INTRODUCTION TO PART II: IND-COHERENT SHEAVES

1. Ind-coherent sheaves vs quasi-coherent sheaves

One of the primary goals of this book is to construct the theory of ind-coherent sheaves as a theory of $\mathcal{O}$-modules on prestacks that exists alongside the theory of quasi-coherent sheaves.

We shall now try to explain what we mean by a 'theory', and highlight the formal features that the two theories have in common and those that set them apart.

1.1. For us QCoh is ultimately a functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}} .$$

I.e., it is a functorial assignment

$$(\mathcal{X} \in \mathrm{PreStk}) \rightsquigarrow (\mathrm{QCoh}(\mathcal{X}) \in \mathrm{DGCat}_{\mathrm{cont}}) \text{ and } (\mathcal{X} \overset{f}{\to} \mathcal{Y}) \rightsquigarrow (f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})).$$

Moreover, the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ has a natural right-lax symmetric monoidal structure, where PreStk is a symmetric monoidal category with respect to the Cartesian product, and $\mathrm{DGCat}_{\mathrm{cont}}$ is symmetric monoidal category with respect to the $\otimes$ tensor product of DG categories.

NB: Here it is of crucial importance that we work with $\mathrm{DGCat}_{\mathrm{cont}}$ (and not DGCat): the operation of tensor product of DG categories is only functorial with respect to continuous (i.e., colimit preserving) functors.

Thus, for $\mathcal{X}, \mathcal{Y} \in \mathrm{PreStk}$, we have a well-defined functor

$$\mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X} \times \mathcal{Y}), \quad \mathcal{F}, \mathcal{G} \mapsto \mathcal{F} \boxtimes \mathcal{G}. \tag{1.1}$$

1.2. The functor (1.1) is an equivalence if $\mathcal{X}$ and $\mathcal{Y}$ are schemes (in fact, it is an equivalence of just one of them is a scheme).

The functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ has the following features:

(i) If $\mathcal{X} \overset{f}{\to} \mathcal{Y}$ is a *schematic and quasi-compact* morphism between prestacks, the above functor f^* admits a *continuous* right adjoint

$$f_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y}).$$

Moreover, if

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g_{\mathcal{X}}} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g_{\mathcal{Y}}} & \mathcal{Y} \end{array}$$

is a Cartesian diagram of prestacks with vertical maps being schematic, the natural transformation of functors

$$g^*_{\mathcal{Y}} \circ f_* \to f'_* \circ g^*_{\mathcal{X}}, \quad \mathrm{QCoh}(\mathcal{X}) \rightrightarrows \mathrm{QCoh}(\mathcal{Y}')$$

Date: January 8, 2016.

that *arises by adjunction* from the isomorphism of functors

$$(f')^* \circ g_{\mathcal{Y}}^* \simeq g_{\mathcal{X}}^* \circ f^*,$$

is an isomorphism.

(ii) If $\mathcal{X} = X \in \mathrm{Sch}$, then the functor

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \overset{\Delta_X^*}{\to} \mathrm{QCoh}(X) \overset{\Gamma(X,-)}{\longrightarrow} \mathrm{Vect}$$

defines the counit of a duality, thereby giving rise to an equivalence

$$\mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \to \mathrm{QCoh}(X).$$

In the above formula $\Gamma(X,-)$ is the functor $(p_X)_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{pt}) = \mathrm{Vect}$, where p_X is the tautological projection $X \to \mathrm{pt}$.

1.3. Here is what the theory of IndCoh will do. First and foremost it will be a functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

I.e., it is a functorial assignment

$$(\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}) \rightsquigarrow (\mathrm{IndCoh}(\mathcal{X}) \in \mathrm{DGCat}_{\mathrm{cont}}) \text{ and } (\mathcal{X} \overset{f}{\to} \mathcal{Y}) \rightsquigarrow (f^! : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X})).$$

As in the case of QCoh, the functor $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ has a natural right-lax symmetric monoidal structure.

If we work over the ground field of characteristic 0 (which is our assumption throughout), then the corresponding functor

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X} \times \mathcal{Y}), \quad \mathcal{F}, \mathcal{G} \mapsto \mathcal{F} \boxtimes \mathcal{G}. \tag{1.2}$$

is an equivalence if either $\mathcal{X}$ or $\mathcal{Y}$ is a scheme.

