.6.2. *An object of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}})$ *belongs to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}})$ *if (and only if) its !-restrictions to the locally closed substacks* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$ *and/or* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda,\mathrm{good\ at}\ \underline{y}}$ *belong to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}})$ *and/or* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda,\mathrm{good\ at}\ \underline{y}})$.

4.6.3. Next, we have the following stabilization result:

Proposition 4.6.4. *For a fixed $\mu \in \Lambda$ there exists a large enough subgroup $N^\alpha_{\underline{y}} \subset \mathfrak{L}_{\underline{y}}(N)$ such that in order to test that an object of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}})$ *belongs to* $\mathrm{Whit}(\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}})$*, it sufficient to test that its pullback to* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot\underline{y}}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$ *belongs to*

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot\underline{y}}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}})^{N^\alpha_{\underline{y}},\chi_{\underline{y}}}.$$

Proof. It suffices to show that there exists $N^\alpha_{\underline{y}} \subset \mathfrak{L}_{\underline{y}}(N)$ which acts transitively on every $\mathfrak{L}_{\underline{y}}(N)$-orbit on $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot\underline{y}}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$.

The existence of such $N^\alpha_{\underline{y}} \subset \mathfrak{L}_{\underline{y}}(N)$ follows from Lemma 4.4.6.

□

Corollary 4.6.5. *The inclusions*

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}}),$$

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}),$$

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\ at}\ \underline{y}})$$

*all admit continuous right adjoints. These right adjoints commute with the corresponding !-pullback and *-pushforward functors.*

Proof. We have:

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}}) \simeq \lim_{\mu\in\Lambda} \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}})$$

(with respect to the !-restriction functors), and

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ at}\ \underline{y}}) \simeq \lim_{\mu\in\Lambda} \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}).$$

So it is enough to prove the assertion of the proposition for a fixed $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$ and the substacks

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu'\cdot x,\mathrm{good\ at}\ \underline{y}} \subset (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu'\cdot x,\mathrm{good\ at}\ \underline{y}} \subset (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$$

for $\mu' \leq \mu$.

However, now the assertion follows from Proposition 4.6.4: the required right adjoint is given on the corresponding

$$N^{-\alpha}_{\underline{y}}\backslash(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot\underline{y}}}_{\leq\mu\cdot x,\mathrm{good\ at}\ \underline{y}}$$

by the functor $\mathrm{Av}_*^{N^\alpha_{\underline{y}}/N^{-\alpha}_{\underline{y}},\chi_{\underline{y}}}$.

□

4.6.6. Let now $\underline{y}$ be equal to $\underline{y}' \sqcup \underline{y}''$. Note that

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}} = (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}'} \cap (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}''}.$$

We claim:

Proposition 4.6.7.

(a) *The restriction functor*

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}'}) \to \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}})$$

sends

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}'}) \to \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}}).$$

(b) *The diagram*

$$\begin{tikzcd}
\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}'}) \arrow[r] & \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}}) \\
\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}'}) \arrow[u] \arrow[r] & \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}}) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are the right adjoints to the inclusions, also commutes.

Proof. To prove point (a), it suffices to show that the action of $\mathfrak{L}_{\underline{y}'}(N)$ along the orbits of $\mathfrak{L}_{\underline{y}}(N)$ on $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot \underline{y}}}_{\infty\cdot x,\text{good at } \underline{y}}$ is transitive. However, this follows from Lemma 4.4.5.

To prove point (b), it is enough to do so for the embedding

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\text{good at } \underline{y}} \hookrightarrow (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x,\text{good at } \underline{y}'}.$$

Now the assertion follows from Lemma 4.4.6: the right adjoint in question is given by the functor $\mathrm{Av}_*^{N^\alpha_{\underline{y}'},\chi_{\underline{y}'}}$ for a large enough subgroup

$$N^\alpha_{\underline{y}'} \subset \mathfrak{L}_{\underline{y}'}(N).$$

□

4.6.8. The above discussion was *not* specific to the fact that we were dealing with a non-degenerate character $\chi_{\underline{y}}$; in particular it equally applies to the case when the character is trivial.

However, the following assertion *is* specific to the non-degenerate case:

Lemma 4.6.9.

(a) *Any object of* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } \underline{y}})$ *supported outside of*

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}} \subset (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } \underline{y}}$$

is zero.

(b) *The category* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } \underline{y}})$ *is zero unless* $\mu + n\cdot\rho \in \Lambda^+_{\mathbb{Q}}$.

For the proof see [FGV, Lemma 6.2.4].

4.7. **Definition of the global Whittaker category.** In this section we will finally define the sought-after category $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$. I.e., we will show how to get rid of the auxiliary point(s) $\underline{y}$.

4.7.1. We define
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$$
to be the full subcategory that consists of objects such that their restriction to
$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } \underline{y}} \subset (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$$
belongs to
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } \underline{y}}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } \underline{y}})$$
for any finite non-empty collection of points $\underline{y}$.

By Proposition 4.6.7, it is enough to check this condition for $\underline{y}$ being singletons $\{y\}$. Note also that every quasi-compact algebraic substack of $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$ is contained in a union of $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good at } y})$ for finitely many points y.

4.7.2. We define the full categories
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x}),$$
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}),$$
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda})$$
by the same principle.

The corresponding !-pullback and *-pushforward functors map these full subcategories to one another. From Lemma 4.6.2 we obtain:

Corollary 4.7.3. *An object of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *belongs to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *if (and only if) its !-restrictions to the locally-closed substacks* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}$ *(resp.,* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda}$*) belong to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x})$ *(resp.,* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{def}=\lambda})$*).*

From Proposition 4.6.7(b) and Corollary 4.6.5, we obtain:

Corollary 4.7.4. *The inclusions*
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}),$$
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq\mu\cdot x}),$$
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x})$$
*all admit continuous right adjoints. These right adjoints commute with the corresponding !-pullback and *-pushforward functors.*

We will denote the right adjoint(s) appearing in the above corollary by $\mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}}$.

4.7.5. The above assertions are *not* specific to the fact that we were dealing with a non-degenerate character. In the non-degenerate case, from Lemma 4.6.9 we obtain:

Corollary 4.7.6.

(a) *The restriction functor* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good at } \underline{y}})$ *is an equivalence for any* $\underline{y}$*.*

(b) *The category* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x})$ *is zero unless* $\mu + n\cdot\rho \in \Lambda^+_{\mathbb{Q}}$*.*

(c) *The restriction functor*
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}) \to (\overline{\mathrm{Bun}}_N)_{=\mu\cdot x,\text{good elswhr}}$$
is an equivalence.

(d) *An object of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *belongs to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *if (and only if) for every* μ*:*

(i) *Its !-restriction to* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x} - (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}$ *is zero;*

(ii) *Its !-restriction to* $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}$ *belongs to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}})$.

Remark 4.7.7. In Theorem 5.2.2(b) we will give an explicit local description of the categories $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}})$.

4.7.8. From Corollary 4.7.6 we obtain:

Corollary 4.7.9.
(a) *For a given* μ*, every object of* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x})$ *is a clean extension from a quasi-compact substack.*
(b) *The category* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *is compactly generated; the forgetful functor*

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$$

sends compacts to compacts.

Proof. Point (a) follows from the fact that for a given μ the set of $\mu' \in \Lambda$ that satisfy

$$\mu' \in \mu - \Lambda \text{ and } \mu' + n\cdot\rho \in \Lambda^+$$

is finite.

To prove point (b), it suffices to show that for a fixed μ, the category $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x,\text{good at } \underline{y}})$ for some/any non-empty $\underline{y}$, is compactly generated. However, by point (a), the latter is equivalent to the category of (twisted) sheaves on a quasi-compact algebraic stack, and the assertion follows from [DrGa].

□

4.8. **Duality for the global Whittaker category.** In this subsection we will show that the global Whittaker category (unlike its local counterpart) is more or less tautologically self-dual, up to replacing χ by its inverse.

4.8.1. As was mentioned above, the algebraic stacks $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}$ are not quasi-compact. Hence, the functor

$$\Gamma((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}, -) : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}) \to \mathrm{Vect}$$

is not continuous, and we *do not* have a Verdier duality identification of $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x})$ with its dual.

However, if $\mathcal{F} \in (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}$ is a *-extension from a quasi-compact substack, the functor

$$\Gamma((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}, \mathcal{F} \overset{!}{\otimes} -) : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}) \to \mathrm{Vect}$$

is continuous.

In particular, it follows from Corollary 4.7.9 that for $\mathcal{F} \in \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$, the functor

$$\Gamma((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}, \mathcal{F} \overset{!}{\otimes} -) : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Vect}$$

is continuous.

4.8.2. We claim:

Proposition 4.8.3. *The category* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *is canonically dual to a similar category defined using the opposite character; this duality is uniquely defined by the property that for* $\mathcal{F} \in \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *and* $\mathcal{F}' \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *we have a functorial isomorphism*

$$\langle \mathcal{F}, \mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}}(\mathcal{F}')\rangle = \Gamma((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'). \tag{4.7}$$

Proof. It suffices to define a contravariant equivalence between the corresponding subcategories of compact objects.

Every compact object $\mathcal{F} \in \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ is supported on some $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x}$, and by Corollary 4.7.9(a) is a clean extension from some quasi-compact open. Hence, $\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})$ is a compact object in $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x})$ and belongs to $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\leq \mu\cdot x})$ with the opposite character.

For $\mathcal{F}' \in \operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ we have a canonical isomorphism

$$\Gamma((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}') \simeq \mathcal{H}om_{\operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})}(\mathbb{D}^{\operatorname{Verdier}}(\mathcal{F}), \mathcal{F}') \simeq$$
$$\simeq \mathcal{H}om_{\operatorname{Whit}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})}(\mathbb{D}^{\operatorname{Verdier}}(\mathcal{F}), \operatorname{Av}^{\operatorname{Whit}}_{*,\operatorname{glob}}(\mathcal{F}')) =: \langle \mathcal{F}, \operatorname{Av}^{\operatorname{Whit}}_{*,\operatorname{glob}}(\mathcal{F}')\rangle,$$

as required.

□

5. The local vs global comparison

In this section we will compare the local and global definitions of the Whittaker category. Our main result, Theorem 5.5.2, will say that they are equivalent.

5.1. **The local-to-global map.** In this subsection we will introduce a map between geometries that will eventually let us compare the local and the global definitions of the Whittaker category.

5.1.1. We will now introduce the twisted versions the group ind-scheme $\mathfrak{L}(N)$ and the ind-scheme $\mathcal{Y} = \mathfrak{L}(G)/K_n$ that it acts on.

Instead of $\mathfrak{L}(N)$, we will use the group ind-scheme $\mathfrak{L}_x(N)$, defined following the recipe in Sect. 4.4.1. We let χ_x denote the canonical character on $\mathfrak{L}_x(N)$.

For $\mathcal{Y}$ we will keep the same notation, but we will mean the moduli space of triples

$$(\mathcal{P}_G, \gamma, \epsilon),$$

where $\mathcal{P}_G$ is a G-bundle on the formal disc around x, γ is an identification of $\mathcal{P}_G$ with $\mathcal{P}_G^{\omega^\rho}$ on the formal punctured disc, and ϵ is a structure of level n at x on $\mathcal{P}_G$.

The group of automorphisms of $\mathcal{P}_G^{\omega^\rho}$ on the formal punctured disc, acts on $\mathcal{Y}$; in particular we have a $\mathfrak{L}_x(N)$ -action on $\mathcal{Y}$.

5.1.2. Recall that according to the Beauville-Laszlo theorem, the data of $(\mathcal{P}_G, \gamma)$ in the definition of $\mathcal{Y}$ can be reinterpreted by letting $\mathcal{P}_G$ be a G-bundle over X and γ an identification of $\mathcal{P}_G$ with $\mathcal{P}_G^{\omega^\rho}$ on $X - x$.

The G-bundle $\mathcal{P}_G^{\omega^\rho}$ comes equipped with a tautological Plücker data (4.1). From here we obtain a map

$$\pi : \mathcal{Y} \to (\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}. \tag{5.1}$$

5.1.3. Our first goal is to prove:

Theorem 5.1.4.
(a) *The functor* $\pi^!$ *sends* $\operatorname{Whit}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *to* $\operatorname{Whit}(\mathcal{Y})$.
(b) *Vice versa, if* $\mathcal{F} \in \operatorname{Shv}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *is such that* $\pi^!(\mathcal{F}) \in \operatorname{Whit}(\mathcal{Y})$*, and its !-restriction to the locally closed subsets*

$$(\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x} - (\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\operatorname{good\,elswhr}}$$

vanishes (for all μ*), then* $\mathcal{F} \in \operatorname{Whit}((\overline{\operatorname{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$.

5.2. **A strata-wise equivalence.** In this subsection we will show that the map π of (5.1) defines a *strata-wise* equivalence between the local and the global Whittaker categories.

The discussion in this subsection applies equally well to the situation with the trivial character.

5.2.1. Let $\mathcal{Y}^\mu$ be as in Sect. 2.3.1. Note that for a given $\mu \in \Lambda$, the map π restricts to a map

$$\pi_\mu : \mathcal{Y}^\mu \to \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}).$$

We will deduce Theorem 5.1.4 from the following more precise assertion:

Theorem 5.2.2.

(a) *For every μ, the functor $\pi_\mu^!$ sends* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}})$ *to* $\mathrm{Whit}(\mathcal{Y}^\mu)$.

(b) *The resulting functor* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}) \to \mathrm{Whit}(\mathcal{Y}^\mu)$ *is an equivalence.*

(c) *If* $\mathcal{F} \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}})$ *is such that* $\pi_\mu^!(\mathcal{F}) \in \mathrm{Whit}(\mathcal{Y}^\mu)$, *then*

$$\mathcal{F} \in \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}).$$

The implication Theorem 5.2.2(a) $\Rightarrow$ Theorem 5.1.4(a) follows from Proposition 2.3.3(c). The implication Theorem 5.2.2(c) $\Rightarrow$ Theorem 5.1.4(b) follows from Corollary 4.7.6(d).

The rest of this subsection is devoted to the proof of Theorem 5.2.2.

5.2.3. Choose a point $y \in X$ different from x. Let N_{X-x} (resp., $N_{X-(x,y)}$) denote the group ind-scheme of sections of N^{ω^ρ} over $X - x$ (resp., $X - (x, y)$).

Restriction to the formal punctured discs around x and y defines embeddings

$$N_{X-x} \hookrightarrow \mathfrak{L}_x(N)$$

and

$$\mathfrak{L}_y(N) \hookleftarrow N_{X-(x,y)} \hookrightarrow \mathfrak{L}_x(N).$$

By the sum of residues formula, we have

$$\chi_x|_{N_{X-(x,y)}} = -\chi_y|_{N_{X-(x,y)}}.$$

Note that the map π extends to a map

$$\mathcal{Y} \hookrightarrow \mathfrak{L}_y(N)/\mathfrak{L}_y^+(N) \times \mathcal{Y} \xrightarrow{\pi_y} (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\text{good elswhr}}.$$

Moreover, the above map π_y lifts to a map

$$\pi_y^{\text{level}} : \mathfrak{L}_y(N) \times \mathcal{Y} \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot y}}_{\infty\cdot x,\text{good elswhr}},$$

which is equivariant with respect to the $\mathfrak{L}_y(N)$-actions: we consider the $\mathfrak{L}_y(N)$-action by *right* multiplication on the $\mathfrak{L}_y(N)$-factor in $\mathfrak{L}_y(N) \times \mathcal{Y}$ and the $\mathfrak{L}_y(N)$-action on $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot y}}_{\infty\cdot x,\text{good elswhr}}$ from Sect. 4.4.3.

Denote by $\pi_{y,\mu}$ and $\pi_{y,\mu}^{\text{level}}$ the corresponding maps

$$\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N) \times \mathcal{Y}^\mu \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}} \text{ and } \mathfrak{L}_y(N) \times \mathcal{Y}^\mu \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot y}}_{=\mu\cdot x,\text{good elswhr}}.$$

We have:

Proposition 5.2.4.

(a) *Pullback along π_μ defines an equivalence*

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}) \simeq \mathrm{Shv}(\mathcal{Y}^\mu)^{N_{X-x}}$$

(b) *Pullback along $\pi_{y,\mu}$ defines an equivalence*

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}}) \simeq \mathrm{Shv}(\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N) \times \mathcal{Y}^\mu)^{N_{X-(x,y)}}.$$

(c) *Pullback along $\pi_{y,\mu}^{\text{level}}$ defines an equivalence*

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}, N\text{-level}_{\infty\cdot y}}_{=\mu\cdot x,\text{good elswhr}}) \simeq \mathrm{Shv}(\mathfrak{L}_y(N) \times \mathcal{Y}^\mu)^{N_{X-(x,y)}}.$$

Proof. For point (a), we claim that the map

$$\mathcal{Y}^\mu \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x},\mathrm{good\ elswhr}}_{=\mu\cdot x}$$

identifies $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}$ with the prestack quotient of $\mathcal{Y}^\mu$ with respect to the action of N_{X-x}.

Indeed, this is just the statement that any N^{ω^ρ}-bundle on {affine test scheme} $\times (X-x)$ admits a trivialization.

Points (b) and (c) are proved similarly.

□

Remark 5.2.5. The same proof shows that the functor $\pi^!$ defines an equivalence

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ elswhr}}) \simeq \mathrm{Shv}(\mathcal{Y})^{N_{X-x}},$$

where

$$(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ elswhr}} := (\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\mathrm{good\ elswhr}} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G^{G\text{-level}_{n\cdot x}},$$

and where $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x,\mathrm{good\ elswhr}}$ is as Remark 4.2.3.

This statement is not as useful for us because $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x,\mathrm{good\ elswhr}}$ is not an algebraic stack, so we cannot say much about the category of sheaves on it.

Let us also observe:

Lemma 5.2.6.
(a) *With respect to the equivalence of of Proposition 5.2.4(c), objects of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x},N\text{-level}_{\infty\cdot y}}_{=\mu\cdot x,\mathrm{good\ elswhr}})$ *that are* $(\mathfrak{L}_y(N),\chi_y)$*-equivariant correspond to objects of* $\mathrm{Shv}(\mathfrak{L}_y(N)\times\mathcal{Y}^\mu)$ *that are* $N_{X-(x,y)}$*-equivariant with respect to the diagonal action* and *that are* $(\mathfrak{L}_y(N),\chi_y)$*-equivariant on the* $\mathfrak{L}_y(N)$*-factor by* right *multiplication.*

(a') *Same as* (a), *but we replace* χ_y *by* $-\chi_y$ *and instead of right multiplication we consider left multiplication.*

(b) *With respect to the equivalence of Proposition 5.2.4(b), objects of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\ elswhr}})$ *that belong to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\ elswhr}})$ *correspond to objects of* $\mathrm{Shv}(\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times\mathcal{Y}^\mu)$ *that are* $N_{X-(x,y)}$*-equivariant with respect to the diagonal action* and *that are* $(\mathfrak{L}_y(N),-\chi_y)$*-equivariant on the* $\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)$*-factor.*

5.2.7. We are now ready to prove Theorem 5.2.2.

Let us observe that for an object

$$\mathcal{F}\in\mathrm{Shv}(\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times\mathcal{Y}^\mu)$$

that is $N_{X-(x,y)}$-equivariant wth respect to the *diagonal* action of $N_{X-(x,y)}$ on $\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times\mathcal{Y}^\mu$, the following extra conditions are equivalent:

(i) $\mathcal{F}$ is $(N_{X-(x,y)},\chi_x)$-equivariant with respect to the action on the $\mathcal{Y}^\mu$-factor;

(ii) $\mathcal{F}$ is $(N_{X-(x,y)},-\chi_y)$-equivariant with respect to the action on the $\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)$-factor;

(iii) Both conditions (i) and (ii);

(iv) The restriction of $\mathcal{F}$ to $1\times\mathcal{Y}^\mu$ is $(N_{X-(x,y)},\chi_x)$-equivariant.

Moreover, restriction as in point (iv) defines an equivalence from the category spanned by such objects to

$$\mathrm{Shv}(\mathcal{Y}^\mu)^{N_{X-(x,y)},\chi_x}.$$

Hence, using Lemma 5.2.6(b), it remains to prove the next assertion:

Proposition 5.2.8.

(a) *The forgetful functor*

$$\mathrm{Shv}(\mathcal{Y}^\mu)^{\mathfrak{L}_x(N),\chi_x} \to \mathrm{Shv}(\mathcal{Y}^\mu)^{N_{X-(x,y)},\chi_x}$$

is an equivalence.

(b) *For any prestack Z, the forgetful functor*

$$\mathrm{Shv}(\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times Z)^{\mathfrak{L}_y(N),-\chi_y} \to \mathrm{Shv}(\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times Z)^{N_{X-(x,y)},-\chi_y}$$

is an equivalence.

