DG INDSCHEMES

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To Igor Frenkel on the occasion of his 60th birthday

Abstract. We develop the notion of indscheme in the context of derived algebraic geometry, and study the categories of quasi-coherent sheaves and ind-coherent sheaves on indschemes. The main results concern the relation between classical and derived indschemes and the notion of formal smoothness.

Contents

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Introduction

0.1. **What is this paper about?** The goal of this paper is to develop the foundations of the theory of indschemes, especially in the context of derived algebraic geometry.

0.1.1. The first question to ask here is "why bother"? For, it is more or less clear what DG indschemes are: functors on the category of affine DG schemes, i.e., ∞-prestacks in the terminology of [GL:Stacks], that can be written as filtered colimits of DG schemes with transition maps being closed embeddings.

The definition of the category of quasi-coherent sheaves on a DG indscheme $\mathcal{X}$ is also automatic: the category $\text{QCoh}(\mathcal{X})$ is defined on any ∞-prestack (see [GL:QCoh, Sect. 1.1] or [Lu1, Sect. 2.7]), and in particular on a DG indscheme.

Here is, however, the question, which started life as a remark in another paper, but answering which in detail was one of the main reasons for writing the present one:

0.1.2. Consider the affine Grassmannian Gr_G corresponding to an algebraic group G. This is an indscheme that figures prominently in the geometric Langlands program. We would like to consider the category $\text{QCoh}(\text{Gr}_G)$ of quasi-coherent sheaves on Gr_G.[1] However, a moment's reflection leads one to conclude that the expression $\text{QCoh}(\text{Gr}_G)$ is ambiguous. Namely, the affine Grassmannian itself can be understood in two, a priori different, ways.

Recall that, as a functor on the category of commutative algebras, Gr_G assigns to a commutative algebra A the groupoid of G-torsors over $\text{Spec}(A[\![t]\!])$ with a trivialization over $\text{Spec}(A(\!(t)\!))$.

Now, we can first take A's to be *classical*, i.e., non-derived, commutative algebras, and thus consider Gr_G as a classical indscheme. Let us denote this version of Gr_G by ${}^{cl}\text{Gr}_G$. As for any classical indscheme, we can consider the category $\text{QCoh}({}^{cl}\text{Gr}_G)$.

The second possibility is to take A's to be DG algebras, and thus consider Gr_G right away as an object of derived algebraic geometry. Thus, we obtain a different version of $\text{QCoh}(\text{Gr}_G)$.

There is a natural functor

$$\text{QCoh}(\text{Gr}_G) \to \text{QCoh}({}^{cl}\text{Gr}_G), \tag{0.1}$$

and our initial question was whether or not it is an equivalence.

If it were not an equivalence, it would signify substantial trouble for the geometric Langlands community: on the one hand, ${}^{cl}\text{Gr}_G$ is a familiar object that people have dealt with for some time now. However, it is clear that the Gr_G is "the right object to consider" if we ever want to mix derived algebraic geometry into our considerations, which we inevitably do. [2]

To calm the anxious reader, let us say that the functor (0.1) is an equivalence, as is guaranteed by Theorem 9.3.4 of the present paper.

In fact, we show that Gr_G is "the same as" ${}^{cl}\text{Gr}_G$, in the sense that the former is obtained from the latter by the natural procedure of turning classical schemes/indschemes/∞-stacks into

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[1] The other main result of this paper, also of direct relevance to geometric Langlands, is described in Sect. 0.3.1 below. It expresses the category $\text{QCoh}(\text{Gr}_G)$ in terms of the corresponding category of ind-coherent sheaves on Gr_G.

[2] One might raise an objection to the relevance of the above question by remarking that for geometric Langlands we mainly consider D-modules on Gr_G, and those only depend on the underlying classical indscheme. However, this is not accurate, since along with D-modules, we consider their global sections as quasi-coherent sheaves, and the latter do depend on the scheme-theoretic structure.

derived ones, [3] which preserves the operation of taking QCoh (see [GL:QCoh, Lemma 1.2.5] for the latter statement).

0.1.3. Another result along these lines, Proposition 6.8.2, concerns formal completions.

Let X be a classical scheme and $Y \subset X$ a Zariski-closed subset. Consider the formal completion $X^\wedge_Y$. By definition, as a functor on commutative algebras, $X^\wedge_Y$ assigns to a ring A the groupoid of maps $\operatorname{Spec}(A) \to X$, such that their image is, set-theoretically, contained in Y.