Already here, there is one piece of difference from the case of QCoh: the functor (1.2) is guaranteed to be an equivalence on a far larger class of algebro-geometric objects. Namely, it suffices to require that $\mathcal{X}$ (or $\mathcal{Y}$) be an *inf-scheme*. We refer the reader to [Chapter III.2] where it is explained what inf-schemes are. Here we will just say that this is a class of prestacks that includes formal schemes and de Rham prestacks of schemes, and is closed under fiber products.

1.4. Here are some features of the functor $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$:

(i) If $\mathcal{X} \overset{f}{\to} \mathcal{Y}$ is a schematic (more generally, *inf-schematic*) morphism between prestacks, we have a well-defined continuous functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y}),$$

and if if

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g_{\mathcal{X}}} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g_{\mathcal{Y}}} & \mathcal{Y} \end{array}$$

is a Cartesian diagram of laft prestacks with vertical maps being schematic (more generally, *inf-schematic*), then we are *given* an isomorphism of functors

$$g_{\mathcal{Y}}^! \circ f_*^{\mathrm{IndCoh}} \to (f')_*^{\mathrm{IndCoh}} \circ g_{\mathcal{X}}^!, \quad \mathrm{QCoh}(\mathcal{X}) \rightrightarrows \mathrm{QCoh}(\mathcal{Y}'). \tag{1.3}$$

However, unlike the case of QCoh, for a general f, the functor f_*^{IndCoh} is not the adjoint of $f^!$ on either side. In particular, the isomorphism (1.3) does not come by adjunction from some *a priori* defined map. So, (1.3) is really an additional piece of data.

That said, if f is an open embedding, it is stipulated that f_*^{IndCoh} should be the *right* adjoint of $f^!$, and in this case, the map $\to$ in (1.3) should come by adjunction from the isomorphism

$$(f')^! \circ g_{\mathcal{Y}}^! \simeq g_{\mathcal{X}}^! \circ f^!.$$

Also, it is stipulated that if f is proper, then f_*^{IndCoh} should be the *left* adjoint of $f^!$, and in this case, the map $\leftarrow$ in (1.3) should come by adjunction from the isomorphism

$$(f')^! \circ g_{\mathcal{Y}}^! \simeq g_{\mathcal{X}}^! \circ f^!.$$

(ii) If $\mathcal{X} = X \in \text{Sch}$ (more generally, $\mathcal{X}$ can be an inf-scheme), then the functor

$$\text{IndCoh}(X) \otimes \text{IndCoh}(X) \simeq \text{IndCoh}(X \times X) \overset{\Delta_X^!}{\to} \text{IndCoh}(X) \overset{\Gamma^{\text{IndCoh}}(X,-)}{\longrightarrow} \text{Vect}$$

defines the counit of a duality, thereby giving rise to an equivalence

$$\mathbf{D}_X^{\text{Serre}} : \text{IndCoh}(X)^\vee \to \text{IndCoh}(X).$$

In the above formula $\Gamma^{\text{IndCoh}}(X, -)$ is the functor

$$(p_X)_*^{\text{IndCoh}} : \text{IndCoh}(X) \to \text{IndCoh}(\text{pt}) = \text{Vect},$$

where p_X is the tautological projection $X \to \text{pt}$.

1.5. To summarise, we can say that the category $\text{IndCoh}(\mathcal{X})$ and the functor f_*^{IndCoh} is guaranteed to be better behaved on a larger class of objects and morphisms (than QCoh and f_*).

But the nature of the relationship between pullbacks and push-forwards for IndCoh is quite different from that of QCoh.

Finally, we should say that there will exist a natural transformation

$$\text{QCoh}^*_{\text{PreStk}} |_{\text{PreStk}_{\text{laft}}} =: \text{QCoh}^*_{\text{PreStk}_{\text{laft}}} \overset{\Upsilon_{\text{PreStk}_{\text{laft}}}}{\longrightarrow} \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}}$$

as (symmetric monoidal) functors

$$(\text{PreStk}_{\text{laft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}.$$

The corresponding functor

$$\Upsilon_{\mathcal{X}} : \text{QCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{X})$$

will, of course, not be an equivalence in general. However:

(a) If $\mathcal{X} = X \in \text{Sch}_{\text{aft}}$, then Υ_X is an equivalence if and only if X is a smooth classical scheme.