Proof of Proposition 5.2.8. We will prove point (a), as point (b) is similar. The idea of the proof is that $N_{X-(x,y)}$ is "dense" in $\mathfrak{L}_x(N)$. Here is how we spell this out:

Let Y^μ be as in Sect. 2.3.5. Since the action of $N_{X-(x,y)}$ on S^μ is transitive (this is one incarnation of the density of $N_{X-(x,y)}$ in $\mathfrak{L}_x(N)$), in Lemma 2.3.6, we obtain that restriction along $Y^\mu \hookrightarrow \mathcal{Y}^\mu$ defines an equivalence

$$\mathrm{Shv}(\mathcal{Y}^\mu)^{N_{X-(x,y)},\chi_x} \to \mathrm{Shv}(Y^\mu)^{N_{X-(x,y)}\cap N^\mu,\chi_x}.$$

Hence, it remains to see that the restriction functor

$$\mathrm{Shv}(Y^\mu)^{N^\mu,\chi_x} \to \mathrm{Shv}(Y^\mu)^{N_{X-(x,y)}\cap N^\mu,\chi_x}$$

is an equivalence.

To show this we note that we can find a normal group sub-scheme $N'' \subset N^\mu$ of finite codimension such that its action on Y^μ is trivial. Hence, it suffices to show that the functor

$$\mathrm{Shv}(Y^\mu)^{N^\mu/N',\chi_x} \to \mathrm{Shv}(Y^\mu)^{N_{X-(x,y)}\cap N^\mu/N_{X-(x,y)}\cap N',\chi_x}$$

is an equivalence.

However, for any N' of finite codimension in N^μ, the map

$$N_{X-(x,y)}\cap N^\mu/N_{X-(x,y)}\cap N' \to N^\mu/N'$$

is an isomorphism (again, by the density of $N_{X-(x,y)}$ in $\mathfrak{L}_x(N)$).

□

For future use, we note that the above argument also proves the following:

Lemma 5.2.9. *Under the equivalence of of Proposition 5.2.4(a), objects of* $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}})$ *that belong to* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\text{good elswhr}})$ *correspond to objects of* $\mathrm{Shv}(\mathcal{Y}^\mu)$*, for which N_{X-x}-equivariance extends to $(\mathfrak{L}_x(N),\chi_x)$-equivariance.*

5.3. **Local-to-global functor and duality.** Above we have considered the functor $\pi^!$ that maps the global Whittaker category to $\mathrm{Whit}(\mathcal{Y})$. In this subsection we will define a functor that maps $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$ to the global version.

5.3.1. Recall that according to Corollary 3.3.3 the dual of $\mathrm{Whit}(\mathcal{Y})$ is the category $\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}}$ (defined using the opposite character). Similarly, according to Proposition 4.8.3, the category dual to $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ is again $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ (defined using the opposite character).

Let us describe the resulting functor

$$\pi_{*,\mathrm{Whit}} : \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} := \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}_x(N),\chi_x} \to \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \tag{5.2}$$

dual to

$$\pi^! : \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}_x(N),\chi_x}.$$

5.3.2. Note that the morphism π is ind-schematic, so the functor

$$\pi_* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$$

is well-defined.

We claim:

Lemma 5.3.3. *The composite*

$$\mathrm{Shv}(\mathcal{Y}) \overset{p^{\mathfrak{L}_x(N),\chi_x}}{\longrightarrow} \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \overset{\pi_{*,\mathrm{Whit}}}{\longrightarrow} \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$$

identifies canonically with the functor $\mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}} \circ \pi_*$.

(We remind that the functor $\mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}}$ that appears in the lemma is the right adjoint to the embedding $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \hookrightarrow \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$.)

Proof. The assertion follows from (4.7) and the fact that for any $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c$, the object

$$\pi_*(\mathcal{F}) \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$$

is *-extended from a quasi-compact substack so that for any $\mathcal{F}' \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ we have

$$\Gamma((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}, \pi_*(\mathcal{F}) \overset{!}{\otimes} \mathcal{F}') \simeq \Gamma(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}').$$

□

Remark 5.3.4. In Sect. 5.4.2(b) we will describe the functor $\mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}} \circ \pi_*$ in terms local at x.

5.4. **Comparing the averaging procedures.** In this subsection we will see that the composite functor

$$\mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathcal{Y})$$

essentially coincides with the functor $\text{Ps-Id}_{\mathrm{Whit}}$ of Sect. 3.4.5.

5.4.1. Recall that N_{X-x} denotes the group ind-scheme of sections of of N^{ω^ρ} over $X - x$. Note, however, that the image of $N_{X-x} \hookrightarrow \mathfrak{L}_x(N)$ is no longer dense. Let $N' \subset \mathfrak{L}_x(N)$ be a large enough group subscheme so that $N' \cdot N_{X-x} = \mathfrak{L}_x(N)$.

We claim:

Proposition 5.4.2.

(a) $\pi^! \circ \mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}} : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathcal{Y})$ *identifies canonically with* $\mathrm{Av}^{N',\chi_x}_* \circ \pi^!$.

(b) *The functor* $\mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}} \circ \pi_*$ *identifies with* $\pi_* \circ \mathrm{Av}^{N',\chi_x}_*$.

Proof. For point (a), it suffices to prove the corresponding assertion for the functor

$$\pi^!_\mu : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\,elswhr}}) \to \mathrm{Whit}(\mathcal{Y}^\mu)$$

for all $\mu \in \Lambda$.

For point (b), it suffices to prove the corresponding assertion for the functor

$$(\pi_\mu)_* : \mathrm{Whit}(\mathcal{Y}^\mu) \to \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x,\mathrm{good\,elswhr}})$$

for all $\mu \in \Lambda$.

Now both assertions follow from the equivalence of Lemma 5.2.9.

□

5.4.3. Consider the action of the group ind-scheme N_{X-x} on $\mathcal{Y}$, and consider the corresponding functor

$$\mathrm{Av}^{N_{X-x}}_{*,\mathrm{ren}} := \mathrm{act}_* \circ p^! : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}),$$

see Remark 3.4.4.

We claim:

Proposition 5.4.4. *The composite* $\pi^! \circ \pi_* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})$ *identifies with the above functor* $\mathrm{Av}^{N_{X-x}}_{*,\mathrm{ren}}$.

Proof. Follows by base change from the fact that the action of N_{X-x} on $\mathcal{Y}$ defines an isomorphism

$$N_{X-x} \times \mathcal{Y} \simeq \mathcal{Y} \underset{(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}}{\times} \mathcal{Y}.$$

□

As a consequence, we obtain:

Corollary 5.4.5. *The functor*

$$\pi^! \circ \pi_{*,\mathrm{Whit}} : \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Whit}(\mathcal{Y})$$

identifies canonically with $\text{Ps-Id}_{\mathrm{Whit}}[-2d]$, *where* $d = \dim(N_0 \backslash \mathfrak{L}_x(N)/N_{X-x})$.

Proof. By Lemma 5.3.3, we need to show that the functor

$$\pi^! \circ \mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}} \circ \pi_*$$

identifies with $\mathrm{Av}^{\mathfrak{L}_x(N),\chi_x}_{*,\mathrm{ren}}$.

Combining Propositions 5.4.2(b) and 5.4.4, we obtain that the functor $\pi^! \circ \mathrm{Av}^{\mathrm{Whit}}_{*,\mathrm{glob}} \circ \pi_*$ is given by

$$\mathrm{Av}^{N',\chi_x}_* \circ \mathrm{Av}^{N_{X-x}}_{*,\mathrm{ren}},$$

where N' is as in Proposition 5.4.2.

However, unwinding the definitions, it is easy to see that $\mathrm{Av}^{N',\chi_x}_* \circ \mathrm{Av}^{N_{X-x}}_{*,\mathrm{ren}}$ identifies canonically with

$$\mathrm{Av}^{\mathfrak{L}_x(N),\chi_x}_{*,\mathrm{ren}}[2d] =: \text{Ps-Id}_{\mathrm{Whit}}[-2d].$$

□

5.5. Statement of the local-to-global equivalence.

5.5. Statement of the local-to-global equivalence. In this subsection we finally state the local-to-global comparison theorem.

5.5.1. We are now ready to state the main result of this paper:

Theorem 5.5.2. *The functor*

$$\pi^! : \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathcal{Y})$$

is an equivalence.

The proof will be given in Sect. 6.3.

5.5.3. Some remarks are in order. Note that for every μ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Whit}(\mathcal{Y}^\mu) \arrow[r] & \mathrm{Whit}(\mathcal{Y}) \\
\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{=\mu\cdot x}) \arrow[u] \arrow[r] & \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}), \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are *-direct image functors. Since the left vertical arrows are equivalences for all μ (by Theorem 5.2.2(b) and Corollary 4.7.6(c)), we see that the functor in the theorem is a "stratum-wise equivalence". So the challenge of the theorem is to show that these strata glue in the same way in the source and the target.

Given Proposition 2.3.3(b), we obtain that Theorem 5.5.2 is equivalent to the statement that the functor

$$\pi^! : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathcal{Y})$$

is fully faithful, when restricted to

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}).$$

5.5.4. From Theorem 5.5.2 and Lemma 5.3.3, we obtain:

Corollary 5.5.5. *The functor*

$$\pi_{*,\mathrm{Whit}} : \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$$

is an equivalence.

Finally, combining with Corollary 5.4.5 we obtain:

Corollary 5.5.6. *The functor* Ps-$\mathrm{Id}_{\mathrm{Whit}}$ *is an equivalence.*

Thus, we obtain a proof of Theorem 3.4.8.

6. Ran version and the proof of the main theorem

The proof of Theorem 5.5.2 is based on considering the *Ran space* version of the local Whittaker category.

We will show that the the pullback functor from the global version to the Ran version is fully faithful (this will be a geometric assertion not related to the specifics of the Whittaker situation). Then we will show that the original local Whittaker category (at one point of the curve) is equivalent to the Ran version.

6.1. **Ran geometry.** In this subsection we recall the definition of the Ran space and various geometric objects associated with it.

6.1.1. Recall that the Ran space of X, denoted $\mathrm{Ran}(X)$, is the functor that associates to an affine test scheme S the *set* of finite non-empty subsets $\mathcal{I} \subset \mathrm{Hom}(S, X)$.

Explicitly,

$$\mathrm{Ran}(X) \simeq \underset{I}{\mathrm{colim}}\, X^I,$$

where the colimit is taken in PreStk, and the index category is opposite to that of finite non-empty subsets and surjective maps; to a surjection $\phi : I_1 \twoheadrightarrow I_2$ we associate the correspondind diagonal map

$$\Delta_\phi : X^{I_2} \to X^{I_1}.$$

6.1.2. We will consider a version of $\mathrm{Ran}(X)$ with a marked point, denoted $\mathrm{Ran}(X)_x$. By definition, $\mathrm{Ran}(X)_x$ associates to an affine test scheme S the set of finite non-empty subsets $\mathcal{I} \subset \mathrm{Hom}(S, X)$ with a distinguished element corresponding to the map

$$S \to \mathrm{pt} \overset{x}{\to} X.$$

Explicitly,

$$\mathrm{Ran}(X)_x \simeq \underset{I}{\mathrm{colim}}\, (X^I \underset{X}{\times} \{x\}),$$

where the colimit is taken in PreStk, and the index category is opposite to that of finite non-empty subsets equipped with a distinguished element and surjective maps that preserve distinguished elements.

Note that we have a map $\mathrm{Ran}(X) \to \mathrm{Ran}(X)_x$, given by adding the distinguished element.

6.1.3. Let $\mathrm{Gr}_{G,\mathrm{Ran}}$ denote the (following slightly twisted version of the) Ran Grassmannian of G:

By definition, $\mathrm{Gr}_{G,\mathrm{Ran}}$ attaches to an affine test scheme S the set of triples $(\mathcal{I}, \mathcal{P}_G, \gamma)$, where:

- $\mathcal{I}$ is a finite non-empty subset of $\mathrm{Hom}(S, X)$;
- $\mathcal{P}_G$ is a G-bundle on $S \times X$;
- γ is an identification of $\mathcal{P}_G$ with the pullback of $\mathcal{P}_G^{\omega^\rho}$ over the open subset of $S \times X$ equal to the complement of the union of graphs of the maps $S \to X$ that comprise $\mathcal{I}$.

6.1.4. Let $S^0_{\mathrm{Ran}} \subset \overline{S}^0_{\mathrm{Ran}}$ be the (locally) closed subfunctors of $\mathrm{Gr}_{\mathrm{Ran}}$ defined by the following conditions:

For $\overline{S}^0_{\mathrm{Ran}}$, we require that the composite *meromorphic* maps

$$(\omega^{\frac{1}{2}})^{\langle \check\lambda, 2\rho\rangle} \to \mathcal{V}^{\check\lambda}_{\mathcal{P}_G^{\omega^\rho}} \overset{\gamma}{\to} \mathcal{V}^{\check\lambda}_{\mathcal{P}_G}$$

be *regular* on $S \times X$ for all $\check\lambda \in \check\Lambda^+$.

For S^0_{Ran} we require that these maps be an injective bundle maps.

6.1.5. We will now introduce a Ran version of the space $\mathcal{Y}$, denoted $\mathcal{Y}_{\mathrm{Ran}}$. By definition $\mathcal{Y}_{\mathrm{Ran}}$ attaches to an affine test scheme S the set of quadruples $(\mathcal{I}, \mathcal{P}_G, \gamma, \epsilon)$, where:

- $\mathcal{I}$ is a finite non-empty subset of $\mathrm{Hom}(S, X)$ with a distinguished element corresponding to the map $S \to \mathrm{pt} \overset{x}{\to} X$;
- $\mathcal{P}_G$ is a G-bundle on $S \times X$;
- γ is an identification of $\mathcal{P}_G$ with the pullback of $\mathcal{P}_G^{\omega^\rho}$ over the open subset of $S \times X$ equal to the complement of the union of graphs of the maps $S \to X$ that comprise $\mathcal{I}$, subject to the condition that the composite *meromorphic* maps
$$(\omega^{\frac{1}{2}})^{\langle \check\lambda, 2\rho\rangle} \to \mathcal{V}^{\check\lambda}_{\mathcal{P}_G^{\omega^\rho}} \overset{\gamma}{\to} \mathcal{V}^{\check\lambda}_{\mathcal{P}_G} \tag{6.1}$$
be *regular* on $S \times (X - x)$ for all $\check\lambda \in \check\Lambda^+$.
- ϵ is a structure of level n on $\mathcal{P}_G$ along $S \times \{x\}$.

6.1.6. As in Sect. 5.1.2, we have a naturally defined map

$$\pi_{\mathrm{Ran}} : \mathcal{Y}_{\mathrm{Ran}} \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}.$$

We have the following basic geometric assertion:

Theorem 6.1.7. *The pullback functor*

$$\pi^!_{\mathrm{Ran}} : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathcal{Y}_{\mathrm{Ran}})$$

is fully faithful.

The proof repeats verbatim the proof of [Ga2, Theorem 3.4.4].

6.2. **The Ran version of the Whittaker category.** In this subsection we state the key result, Theorem 6.2.5, which says that the Ran version of the Whittaker category is (essentially) equivalent to the local one (at point point $x \in X$).

6.2.1. The definition of the Whittaker category $\mathrm{Whit}(\mathcal{Y})$ has a Ran version, denoted

$$\mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}}) := \mathrm{Shv}(\mathcal{Y}_{\mathrm{Ran}})^{\mathfrak{L}_{\mathrm{Ran}}(N),\chi_{\mathrm{Ran}}}.$$

We refer the reader to [Ga2, Sect. 1.2], where the definition is spelled out for the trivial character; the case of the non-degenerate character is no different. See also Sect. 6.4, below.

As in Theorem 5.1.4 one shows:

Proposition 6.2.2. *The functor* $\pi^!_{\mathrm{Ran}}$ *sends* $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ *to* $\mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}})$.

Combined with Theorem 6.1.7, we obtain:

Corollary 6.2.3. *The functor*

$$\pi^!_{\mathrm{Ran}} : \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}})$$

is fully faithful.

6.2.4. Note now that we have a naturally defined map (in fact, a closed embedding)

$$\mathrm{unit}_{\mathrm{Ran}} : \mathrm{Ran}(X)_x \times \mathcal{Y} \to \mathcal{Y}_{\mathrm{Ran}}.$$

Observe also that the definition of $\mathrm{Whit}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$ has a variant

$$\mathrm{Whit}(\mathcal{Z} \times \mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Z} \times \mathcal{Y})$$

for an arbitrary prestack $\mathcal{Y}$, where we use the action of $\mathfrak{L}_x(N)$ on $\mathcal{Z} \times \mathcal{Y}$ coming from the $\mathcal{Y}$-factor.

In Sect. 6.4 we will prove:

Theorem 6.2.5. *The functor*

$$\mathrm{unit}^!_{\mathrm{Ran}} : \mathrm{Shv}(\mathcal{Y}_{\mathrm{Ran}}) \to \mathrm{Shv}(\mathrm{Ran}(X)_x \times \mathcal{Y})$$

defines an equivalence

$$\mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}}) \simeq \mathrm{Whit}(\mathrm{Ran}(X)_x \times \mathcal{Y}).$$

6.3. **Proof of the main theorem.** In this subsection we will deduce Theorem 5.5.2 from Theorem 6.2.5.

6.3.1. As was explained in Sect. 5.5.3, it suffices to show that the functor

$$\pi^! : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Shv}(\mathcal{Y})$$

is fully faithful, when restricted to

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}).$$

6.3.2. Note that the map $\pi : \mathcal{Y} \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$ equals the composition

$$\mathcal{Y} \to \mathrm{Ran}(X)_x \times \mathcal{Y} \xrightarrow{\mathrm{unit}_{\mathrm{Ran}}} \mathcal{Y}_{\mathrm{Ran}} \xrightarrow{\pi_{\mathrm{Ran}}} (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x},$$

where the first arrow corresponds to the tautological map

$$\mathrm{pt} \overset{\{x\}}{\to} \mathrm{Ran}(X)_x.$$

Hence, the functor $\pi^!$, restricted to $\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})$ is the composition

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \xrightarrow{\pi^!_{\mathrm{Ran}}} \mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}}) \xrightarrow{\mathrm{unit}^!_{\mathrm{Ran}}} \mathrm{Whit}(\mathrm{Ran}(X)_x \times \mathcal{Y}) \to \mathrm{Whit}(\mathcal{Y}). \tag{6.2}$$

According to Theorem 6.1.7, the first arrow in (6.2) is fully faithful, and the second arrow is an equivalence by Theorem 6.2.5. Hence, the functor

$$(\pi_{\mathrm{Ran}} \circ \mathrm{unit}_{\mathrm{Ran}})^! : \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathrm{Ran}(X)_x \times \mathcal{Y})$$

is fully faithful.

6.3.3. Note now that the map
$$\pi_{\mathrm{Ran}} \circ \mathrm{unit}_{\mathrm{Ran}} : \mathrm{Ran}(X)_x \times \mathcal{Y} \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}$$
factors as
$$\mathrm{Ran}(X)_x \times \mathcal{Y} \to \mathcal{Y} \xrightarrow{\pi} (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x},$$
where the first map is the projection on the $\mathcal{Y}$-factor.

Hence, the functor $\pi^!_{\mathrm{Ran}}$ is a *retract* of the functor $(\pi_{\mathrm{Ran}} \circ \mathrm{unit}_{\mathrm{Ran}})^!$. In particular, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathrm{Ran}(X)_x \times \mathcal{Y})$, the map
$$\mathcal{H}om_{\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})}(\mathcal{F}_1, \mathcal{F}_2) \to \mathcal{H}om_{\mathrm{Whit}(\mathcal{Y})}(\pi^!(\mathcal{F}_1), \pi^!(\mathcal{F}_2))$$
is a *retract* of the map
$$\mathcal{H}om_{\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x})}(\mathcal{F}_1, \mathcal{F}_2) \to \mathcal{H}om_{\mathrm{Whit}(\mathrm{Ran}(X)_x\times\mathcal{Y})}((\pi_{\mathrm{Ran}} \circ \mathrm{unit}_{\mathrm{Ran}})^!(\mathcal{F}_1), (\pi_{\mathrm{Ran}} \circ \mathrm{unit}_{\mathrm{Ran}})^!(\mathcal{F}_2)).$$

Hence, since the latter is an is isomorphism, so is the former.

□[Theorem 5.5.2]

Remark 6.3.4. Instead of using the retract argument, we could have finished the proof differently, using the fact that the marked Ran space $\mathrm{Ran}(X)_x$ is contractible. Indeed, the latter implies that the functor
$$\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathrm{Ran}(X)_x \times \mathcal{Y})$$
is fully faithful.

Hence, the fact that the composite
$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{n\cdot x}}_{\infty\cdot x}) \xrightarrow{\pi^!} \mathrm{Whit}(\mathcal{Y}) \to \mathrm{Whit}(\mathrm{Ran}(X)_x \times \mathcal{Y})$$
is fully faithful implies that the first arrow is fully faithful.

6.4. **Unital structure.** The goal of this subsection is to supply a crucial ingredient that will be used in the proof of Theorem 6.2.5. It will amount to a *unital structure* on $\mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}})$ in the world of factorization categories and modules over them.

6.4.1. For I an object in category of finite sets with a marked point (see Sect. 6.1.2), denote
$$\mathcal{Y}_I := X^I \underset{\mathrm{Ran}(X)_x}{\times} \mathcal{Y}.$$

We have
$$\mathcal{Y}_{\mathrm{Ran}} \simeq \underset{I}{\mathrm{colim}}\, \mathcal{Y}_I,$$
and hence
$$\mathrm{Shv}(\mathcal{Y}_{\mathrm{Ran}}) \simeq \underset{I}{\lim}\, \mathrm{Shv}(\mathcal{Y}_I). \tag{6.3}$$

We have the corresponding full subcategories
$$\mathrm{Whit}(\mathcal{Y}_I) \subset \mathrm{Shv}(\mathcal{Y}_I).$$
and under the equivalence (6.3), the full subcategory
$$\underset{I}{\lim}\, \mathrm{Whit}(\mathcal{Y}_I) \subset \underset{I}{\lim}\, \mathrm{Shv}(\mathcal{Y}_I)$$
corresponds to $\mathrm{Whit}(\mathcal{Y}_{\mathrm{Ran}}) \subset \mathrm{Shv}(\mathcal{Y}_{\mathrm{Ran}})$.