However, again there are two ways to understand $X^\wedge_Y$: as a classical indscheme, which we then turn into a DG indscheme by the procedure mentioned above. Or, we can consider it as a functor of DG algebras, obtaining a DG indscheme right away.

In Proposition 6.8.2 we show that, under the assumption that X is Noetherian, the above two versions of $X^\wedge_Y$ are isomorphic.

So, by and large, this paper is devoted to developing the theory in order to prove the above and similar results.

0.2. **What is done in this paper.** We shall presently proceed to review the main results of this paper (not necessarily in the order in which they appear in the paper).

We should say that none of these results is really surprising. Rather, they are all in the spirit of "things work as they should." [4]

0.2.1. *DG indschemes via deformation theory.* The first theorem of this paper, Theorem 5.1.1, addresses the following issue. Let $\mathcal{X}$ be an ∞-prestack, such that the underlying classical ∞-prestack is a classical indscheme. What are the conditions that would guarantee that $\mathcal{X}$ is itself a DG indscheme?

There is a natural guess: since DG algebras can be thought of as infinitesimal deformations of classical algebras, if we know the behavior of the functor $\mathcal{X}$ on the latter, its behavior on the former should be governed by deformation theory.

By deformation theory we mean the following: if an algebra A' is the extension of an algebra A by a square-zero ideal $\mathcal{I}$, then the groupoid of extensions of a given map $x : \operatorname{Spec}(A) \to \mathcal{X}$ to a map $x' : \operatorname{Spec}(A') \to \mathcal{X}$ is determined by the *cotangent space* to $\mathcal{X}$ at x, denoted $T^*_x\mathcal{X}$, which is understood just as a functor on the category of $\mathcal{I}$'s, i.e., on A-mod.

If we expect $\mathcal{X}$ to be a DG indscheme, then the functor

$$T^*_x\mathcal{X} : A\text{-mod} \to \infty\text{-Grpd} \tag{0.2}$$

must have certain properties: for a given algebra A, as well as for algebra homomorphisms $A \to B$. If an abstract ∞-prestack $\mathcal{X}$ has these properties, we shall say that $\mathcal{X}$ *admits connective deformation theory.*

Our Theorem 5.1.1 asserts that if $\mathcal{X}$ is such that its underlying classical ∞-prestack is a classical indscheme, and if $\mathcal{X}$ admits connective deformation theory, then it is a DG indscheme.

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[3]This procedure is the left Kan extension along the embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$, followed by sheafification in the fppf topology.

[4]For the duration of the paper we make the technical assumption that our DG indschemes are what one could call "ind-quasi compact and "ind-quasi separated."

0.2.2. *Formal smoothness.* Let us recall the notion of formal smoothness for a classical scheme, or more generally for a classical ∞-prestack, i.e., a functor

$$\mathcal{X} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}\,. \tag{0.3}$$

We say that $\mathcal{X}$ is *formally smooth* if whenever $S \to S'$ is a nilpotent embedding (i.e., a closed embedding with a nilpotent ideal), then the restriction map

$$\pi_0(\mathcal{X}(S')) \to \pi_0(\mathcal{X}(S))$$

is surjective.

The notion of formal smoothness in the DG setting is less evident. We formulate it as follows. Let $\mathcal{X}$ be an ∞-prestack, i.e., just a functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}\,.$$

We say that it is formally smooth if:

- When we restrict $\mathcal{X}$ to classical affine schemes, the resulting functor as in (0.3), is formally smooth in the classical sense.
- For an affine DG scheme $S = \mathrm{Spec}(A)$, the i-th homotopy group of the ∞-groupoid $\mathcal{X}(S)$ depends only on the truncation $\tau^{\geq -i}(A)$ (i.e., a map $A_1 \to A_2$ that induces an isomorphisms of the i-th truncations should induce an isomorphism of π_i's of $\mathcal{X}(\mathrm{Spec}(A_1))$ and $\mathcal{X}(\mathrm{Spec}(A_2))$. [5]

It is well-known that if a classical scheme *of finite type* is classically formally smooth, then it is actually smooth. This implies that it is formally smooth also when viewed as a derived scheme. [6]

The question we consider is whether the same is true for indschemes. Namely, if $\mathcal{X}$ is a classical indscheme, which is classically formally smooth, and *locally of finite type*, is it true that it will be formally smooth also as a DG indscheme? (By "as a DG indscheme" we mean the procedure of turning classical ∞-stacks into derived ones by the procedure mentioned above.)