(b) If $\mathcal{X} = X_{\text{dR}}$, for $X \in \text{Sch}_{\text{aft}}$, the functor $\Upsilon_{X_{\text{dR}}}$ is always an equivalence.

2. How to construct IndCoh?

One should say that it is quite a long way to construct IndCoh having the above pieces of structure: it will take us all of Parts II and III of this book to do so. Here we will outline the strategy of how this is done.

2.1. In [Chapter II.1] we begin by constructing the category $\mathrm{IndCoh}(X)$ for an individual object $X \in \mathrm{Sch}_{\mathrm{aft}}$.

We start with the usual category $\mathrm{QCoh}(X)$ and consider its (non-cocomplete) subcategory $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ consisting of bounded complexes with coherent cohomologies. We let $\mathrm{IndCoh}(X)$ to be the ind-completion of of $\mathrm{Coh}(X)$.

We obtain that $\mathrm{IndCoh}(X)$ is a compactly generated category, equipped with a t-structure and a tautologically defined t-exact functor

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$

that induces an equivalence on the eventually coconnective subcategories, i.e., the corresponding functors

$$\mathrm{IndCoh}(X)^{\geq -n} \to \mathrm{QCoh}(X)^{\geq -n}$$

are equivalences for any n.

Thus, $\mathrm{IndCoh}(X)$ begins life as a 1small modification' of $\mathrm{QCoh}(X)$–the two categories only differ at $-\infty$. But once we construct IndCoh as a full-fledged theory, it will be quite different from QCoh, as was explained in Sect. 1 above.

2.2. Having defined the category $\mathrm{IndCoh}(X)$ for an individual object $X \in \mathrm{Sch}_{\mathrm{aft}}$ we proceed to defining the *-push forward functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

for a morphism $f : X \to Y$ between schemes.

The functor f_*^{IndCoh} is essentially inherited from QCoh: it is uniquely determined by the requirement that it should be left t-exact and make the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\Psi_X} & \mathrm{QCoh}(X) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \mathrm{QCoh}(Y) \end{array}$$

commute.

Furthermore, we show that the assigment

$$X \rightsquigarrow \mathrm{IndCoh}(X), \quad (X \overset{f}{\to} Y) \rightsquigarrow f_*^{\mathrm{IndCoh}}$$

naturally extends to a functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}\,. \tag{2.1}$$

2.3. Our subsequent task is to construct the !-pullback functors for IndCoh, equipped with base change isomorphisms (1.3) against *-push forwards.

When a map $X \overset{f}{\to} Y$ is proper, we define $f^!$ to be the *right* adjoint of f_*^{IndCoh}, and when it is an open embedding, we define $f^!$ to be the *left* adjoint of f_*^{IndCoh}.

In each of these cases, base change against *-push forwards is a property and not an additional piece of structure, because the corresponding map in one direction[1] comes by adjunction from a tautological isomorphism.

[1]But the direction of the map is different for proper maps and open embeddings.

For a general f, we decompose it as a composition

$$f = f_1 \circ f_2 \tag{2.2}$$

with f_1 an open embedding and f_2 a proper map, and we wish to define $f^!$ to be $f_2^! \circ f_1^!$. The challenge is to show that definition is canonically independent of the decomposition (2.2), and that it is functorial with respect to compositions of maps.

Furthermore, we need to show that $f^!$ thus defined is equipped with base change isomorphisms (1.3), and that these isomorphisms are compatible with compositions etc. However, before proving these compatibilities, we need to formulate them in the ∞-categorical level, and this brings us to the paradigm of the *category of correspondences*.

2.4. In [Chapter II.2] we introduce, following a suggestion of J. Lurie, an $(\infty, 2)$-category, denoted $\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}$.