Let unit_I denote the corresponding map
$$X^I \times \mathcal{Y} \to \mathcal{Y}_I.$$

To prove Theorem 6.2.5, it suffices to prove its version for every I individually:

Theorem 6.4.2. *For every I, the functor $\mathrm{unit}^!_I$ induces an equivalence*
$$\mathrm{Whit}(\mathcal{Y}_I) \to \mathrm{Whit}(X^I \times \mathcal{Y}).$$

The assertion of Theorem 6.4.2 is Zariski-local in X, so from now on we will assume that X is affine. (This is done for notational convenience, when manipulating the two versions of the parameterized formal disc, $\widehat{\mathcal{D}}_{x_I}$ and $\mathcal{D}_{x_I}$ below.)

6.4.3. For an affine test-scheme S and an S-point x_I of X^I (i.e., an I-tuple of maps $S \to X$), let $\widehat{\mathcal{D}}_{x_I}$ be the formal completion of $S \times X$ along the union of the graphs of maps that comprise x_I. We consider $\widehat{\mathcal{D}}_{x_I}$ as an ind-scheme.

The assumption that X be affine implies that $\widehat{\mathcal{D}}_{x_I}$ is *ind-affine*; in particular, it gives rise to a well-defined ind-object in the category of affine schemes. Let $\mathcal{D}_{x_I}$ denote the colimit of $\widehat{\mathcal{D}}_{x_I}$, taken in the category of affine schemes. I.e., if

$$\widehat{\mathcal{D}}_{x_I} = \underset{\alpha}{\mathrm{colim}}\ \mathrm{Spec}(A_\alpha),$$

where the colimit is taken in PreStk, then $\mathcal{D}_{x_I} = \mathrm{Spec}(A)$, where

$$A := \lim_{\alpha} A_\alpha,$$

where the limit is taken in the category of commutative algebras.

The points comprising x_I give rise to an I-tuple of maps $S \to \mathcal{D}_{x_I}$. Let $\overset{\circ}{\mathcal{D}}_{x_I}$ be the (affine) open subscheme of $\mathcal{D}_{x_I}$ obtained by removing the union of the graphs of the above maps $S \to \mathcal{D}_{x_I}$. We will also consider the larger (affine) open subscheme

$$\mathcal{D}'_{x_I} := \mathcal{D}_{x_I} - (S \times \{x\}).$$

That is, instead of all maps that comprise x_I, we only take the distinguished constant map $S \to \mathrm{pt} \overset{x}{\to} X$.

When instead of x_I we just use the distinguished map to x, we obtain the corresponding versions of the formal (resp., formal punctured) disc around x:

$$S \times \widehat{\mathcal{D}}_x,\ S \widehat{\times} \mathcal{D}_x := \mathrm{colim}(S \times \widehat{\mathcal{D}}_x) \text{ and } S \widehat{\times} \overset{\circ}{\mathcal{D}}_x := S \widehat{\times} \mathcal{D}_x - (S \times \{x\}).$$

We have the naturally defined maps

$$S \times \widehat{\mathcal{D}}_x \to \widehat{\mathcal{D}}_{x_I},\ S \widehat{\times} \mathcal{D}_x \to \mathcal{D}_{x_I} \text{ and } S \widehat{\times} \overset{\circ}{\mathcal{D}}_x \to \mathcal{D}'_{x_I}.$$

6.4.4. For every I, let $\mathfrak{L}^+_I(N)$ (resp., $\mathfrak{L}^+_I(N)' \subset \mathfrak{L}_I(N)$) denote the following group-schemes (resp., group ind-schemes) over X^I:

- A lift of x_I to a map $S \to \mathfrak{L}^+_I(N)$ is a map $\mathcal{D}_{x_I} \to N^{\omega^\rho}$ (or, equivalently, a map $\widehat{\mathcal{D}}_{x_I} \to N^{\omega^\rho}$), compatible with a projection to X.
- A lift of x_I to a map $S \to \mathfrak{L}^+_I(N)$ is a map $\overset{\circ}{\mathcal{D}}_{x_I} \to N^{\omega^\rho}$, compatible with a projection to X.
- A lift of x_I to a map $S \to \mathfrak{L}^+_I(N)'$ is a map $\mathcal{D}'_{x_I} \to N^{\omega^\rho}$, compatible with a projection to X.

We have the closed embeddings

$$\mathfrak{L}^+_I(N) \subset \mathfrak{L}^+_I(N)' \subset \mathfrak{L}_I(N),$$

and the projections

$$\mathfrak{L}^+_I(N) \to X^I \times \mathfrak{L}^+_x(N) \text{ and } \mathfrak{L}^+_I(N)' \to X^I \times \mathfrak{L}_x(N).$$

Remark 6.4.5. Let a k-point x_I be given by a collection of I distinct points y_i of X, and the distinguished point x. Then the fibers of $\mathfrak{L}^+_I(N)$, $\mathfrak{L}^+_I(N)'$ and $\mathfrak{L}_I(N)$ over such x_I are given, respectively, by

$$\prod_i \mathfrak{L}^+_{y_i}(N) \times \mathfrak{L}^+_x(N),\ \prod_i \mathfrak{L}^+_{y_i}(N) \times \mathfrak{L}_x(N) \text{ and } \prod_i \mathfrak{L}_{y_i}(N) \times \mathfrak{L}_x(N).$$

6.4.6. By definition

$$\mathrm{Whit}(\mathcal{Y}_I) = \mathrm{Shv}(\mathcal{Y}_I)^{\mathfrak{L}_I(N),\chi_I}.$$

We note that the map unit_I is compatible with the action of $\mathfrak{L}_I(N)'$, where $\mathfrak{L}_I(N)'$ acts on $X^I \times \mathcal{Y}$ via the projection

$$\mathfrak{L}_I^+(N)' \to X^I \times \mathfrak{L}_x(N) \tag{6.4}$$

and the $\mathfrak{L}_x(N)$-action on $\mathcal{Y}$.

Hence, the functor $\mathrm{unit}_I^!$ gives rise to a functor

$$\mathrm{Whit}(\mathcal{Y}_I) = \mathrm{Shv}(\mathcal{Y}_I)^{\mathfrak{L}_I(N),\chi_I} \to \mathrm{Shv}(X^I \times \mathcal{Y})^{\mathfrak{L}_I^+(N)',\chi_I},$$

while the functor

$$\mathrm{Whit}(X^I \times \mathcal{Y}) = \mathrm{Shv}(X^I \times \mathcal{Y})^{\mathfrak{L}_x(N),\chi_x} \to \mathrm{Shv}(X^I \times \mathcal{Y})^{\mathfrak{L}_I^+(N)',\chi_I}$$

is an equivalence, since the kernel of (6.4) is pro-unipotent.

This shows that $\mathrm{unit}_I^!$ gives rise to a well-defined functor

$$\mathrm{Whit}(\mathcal{Y}_I) \to \mathrm{Whit}(X^I \times \mathcal{Y}).$$

6.4.7. We now claim:

Theorem 6.4.8. *The functor*

$$\mathrm{unit}_I^! : \mathrm{Whit}(\mathcal{Y}_I) \to \mathrm{Whit}(X^I \times \mathcal{Y})$$

admits a left adjoint. Moreover, this left adjoint respects the actions of $\mathrm{Shv}(X^I)$ *on the two sides, given by the operation of !-pullback and* $\overset{!}{\otimes}$.

The proof of Theorem 6.4.8 will be given in Sect. 6.6.

6.5. **Proof of Theorem 6.4.2.** In this subsection we will show how Theorem 6.4.8 implies Theorem 6.4.2.

6.5.1. Consider the stratification of X^I according to the pattern of collision of points (including the distinguished point x). The strata are enumerated by equivalence relations on X (partitions of X as a disjoint union of subsets). For each partition $\mathfrak{P}$, let $X^{\mathfrak{P}}$ denote the corresponding locally closed subset of X^I. Denote

$$\mathcal{Y}_{\mathfrak{P}} := X^{\mathfrak{P}} \underset{X^I}{\times} \mathcal{Y}_{\mathrm{Ran}}.$$

Consider the corresponding categories

$$\mathrm{Whit}(\mathcal{Y}_{\mathfrak{P}}) \subset \mathrm{Shv}(\mathcal{Y}_{\mathfrak{P}}) \text{ and } \mathrm{Whit}(X^{\mathfrak{P}} \times \mathcal{Y}) \subset \mathrm{Shv}(X^{\mathfrak{P}} \times \mathcal{Y}).$$

The map unit_I induces a map

$$\mathrm{unit}_{\mathfrak{P}} : X^{\mathfrak{P}} \times \mathcal{Y} \to \mathcal{Y}_{\mathfrak{P}}.$$

We will prove:

Proposition 6.5.2. *The functor*

$$\mathrm{unit}_{\mathfrak{P}}^! : \mathrm{Whit}(\mathcal{Y}_{\mathfrak{P}}) \to \mathrm{Whit}(X^{\mathfrak{P}} \times \mathcal{Y})$$

is an equivalence for every $\mathfrak{P}$.

6.5.3. Let us deduce Theorem 6.4.2 from Proposition 6.5.2, combined with Theorem 6.4.8. First off, Proposition 6.5.2 implies that the functor $\mathrm{unit}^!_I$ is conservative. Hence, it remains to show that the unit of the adjunction

$$\mathcal{F} \to \mathrm{unit}^!_I \circ (\mathrm{unit}^!_I)^L(\mathcal{F}), \quad \mathcal{F} \in \mathrm{Whit}(X^I \times \mathcal{Y})$$

is an isomorphism.

Let $\iota_{\mathfrak{P}}$ denote the locally closed embedding $X^{\mathfrak{P}} \to X^I$. It suffices to show that each of the maps

$$(\iota_{\mathfrak{P}})_* \circ \iota^!_{\mathfrak{P}}(\mathcal{F}) \to (\iota_{\mathfrak{P}})_* \circ \iota^!_{\mathfrak{P}} \circ \mathrm{unit}^!_I \circ (\mathrm{unit}^!_I)^L(\mathcal{F})$$

is an isomorphism.

We have a commutative diagram

$$\begin{CD}
(\iota_{\mathfrak{P}})_* \circ \iota^!_{\mathfrak{P}}(\mathcal{F}) @>>> (\iota_{\mathfrak{P}})_* \circ (\iota_{\mathfrak{P}})^! \circ \mathrm{unit}^!_I \circ (\mathrm{unit}^!_I)^L(\mathcal{F}) \\
@A{\mathrm{Id}}AA @AAA \\
(\iota_{\mathfrak{P}})_* \circ \iota^!_{\mathfrak{P}}(\mathcal{F}) @>>> (\iota_{\mathfrak{P}})_* \circ \mathrm{unit}^!_{\mathfrak{P}} \circ (\mathrm{unit}^!_{\mathfrak{P}})^L \circ (\iota_{\mathfrak{P}})^!(\mathcal{F}).
\end{CD}$$

The bottom horizontal arrow in this diagram is an isomorphism by Proposition 6.5.2. Hence, it remains to show that the right vertical arrow is an isomorphism.

We have

$$\iota^!_{\mathfrak{P}} \circ \mathrm{unit}^!_I \simeq \mathrm{unit}^!_{\mathfrak{P}} \circ \iota^!_{\mathfrak{P}} \text{ and } (\iota_{\mathfrak{P}})_* \circ \mathrm{unit}^!_{\mathfrak{P}} \simeq \mathrm{unit}^!_I \circ (\iota_{\mathfrak{P}})_*.$$

So it suffices to show that the map

$$(\mathrm{unit}^!_{\mathfrak{P}})^L \circ (\iota_{\mathfrak{P}})^!(\mathcal{F}) \to (\iota_{\mathfrak{P}})^! \circ (\mathrm{unit}^!_I)^L(\mathcal{F})$$

is an isomorphism in $\mathrm{Whit}(\mathcal{Y}_{\mathfrak{P}})$.

Since $\iota_{\mathfrak{P}}$ is a locally closed embedding, the functor $(\iota_{\mathfrak{P}})_*$ is fully faithful. Hence, for any $\mathcal{F}' \in \mathrm{Whit}(\mathcal{Y}_{\mathfrak{P}})$ we have

$$\begin{aligned}
&\mathcal{H}om((\mathrm{unit}^!_{\mathfrak{P}})^L \circ (\iota_{\mathfrak{P}})^!(\mathcal{F}), \mathcal{F}') \simeq \\
&\simeq \mathcal{H}om((\iota_{\mathfrak{P}})^!(\mathcal{F}), \mathrm{unit}^!_{\mathfrak{P}}(\mathcal{F}')) \simeq \mathcal{H}om((\iota_{\mathfrak{P}})_* \circ (\iota_{\mathfrak{P}})^!(\mathcal{F}), (\iota_{\mathfrak{P}})_* \circ \mathrm{unit}^!_{\mathfrak{P}}(\mathcal{F}')) \simeq \\
&\simeq \mathcal{H}om((\iota_{\mathfrak{P}})_* \circ (\iota_{\mathfrak{P}})^!(\mathcal{F}), \mathrm{unit}^!_I \circ (\iota_{\mathfrak{P}})_*(\mathcal{F}')) \simeq \mathcal{H}om((\iota_{\mathfrak{P}})_*(\omega_{X^{\mathfrak{P}}}) \overset{!}{\otimes} \mathcal{F}, \mathrm{unit}^!_I \circ (\iota_{\mathfrak{P}})_*(\mathcal{F}')) \simeq \\
&\simeq \mathcal{H}om((\mathrm{unit}^!_I)^L((\iota_{\mathfrak{P}})_*(\omega_{X^{\mathfrak{P}}}) \overset{!}{\otimes} \mathcal{F}), (\iota_{\mathfrak{P}})_*(\mathcal{F}')) \simeq \mathcal{H}om((\iota_{\mathfrak{P}})_*(\omega_{X^{\mathfrak{P}}}) \overset{!}{\otimes} (\mathrm{unit}^!_I)^L(\mathcal{F}), (\iota_{\mathfrak{P}})_*(\mathcal{F}')) \simeq \\
&\simeq \mathcal{H}om((\iota_{\mathfrak{P}})_* \circ (\iota_{\mathfrak{P}})^! \circ (\mathrm{unit}^!_I)^L(\mathcal{F}), (\iota_{\mathfrak{P}})_*(\mathcal{F}')) \simeq \mathcal{H}om((\iota_{\mathfrak{P}})^! \circ (\mathrm{unit}^!_I)^L(\mathcal{F}), \mathcal{F}'),
\end{aligned}$$

as desired, where the only non-trivial isomorphism is that on the fourth line, and it takes place due to the fact the functor $(\mathrm{unit}^!_I)^L$ commutes with !-tensor products with objects of $\mathrm{Shv}(X^I)$ (by Theorem 6.4.8). □[Theorem 6.4.2]

6.5.4. The rest of this subsection is devoted to the proof of Proposition 6.5.2. Let k be the number of elements in the partition $\mathfrak{P}$, not counting the element containing the distinguished point. Then $X^{\mathfrak{P}} \simeq (X-x)^k - \mathrm{Diag}$, where $\mathrm{Diag} \subset (X-x)^k$ is the diagonal advisor.

We have

$$\mathcal{Y}_{\mathfrak{P}} \simeq \left(((X-x)^k - \mathrm{Diag}) \underset{\mathrm{Ran}(X)}{\times} \overline{S}^0_{\mathrm{Ran}} \right) \times \mathcal{Y}$$

Note also that we have a canonical isomorphism

$$\mathfrak{L}_{\mathfrak{P}}(N) := X^{\mathfrak{P}} \underset{X^I}{\times} \mathfrak{L}_I(N) \simeq \left(((X-x)^k - \mathrm{Diag}) \underset{\mathrm{Ran}(X)}{\times} \mathfrak{L}_{\mathrm{Ran}}(N) \right) \times \mathfrak{L}_x(N).$$

Consider the the open subset

$$\left(((X-x)^k - \mathrm{Diag}) \underset{\mathrm{Ran}(X)}{\times} S^0_{\mathrm{Ran}} \right) \times \mathcal{Y} \subset \left(((X-x)^k - \mathrm{Diag}) \underset{\mathrm{Ran}(X)}{\times} \overline{S}^0_{\mathrm{Ran}} \right) \times \mathcal{Y}.$$

The assertion of Proposition 6.5.2 follows from the combination of the next two lemmas:

Lemma 6.5.5. *Restriction defines an equialence*

$$\mathrm{Shv}\left(\left(((X-x)^k-\mathrm{Diag})\underset{\mathrm{Ran}(X)}{\times}\overline{S}^0_{\mathrm{Ran}}\right)\times\mathcal{Y}\right)^{\mathfrak{L}_{\mathfrak{P}}(N),\chi_{\mathrm{Ran}}}\to$$
$$\to\mathrm{Shv}\left(\left(((X-x)^k-\mathrm{Diag})\underset{\mathrm{Ran}(X)}{\times}S^0_{\mathrm{Ran}}\right)\times\mathcal{Y}\right)^{\mathfrak{L}_{\mathfrak{P}}(N),\chi_{\mathrm{Ran}}}.$$

Lemma 6.5.6. *Restriction along* $\mathrm{unit}_{\mathfrak{P}}$ *defines an equivalence*

$$\mathrm{Whit}(\mathcal{Y}_{\mathfrak{P}})=\mathrm{Shv}\left(\left(((X-x)^k-\mathrm{Diag})\underset{\mathrm{Ran}(X)}{\times}S^0_{\mathrm{Ran}}\right)\times\mathcal{Y}\right)^{\mathfrak{L}_{\mathfrak{P}}(N),\chi_{\mathrm{Ran}}}\simeq$$
$$\simeq\mathrm{Shv}\left(((X-x)^k-\mathrm{Diag})\times\mathcal{Y}\right)^{\mathfrak{L}_x(N),\chi_x}=\mathrm{Whit}(X^{\mathfrak{P}}\times\mathcal{Y}).$$

Proof of Lemma 6.5.5. We claim that the category

$$\mathrm{Shv}\left(\left(((X-x)^k-\mathrm{Diag})\underset{\mathrm{Ran}(X)}{\times}(\overline{S}^0_{\mathrm{Ran}}-S^0_{\mathrm{Ran}})\right)\times\mathcal{Y}\right)^{\mathfrak{L}_{\mathfrak{P}}(N),\chi_{\mathrm{Ran}}}$$

is zero. This follows in the same way as Proposition 2.3.3(a).

□

Proof of Lemma 6.5.6. Follows from the fact that $\mathfrak{L}_{\mathfrak{P}}(N)'\subset\mathfrak{L}_{\mathfrak{P}}(N)$ identifies with

$$\left(((X-x)^k-\mathrm{Diag})\underset{\mathrm{Ran}(X)}{\times}\mathfrak{L}^+_{\mathrm{Ran}}(N))\right)\times\mathfrak{L}_x(N)\subset\left(((X-x)^k-\mathrm{Diag})\underset{\mathrm{Ran}(X)}{\times}\mathfrak{L}_{\mathrm{Ran}}(N))\right)\times\mathfrak{L}_x(N).$$

□

6.6. **Proof of Theorem 6.4.8.** We are going to show that Theorem 6.4.8 follows from a Ran version of Theorem 2.5.5.

6.6.1. The desired left adjoint is given as the composition of

$$\mathrm{Whit}(X^I\times\mathcal{Y})\hookrightarrow\mathrm{Shv}(X^I\times\mathcal{Y})\overset{(\mathrm{unit}_I)_!}{\hookrightarrow}\mathrm{Shv}(\mathcal{Y}_I),\tag{6.5}$$

and the partially defined functor $\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}$.

We need to show that $\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}$ is defined on the essential image of (6.5), and commutes with !-tensoring by pullback of objects of $\mathrm{Shv}(X^I)$.

Remark 6.6.2. Note that Theorem 6.4.8 formally follows from Theorem 6.4.2.

Let us also note that we can use an appropriately defined functor $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}_I(N),\chi_I}$ to construct a left inverse of the functor $\mathrm{unit}_I^!$; the functor $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}_I(N),\chi_I}$ commutes with !-tensoring by objects of $\mathrm{Shv}(X^I)$ by construction. What is not a priori clear is that $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}_I(N),\chi_I}\circ(\mathrm{unit}_I)_!$ is the left adjoint adjoint of $\mathrm{unit}_I^!$. However, once we know Theorem 6.4.2, we will obtain an isomorphism

$$\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}_I(N),\chi_I}\circ(\mathrm{unit}_I)_!\simeq\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}\circ(\mathrm{unit}_I)_!.$$

6.6.3. For $j\in\mathbb{Z}^{\geq 0}$, let $I^j\subset\mathfrak{L}_x(G)$ be the subgroup defined in Sect. 2.5.1. As in Sect. 2.5.7, the !-averaging functor

$$\mathrm{Av}_!^{\mathfrak{L}_x(N),\chi_x}:\mathrm{Shv}(X^I\times\mathcal{Y})\to\mathrm{Whit}(X^I\times\mathcal{Y})$$

is defined on the essential image of

$$\mathbf{oblv}_{I^j,\chi_x}:\mathrm{Shv}(X^I\times\mathcal{Y})^{I^j,\chi_x}\to\mathrm{Shv}(X^I\times\mathcal{Y}),$$

and the essential images of the functors $\mathrm{Av}_!^{\mathfrak{L}_x(N),\chi_x}\circ\mathbf{oblv}_{I^j,\chi_x}$ generate $\mathrm{Whit}(X^I\times\mathcal{Y})$.

Moreover, the proof of Theorem 2.5.5 shows that the functor $\mathrm{Av}_!^{\mathfrak{L}_x(N),\chi_x} \circ \mathbf{oblv}_{I^j,\chi_x}$ commutes with !-tensoring by pullback of objects of $\mathrm{Shv}(X^I)$. Hence, it suffices to check that for every j, the functor $\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}$ is defined on the essential image of

$$(\mathrm{unit}_I)_! \circ \mathbf{oblv}_{I^j,\chi_x} : \mathrm{Shv}(X^I \times \mathcal{Y})^{I^j,\chi_x} \to \mathrm{Shv}(X^I \times \mathcal{Y}), \tag{6.6}$$

and commutes with !-tensoring by pullback of objects of $\mathrm{Shv}(X^I)$.

6.6.4. Let G^{ω^ρ} be the group-scheme over X corresponding to automorphisms of the G-bundle $\mathcal{P}_G^{\omega^\rho}$. I.e., G^{ω^ρ} is the twist of the constant group-scheme with fiber G by the G-torsor $\mathcal{P}_G^{\omega^\rho}$ using the adjoint action.

Let $\mathfrak{L}_I^+(G)$ (resp., $\mathfrak{L}_I^+(G)' \subset \mathfrak{L}_I(G)$) be the group-scheme (resp., group ind-scheme) over X^I, defined in the same way as in Sect. 6.4.4 with N^{ω^ρ} replaced by G^{ω^ρ}.

We have the projection $\mathfrak{L}_I^+(G)' \to \mathfrak{L}_x(G)$, and let $\mathfrak{L}_I^+(G)^j$ be the group subscheme of $\mathfrak{L}_I^+(G)'$ equal to the preimage of $I^j \subset \mathfrak{L}_x(G)$.

Remark 6.6.5. For a k-point x_I given by a collection of I distinct points y_i of X, and the distinguished point x, the fiber of $\mathfrak{L}_I^+(G)^j$ over such x_I is given by

$$\prod_i \mathfrak{L}_{y_i}^+(G) \times I^j.$$

6.6.6. Let $\mathcal{Y}_I^{\mathrm{big}}$ be the following ind-scheme over X^I: it classifies quadruples $(x_I, \mathcal{P}_G, \gamma, \epsilon)$, where:

- x_I is a point of X^I:
- $\mathcal{P}_G$ is a G-bundle on X;
- γ is an identification of $\mathcal{P}_G$ with $\mathcal{P}_G^{\omega^\rho}$ over the complement of x_I;
- ϵ is a structure of level n on $\mathcal{P}_G$ at x.

In other words, the difference between $\mathcal{Y}_I^{\mathrm{big}}$ and $\mathcal{Y}_I$ is that we no longer require that the maps (6.1) be regular away from x_I. The ind-scheme $\mathcal{Y}_I^{\mathrm{big}}$ is acted on by $\mathfrak{L}_I(G)$.

We have a closed embedding

$$\mathcal{Y}_I \hookrightarrow \mathcal{Y}_I^{\mathrm{big}}.$$

The action of $\mathfrak{L}_I^+(G)'$ preserves the image of the composition

$$X^I \times \mathcal{Y} \overset{\mathrm{unit}_I}{\hookrightarrow} \mathcal{Y}_I \to \mathcal{Y}_I^{\mathrm{big}},$$

and the resulting action of $\mathfrak{L}_I^+(G)'$ on $X^I \times \mathcal{Y}$ factors through the projection $\mathfrak{L}_I^+(G)' \to \mathfrak{L}_x(G)$, and the $\mathfrak{L}_x(G)$-action on $X^I \times \mathcal{Y}$ via the $\mathcal{Y}$-factor.

Hence, the essential image of the functor (6.6), composed with $\mathrm{Shv}(X^I \times \mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}})$, factors as

$$\mathrm{Shv}(X^I \times \mathcal{Y})^{I^j,\chi_x} \to \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}})^{\mathfrak{L}_I^+(G)^j,\chi_x} \to \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}}),$$

where the second arrow is the forgetful functor.

6.6.7. We define the full subcategory

$$\mathrm{Whit}(\mathcal{Y}_I^{\mathrm{big}}) \subset \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}})$$

by the same procedure as for $\mathcal{Y}_I$.

We now have the following extension of Theorem 2.5.5:

Theorem 6.6.8. *The partially defined functor*

$$\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I} : \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}}) \to \mathrm{Whit}(\mathcal{Y}_I^{\mathrm{big}}),$$

left adjoint to the forgetful functor $\mathrm{Whit}(\mathcal{Y}_I^{\mathrm{big}}) \to \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}})$*, is defined on the essential image of the forgetful functor*

$$\mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}})^{\mathfrak{L}_I^+(G)^j,\chi_x} \to \mathrm{Shv}(\mathcal{Y}_I^{\mathrm{big}}),$$

and commutes with !-tensoring by pullback of objects of $\mathrm{Shv}(X^I)$.

In particular, Theorem 6.6.8 implies that $\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}$ is defined on the essential image of $(\mathrm{unit}_I)_! \circ \mathbf{oblv}_{I^j,\chi_x}$ and commutes with !-tensoring by pullback of objects of $\mathrm{Shv}(X^I)$.

A proof of Theorem 6.6.8 is a straightforward adaptation of the proof of Theorem 2.5.5, given in Sect. A.

7. Generalizations

7.1. **Full level structure.** In this subsection we will show that, when considering the Whittaker category, one can replace $\mathcal{Y} := \mathfrak{L}(G)/K$ by the loop group $\mathfrak{L}(G)$ itself.

7.1.1. Consider the ind-scheme $\mathfrak{L}(G)$, as acted on by itself on the left. Consider the category $\mathrm{Shv}(\mathfrak{L}(G))$, see Sect. C.3.2.

Note that for any group subscheme $N' \subset \mathfrak{L}(N)$, the functor

$$\mathrm{Av}_*^{N',\chi} : \mathrm{Shv}(\mathfrak{L}(G)) \to \mathrm{Shv}(\mathfrak{L}(G))$$

makes sense.

Hence, we can define

$$\mathrm{Whit}(\mathfrak{L}(G)) \subset \mathrm{Shv}(\mathfrak{L}(G))$$

as full subcategory consisting of objects $\mathcal{F}$, for which the map

$$\mathrm{Av}_*^{N',\chi}(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism for any N'.

7.1.2. Here is a more explicit description of the subcategory. By Sect. C.2.9,

$$\mathrm{Shv}(\mathfrak{L}(G)) \simeq \lim_n \mathrm{Shv}(\mathfrak{L}(G)/K_n), \tag{7.1}$$

where the transition functors

$$\mathrm{Shv}(\mathfrak{L}(G)/K_{n''}) \to \mathrm{Shv}(\mathfrak{L}(G)/K_{n'}), \quad n'' \geq n' \tag{7.2}$$

are given by the operation of *-direct image.

In terms of this identification, we have

$$\mathrm{Whit}(\mathfrak{L}(G)) = \lim_n \mathrm{Whit}(\mathfrak{L}(G)/K_n) \subset \lim_n \mathrm{Shv}(\mathfrak{L}(G)/K_n). \tag{7.3}$$

7.1.3. Note that for a DG category $\mathbf{C}$, the full subcategory

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathfrak{L}(G)), \mathbf{C})^{\mathfrak{L}(N),\chi} \subset \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Shv}(\mathfrak{L}(G)), \mathbf{C})$$

makes sense.

Hence, we can also define

$$\mathrm{Whit}(\mathfrak{L}(G))_{\mathrm{co}} := \mathrm{Shv}(\mathfrak{L}(G))_{\mathfrak{L}(N),\chi}.$$

7.1.4. Here is a more explicit description of $\mathrm{Whit}(\mathfrak{L}(G))_{\mathrm{co}}$. Recall that according to Sect. 1.3.3, in addition to the realization of $\mathrm{Shv}(\mathfrak{L}(G))$ given by (7.1), we also have an identification

$$\mathrm{Shv}(\mathfrak{L}(G)) \simeq \mathrm{colim}_n \mathrm{Shv}(\mathfrak{L}(G)/K_n), \tag{7.4}$$

where the transition functors

$$\mathrm{Shv}(\mathfrak{L}(G)/K_{n'}) \to \mathrm{Shv}(\mathfrak{L}(G)/K_{n''}), \quad n'' \geq n' \tag{7.5}$$

are given by the operation of *-pullback.

It follows that we have a canonical equivalence:

$$\mathrm{Whit}(\mathfrak{L}(G))_{\mathrm{co}} \simeq \mathrm{colim}_n \mathrm{Whit}(\mathfrak{L}(G)/K_n)_{\mathrm{co}}, \tag{7.6}$$

where the transition functors are induced by (7.5).

7.1.5. Note now that the functor Ps-Id (see Sect. 3.4) makes sense also as a functor

$$\mathrm{Whit}(\mathfrak{L}(G))_{\mathrm{co}} \to \mathrm{Whit}(\mathfrak{L}(G)).$$

We claim:

Theorem 7.1.6. *The functor* $\text{Ps-Id} : \mathrm{Whit}(\mathfrak{L}(G))_{\mathrm{co}} \to \mathrm{Whit}(\mathfrak{L}(G))$ *is an equivalence.*

Proof. Note that the *-pushforward functors (7.2) induce functors

$$\mathrm{Whit}(\mathfrak{L}(G)/K_{n''})_{\mathrm{co}} \to \mathrm{Whit}(\mathfrak{L}(G)/K_{n'})_{\mathrm{co}},$$

which provide right adjoints to the *-pullback functors

$$\mathrm{Whit}(\mathfrak{L}(G)/K_{n'})_{\mathrm{co}} \to \mathrm{Whit}(\mathfrak{L}(G)/K_{n''})_{\mathrm{co}}.$$

Hence, by Sect. 1.3.3, we can realize $\mathrm{Whit}(\mathfrak{L}(G))_{\mathrm{co}}$ as

$$\lim_n \mathrm{Whit}(\mathfrak{L}(G)/K_n)_{\mathrm{co}},$$

with the transition functors induced by (7.2).

In terms of this identification and (7.3), the fuctor Ps-Id is given by the family of functors

$$\text{Ps-Id} : \mathrm{Whit}(\mathfrak{L}(G)/K_n)_{\mathrm{co}} \to \mathrm{Whit}(\mathfrak{L}(G)/K_n),$$

while the latter are isomorphisms by Theorem 3.4.8.

□

7.1.7. By the same token, we can consider the prestack $(\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x}}_{\infty\cdot x}$ (where we equip our G-bundle with a full level structure at x), the category $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x}}_{\infty\cdot x})$ and its full subcategory

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x}}_{\infty\cdot x}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x}}_{\infty\cdot x}).$$

Passing to the limit in Theorem 5.5.2, we obtain:

Theorem 7.1.8. *The !-pullback functor along* $\mathfrak{L}_x(G) \to (\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x}}_{\infty\cdot x}$ *defines an equivalence*

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G\text{-level}_{\infty\cdot x}}_{\infty\cdot x}) \to \mathrm{Whit}(\mathfrak{L}_x(G)).$$

7.2. **Multi-point version.** The local Whittaker category we have defined is attached to the formal disc $\widehat{\mathcal{D}}_x$ for some point x on a curve X. We will now show how to generalize this by considering a *parameterized multi-disc* $\widehat{\mathcal{D}}_{x_I}$, which lives over X^I.

7.2.1. Fix a finite set I and a map

$$n_I : I \to \mathbb{Z}^{\geq 0}, \quad i \mapsto n_i.$$

Let $\mathcal{Y}$ denote the following ind-scheme over X^I. For an affine test-scheme S, an S-point of $\mathcal{Y}$ is a datum of $(x_I, \mathcal{P}_G, \gamma, \epsilon)$, where:

- x_I is an S-point of X^I (i.e., an I-tuple of S-points x_i of X);
- $\mathcal{P}_G$ is a G-bundle on $\mathcal{D}_{x_I}$ (equivalently, on $\widehat{\mathcal{D}}_{x_I}$);
- γ is an identification between $\mathcal{P}_G$ and $\mathcal{P}_G^{\omega^\rho}$ over $\overset{\circ}{\mathcal{D}}_{x_I}$;
- ϵ is a trivialization of the restriction of $\mathcal{P}_G$ to the subscheme $\Sigma\, n_i \cdot \mathrm{Graph}_{x_i} \subset \widehat{\mathcal{D}}_{x_I}$ (we view each Graph_{x_i} as a Cartier divisor on $S \times X$).

7.2.2. The above ind-scheme $\mathcal{Y}$ is acted on by $\mathfrak{L}_I(G)$ and, in particular, by $\mathfrak{L}_I(N)$. Proceeding as in the single-point case, we can introduce the corresponding categories

$$\mathrm{Whit}(\mathcal{Y}) := \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}_I(N),\chi_I} \text{ and } \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} := \mathrm{Shv}(\mathcal{Y})_{\mathfrak{L}_I(N),\chi_I},$$

and the functor

$$\text{Ps-Id} : \mathrm{Whit}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Whit}(\mathcal{Y}). \tag{7.7}$$

We also have a relative (over X^I) version of the ind-algebraic stack $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}^{G\text{-level}_{n\cdot x}}$, denoted

$$(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}^{G\text{-level}_{n_I\cdot x_I}},$$

and the corresponding full subcategory

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x}^{G\text{-level}_{n_I\cdot x_I}}) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x}^{G\text{-level}_{n_I\cdot x_I}}).$$

A straightforward generalization of Theorem 5.5.2 gives:

Theorem 7.2.3. *Pullback along* $\mathcal{Y} \to (\overline{\mathrm{Bun}}_N)_{\infty\cdot x}^{G\text{-level}_{n_I\cdot x_I}}$ *defines an equivalence*

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x}^{G\text{-level}_{n_I\cdot x_I}})) \to \mathrm{Whit}(\mathcal{Y}).$$

From here, as in the case of a single point, we obtain:

Theorem 7.2.4. *The functor* (7.7) *is an equivalence.*

7.2.5. As in Sect. 7.1, the above constructions and assertions can be generalized to the case when the function n^I is allowed take value ∞ on some elements of I.

The resulting geometric objects are inverse limits of the corresponding objects for finite values of n_I.

The resulting Whittaker categories can be realized both as limits and as colimits of one corresponding to finite values of n_I.

7.3. **"Abstract" Whittaker categories.** In this subsection we will study various versions of the Whittaker model of an *abstract* category $\mathbf{C}$, equipped with an action of $\mathfrak{L}(G)$.

7.3.1. Let $\mathbf{C}$ be a category acted on by $\mathfrak{L}(G)$; see Sect. D.1.1. For any group-subscheme $N' \subset \mathfrak{L}(N)$, we can consider the functor

$$\mathrm{Av}_*^{N',\chi} : \mathbf{C} \to \mathbf{C}.$$

Hence, as in Sect. 7.1, we can consider the full category

$$\mathrm{Whit}(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}(N),\chi} \simeq \lim_{\alpha} \mathbf{C}^{N^\alpha,\chi} = \bigcap_{\alpha} \mathbf{C}^{N^\alpha,\chi} \subset \mathbf{C},$$

where N^α are as on (1.1).

In addition, we can consider

$$\mathrm{Whit}(\mathbf{C})_{\mathrm{co}} := \mathbf{C}_{\mathfrak{L}(N),\chi} \simeq \operatorname*{colim}_{\alpha} \mathbf{C}_{N^\alpha,\chi} \simeq \operatorname*{colim}_{\alpha} \mathbf{C}^{N^\alpha,\chi},$$

where the last colimit is formed using the transition functors

$$\mathrm{Av}_*^{N^{\alpha''}/N^{\alpha'},\chi} : \mathbf{C}^{N^{\alpha'},\chi} \to \mathbf{C}^{N^{\alpha''},\chi}, \quad N^{\alpha'} \subset N^{\alpha''}.$$

In addition, we have a well-defined functor

$$\text{Ps-Id} : \mathrm{Whit}(\mathbf{C})_{\mathrm{co}} \to \mathrm{Whit}(\mathbf{C}).$$

7.3.2. We will prove:

Theorem 7.3.3.
(a) *For any* $\mathbf{C}$*, equipped with an action of* $\mathfrak{L}(G)$*, the functor* $\text{Ps-Id} : \text{Whit}(\mathbf{C})_{\text{co}} \to \text{Whit}(\mathbf{C})$ *is an equivalence.*
(b) *The functor*

$$\mathbf{C} \mapsto \text{Whit}(\mathbf{C}), \quad \mathfrak{L}(G)\text{-}\mathbf{mod} \to \text{DGCat}_{\text{cont}}$$

commutes with limits and colimits, and for an abstract DG category $\mathbf{C}_0$*, the naturally defined functor*

$$\text{Whit}(\mathbf{C}) \otimes \mathbf{C}_0 \to \text{Whit}(\mathbf{C} \otimes \mathbf{C}_0)$$

is an equivalence.
(c) *If* $\mathbf{C}$ *is dualizable, then so is* $\text{Whit}(\mathbf{C})$ *and we have a canonical equivalence*

$$\text{Whit}(\mathbf{C})^\vee \simeq \text{Whit}(\mathbf{C}^\vee).$$

7.3.4. The proof of Theorem 7.3.3 is based on the following observation. For a pair of categories $\mathbf{C}_1$ and $\mathbf{C}_2$ acted on by $\mathfrak{L}(G)$ we can consider the category.

$$(\mathbf{C}_1 \otimes \mathbf{C}_2)^{\mathfrak{L}(G)}.$$

It is a basic fact (see Theorem D.1.4) that for G reductive, the functor

$$\mathbf{C}_1, \mathbf{C}_2 \mapsto (\mathbf{C}_1 \otimes \mathbf{C}_2)^{\mathfrak{L}(G)}$$

commutes with colimits in each variable.

Remark 7.3.5. In fact the above functor is canonically isomorphic to the functor

$$\mathbf{C}_1, \mathbf{C}_2 \mapsto \mathbf{C}_1 \underset{\text{Shv}(\mathfrak{L}(G))}{\otimes} \mathbf{C}_2,$$

see Theorem D.1.4(b).

7.3.6. Note that for a category $\mathbf{C}$ acted on by $\mathfrak{L}(G)$, we have a canonical identification

$$\mathbf{C} \simeq (\text{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})^{\mathfrak{L}(G)},$$

where we consider $\text{Shv}(\mathfrak{L}(G))$ as acted on by $\mathfrak{L}(G)$ via right multiplication. The above equivalence is an equivalence of categories acted on by $\mathfrak{L}(G)$, where we endow $\text{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})^{\mathfrak{L}(G)}$ with a $\mathfrak{L}(G)$-action via left multiplication.

We will prove:

Proposition 7.3.7. *The natural map*

$$\text{Whit}(\mathbf{C})_{\text{co}} \simeq \text{Whit}((\text{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})^{\mathfrak{L}(G)})_{\text{co}} \to (\text{Whit}(\text{Shv}(\mathfrak{L}(G)))_{\text{co}} \otimes \mathbf{C})^{\mathfrak{L}(G)}$$

is an equivalence.

Proof. We have:

$$\text{Whit}(\mathbf{C})_{\text{co}} := \underset{\alpha}{\text{colim}}\, \mathbf{C}^{N^\alpha, \chi},$$

while

$$(\text{Whit}(\text{Shv}(\mathfrak{L}(G)))_{\text{co}} \otimes \mathbf{C})^{\mathfrak{L}(G)} := \left((\underset{\alpha}{\text{colim}}\, \text{Shv}(\mathfrak{L}(G))_{N^\alpha, \chi}) \otimes \mathbf{C} \right)^{\mathfrak{L}(G)} \simeq$$

$$\simeq \left(\underset{\alpha}{\text{colim}}\, (\text{Shv}(\mathfrak{L}(G))_{N^\alpha, \chi} \otimes \mathbf{C}) \right)^{\mathfrak{L}(G)} \overset{\text{commutation with colimits}}{\simeq} \underset{\alpha}{\text{colim}} \left((\text{Shv}(\mathfrak{L}(G))_{N^\alpha, \chi} \otimes \mathbf{C})^{\mathfrak{L}(G)} \right) \simeq$$

$$\simeq \underset{\alpha}{\text{colim}}\, (\text{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})^{(N^\alpha, \chi), \mathfrak{L}(G)} \simeq \underset{\alpha}{\text{colim}} \left((\text{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})^{\mathfrak{L}(G)} \right)_{N^\alpha, \chi} \simeq \underset{\alpha}{\text{colim}}\, \mathbf{C}_{N^\alpha, \chi},$$

as desired.