Its objects are $X \in \text{Sch}_{\text{aft}}$. The $(\infty, 1)$-category of morphisms between X_0 and X_1 has as objects diagrams

$$\begin{array}{ccc} X_{0,1} & \xrightarrow{g} & X_0 \\ {\scriptstyle f}\downarrow & & \\ X_1. & & \end{array}$$

and as morphisms (i.e., 2-morphisms in $\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}$) diagrams

$$\begin{array}{ccccc} X^s_{0,1} & & \xrightarrow{f^s} & & X_0 \\ & \overset{h}{\searrow} & & \nearrow{\scriptstyle f^t} & \\ {\scriptstyle g^s}\downarrow & & X^t_{0,1} & & \\ & & \downarrow{\scriptstyle g^t} & & \\ & \searrow & X_1, & & \end{array}$$

where h is proper and the superscripts 's' and 't' stand for 'source' and 'target', respectively.

This $(\infty, 2)$-category is equipped with 1-fully faithful functors

$$\text{Sch}_{\text{aft}} \to \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}} \leftarrow (\text{Sch}_{\text{aft}})^{\text{op}}. \tag{2.3}$$

2.5. We refer the reader to the introduction to [Chapter II.2], where it is explained that a proper way to encode IndCoh equipped with both functorialities (!-pullback and *-pushforward) is a functor

$$\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}} : \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}} \to \text{DGCat}_{\text{cont}}, \tag{2.4}$$

whose restriction to Sch_{aft} (under the functor $\to$ in (2.3)) is the functor

$$\text{IndCoh}_{\text{Sch}_{\text{aft}}} : \text{Sch}_{\text{aft}} \to \text{DGCat}_{\text{cont}}$$

of (2.1), and whose restriction to $(\text{Sch}_{\text{aft}})^{\text{op}}$ (under the functor $\leftarrow$ in (2.3)) is the functor

$$\text{IndCoh}^!_{\text{Sch}_{\text{aft}}} : (\text{Sch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}},$$

encoding the !-pullback.

Thus, in order to construct the theory of IndCoh on schemes, we need to extend the functor (2.1) to a functor (2.4). We prove in [Chapter II.2, Theorem 2.1.4] that such an extension exists and is unique.

2.6. Having thus constructed the theory of IndCoh on schemes, we need to extend it to prestacks, so that is satisfies (i) from Sect. 1.4.

This is done by the procedure of right Kan extension on the suitable categories of correspondences.

The extension from schemes to inf-schemes (resp., allowing inf-schematic maps between prestacks instead of schematic ones) requires quite a bit more work, and will be the subject of Part III of the book.

2.7. Finally, we show that the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}}$ has a natural symmetric monoidal structure.

From here we formally deduce the Serre duality structure on $\mathrm{IndCoh}(X)$ for $X \in \mathrm{Sch}_{\mathrm{aft}}$, mentioned in (ii) from Sect. 1.4.

2.8. By the construction of $\mathrm{IndCoh}(X)$ for a scheme X, it carries an action of the (symmetric) monoidal category $\mathrm{QCoh}(X)$.

In [Chapter II.3] we formulate and prove how this structure is compatible with the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}}$ of (2.4).

One consequence of this compatibility is the canonically defined natural transformation

$$\Upsilon_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$$

that *right-Kan-extends* to the natural transformation

$$\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}} : \mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$$

mentioned in Sect. 1.5.

NB: for a scheme X we have a pair of functors

$$\mathrm{IndCoh}(X) \overset{\Psi_X}{\to} \mathrm{QCoh}(X) \text{ and } \mathrm{QCoh}(X) \overset{\Upsilon_X}{\to} \mathrm{IndCoh}(X).$$

We will show that these functors are *mutually dual*, where we identify

$$\mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X)$$

via the functors $\mathbf{D}_X^{\mathrm{naive}}$ and $\mathbf{D}_X^{\mathrm{Serre}}$, respectively.

We note also that whereas the functor

$$\Upsilon_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})$$

is defined for any prestack $\mathcal{X}$, the functor $\Psi_{\mathcal{X}}$ is *not*; the latter is really a feature of schemes (or, more generally, Artin stacks). So, the functor Ψ_X that was so necessary for the initial stages of the development of IndCoh in a sense loses its significance further along the development of the theory.