∎

Proof of Theorem 7.3.3. Note that in addition to the equivalence of Proposition 7.3.7, we have the tautological equivalence

$$\mathrm{Whit}(\mathbf{C}) \simeq \mathrm{Whit}\left((\mathrm{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})^{\mathfrak{L}(G)}\right) \simeq (\mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C}))^{\mathfrak{L}(G)}.$$

We have a commutative diagram

$$\begin{CD}
\mathrm{Whit}(\mathbf{C})_{\mathrm{co}} @>{\text{Proposition 7.3.7}}>> (\mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)))_{\mathrm{co}} \otimes \mathbf{C})^{\mathfrak{L}(G)} @>{\sim}>> (\mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C})_{\mathrm{co}})^{\mathfrak{L}(G)} \\
@VVV @. @VVV \\
\mathrm{Whit}(\mathbf{C}) @. \xrightarrow{\sim} @. (\mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C}))^{\mathfrak{L}(G)}.
\end{CD}$$

Now, the equivalence of Theorem 7.1.6 extends to an equivalence

$$\mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C}')_{\mathrm{co}} \to \mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)) \otimes \mathbf{C}')$$

for any test DG category $\mathbf{C}'$. Hence, the right vertical arrow in the above diagram is an equivalence. Therefore, so is the left vertical arrow.

This proves point (a) of the theorem.

Point (b) follows from point (a): the functor

$$\mathbf{C} \mapsto \mathrm{Whit}(\mathbf{C})$$

manifestly commutes with limits (given the fact that the forgetful functor $\mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$ commutes with limits), while the functor

$$\mathbf{C} \mapsto \mathrm{Whit}(\mathbf{C})_{\mathrm{co}}$$

manifestly commutes with limits (given the fact that the forgetful functor $\mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$ commutes with colimits).

To prove point (b), given (a), it suffices to establish a canonical isomorphism

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Whit}(\mathbf{C})_{\mathrm{co}}, \mathbf{C}_0) \simeq \mathrm{Whit}(\mathbf{C}^\vee) \otimes \mathbf{C}_0$$

that functorially depends on the test DG category $\mathbf{C}_0$.

By definition, we have

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Whit}(\mathbf{C})_{\mathrm{co}}, \mathbf{C}_0) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathbf{C}_0)^{\mathfrak{L}(N),\chi} \simeq \mathrm{Whit}(\mathbf{C}^\vee \otimes \mathbf{C}_0),$$

and the assertion follows from point (b). □

Let is also note:

Corollary 7.3.8. *The natural map*

$$\mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G))) \underset{\mathrm{Shv}(\mathfrak{L}(G))}{\otimes} \mathbf{C} \to \mathrm{Whit}(\mathrm{Shv}(\mathfrak{L}(G)) \underset{\mathrm{Shv}(\mathfrak{L}(G))}{\otimes} \mathbf{C}) \simeq \mathrm{Whit}(\mathbf{C})$$

is an equivalence.

Proof. Follows from Theorem 7.3.3 as the assertion is manifestly true for $\mathrm{Whit}(-)$ replaced by $\mathrm{Whit}(-)_{\mathrm{co}}$, □

7.3.9. *The ultimate generalization.* We now fix a finite set I, and consider the group ind-scheme $\mathfrak{L}_I(G)$ over X^I. Consider the category $\mathfrak{L}_I(G)\text{-}\mathbf{mod}$, whose objects are DG categories $\mathbf{C}$ acted on by the monoidal category $\mathrm{Shv}(\mathfrak{L}_I(G))$.

Proceeding as above for $\mathbf{C} \in \mathfrak{L}_I(G)\text{-}\mathbf{mod}$, we define the categories

$$\mathrm{Whit}(\mathbf{C}) \text{ and } \mathrm{Whit}(\mathbf{C})_{\mathrm{co}}$$

and the functor

$$\text{Ps-Id} : \mathrm{Whit}(\mathbf{C})_{\mathrm{co}} \to \mathrm{Whit}(\mathbf{C}).$$

Using Sect. 7.2.5, we prove the corresponding version of Theorem 7.3.3 in this situation.

Appendix A. Proof of Theorem 2.5.5

We will essentially copy the proof from [Ras, Sect. 2.12], adding a few details.

A.1. **Compactification of the action map.**

A.1.1. Denote $\mathcal{Y} := \mathfrak{L}(G)/K_n$; we will consider it as an ind-scheme equipped with an action of $\mathfrak{L}(G)$. Let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathcal{Y})^{I^j\cap\mathfrak{L}(N),\chi}$. Then we have a well-defined

$$\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F} \in \mathrm{Shv}(\mathfrak{L}(N) \overset{I^j\cap\mathfrak{L}(N)}{\times} \mathcal{Y}),$$

where $\overset{H}{\times}$ means "divide by the diagonal action of H".

The action of $\mathfrak{L}(N)$ on $\mathcal{Y}$ defines a map

$$\mathrm{act} : \mathfrak{L}(N) \overset{I^j\cap\mathfrak{L}(N)}{\times} \mathcal{Y} \to \mathcal{Y}.$$

The object $\mathcal{F}$ is $(\mathfrak{L}(N),\chi)$-adapted if the partially defined left adjoint of

$$\mathrm{act}^! \otimes \mathrm{Id}_{\mathbf{C}} : \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C} \to \mathrm{Shv}(\mathfrak{L}(N) \overset{I^j\cap\mathfrak{L}(N)}{\times} \mathcal{Y}) \otimes \mathbf{C}$$

is defined on objects of the form $(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}) \otimes \mathbf{c}$ for all $\mathbf{c} \in \mathbf{C}$ and equals $\mathrm{act}_!(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}) \otimes \mathbf{c}$.

A.1.2. Let us introduce some short-hand notation: for $H \subset \mathfrak{L}(G)$ denote

$$H^j := \mathrm{Ad}_{\mathrm{Ad}_{-j\check{\rho}(t)}(H)}\,.$$

Note that

$$\mathfrak{L}^+(N)^j = I^j \cap \mathfrak{L}(N).$$

Consider the ind-scheme $\mathfrak{L}(G)$ equipped with the projection

$$\mathfrak{L}(G) \to \mathfrak{L}(G)/\mathfrak{L}^+(G)^j, \tag{A.1}$$

where we note that $\mathfrak{L}(G)/\mathfrak{L}^+(G)^j$ is isomorphic to the affine Grassmannian.

Taking the preimage of the $\mathfrak{L}(N)$-orbit through the origin in $\mathfrak{L}(G)/\mathfrak{L}^+(G)^j$, we obtain a locally closed ind-subscheme, denoted

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \subset \mathfrak{L}(G),$$

equipped with a free action of $\mathfrak{L}^+(G)^j$. Note that we have an identification

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \simeq \mathfrak{L}(N) \overset{\mathfrak{L}^+(N)^j}{\times} \mathfrak{L}^+(G)^j.$$

We can form the fiber product

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y},$$

equipped with a locally closed embedding into

$$\mathfrak{L}(G) \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}.$$

In particular, $\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}$ is an ind-scheme of ind-finite type, and we have an isomorphism

$$\mathfrak{L}(N) \overset{\mathfrak{L}^+(N)^j}{\times} \mathcal{Y} \simeq \mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}.$$

Under the above identification, the map

$$\mathrm{act} : \mathfrak{L}(N) \overset{\mathfrak{L}^+(N)^j}{\times} \mathcal{Y} \to \mathcal{Y}$$

equals the composition

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y} \to \mathfrak{L}(G) \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y} \to \mathcal{Y},$$

where the second arrow is given by the action of $\mathfrak{L}(G)$ on $\mathcal{Y}$.

A.1.3. Let
$$\overline{\mathfrak{L}(N)/\mathfrak{L}^+(N)^j} \subset \mathfrak{L}(G)/\mathfrak{L}^+(G)^j$$
denote the closure of the $\mathfrak{L}(N)$-orbit $\mathfrak{L}(N)/\mathfrak{L}^+(N)^j$ through the origin in $\mathfrak{L}(G)/\mathfrak{L}^+(G)^j$.

Let
$$\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \subset \mathfrak{L}(G)$$
denote the preimage of $\overline{\mathfrak{L}(N)/\mathfrak{L}^+(N)^j}$ under the projection (A.1). In other words, $\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}$ is the closure of $\mathfrak{L}(N)\mathfrak{L}^+(G)^j$ in $\mathfrak{L}(G)$.

The group-scheme $\mathfrak{L}^+(G)^j$ still acts freely on $\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}$ by right multiplication. We can form
$$\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y},$$
which is a closed ind-subscheme of $\mathfrak{L}(G) \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}$ (and thus is an ind-scheme of ind-finite type).

We have an open embedding
$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y} \overset{j}{\hookrightarrow} \overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}$$
and a map
$$\overline{\mathrm{act}} : \overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y} \to \mathcal{Y}.$$

A.1.4. We will deduce Theorem 2.5.5 from the following observation:

Proposition A.1.5. *If $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N)^j,\chi}$ lies in essential image of the forgetful functor*
$$\mathbf{oblv}_{I^j/\mathfrak{L}(N)^j,\chi} : \mathrm{Shv}(\mathcal{Y})^{I^j,\chi} \to \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N)^j,\chi},$$
then for any $\mathbf{C}$ and any $\mathbf{c} \in \mathbf{C}$, the functor left adjoint to
$$(j^! \otimes \mathrm{Id}_{\mathbf{C}}) : \mathrm{Shv}(\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}) \otimes \mathbf{C} \to \mathrm{Shv}(\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}) \otimes \mathbf{C}$$
is defined on
$$(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}) \otimes \mathbf{c} \in \mathrm{Shv}(\mathfrak{L}(N) \overset{\mathfrak{L}^+(N)^j}{\times} \mathcal{Y}) \otimes \mathbf{C} \simeq \mathrm{Shv}(\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}) \otimes \mathbf{C}$$
and equals
$$j_*(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}) \otimes \mathbf{c}.$$

Note that Proposition A.1.5 says, in particular, that the object
$$\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F} \in \mathrm{Shv}(\mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y})$$
is *clean* with respect to the map j.

A.1.6. Let us assume Proposition A.1.5 and deduce that for $\mathcal{F}$ in the essential image of $\mathbf{oblv}_{I^j/\mathfrak{L}(N)^j,\chi}$, the object $\mathrm{act}_!(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F})$ exists and
$$\mathrm{act}_!(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}) \otimes \mathbf{c}$$
provides the value of the left adjoint to $\mathrm{act}^! \otimes \mathrm{Id}_{\mathbf{C}}$ on $(\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}) \otimes \mathbf{c}$.

Using Proposition A.1.5, it suffices to show that that for *any*
$$\widetilde{\mathcal{F}} \in \mathrm{Shv}(\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}),$$
the object
$$\overline{\mathrm{act}}_!(\widetilde{\mathcal{F}}) \in \mathrm{Shv}(\mathcal{Y})$$
exists, and
$$\overline{\mathrm{act}}_!(\widetilde{\mathcal{F}}) \otimes \mathbf{c} \in \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C}$$

provides the value of the left adjoint to

$$\overline{\mathrm{act}}^! \otimes \mathrm{Id}_{\mathbf{C}} : \mathrm{Shv}(\mathcal{Y}) \otimes \mathbf{C} \to \mathrm{Shv}(\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}) \otimes \mathbf{C}$$

on $\widetilde{\mathcal{F}} \otimes \mathbf{C}$.

A.1.7. In fact we claim that the left adjoint to $\overline{\mathrm{act}}^!$ is given by $\overline{\mathrm{act}}_*$ (which implies that the left adjoint to $\overline{\mathrm{act}}^! \otimes \mathrm{Id}_{\mathbf{C}}$ is given by $\overline{\mathrm{act}}_* \otimes \mathrm{Id}_{\mathbf{C}}$).

Indeed, we claim that the morphism $\overline{\mathrm{act}}$ is ind-proper. To show this, it is enough to show that the action morphism

$$\mathfrak{L}(G) \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y} \to \mathcal{Y} \tag{A.2}$$

is ind-proper.

For this, we note that the automorphism

$$(g, y) \mapsto (g, g \cdot y)$$

isomorphes $\mathfrak{L}(G) \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}$ to the product $\mathfrak{L}(G)/\mathfrak{L}^+(G)^j \times \mathcal{Y}$, and the action morphism (A.2) gets transformed to the projection on the second factor.

Now the assertion follows from the fact that the ind-scheme $\mathfrak{L}(G)/\mathfrak{L}^+(G)^j$ is ind-proper (being isomorphic to the affine Grassmannian).

A.2. **Proof of the cleanness statement.** In this subsection we will prove Proposition A.1.5. In order to unburden the notation we will take $\mathbf{C} = \mathrm{Vect}$ and $\mathbf{c} = \mathsf{e}$; the proof in the general case is literally the same.

A.2.1. We need to show that objects of the form

$$\chi^!(\text{A-Sch}) \widetilde{\boxtimes} \mathcal{F}$$

for $\mathcal{F}$ in the essential image of

$$\mathbf{oblv}_{I^j/\mathfrak{L}(N)^j,\chi} : \mathrm{Shv}(\mathcal{Y})^{I^j,\chi} \to \mathrm{Shv}(\mathcal{Y})^{\mathfrak{L}(N)^j,\chi}$$

are clean with respect to j.

With no restriction of generality we can assume that $\mathcal{F}$ is supported on a $\mathfrak{L}^+(G)^j$-stable (finite-dimensional) subscheme $\mathcal{Y}' \subset \mathcal{Y}$. The action of $\mathfrak{L}^+(G)^j$ on such $\mathcal{Y}'$ factors through a quotient by a normal subgroup $H \subset \mathfrak{L}^+(G)^j$.

In what follows, when we write

$$\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y}$$

we will actually mean

$$\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}/H \overset{\mathfrak{L}^+(G)^j/H}{\times} \mathcal{Y}'.$$

When we will write

$$\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \times \mathcal{Y}$$

we will actually mean

$$\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}/H \times \mathcal{Y}'.$$

We perform this manipulation in order to emphasize that we are dealing with ind-schemes of ind-finite type. However, we will omit H and $\mathcal{Y}'$ from the notation in order to unburden the formulas.

A.2.2. Consider the pullback of $\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}$ to

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \times \mathcal{Y}$$

(see the above conventions) along the projection

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \times \mathcal{Y} \to \mathfrak{L}(N)\mathfrak{L}^+(G)^j \overset{\mathfrak{L}^+(G)^j}{\times} \mathcal{Y} \simeq \mathfrak{L}(N) \overset{\mathfrak{L}^+(N)^j}{\times} \mathcal{Y}.$$

Recall that K_j denotes the j-th congruence subgroup in $\mathfrak{L}^+(G)$; and recall our notation

$$K_j^j \subset \mathfrak{L}^+(G)^j.$$

The assertion of Proposition A.1.5 is obtained as a combination of the following two statements:

Proposition A.2.3. *For $\mathcal{F}$ in the essential image of $\mathbf{oblv}_{I^j/\mathfrak{L}(N)^j,\chi}$, the pullback of $\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}$ to $\mathfrak{L}(N)\mathfrak{L}^+(G)^j \times \mathcal{Y}$ is $(\mathfrak{L}(N), \chi)$-equivariant with respect to the action*

$$n \cdot (n_1 g, y) = (nn_1 g, g)$$

and is K_j^j-equivariant with respect to the action

$$k \cdot (n_1 g, y) = (n_1 g k^{-1}, y).$$

Proposition A.2.4. *For any (ind-scheme) $\mathcal{Y}$, any $\widetilde{\mathcal{F}} \in \operatorname{Shv}((\mathfrak{L}(N)\mathfrak{L}^+(G)^j) \times \mathcal{Y})$ with the equivariance properties specified in Proposition A.2.3, its *-extension to $\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} \times \mathcal{Y}$ equals the !-extension.*

Proof of Proposition A.2.3. Let $\mathcal{F}'$ denote the pullback of $\mathcal{F}$ to $\mathfrak{L}^+(G)^j \times \mathcal{Y}$ along the action map

$$\mathfrak{L}^+(G)^j \times \mathcal{Y} \to \mathcal{Y}$$

(see our conventions).

We can write

$$\mathfrak{L}(N)\mathfrak{L}^+(G)^j \times \mathcal{Y} \simeq \mathfrak{L}(N) \overset{\mathfrak{L}^+(N)^j}{\times} (\mathfrak{L}^+(G)^j \times \mathcal{Y}),$$

and with respect to this identification, the pullback of $\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}$ to $\mathfrak{L}(N)\mathfrak{L}^+(G)^j \times \mathcal{Y}$ goes over to

$$\chi^!(\text{A-Sch})\widetilde{\boxtimes}\mathcal{F}'.$$

This makes the assertion about $(\mathfrak{L}(N), \chi)$-equivariance is immediate. For the assertion regarding K_j^j-equivariance, it suffices to show that $\mathcal{F}'$ is K_j^j-equivariant with respect to the action

$$k \cdot (g, y) = (g \cdot k^{-1}, y).$$

Note that $K_j^j \subset I^j$ and $\chi|_{K_j^j}$ is trivial. Hence, $\mathcal{F}$ is obtained as pullback of an object $\mathcal{F}''$ on the quotient stack $K_j^j\backslash\mathcal{Y}$. Our $\mathcal{F}'$ is thus the pullback of $\mathcal{F}''$ along the composite map

$$\mathfrak{L}^+(G)^j \times \mathcal{Y} \to \mathcal{Y} \to K_j^j\backslash\mathcal{Y}.$$

Hence, it suffices to show that the above composite map is K_j^j-*invariant* for the above action of K_j^j on $\mathfrak{L}^+(G)^j \times \mathcal{Y}$. However, this follows from the normality of K_j^j in $\mathfrak{L}^+(G)^j$.

□

A.3. **Proof of Proposition A.2.4.**

A.3.1. The idea of the proof is that the category

$$\mathrm{Shv}((\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} - \mathfrak{L}(N)\mathfrak{L}^+(G)^j)/K_j^j \times \mathcal{Y})^{\mathfrak{L}(N),\chi}$$

is zero (which follows from Lemma A.3.3 below).

However, that fact on its own does not seem to suffice for the proof of the cleanness statement, because the functor

$$\mathrm{Av}_*^{\mathfrak{L}(N),\chi} : \mathrm{Shv}(\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}/K_j^j \times \mathcal{Y}) \to \mathrm{Shv}(\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}/K_j^j \times \mathcal{Y})^{\mathfrak{L}(N),\chi}$$

does not a priori preserve the subcategory of objects supported on

$$(\overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j} - \mathfrak{L}(N)\mathfrak{L}^+(G)^j)/K_j^j \subset \overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}/K_j^j.$$

So we need a employ more delicate analysis.

A.3.2. Denote

$$\mathcal{Z} := \overline{\mathfrak{L}(N)\mathfrak{L}^+(G)^j}/K_j^j \times \mathcal{Y} \text{ and } \mathcal{Z}^0 := \mathfrak{L}(N)\mathfrak{L}^+(G)^j/K_j^j \times \mathcal{Y}.$$

We will argue by contradiction, so let us be given a non-zero map

$$j_*(\mathcal{F}) \to \mathcal{F}', \tag{A.3}$$

where $\mathcal{F}_1$ is supported on $\mathcal{Z} - \mathcal{Z}^0$.

Let $Z \subset \mathcal{Z}$ be a (finite-dimensional) subscheme of $\mathcal{Z}$ such that the resulting map

$$j_*(\mathcal{F})|_Z \to \mathcal{F}'|_Z \tag{A.4}$$

is non-zero.

Consider the intersection $Z \cap (\mathcal{Z} - \mathcal{Z}^0)$. We will prove:

Lemma A.3.3. *There exists a large enough subgroup $N^\alpha \subset \mathfrak{L}(N)$ so that for every point z in the intersection $Z \cap (\mathcal{Z} - \mathcal{Z}^0)$, the restriction of the character χ to*

$$\mathrm{Stab}_{N^\alpha}(z) \subset N^\alpha \subset \mathfrak{L}(N)$$

is non-trivial.

The lemma will be proved below. Let is proceed with the proof of Proposition A.2.4.

A.3.4. Since $\mathcal{F}$ is $(\mathfrak{L}(N), \chi)$-equivariant, and in particular (N^α, χ)-equivariant, the map (A.3) factors as

$$j_*(\mathcal{F}) \to \mathrm{Av}_*^{N^\alpha}(\mathcal{F}') \to \mathcal{F}'.$$

In particular, the map (A.4) factors as

$$j_*(\mathcal{F})|_Z \to \mathrm{Av}_*^{N^\alpha}(\mathcal{F}')|_Z \to \mathcal{F}'|_Z.$$

We will arrive to a contradiction by showing that

$$\mathrm{Av}_*^{N^\alpha}(\mathcal{F}')|_Z = 0.$$

Indeed, Lemma A.3.3 implies that for any $\mathcal{F}'' \in \mathrm{Shv}(\mathcal{Z} - \mathcal{Z}^0)^{N^\alpha,\chi}$, the restriction $\mathcal{F}''|_Z$ vanishes.

A.4. Proof of Lemma A.3.3.

A.4.1. *Step 1.* Note that for any $z \in \mathcal{Z}$, its stabilizer $\mathrm{Stab}_{\mathfrak{L}(N)}(z)$ is a bounded subgroup in $\mathfrak{L}(N)$. Hence, given a finite-dimensional $Z \subset \mathcal{Z}$, there exists a large enough subgroup $N^\alpha \subset \mathfrak{L}(N)$ so that

$$\mathrm{Stab}_{\mathfrak{L}(N)}(z) \subset N^\alpha, \ \forall z \in Z.$$

Hence, to prove the lemma, it suffices to show that for any $z \in \mathcal{Z} - \mathcal{Z}^0$, the restriction of χ to $\mathrm{Stab}_{\mathfrak{L}(N)}(z)$ is non-trivial.

A.4.2. *Step 2.* Note that for the analysis of the stabilizers, the $\mathcal{Y}$ factor is irrelevant. Thus, let z belong to

$$\mathfrak{L}(N)\lambda(t)\mathfrak{L}^+(G)^j/K_j^j$$

with $0 \neq \lambda \in -\Lambda^{\mathrm{pos}}$. In particular, λ is non-dominant.

Conjugating by an element of $\mathfrak{L}(N)$, we can further assume that $z \in \lambda(t)\mathfrak{L}^+(G)^j/K_j^j \times \mathcal{Y}$. Furthermore, since K_j^j is normal in $\mathfrak{L}^+(G)^j$. Hence, we can assume that $z = \lambda(t)$.

A.4.3. *Step 3.* Note that

$$\mathfrak{L}^+(N) \subset K_j^j,$$

hence

$$\mathrm{Ad}_{-\lambda(t)}(\mathfrak{L}^+(N)) \subset \mathrm{Stab}_{\mathfrak{L}(N)}(z).$$

However, it is clear that for λ non-dominant, the restriction of χ to $\mathrm{Ad}_{-\lambda(t)}(\mathfrak{L}^+(N))$ is non-trivial.

Appendix B. Invariants vs coinvariants for group actions

B.1. **The statement.**

B.1.1. Let H be an algebraic group (of finite type). Let $\mathbf{C}$ be a DG category equipped with an action of H, which by definition means an action of the monoidal category $\mathrm{Shv}(H)$ (the monoidal structure is given by convolution).

Consider the functor

$$\mathrm{Av}_*^H : \mathbf{C} \to \mathbf{C}^H. \tag{B.1}$$

The goal of this appendix is to prove the following result:

Theorem B.1.2. *The functor* (B.1) *is universal among H-invariant functors from* $\mathbf{C}$ *to categories equipped with the* trivial *H-action.*

Another way to state Theorem B.1.2 is that the H-invariant functor (B.1) defines an equivalence

$$\mathbf{C}_H \simeq \mathbf{C}^H. \tag{B.2}$$

B.1.3. *An example.* Take $\mathbf{C} = \mathrm{Shv}(H)$. We have

$$\mathrm{Vect} \simeq \mathrm{Shv}(H)^H, \quad \mathsf{e} \mapsto \mathsf{e}_H.$$

The functor $\mathrm{Av}_*^H : \mathrm{Shv}(H) \to \mathrm{Shv}(H)^H \simeq \mathrm{Vect}$ identifies with

$$\mathcal{F} \mapsto \mathrm{C}^\cdot(H, \mathcal{F}).$$

This makes the assertion of Theorem B.1.2 manifest in this case.

B.1.4. As a formal corollary of Theorem B.1.2, we obtain:

Corollary B.1.5.
(a) *The functor*

$$\mathbf{C} \mapsto \mathbf{C}^H, \quad H\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with colimits.
(b) *The functor*

$$\mathbf{C} \mapsto \mathbf{C}_H, \quad H\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits.

Proof. The assertion about colimits is obvious for the functor $\mathbf{C} \mapsto \mathbf{C}_H$ and about limits for the functor $\mathbf{C} \mapsto \mathbf{C}^H$. Now apply (B.2). □

B.1.6. Let also note that the conclusion of either point (a) or (b) of Corollary B.1.5 implies Theorem B.1.2. Let us prove this for point (b):

Proof. For any $\mathbf{C}$ acted on by H, we have

$$\mathbf{C} \simeq (\operatorname{Shv}(H) \otimes \mathbf{C})^H,$$

where H-invariants are taken with respect to the diagonal action on $\mathbf{C}$ and the action on $\operatorname{Shv}(H)$ by right translations. The above equivalence respects the H-actions, where the action on the RHS comes from the action on $\operatorname{Shv}(H)$ be left translations.

In other words, we obtain that $\mathbf{C}$ is isomorphic to the totalization of the cosimplicial DG category acted on by H with terms

$$\mathbf{C}^n := \operatorname{Shv}(H) \otimes \mathbf{C} \otimes \operatorname{Shv}(H)^{\otimes n}, \quad n \geq 0.$$

As in Example B.1.3, the functor

$$(\mathbf{C}^n)_H \to (\mathbf{C}^n)^H$$

is an equivalence for every n.

We have a commutative diagram

$$\begin{tikzcd}
\mathbf{C}_H \arrow[r] \arrow[d, "\sim"'] & \mathbf{C}^H \arrow[d, "\sim"] \\
\operatorname{Tot}(\mathbf{C}^\bullet)_H \arrow[r] \arrow[d] & \operatorname{Tot}(\mathbf{C}^\bullet)^H \arrow[d, "\sim"] \\
\operatorname{Tot}((\mathbf{C}^\bullet)_H) \arrow[r, "\sim"] & \operatorname{Tot}((\mathbf{C}^\bullet)^H).
\end{tikzcd}$$

Assuming Corollary B.1.5(b), we obtain that the lower left vertical arrow is an equivalence. Hence, $\mathbf{C}_H \to \mathbf{C}^H$ is also an equivalence, as desired.

□

B.2. **Locally constant actions.**

B.2.1. For a scheme Y, let

$$\operatorname{Shv}(Y)^0 \subset \operatorname{Shv}(Y)$$

be the full subcategory generated by the constant sheaf $\mathsf{e}_Y \in \operatorname{Shv}(Y)$. Since e_Y is compact, the tautological embedding

$$\operatorname{Shv}(Y)^0 \hookrightarrow \operatorname{Shv}(Y)$$

admits a continuous right adjoint.

Let $\mathrm{C}^\cdot(Y)$ denote the (commutative) algebra of cochains on H (in our sheaf theory). I.e.,

$$\mathrm{C}^\cdot(Y) := \mathcal{E}nd_{\operatorname{Shv}(Y)}(\mathsf{e}_Y, \mathsf{e}_Y).$$

We have a canonical equivalence

$$\operatorname{Shv}(Y)^0 \simeq \mathrm{C}^\cdot(Y)\text{-mod}, \quad \mathcal{F} \mapsto \mathcal{H}om_{\operatorname{Shv}(Y)}(\mathsf{e}_Y, \mathcal{F}) \simeq \mathrm{C}^\cdot(Y, \mathcal{F}).$$

B.2.2. We take $Y = H$. Note that the subcategory

$$\mathrm{Shv}(H)^0 \hookrightarrow \mathrm{Shv}(H)$$

is preserved by the monoidal operation on $\mathrm{Shv}(H)$. Hence, $\mathrm{Shv}(H)^0$ acquires a monoidal structure.

In terms of the identification

$$\mathrm{Shv}(H)^0 \simeq \mathrm{C}^\cdot(H)\text{-mod},$$

this monoidal sructure corresponds to the structure of (commutative) Hopf algebra on $\mathrm{C}^\cdot(H)$, given by the group law on H.

We note that the functor

$$\mathrm{Shv}(H)^0 \twoheadleftarrow \mathrm{Shv}(H), \tag{B.3}$$

which is a priori lax-monoidal, is actually monoidal. This follows, e.g., from the fact that

$$\mathrm{C}^\cdot(H, \mathcal{F}_1 \star \mathcal{F}_2) \simeq \mathrm{C}^\cdot(H, \mathcal{F}_1) \otimes \mathrm{C}^\cdot(H, \mathcal{F}_2).$$

In particular, we obtain that $\mathrm{Shv}(H)^0$ is unital.

B.2.3. We obtain that $\mathrm{Shv}(H)^0$ is a retract of $\mathrm{Shv}(H)$ as a category acted on by H. In particular, we obtain that $\mathrm{Shv}(H)^0$ is dualizable *as a* $\mathrm{Shv}(H)$-module category (see [GR1, Chapter 1, Sect. 8.6] for what this means).

In particular, we obtain that the functor

$$\mathbf{C} \mapsto \mathrm{Shv}(H)^0 \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}, \quad \mathrm{Shv}(H)\text{-}\mathbf{mod} \to \mathrm{Shv}(H)^0\text{-}\mathbf{mod},$$

left adjoint to the restriction functor

$$\mathrm{Shv}(H)^0\text{-}\mathbf{mod} \to \mathrm{Shv}(H)\text{-}\mathbf{mod} =: H\text{-}\mathbf{mod}, \tag{B.4}$$

commutes with *limits*.

Remark B.2.4. Since the functor (B.3) is a colocalization, we obtain that the functor (B.4), and its left and right adjoints are isomorphic. Indeed, the right adjoint in question is given by

$$\mathbf{C} \mapsto \mathrm{Funct}_{\mathrm{Shv}(H)\text{-}\mathbf{mod}}(\mathrm{Shv}(H)^0\text{-}\mathbf{mod}, \mathbf{C}).$$

However, the self-duality of $\mathrm{Shv}(H)$ as a left/right module category over itself implies that the dual of $\mathrm{Shv}(H)^0$ as a left $\mathrm{Shv}(H)$-module identifies with $\mathrm{Shv}(H)^0$ as a right $\mathrm{Shv}(H)$-module, so

$$\mathbf{C} \mapsto \mathrm{Funct}_{\mathrm{Shv}(H)\text{-}\mathbf{mod}}(\mathrm{Shv}(H)^0\text{-}\mathbf{mod}, \mathbf{C}) \simeq \mathrm{Shv}(H)^0 \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}.$$

Remark B.2.5. Note also that for $\mathbf{C}$ as above, $\mathrm{Shv}(H)^0 \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}$ is the colocalization of $\mathbf{C}$, and is the maximal full subcategory on which the action of $\mathrm{Shv}(H)$ factors through $\mathrm{Shv}(H)^0$.

B.2.6. Since the functor

$$\mathrm{Vect} \to \mathrm{Shv}(H), \quad \mathsf{e} \mapsto \mathsf{e}_H$$

factors through $\mathrm{Shv}(H)^0$, the augmentation functor

$$\mathrm{Shv}(H) \to \mathrm{Vect}, \quad \mathcal{F} \mapsto \mathrm{C}^\cdot(H, \mathcal{F})$$

factors as

$$\mathrm{Shv}(H) \twoheadrightarrow \mathrm{Shv}(H)^0 \to \mathrm{Vect}.$$

According to Sect. B.1.6, in order to prove Theorem B.1.2, it suffices to show that the functor

$$\mathbf{C} \mapsto \mathrm{Vect} \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}, \quad H\text{-}\mathbf{mod} \to \mathrm{Vect}$$

commutes with limits. We rewrite

$$\mathrm{Vect} \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C} \simeq \mathrm{Vect} \underset{\mathrm{Shv}(H)^0}{\otimes} (\mathrm{Shv}(H)^0 \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}).$$

Hence, by B.2.3, in order to prove Theorem B.1.2, it suffices to show that the functor

$$\mathbf{C}' \mapsto \mathrm{Vect} \underset{\mathrm{Shv}(H)^0}{\otimes} \mathbf{C}', \quad \mathrm{Shv}(H)^0\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits. We will prove:

Proposition B.2.7. *The functor*

$$\mathbf{C}' \mapsto \mathbf{C}^r \underset{\mathrm{Shv}(H)^0}{\otimes} \mathbf{C}', \quad \mathrm{Shv}(H)^0\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits for any right $\mathrm{Shv}(H)^0$*-module category* $\mathbf{C}^r$ *which is dualizable as a plain DG category.*

B.3. **Rigidity.**

B.3.1. For a scheme Y, let

$$\mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{fin.dim}} \subset \mathrm{C}^{\cdot}(Y)\text{-mod}$$

be the full (but not cocomplete) subcategory consisting of modules are that are finite-dimensional over the field of coefficients e.

Let $\mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{ren}}$ denote the ind-completion of $\mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{fin.dim}}$. The tautological embedding

$$\mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{fin.dim}} \hookrightarrow \mathrm{C}^{\cdot}(Y)\text{-mod}$$

gives rise to a continuous functor

$$\Psi : \mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{ren}} \to \mathrm{C}^{\cdot}(Y)\text{-mod}.$$

Since $\mathrm{C}^{\cdot}(Y)$ is finite-dimensional, the functor Ψ admits a left adjoint, denoted Ξ, given by sending the compact generator

$$\mathrm{C}^{\cdot}(Y) \in \mathrm{C}^{\cdot}(Y)\text{-mod} \tag{B.5}$$

to $\mathrm{C}^{\cdot}(Y)$ viewed as an object of $\mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{fin.dim}} \subset \mathrm{C}^{\cdot}(Y)\text{-mod}^{\mathrm{ren}}$.

It is clear that the co-unit of the adjunction

$$\mathrm{Id} \to \Psi \circ \Xi$$

is an isomorphism when evaluated on the generator (B.5). Hence Ξ is fully faithful, and so Ψ is a colocalization.

B.3.2. Take $Y = H$. The subcategory

$$\mathrm{C}^{\cdot}(H)\text{-mod}^{\mathrm{fin.dim}} \subset \mathrm{C}^{\cdot}(H)\text{-mod}$$

is preserved by the monoidal operation. Hence, $\mathrm{C}^{\cdot}(H)\text{-mod}^{\mathrm{ren}}$ acquires a monoidal structure so that the functor Ψ is monoidal.

Hence, the restriction functor

$$(\mathrm{C}^{\cdot}(Y)\text{-mod})\text{-}\mathbf{mod} \to (\mathrm{C}^{\cdot}(H)\text{-mod}^{\mathrm{ren}})\text{-}\mathbf{mod}$$

is fully faithful, and for a pair of a left/right $\mathrm{C}^{\cdot}(Y)$-mod-module categories $\mathbf{C}^l$ and $\mathbf{C}^r$, we have

$$\mathbf{C}^r \underset{\mathrm{C}^{\cdot}(H)\text{-mod}}{\otimes} \mathbf{C}^l \simeq \mathbf{C}^r \underset{\mathrm{C}^{\cdot}(H)\text{-mod}^{\mathrm{ren}}}{\otimes} \mathbf{C}^l.$$

B.3.3. Hence, in order to prove Proposition B.2.7, it suffices to show that the functor

$$\mathbf{C}^l \mapsto \mathbf{C}^r \underset{\mathrm{Shv}(H)^0}{\otimes} \mathbf{C}^l, \quad (\mathrm{C}^{\cdot}(H)\text{-mod}^{\mathrm{ren}})\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits for any $\mathbf{C}^r$ which is dualizable as a plain DG category.

However, this follows from the fact that the monoidal category $\mathrm{C}^{\cdot}(H)\text{-mod}^{\mathrm{ren}}$ is *rigid*, see [GR1, Chapter 1, Prop. 9.5.3].

□[Theorem B.1.2]

B.4. **Recovering the category category from invariants.**

B.4.1. The functor

$$\mathbf{C} \mapsto \mathbf{C}^H, \quad H\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

naturally upgrades to a functor

$$\mathbf{C} \mapsto \mathbf{C}^{H,\mathrm{enh}}, \quad H\text{-}\mathbf{mod} \to \mathrm{Shv}(\mathrm{pt}/H)\text{-}\mathbf{mod}, \tag{B.6}$$

where we note that

$$\mathrm{Shv}(\mathrm{pt}/H)\text{-}\mathbf{mod} \simeq \mathrm{Funct}_H(\mathrm{Vect}, \mathrm{Vect}).$$

The functor (B.6) is clearly not conservative. However, by construction, it factors as the composition

$$H\text{-}\mathbf{mod} \to \mathrm{Shv}(H)^0\text{-}\mathbf{mod}, \quad \mathbf{C} \mapsto \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathrm{Shv}(H)^0, \mathbf{C}) \simeq \mathrm{Shv}(H)^0 \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}$$

and the functor

$$\mathrm{Shv}(H)^0\text{-}\mathbf{mod} \to \mathrm{Vect}$$

equal to the restriction of (B.6) along the fully faithful embedding $\mathrm{Shv}(H)^0\text{-}\mathbf{mod} \to H\text{-}\mathbf{mod}$.

B.4.2. We claim:

Theorem B.4.3. *The functor*

$$\mathbf{C} \mapsto \mathbf{C}^{H,\mathrm{enh}}, \quad \mathrm{Shv}(H)^0\text{-}\mathbf{mod} \to \mathrm{Shv}(\mathrm{pt}/H)\text{-}\mathbf{mod} \tag{B.7}$$

is an equivalence of categories.

The rest of this subsection is devoted to the proof of Theorem B.4.3

B.4.4. The functor left adjoint to (B.6) is given by

$$\mathbf{D} \mapsto \mathrm{Vect} \underset{\mathrm{Shv}(\mathrm{pt}/H)}{\otimes} \mathbf{D}. \tag{B.8}$$

We claim that its essential image belongs to $\mathrm{Shv}(H)^0\text{-}\mathbf{mod} \subset H\text{-}\mathbf{mod}$. Indeed, since (B.8) commutes with colimits, it is enough to show this for the generator, i.e., $\mathbf{D} = \mathrm{Shv}(\mathrm{pt}/H)$, and in this case the assertion is clear.

We will show that both the unit and the counit of the adjunction are isomorphisms.

B.4.5. For $\mathbf{D}$ as above, the unit of the adjunction is the canonical map

$$\mathbf{D} \simeq \mathrm{Shv}(\mathrm{pt}/H) \underset{\mathrm{Shv}(\mathrm{pt}/H)}{\otimes} \mathbf{D} \simeq \mathrm{Vect}^H \underset{\mathrm{Shv}(\mathrm{pt}/H)}{\otimes} \mathbf{D} \to (\mathrm{Vect} \underset{\mathrm{Shv}(\mathrm{pt}/H)}{\otimes} \mathbf{D})^H.$$

Now, by Corollary B.1.5(a), the last arrow in the above composition is an equivalence, as desired.

B.4.6. Since the functor (B.6) commutes with colimits, in order to show that that the counit of the adjunction is an isomorphism, it is enough to do so when evaluated on $\mathbf{C} = \mathrm{Shv}(H)^0$. In this case, the assertion amounts to the fact that the functor

$$\mathrm{Vect} \underset{\mathrm{Shv}(\mathrm{pt}/H)}{\otimes} \mathrm{Vect} \to \mathrm{Shv}(H)^0 \tag{B.9}$$

is an equivalence.

B.4.7. We have:

$$\mathrm{Shv}(\mathrm{pt}/H) \simeq \mathrm{C}_\cdot(H)\text{-mod},$$

where the algebra structure on $\mathrm{C}_\cdot(H)$ is given by the group law on H. The (symmetric) monoidal structure on $\mathrm{C}_\cdot(H)$-mod is given by the structure of (cocommutative) Hopf algebra on $\mathrm{C}_\cdot(H)$.

As in Sect. B.3.1, we can write $\mathrm{C}_\cdot(H)$-mod as a (symmetric) monoidal colocalization of the category $\mathrm{C}_\cdot(H)\text{-mod}^{\mathrm{ren}}$, which is equivalent to $\mathrm{C}^\cdot(\mathrm{pt}/H)$-mod.

Write $\mathrm{C}^\cdot(\mathrm{pt}/H) \simeq \mathrm{Sym}(\mathfrak{a})$-mod. Then

$$\mathrm{Vect} \underset{\mathrm{Shv}(\mathrm{pt}/H)}{\otimes} \mathrm{Vect} \simeq \mathrm{Vect} \underset{\mathrm{Sym}(\mathfrak{a})\text{-mod}}{\otimes} \mathrm{Vect} \simeq \mathrm{Sym}(\mathfrak{a}[1])\text{-mod}.$$

The desired equivalence (B.9) follows from the identification

$$\mathrm{C}^\cdot(H) \simeq \mathrm{Sym}(\mathfrak{a}[1]),$$

given by transgression.

□[Theorem B.4.3]

B.5. **The maximal subcategory with a locally constant action.**

B.5.1. Let $\mathbf{C}$ be equipped with an action of G. Set

$$\mathbf{C}_{\mathrm{l.c.}} := \mathrm{Shv}(H)^0 \underset{\mathrm{Shv}(H)}{\otimes} \mathbf{C}.$$

The adjunction

$$\mathrm{Shv}(H)^0 \rightleftarrows \mathrm{Shv}(H)$$

as H-module categories defines an adjunction

$$\mathbf{C}_{\mathrm{l.c.}} \rightleftarrows \mathbf{C} \tag{B.10}$$

as H-module categories.

In particular, we obtain that $\mathbf{C}_{\mathrm{l.c.}}$ is a colocalization of $\mathbf{C}$ as a plain DG category.

Thus, we can think of $\mathbf{C}_{\mathrm{l.c.}}$ as the maximal sub/quotient category of $\mathbf{C}$ on which the action of H is locally constant.

B.5.2. Note that the functors in (B.10) induce equivalences on the corresponding categories of H-coinvariants, and hence invariants

$$\mathbf{C}_{\mathrm{l.c.}}^H \simeq \mathbf{C}^H.$$

It is is easy to see from the constructions that the resulting colocalization functor on $\mathbf{C}$ can be explicitly described as follows

$$\mathbf{c} \mapsto \mathsf{e} \underset{\mathrm{C}^\cdot(H)}{\otimes} \mathbf{oblv}_H \circ \mathrm{Av}_*^H(\mathbf{c}). \tag{B.11}$$

Indeed, by construction, the functor (B.11) takes values in $\mathbf{C}_{\mathrm{l.c.}} \subset \mathbf{C}$; hence by Theorem B.4.3, it suffices to show that it induces the identity endo-functor on $\mathbf{C}^H$, which is immediate.

Appendix C. Sheaf theory in infinite type

In this section we collect miscellanea related to the definition of the category of sheaves on "infinite-dimensional" algebro-geometric objects. We will the use this to define the notion of action of the loop group $\mathfrak{L}(G)$ on a DG category.

C.1. **Placid (ind-)schemes.** Although one can, in principle, define the category $\mathrm{Shv}(\mathcal{Z})$ for *any* k-scheme (or even prestack) $\mathcal{Z}$, the result would be rather unwieldy. In this subsection we single out a certain class of schemes (we call them *placid*), and for which the category $\mathrm{Shv}(\mathcal{Z})$ is manageable.

The main point of the notion of placidity is that it is a *property* and *not* extra structure on a scheme.

C.1.1. Let $\mathcal{Z}$ be a scheme over k, but not necessarily of finite type. We shall say that $\mathcal{Z}$ is *placid* if $\mathcal{Z}$ can be written as filtered limit

$$\mathcal{Z} \simeq \lim_{\alpha} Z_\alpha, \tag{C.1}$$

where Z_α are schemes of finite type, and the transition maps $Z_\alpha \to Z_\beta$ are smooth and surjective.

C.1.2. It is not difficult to show show that if $\mathcal{Z}$ is placid, then the category of presentations of $\mathcal{Z}$ as (C.1) has an initial object, and in particular is contractible. So any two presentations (C.1) are essentially equivalent.

C.1.3. Let $\mathcal{Z}$ be a placid scheme and $\mathcal{Z}' \subset \mathcal{Z}$ a closed subscheme. We shall see that this closed embedding is placid if for some/any presentation of $\mathcal{Z}$ as (C.1), there exists an index i and a closed subscheme $Z'_\alpha \subset Z_\alpha$ so that

$$\mathcal{Z}' = Z'_\alpha \underset{Z_\alpha}{\times} \mathcal{Z}.$$

C.1.4. Let $\mathcal{Y}$ be an ind-scheme (not necessarily of ind-finite type). We shall say that $\mathcal{Y}$ is placid if it can be written as a filtered colimit

$$\operatorname*{colim}_{i} \mathcal{Z}_i, \tag{C.2}$$

where $\mathcal{Z}_j$ are placid schemes, and the transition maps $\mathcal{Z}_i \to \mathcal{Z}_j$ are placid closed embeddings.

C.1.5. It is not difficult to show that the category of presentations of $\mathcal{Y}$ as (C.2) has a final object, and in particular is contractible. So any two presentations (C.2) are essentially equivalent.

C.1.6. Let $\alpha \mapsto \mathcal{Y}_\alpha$ be a filtered family of ind-schemes of ind-finite type with transition maps $f_{\alpha,\beta}$: $\mathcal{Y}_\alpha \to \mathcal{Y}_\beta$ smooth and surjective. With no restriction of generality, we can assume that the index category A has an initial object α_0.

Set

$$\mathcal{Y}_{\alpha_0} \simeq \operatorname*{colim}_{i\in I} Y_i,$$

for a filtered category I, where Y_i are schemes of finite type, and the transition maps $Y_i \to Y_j$ are closed embeddings.

Set $\mathcal{Z}_i := \lim\limits_{\alpha} Y_i \underset{\mathcal{Y}_{\alpha_0}}{\times} \mathcal{Y}_\alpha$. Then $\mathcal{Z}_i$ is a placid scheme, and for $i \le j$, the corresponding map $\mathcal{Z}_i \to \mathcal{Z}_j$ is a placid closed embedding.

Set

$$\mathcal{Y} := \operatorname*{colim}_{i\in I} \mathcal{Z}_i.$$

Then $\mathcal{Y}$ is a placid ind-scheme.

C.2. The category of sheaves on a placid (ind-)scheme.

C.2.1. For a placid scheme $\mathcal{Z}$ presented as in (C.1) we let

$$\mathrm{Shv}(\mathcal{Z}) := \operatorname*{colim}_{\alpha} \mathrm{Shv}(Z_\alpha),$$

where for a $Z_\alpha \to Z_\beta$, the corresponding functor

$$\mathrm{Shv}(Z_\beta) \to \mathrm{Shv}(Z_\alpha)$$

is the *-pullback.

By Sect. C.1.2, this definition is canonically independent of the presentation.

Remark C.2.2. As the functor of *-pullback is not t-exact (in the perverse t-structure), the category $\mathrm{Shv}(\mathcal{Z})$ does *not* come equipped with a t-structure. However, since we are taking *-pullbacks with respect to smooth maps, which are t-exact up to a cohomological shift, one can define a certain $\mathbb{Z}$-gerbe on $\mathcal{Z}$, called *the dimension gerbe*, such that a choice of its trivializations gives rise to a t-structure on $\mathcal{Z}$. We will not pursue this in the present paper.

Similarly, if our sheaf theory is that of D-modules, one may wish to construct the category $\mathrm{D\text{-}mod}(\mathcal{Y})$ equipped with a forgetful functor to an appropriately defined version of the category $\mathrm{IndCoh}(\mathcal{Y})$. A choice of such a functor involves a trivialization of certain Picard gerbe. We will not pursue this in the present paper either.

C.2.3. Note that by Sect. 1.3.3, we have a canonical isomorphism

$$\mathrm{Shv}(\mathcal{Z}) := \lim_{\alpha} \mathrm{Shv}(Z_\alpha),$$

where for $Z_\alpha \to Z_\beta$, the corresponding functor $\mathrm{Shv}(Z_\alpha) \to \mathrm{Shv}(Z_\beta)$ is the *-pushforward.

C.2.4. The latter presentation that the assignment

$$\mathcal{Z} \mapsto \mathrm{Shv}(\mathcal{Z})$$

is functorial with respect to *-pushforwards. Explicitly, for a map $f : \mathcal{Z} \to \mathcal{Z}'$ written as

$$\mathcal{Z} \simeq \lim_{\alpha} Z_\alpha \text{ and } \mathcal{Z}' \simeq \lim_{\alpha'} Z'_{\alpha'},$$

respectively, the corresponding functor

$$f_* : \mathrm{Shv}(\mathcal{Z}) \to \mathrm{Shv}(\mathcal{Z}')$$

is characterized by the property that for every i' the composition

$$\mathrm{Shv}(\mathcal{Z}) \to \mathrm{Shv}(\mathcal{Z}') \to \mathrm{Shv}(Z'_{\alpha'})$$

equals

$$\mathrm{Shv}(\mathcal{Z}) \to \mathrm{Shv}(Z_\alpha) \overset{(f_{\alpha,\alpha'})_*}{\longrightarrow} \mathrm{Shv}(Z'_{\alpha'}),$$

where i is some/any index such that the composite $\mathcal{Z} \overset{f}{\to} \mathcal{Z}' \to Z'_{\alpha'}$ factors as

$$\mathcal{Z} \to Z_\alpha \overset{f_{\alpha,\alpha'}}{\longrightarrow} Z'_{\alpha'}.$$

C.2.5. Let $f : \mathcal{Z}' \to \mathcal{Z}$ be a placid closed embedding. It follows from base change that the functor

$$f_* : \mathrm{Shv}(\mathcal{Z}') \to \mathrm{Shv}(\mathcal{Z})$$

admits a continuos right adjoint, to be denoted $f^!$.

Explicitly, if $\mathcal{Z}' = Z'_{\alpha_0} \underset{Z_{\alpha_0}}{\times} \mathcal{Z}$ for some index i, then $f^!$ is given by the compatible family of functors

$$f^!_\alpha : \mathrm{Shv}(Z_\alpha) \to \mathrm{Shv}(Z'_\alpha), \quad Z'_\alpha := Z'_{\alpha_0} \underset{Z_{\alpha_0}}{\times} Z_\alpha, \quad \alpha \geq \alpha_0.$$

C.2.6. Let $\mathcal{Y}$ be a placid ind-scheme, presented as in (C.2). We define

$$\mathrm{Shv}(\mathcal{Y}) := \lim_{i} \mathrm{Shv}(\mathcal{Z}_i),$$

with respect to the !-pullback functors. By Sect. C.1.5, the category $\mathrm{Shv}(\mathcal{Y})$ defined in this way does not depend on the choice of presentation (C.2).

By Sect. 1.3.3 we can also write

$$\mathrm{Shv}(\mathcal{Y}) := \mathrm{colim}_{i}\, \mathrm{Shv}(\mathcal{Z}_i), \tag{C.3}$$

with respect to the *-pushforward functors.

In particular, we obtain that $\mathrm{Shv}(\mathcal{Y})$ is compactly generated.

C.2.7. The presentation (C.3) implies that if $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a map between placid ind-schemes, we have a well-defined functor

$$f_* : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2).$$

C.2.8. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be two placid ind-schemes. In this case, $\mathcal{Y}_1 \times \mathcal{Y}_2$ is also placid, and we have a canonically defined fully faithful functor

$$\mathrm{Shv}(\mathcal{Y}_1) \otimes \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2),$$

which preserves compactness.

C.2.9. Let us be in the situation of Sect. C.1.6. The direct image functors

$$(f_{\alpha,\beta})_* : \mathrm{Shv}(\mathcal{Y}_\alpha) \to \mathrm{Shv}(\mathcal{Y}_\beta)$$

admit left adjoints, $f^*_{\alpha,\beta}$.

By swapping the order of limits, we obtain that the projections

$$f_\alpha : \mathcal{Y} \to \mathcal{Y}_\alpha,$$

and the corresponding functors

$$(f_\alpha)_* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}_\alpha)$$

give rise to an equivalence

$$\mathrm{Shv}(\mathcal{Y}) \simeq \lim_{\alpha} \mathrm{Shv}(\mathcal{Y}_\alpha),$$

where the limit is taken with respect to $(f_{\alpha,\beta})_*$ as transition functors.

By Sect. 1.3.3, we obtain that we also have an equivalence

$$\mathrm{Shv}(\mathcal{Y}) \simeq \operatorname*{colim}_{\alpha} \mathrm{Shv}(\mathcal{Y}_\alpha),$$

where the colimit is taken with respect to $(f_{\alpha,\beta})^*$ as transition functors.

In particular, we have well-defined functors

$$f^*_\alpha : \mathrm{Shv}(\mathcal{Y}_\alpha) \to \mathrm{Shv}(\mathcal{Y}).$$

C.3. **Sheaves on the loop group.**

C.3.1. Consider the group ind-scheme $\mathfrak{L}(G)$. We claim that it is placid as an ind-scheme. Namely, we claim that it falls in the paradigm of Sect. C.1.6.

Indeed, we take the category A to be natural numbers, and we set

$$\mathcal{Y}_n := \mathfrak{L}(G)/K_n.$$

C.3.2. In particular, we obtain that we have a well-defined category $\mathrm{Shv}(\mathfrak{L}(G))$.

Remark C.3.3. We emphasize again that being a placid ind-scheme is a property and not extra structure. So, accessing $\mathfrak{L}(G)$ via the schemes $K_n\backslash\mathfrak{L}(G)$ will lead to an equivalent definition of the category of sheaves.

C.3.4. By virtue of Sects. C.2.8 and C.2.7, the group structure on $\mathfrak{L}(G)$ defines on $\mathrm{Shv}(\mathcal{Y})$ a structure of monoidal category,

Furthermore, if $\mathcal{Y}$ is another placid ind-scheme equipped with an action of $\mathfrak{L}(G)$, the category $\mathrm{Shv}(\mathcal{Y})$ acquires an action of $\mathrm{Shv}(\mathfrak{L}(G))$.

C.3.5. The monoidal category $\mathrm{Shv}(\mathfrak{L}(G))$ is unital, where the unit object is δ_1, i.e., the direct image of e under the unit map $\mathrm{pt} \to \mathfrak{L}(G)$.

Note, however, that we have a canonical identification

$$\delta_1 \simeq \operatorname*{colim}_{n} \mathsf{e}_{K_n}, \tag{C.4}$$

where by a slight abuse of notation we denote by e_{K_n} the direct image of the constant sheaf under the tautological map $K_n \to \mathfrak{L}(G)$.

Appendix D. Invariants and coinvariants for loop group actions

D.1. Categories acted on by the loop group.

D.1.1. By a category acted on by $\mathfrak{L}(G)$ we will mean a module category over $\mathrm{Shv}(\mathfrak{L}(G))$. We will denote the totality of such categories by $\mathfrak{L}(G)$-**mod**.

We will use a similar notation for $\mathfrak{L}^+(G)$ or K_n. We have the natural restriction functors

$$\mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathfrak{L}^+(G)\text{-}\mathbf{mod} \to K_n\text{-}\mathbf{mod}.$$

D.1.2. Note that for every $n \geq 1$, the object $\mathsf{e}_{K_n} \in \mathrm{Shv}(\mathfrak{L}(G))$ is an idempotent. For $\mathbf{C} \in \mathfrak{L}(G)$-**mod**, let $\mathbf{C}^{K_n}$ the image of that idempotent.

From (C.4), we obtain that

$$\mathbf{C} \simeq \underset{n}{\mathrm{colim}}\, \mathbf{C}^{K_n},$$

where the transition maps are the natural inclusions.

By Sect. 1.3.3 we can also write

$$\mathbf{C} \simeq \lim_{n} \mathbf{C}^{K_n},$$

where the transition maps

$$\mathbf{C}^{K_n} \to \mathbf{C}^{K_m}$$

are the averaging functors $\mathrm{Av}_*^{K_m/K_n}$.

D.1.3. The basic example of an object of $\mathfrak{L}(G)$-**mod** is Vect, which is acted upon by $\mathrm{Shv}(\mathfrak{L}(G))$ via the augmentation functor

$$\mathrm{Shv}(\mathfrak{L}(G)) \to \mathrm{Shv}(\mathrm{pt}) = \mathrm{Vect},$$

given by direct image.

For $\mathbf{C} \in \mathfrak{L}(G)$-**mod**, set

$$\mathbf{C}^{\mathfrak{L}(G)} := \mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C}).$$

Similarly, let $\mathbf{C}_{\mathfrak{L}(G)}$ be the universal recipient of a $\mathfrak{L}(G)$-invariant functor.

We have the following result:

Theorem D.1.4. *Let G be reductive. Then:*

(a) *The functor*

$$\mathbf{C} \mapsto \mathbf{C}^{\mathfrak{L}(G)}, \quad \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits.

(b) *There exists a canonical isomorphism $\mathbf{C}^{\mathfrak{L}(G)} \simeq \mathbf{C}_{\mathfrak{L}(G)}$.*

(c) *If $\mathbf{C}$ is dualizable as a plain category, then so is $\mathbf{C}^{\mathfrak{L}(G)}$, and we have a canonical equivalence $(\mathbf{C}^{\mathfrak{L}(G)})^\vee \simeq (\mathbf{C}^\vee)^{\mathfrak{L}(G)}$.*

The rest of this section is devoted to the proof of Theorem D.1.4. First, let us note that points (a) and (c) both follow from point (b).

D.2. Proof of Theorem D.1.4.

D.2.1. Consider the functor

$$\mathbf{C} \mapsto \mathbf{C}^{\mathfrak{L}^+(G)}, \quad \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{D.1}$$

The first observation is that this functor commutes with colimits. Indeed, the functor

$$\mathbf{C} \mapsto \mathbf{C}^{K_1}, \quad \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with colimits, because it is given by the image of an idempotent. Now, the above functor naturally lifts to a functor

$$\mathfrak{L}(G)\text{-}\mathbf{mod} \to G\text{-}\mathbf{mod},$$

and we have

$$\mathbf{C}^{\mathfrak{L}^+(G)} \simeq (\mathbf{C}^{K_1})^G.$$

Hence, the commutation with colimits follows from Corollary B.1.5(a).

D.2.2. Note that the functor (D.1) can also be interpreted as

$$\mathbf{C} \mapsto \mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathrm{Shv}(\mathfrak{L}(G)/\mathfrak{L}^+(G)), \mathbf{C}).$$

Set

$$\mathbb{H} := \mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathrm{Shv}(\mathfrak{L}(G)/\mathfrak{L}^+(G)), \mathrm{Shv}(\mathfrak{L}(G)/\mathfrak{L}^+(G))).$$

This is the Hecke category of $\mathfrak{L}(G)$ with respect to $\mathfrak{L}^+(G)$. As a plain DG category we can identify it with

$$\mathrm{Shv}(\mathfrak{L}(G))^{\mathfrak{L}^+(G)\times\mathfrak{L}^+(G)},$$

with the monoidal structure given by convolution.

Hence, the functor (D.1) upgrades to a functor

$$\mathbf{C} \mapsto \mathbf{C}^{\mathfrak{L}^+(G),\mathrm{enh}}, \quad \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathbb{H}\text{-}\mathbf{mod}. \tag{D.2}$$

D.2.3. The functor (D.2) admits a left adjoint, given by

$$\mathbf{C}' \mapsto \mathrm{Shv}(\mathfrak{L}(G)/\mathfrak{L}^+(G)) \underset{\mathbb{H}}{\otimes} \mathbf{C}'. \tag{D.3}$$

We claim:

Proposition D.2.4. *The functor*

$$\mathbf{C}' \mapsto \mathrm{Funct}_{\mathbb{H}\text{-}\mathbf{mod}}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)), \mathbf{C}'), \tag{D.4}$$

provides a right adjoint to (D.2).

Proof. We need to show that for $\mathbf{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$, there exists a canonical equivalence

$$\mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathbf{C}, \mathrm{Funct}_{\mathbb{H}\text{-}\mathbf{mod}}(\mathrm{Shv}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)), \mathbf{C}')) \simeq \mathrm{Funct}_{\mathbb{H}\text{-}\mathbf{mod}}(\mathbf{C}^{\mathfrak{L}^+(G),\mathrm{enh}}, \mathbf{C}').$$

We rewrite the LHS as

$$\mathrm{Funct}_{\mathbb{H}\text{-}\mathbf{mod}}(\mathrm{Shv}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)) \underset{\mathrm{Shv}(\mathfrak{L}(G)}{\otimes} \mathbf{C}, \mathbf{C}'),$$

hence it remains to establish an equivalence

$$\mathrm{Shv}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)) \underset{\mathrm{Shv}(\mathfrak{L}(G)}{\otimes} \mathbf{C} \simeq \mathbf{C}^{\mathfrak{L}^+(G),\mathrm{enh}}$$

as $\mathbb{H}$-modules.

We rewrite $\mathrm{Shv}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)) \simeq \mathrm{Shv}(\mathfrak{L}(G))^{\mathfrak{L}^+(G),\mathrm{enh}}$ as categories acted on by $\mathfrak{L}(G)$ on the right and by $\mathbb{H}$ on the left. We have a map

$$\mathrm{Shv}(\mathfrak{L}(G))^{\mathfrak{L}^+(G),\mathrm{enh}} \underset{\mathrm{Shv}(\mathfrak{L}(G)}{\otimes} \mathbf{C} \to \left(\mathrm{Shv}(\mathfrak{L}(G)) \underset{\mathrm{Shv}(\mathfrak{L}(G)}{\otimes} \mathbf{C}\right)^{\mathfrak{L}^+(G),\mathrm{enh}} \simeq \mathbf{C}^{\mathfrak{L}^+(G),\mathrm{enh}}.$$

To show that this map is an equivalence, we need to show that the first arrow is an equivalence at the level of the underlying DG categories. However, this follows from the commutation of the functor (D.1) with colimits.

$\square$

Remark D.2.5. The commutation of the functor (D.1) with colimits implies that the functor (D.3) is fully faithful. It follows formally that the functor (D.4) is fully faithful as well.

D.2.6. Consider $\mathrm{Vect}^{\mathfrak{L}^+(G)}$ as an object of $\mathbb{H}$-**mod**. We claim:

Proposition D.2.7.

(a) *The functor* (D.3) *sends* $\mathrm{Vect}^{\mathfrak{L}^+(G)} \in \mathbb{H}\text{-}\mathbf{mod}$ *to* $\mathrm{Vect} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$.

(b) *The functor* (D.4) *sends* $\mathrm{Vect}^{\mathfrak{L}^+(G)} \in \mathbb{H}\text{-}\mathbf{mod}$ *to* $\mathrm{Vect} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$.

We will prove Proposition D.2.7 in Sect. D.4. We proceed with the proof of Theorem D.1.4.

Corollary D.2.8.

(a) *For* $\mathbf{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$*, we have a canonical isomorphism*

$$\mathbf{C}^{\mathfrak{L}(G)} \simeq \mathrm{Funct}_{\mathbb{H}\text{-}\mathbf{mod}}(\mathrm{Vect}^{\mathfrak{L}^+(G)}, \mathbf{C}^{\mathfrak{L}^+(G),\mathrm{enh}}).$$

(b) *For* $\mathbf{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$*, we have a canonical isomorphism*

$$\mathbf{C}_{\mathfrak{L}(G)} \simeq \mathbf{C}^{\mathfrak{L}^+(G),\mathrm{enh}} \underset{\mathbb{H}}{\otimes} \mathrm{Vect}^{\mathfrak{L}^+(G)}.$$

Thus, from the above corollary we obtain that in order to prove Theorem D.1.4(b), it remains to show the following:

Proposition D.2.9. *The object* $\mathrm{Vect}^{\mathfrak{L}^+(G)} \in \mathbb{H}\text{-}\mathbf{mod}$ *is dualizable and self-dual.*

D.3. **Proof of Proposition D.2.9.**

D.3.1. Before we begin the proof, let us note that the contents of Sect. D.2 up until Proposition D.2.9 were *not* specific to the case of the loop group $\mathfrak{L}(G)$ for G reductive. In fact, Propositions D.2.4, D.2.7 and Corollary D.2.8 remain valid for any pair $\mathcal{G}^+ \subset \mathcal{G}$, where:

- $\mathcal{G}$ is a placid group ind-scheme;
- $\mathcal{G}^+ \subset \mathcal{G}$ is a closed placid group-subscheme;
- $\mathcal{G}^+$ admits a homomorphism to a group-scheme of finite type with a pro-unipotent kernel.

By contrast, Proposition D.2.9 (and with it, Theorem D.1.4) are specific to the situation when $\mathcal{G} = \mathfrak{L}(G)$ with G reductive and $\mathcal{G}^+ = \mathfrak{L}^+(G)$. The key feature of this situation is that the ind-scheme $\mathcal{G}/\mathcal{G}^+$ (which is of ind-finite type by assumption) is *ind-proper*.

We will prove Proposition D.2.9 in this slightly more general context.

D.3.2. Let $\mathbb{H}^{\mathrm{loc.fin}} \subset \mathbb{H}$ be the full (but not cocomplete) category consisting of objects which get sent to compact objects in $\mathrm{Shv}(\mathcal{G})$ under the forgetful functor

$$\mathbb{H} \simeq \mathrm{Shv}(\mathcal{G})^{\mathcal{G}^+\times\mathcal{G}^+} \to \mathrm{Shv}(\mathcal{G}).$$

Let

$$\mathbb{H}^{\mathrm{ren}}$$

be the ind-completion of $\mathbb{H}^{\mathrm{loc.fin}}$. Ind-extending the tautological embedding, we obtain a functor

$$\Psi : \mathbb{H}^{\mathrm{ren}} \to \mathbb{H}.$$

This functor admits a left adjoint, denoted Ξ, given by ind-extending the inclusion

$$\mathbb{H}^c \subset \mathbb{H}^{\mathrm{loc.fin}}.$$

It is clear that the unit of the adjunction

$$\mathrm{Id} \to \Psi \circ \Xi$$

is an isomorphism. Hence, Ξ is fully faithful, and Ψ is a localization.

D.3.3. The assumption that $\mathcal{G}/\mathcal{G}^+$ is proper implies that the monoidal operation on $\mathbb{H}$ preserves both $\mathbb{H}^{\mathrm{loc.fin}}$. Ind-extending, we obtain that $\mathbb{H}^{\mathrm{ren}}$ acquires a monoidal structure, for which the functor Ψ is monoidal.

Thus, we obtain that $\mathbb{H}$ is a monoidal localization of $\mathbb{H}^{\mathrm{ren}}$. In particular, for a right (resp., left) $\mathbb{H}$-module category $\mathbf{C}^r$ (resp., $\mathbf{C}^l$), the functor

$$\mathbf{C}^r \underset{\mathbb{H}^{\mathrm{ren}}}{\otimes} \mathbf{C}^l \to \mathbf{C}^r \underset{\mathbb{H}}{\otimes} \mathbf{C}^l$$

is an equivalence.

Hence, in order to prove Proposition D.2.9, it suffices to show that the dual of $\mathrm{Vect}^{\mathcal{G}^+}$ considered as a left $\mathbb{H}^{\mathrm{ren}}$-module is $\mathrm{Vect}^{\mathcal{G}^+}$ identifies canonically with $\mathrm{Vect}^{\mathcal{G}^+}$ considered as a right $\mathbb{H}^{\mathrm{ren}}$-module.

D.3.4. The key observation now is that the ind-properness assumption on $\mathcal{G}/\mathcal{G}^+$ implies that $\mathbb{H}^{\mathrm{ren}}$ is rigid. Indeed, this follows from [GR1, Chapter 1, Lemma 9.1.5]. Explicitly, the monoidal dual of an object

$$\mathcal{F} \in \mathbb{H}^{\mathrm{loc.fin}} \simeq \mathrm{Shv}(\mathcal{G})^{\mathcal{G}^+\times\mathcal{G}^+} \simeq \mathrm{Shv}(\mathcal{G}/\mathcal{G}^+)^{\mathcal{G}^+}$$

is

$$\tau(\mathbb{D}(\mathcal{F})),$$

where $\mathbb{D}$ is Verdier duality on $\mathcal{G}/\mathcal{G}^+$ and τ is the involution on $\mathrm{Shv}(\mathcal{G})^{\mathcal{G}^+\times\mathcal{G}^+}$, given by the inversion on $\mathcal{G}$ (note that it swaps the two factors in $\mathcal{G}^+ \times \mathcal{G}^+$).

D.3.5. Hence, the required self-duality of $\mathrm{Vect}^{\mathcal{G}^+}$ follows from [GR1, Chapter 1, Proposition 9.5.3].

□

D.4. **Proof of Proposition D.2.7.** We will prove point (a). Point (b) is obtained by considering maps from both sides to Vect.

In order to unburden the notation, we will write $\mathcal{G}$ for $\mathfrak{L}(G)$ and $\mathcal{G}^+$ for $\mathfrak{L}^+(G)$.

D.4.1. We need to show that the tautological functor

$$\Phi : \mathrm{Shv}(\mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \mathrm{Vect}^{\mathcal{G}^+} \to \mathrm{Vect} \tag{D.5}$$

is an equivalence.

First, the commutation of (D.1) with colimits implies that the functor (D.5) induces an equivalence after taking $\mathcal{G}^+$-invariants. Hence, by Theorem B.4.3, we obtain that (D.5) induces an equivalence on the full subcategories of both sides, on which the action of $\mathrm{Shv}(\mathcal{G}^+)$ factors through an action of $\mathrm{Shv}(\mathcal{G}^+)^0$, see Remark B.2.5.

In particular, we obtain that the functor (D.5) admits a fully faithful left adjoint, to be denoted Ψ, (which is also a right inverse), compatible with the actions of $\mathcal{G}^+$. A priori, the functor Ψ is *co-lax* compatible with the action of $\mathcal{G}$. I.e., for $\mathcal{F} \in \mathrm{Shv}(\mathcal{G})$ we have a map

$$\Psi(\mathcal{F} \star \mathsf{e}) \to \mathcal{F} \star \Psi(\mathcal{F}). \tag{D.6}$$

We claim, however, that this co-lax compatibility is strict, i.e., the maps (D.6) are isomorphisms. Let us assume that for a moment and finish the proof of Proposition D.2.7.

D.4.2. To prove that the functors Ψ and Φ are mutually inverse equivalences, it suffices to show that $\ker(\Phi)=0$. However, the embedding

$$\ker(\Phi)\hookrightarrow \mathrm{Shv}(\mathcal{G}/\mathcal{G}^+)\underset{\mathbb{H}}{\otimes}\mathrm{Vect}^{\mathcal{G}^+}$$

admits a $\mathcal{G}$-invariant left inverse given by

$$\mathrm{coFib}(\Psi\circ\Phi\to\mathrm{Id}).$$

Hence, it suffices to show that for any $\mathbf{C}\in\mathcal{G}\text{-}\mathbf{mod}$, a $\mathcal{G}$-invariant functor

$$\mathrm{Shv}(\mathcal{G}/\mathcal{G}^+)\underset{\mathbb{H}}{\otimes}\mathrm{Vect}^{\mathcal{G}^+}\to\mathbf{C},$$

whose composition with Ψ vanshes, is actually zero.

However, for any functor as above, the resulting functor

$$\mathrm{Vect}^{\mathcal{G}^+}\simeq(\mathrm{Shv}(\mathcal{G}/\mathcal{G}^+)\underset{\mathbb{H}}{\otimes}\mathrm{Vect}^{\mathcal{G}^+})^{\mathcal{G}^+}\to\mathbf{C}^{\mathcal{G}^+}$$

is zero. Hence, the original functor vanishes by adjunction.

D.4.3. We will now prove that the maps (D.6) are isomorphisms. This will be done in the following general framework, whose slogan is "a functor lax-compatible with an action of a group is actually strictly compatible". First, we show that this principle literally applies when we work with $\mathrm{Shv}(-)=\text{D-mod}(-)$.

Lemma D.4.4. *Let $F:\mathbf{C}_1\to\mathbf{C}_2$ be a functor between categories acted on by $\mathcal{G}$. Suppose that F is equipped with a structure of lax/co-lax compatibility with the action of $\mathcal{G}$. Then this compatibility is actually strict.*

Proof. We will consider the co-lax case; the lax case is similar. By assumption, we are given a natural transformation

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_1 \arrow[r, "\mathrm{act}"] \arrow[d, "\mathrm{Id}\otimes\Psi"'] & \mathbf{C}_1 \arrow[d, "\Psi"] \arrow[dl, Rightarrow, "\alpha"] \\
\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_2 \arrow[r, "\mathrm{act}"] & \mathbf{C}_2,
\end{tikzcd}$$

i.e.,

$$\alpha:\Psi\circ\mathrm{act}\to\mathrm{act}\circ(\mathrm{Id}_{\mathrm{Shv}(\mathcal{G})}\otimes\Psi).$$

We will explicitly construct an inverse of this natural transformation.

Let $i:\mathcal{G}\to\mathcal{G}\times\mathcal{G}\times\mathcal{G}$ be the map

$$g\mapsto(g,g^{-1},g).$$

Using the identification

$$\mathrm{Shv}(\mathcal{G})\otimes\mathrm{Shv}(\mathcal{G})\otimes\mathrm{Shv}(\mathcal{G})\simeq\mathrm{Shv}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}),\tag{D.7}$$

we obtain a functor

$$\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_1\overset{i_*\otimes\mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow}\mathrm{Shv}(\mathcal{G}\times\mathcal{G}\times\mathcal{G})\otimes\mathbf{C}_1\overset{\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{act}}{\longrightarrow}\tag{D.8}$$
$$\to\mathrm{Shv}(\mathcal{G}\times\mathcal{G})\otimes\mathbf{C}_1\overset{\mathrm{id}_{\mathcal{G}}\otimes\mathrm{act}}{\longrightarrow}\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_1\overset{\mathrm{Id}_{\mathrm{Shv}(\mathcal{G})}\otimes\Psi}{\longrightarrow}\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_2\overset{\mathrm{act}}{\longrightarrow}\mathbf{C}_2.$$

On the one hand, we claim the composition (D.8) identifies with

$$\mathrm{act}\circ(\mathrm{Id}_{\mathrm{Shv}(\mathcal{G})}\otimes\Psi).$$

Indeed, we claim that the composition of the first three arrows in (D.8) , i.e.,

$$\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_1\overset{i_*\otimes\mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow}\mathrm{Shv}(\mathcal{G}\times\mathcal{G}\times\mathcal{G})\otimes\mathbf{C}_1\overset{\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{act}}{\longrightarrow}\mathrm{Shv}(\mathcal{G}\times\mathcal{G})\otimes\mathbf{C}_1\overset{\mathrm{id}_{\mathcal{G}}\otimes\mathrm{act}}{\longrightarrow}\mathrm{Shv}(\mathcal{G})\otimes\mathbf{C}_1\tag{D.9}$$

is the identity functor. Indeed, we rewrite (D.9) as

$$\mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{i_* \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{mult} \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1,$$

which is the same as

$$\mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{unit} \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1,$$

and the latter is indeed the identity functor.

On the other hand, the natural transformation α defines a map from (D.8) to

$$\text{(D.10)} \quad \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{i_* \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow}$$
$$\to \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Psi}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_2 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_2 \overset{\mathrm{act}}{\longrightarrow} \mathbf{C}_2.$$

Now, we claim that the composition (D.10) identifies with $\Psi \circ \mathrm{act}$. To prove this, we first rewrite (D.10) as

$$\text{(D.11)} \quad \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{i_* \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow}$$
$$\to \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Psi}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_2 \overset{\mathrm{mult} \otimes \mathrm{id}_{\mathbf{C}_2}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_2 \overset{\mathrm{act}}{\longrightarrow} \mathbf{C}_2.$$

Next we note that we have a commutative square

$$\text{(D.12)} \quad \begin{tikzcd} \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \arrow[r, "\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Psi"] \arrow[d, "\mathrm{mult} \otimes \mathrm{id}_{\mathbf{C}_1}"'] & \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_2 \arrow[d, "\mathrm{mult} \otimes \mathrm{id}_{\mathbf{C}_2}"] \\ \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \arrow[r, "\mathrm{id}_{\mathcal{G}} \otimes \Psi"] & \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_2. \end{tikzcd}$$

Hence, we can rewrite (D.11) as

$$\text{(D.13)} \quad \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{i_* \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow}$$
$$\to \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{mult} \otimes \mathrm{id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \Psi}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_2 \overset{\mathrm{act}}{\longrightarrow} \mathbf{C}_2.$$

Now, the composition of the first three arrows in (D.13), i.e.,

$$\mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{i_* \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{act}}{\longrightarrow} \mathrm{Shv}(\mathcal{G} \times \mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{mult} \otimes \mathrm{id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1$$

is the functor

$$\mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{act}}{\longrightarrow} \mathbf{C}_1 \overset{\mathrm{unit} \otimes \mathrm{Id}_{\mathbf{C}_1}}{\longrightarrow} \mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1.$$

Hence, the composition in (D.13) is indeed isomorphic to

$$\mathrm{Shv}(\mathcal{G}) \otimes \mathbf{C}_1 \overset{\mathrm{act}}{\longrightarrow} \mathbf{C}_1 \overset{\Psi}{\otimes} \mathbf{C}_2,$$

as claimed.

By unwinding the construction, one checks that the natural transformation

$$\mathrm{act} \circ (\mathrm{Id}_{\mathrm{Shv}(\mathcal{G})} \otimes \Psi) \to \Psi \circ \mathrm{act}$$

constructed above is indeed the inverse of α.

$\square$

D.4.5. We will now adapt this argument in order to deduce the fact that the maps (D.6) are isomorphisms. For a scheme $\mathcal{Y}$ consider the action functor

$$\Phi_{\mathcal{Y}} : \operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \operatorname{Vect}^{\mathcal{G}^+} \to \operatorname{Shv}(\mathcal{Y}). \tag{D.14}$$

Let us denote by $\Psi_{\mathcal{Y}}$ the functor

$$\operatorname{Shv}(\mathcal{Y}) \overset{\operatorname{Id}_{\operatorname{Shv}(\mathcal{Y})} \otimes \Psi}{\longrightarrow} \operatorname{Shv}(\mathcal{Y}) \otimes \operatorname{Shv}(\mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \operatorname{Vect}^{\mathcal{G}^+} \to \operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \operatorname{Vect}^{\mathcal{G}^+}. \tag{D.15}$$

We claim that $(\Psi_{\mathcal{Y}}, \Phi_{\mathcal{Y}})$ form an adjoint pair. Indeed, let us denote by i the external tensor product functor

$$\operatorname{Shv}(\mathcal{Y}) \boxtimes \operatorname{Shv}(\mathcal{G}/\mathcal{G}^+) \to \operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+).$$

The functor i preserves compactness; hence it admits a continuous right adjoint, to be denoted i^R. We have a tautological isomorphism

$$\operatorname{Id}_{\operatorname{Shv}(\mathcal{Y})} \otimes \Phi \simeq \Phi_{\mathcal{Y}} \otimes (i \underset{\mathbb{H}}{\otimes} \operatorname{Id}_{\operatorname{Vect}^{\mathcal{G}^+}}).$$

From here we obtain a natural transformation

$$(\operatorname{Id}_{\operatorname{Shv}(\mathcal{Y})} \otimes \Phi) \circ (i^R \underset{\mathbb{H}}{\otimes} \operatorname{Id}_{\operatorname{Vect}^{\mathcal{G}^+}}) \to \Phi_{\mathcal{Y}}. \tag{D.16}$$

We have to show that (D.16) is an isomorphism.

D.4.6. To prove this, it suffices to show that the corresponding natural transformation becomes an isomorphism after precomposition with

$$\operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \overset{\operatorname{Id}_{\operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+)} \otimes \mathsf{e}}{\longrightarrow} \operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \otimes \operatorname{Vect}^{\mathcal{G}^+} \to \operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \operatorname{Vect}^{\mathcal{G}^+}. \tag{D.17}$$

The precomposition of the LHS of (D.16) with (D.17) is the functor

$$\operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \overset{(p_{\mathcal{Y}})_*}{\longrightarrow} \operatorname{Shv}(\mathcal{Y}), \tag{D.18}$$

where $p_{\mathcal{Y}}$ denotes the projection $\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+ \to \mathcal{Y}$.

The precomposition of the LHS of (D.16) with (D.17) is the functor

$$\operatorname{Shv}(\mathcal{Y} \times \mathcal{G}/\mathcal{G}^+) \overset{i^R}{\to} \operatorname{Shv}(\mathcal{Y}) \boxtimes \operatorname{Shv}(\mathcal{G}/\mathcal{G}^+) \overset{\operatorname{Id} \otimes \operatorname{C}^\cdot(\mathcal{G}/\mathcal{G}^+, -)}{\longrightarrow} \operatorname{Shv}(\mathcal{Y}). \tag{D.19}$$

D.4.7. We claim that the latter isomorphism takes place for $\mathcal{G}/\mathcal{G}^+$ replaced by any ind-scheme of ind-finite type

$$Z = \underset{\alpha}{\operatorname{colim}}\, Z_\alpha.$$

Indeed, for $\mathcal{F}' \in \operatorname{Shv}(\mathcal{Y} \times Z)$ and $\mathcal{F}'' \in \operatorname{Shv}(\mathcal{Y})^c$, we have:

$$\mathcal{H}om_{\operatorname{Shv}(\mathcal{Y})}(\mathcal{F}'', (p_{\mathcal{Y}})_*(\mathcal{F}')) \simeq \underset{\alpha}{\operatorname{colim}}\, \mathcal{H}om_{\operatorname{Shv}(\mathcal{Y} \times Z)}(\mathcal{F}'' \boxtimes \mathsf{e}_{Z_\alpha}, \mathcal{F}'),$$

and

$$\mathcal{H}om_{\operatorname{Shv}(\mathcal{Y})}\Big(\mathcal{F}'', (\operatorname{Id} \otimes \operatorname{C}^\cdot(\mathcal{G}/\mathcal{G}^+, -)) \circ i^R(\mathcal{F}')\Big) \simeq \underset{\alpha}{\operatorname{colim}}\, \mathcal{H}om_{\operatorname{Shv}(\mathcal{Y}) \boxtimes \operatorname{Shv}(Z)}(\mathcal{F}'' \otimes \mathsf{e}_{Z_\alpha}, i^R(\mathcal{F}')) \simeq$$
$$\simeq \underset{\alpha}{\operatorname{colim}}\, \mathcal{H}om_{\operatorname{Shv}(\mathcal{Y} \times Z)}(\mathcal{F}'' \boxtimes \mathsf{e}_{Z_\alpha}, \mathcal{F}'),$$

establishing the desired isomorphism.

D.4.8. We finally return to the proof of Proposition D.2.7. Consider the simplicial categories

$$\mathbf{C}_1^\bullet := \mathrm{Shv}(\mathcal{G}^\bullet) \text{ and } \mathbf{C}_2^\bullet := \mathrm{Shv}(\mathcal{G}^\bullet \times \mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \mathrm{Vect}^{\mathcal{G}^+}.$$

We have a naturally defined simplicial functor $\Phi^\bullet : \mathbf{C}_2^\bullet \to \mathbf{C}_1^\bullet$. By Sect. D.4.5, the functor $\Phi^\bullet$ admits a term-wise left adjoint, to be denoted $\Psi^\bullet$.

The argument proving Lemma D.4.4 shows that the natural transformation

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{G}) \arrow[rr, "\mathrm{act}"] \arrow[d, "\Psi^1"'] & & \mathrm{Vect} \arrow[d, "\Psi^0"] \arrow[dll, Rightarrow, "\alpha"] \\
\mathrm{Shv}(\mathcal{G} \times \mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \mathrm{Vect}^{\mathcal{G}^+} \arrow[rr, "\mathrm{act}"] & & \mathrm{Shv}(\mathcal{G}/\mathcal{G}^+) \underset{\mathbb{H}}{\otimes} \mathrm{Vect}^{\mathcal{G}^+}
\end{tikzcd}$$

is an isomorphism. Indeed, the only non-formal part of the argument was the commutation of the square (D.12), which commutes in our case due to the shape of $\Psi^\bullet$ established in Sect. D.4.5.

In addition, due to the shape of $\Psi^\bullet$, we obtain a commutative diagram of functors

$$\begin{tikzcd}
\Psi \circ \mathrm{act} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{act} \circ (\mathrm{Id}_{\mathrm{Shv}(\mathcal{G})} \otimes \Psi) \arrow[d, "\sim"] \\
\Psi \circ \mathrm{act} \arrow[r] & \mathrm{act} \circ \Psi^1
\end{tikzcd} \tag{D.20}$$

Knowing that the bottom horizontal arrow in (D.20) is an isomorphism, we conclude that the natural transformation

$$\Psi \circ \mathrm{act} \to \mathrm{act} \circ (\mathrm{Id}_{\mathrm{Shv}(\mathcal{G})} \otimes \Psi)$$

is an isomorphism, as desired.

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