The answer turns out to be "yes", under some additional technical hypotheses, see Theorem 9.1.2.

Moreover, the above theorem formally implies that (under the same additional hypotheses), every formally smooth DG indscheme is classical, i.e., is obtained by the above procedure from a classical formally smooth indscheme.

The theorem about the affine Grassmannian mentioned above is an easy corollary of this result.

0.2.3. *Loop spaces.* We don't know whether Theorem 9.1.2 remains valid if one omits the locally finite type hypothesis. It is quite possible that this hypothesis is essential. However, we do propose the following conjecture:

Let Z be a classical affine scheme of finite type, which is smooth. Consider the corresponding DG indscheme $Z((t))$ (see Sect. 9.2 for the definition). It is easy to see that it is formally smooth.

We conjecture that, although $Z((t))$ is not locally of finite type, it is classical. The evidence for this is provided by [Dr, Theorem 6.4]. This theorem says that $Z((t))$ violates the locally finite

[5] It is quite possible that a more reasonable definition in both the classical and derived contexts is when the corresponding properties take place not "on the nose", but after Zariski/Nisnevich/étale localization. It is likely that the notion of formal smoothness defined as above is only sensible for ∞-prestacks that are "locally of finite type", or more generally of Tate type.

[6] We do not know whether the latter is true in general without the finite type hypothesis.

type condition by factors isomorphic to the infinite-dimensional affine space, and the latter does not affect the property of being classical.

We prove this conjecture in the special case when Z is an algebraic group G.

0.3. **Quasi-coherent and ind-coherent sheaves on indschemes.** With future applications in mind, the focus of this paper is the categories $\text{IndCoh}(\mathcal{X})$ and $\text{QCoh}(\mathcal{X})$ of ind-coherent and quasi-coherent on a DG indscheme $\mathcal{X}$. [7]

We shall now proceed to state the main result of this paper.

0.3.1. *Comparison of* QCoh *and* IndCoh *on the loop group.* Let us return to the situation of the affine Grassmannian Gr_G, or rather, the loop group $G((t))$. As we now know, both of these DG indschemes are classical.

In the study of local geometric Langlands, one considers the notion of category acted on by the loop group $G((t))$. This notion may be defined in two, a priori, different ways:

(a) As a *co-action* of the *co-monoidal* category $\text{QCoh}(G((t)))$, where the co-monoidal structure is given by pullback with respect to the multiplication map on $G((t))$.

(b) As an *action* of the *monoidal* category $\text{IndCoh}(G((t)))$, where the monoidal structure is given by push-forward with respect to the same multiplication map. [8]

Obviously, one would like these two notions to coincide. This leads one to believe that the corresponding categories $\text{QCoh}(G((t)))$ and $\text{IndCoh}(G((t)))$ are duals of one another (duality is understood here in the sense of [GL:DG, Sect. 2.1]).

Moreover, unless we prove something about $\text{QCoh}(G((t)))$, it would be a rather unwieldy object, as $\text{QCoh}(\mathcal{X})$ is for a general DG indscheme $\mathcal{X}$. For instance, we would not know that it is compactly generated, etc.

0.3.2. To formulate a precise statement, we shall return to the case of the affine Grassmannian. We claim that the functor

$$\text{QCoh}(\text{Gr}_G) \to \text{IndCoh}(\text{Gr}_G)$$

given by tensoring with the dualizing sheaf $\omega_{\text{Gr}_G} \in \text{IndCoh}(\text{Gr}_G)$ is an equivalence.

In fact, we prove Theorem 10.1.1 that asserts that a similarly defined functor is an equivalence for any formally smooth DG indscheme locally of finite type (with an additional technical hypothesis).

This theorem was originally stated and proved by J. Lurie in 2008.

We give a different proof, but it should be noted that Lurie's original proof was much more elegant. The reason we do not reproduce it here is that it uses some not yet documented facts about Ext computations on indschemes.

[7] We the refer the reader to [GL:IndCoh] where the category $\text{IndCoh}(\mathcal{X})$ on a prestack $\mathcal{X}$ is studied. For it to be defined, $\mathcal{X}$ needs to be locally almost of finite type (see [GL:Stacks, Sect. 1.3.9] for what the latter means).

[8] We should remark that when talking about $\text{IndCoh}(G((t)))$, we are leaving the realm of documented mathematics, as $G((t))$ is not locally of finite type. However, it is not difficult to give a definition of IndCoh "by hand" in the particular case of $G((t))$, using the affine Grassmannian.

0.3.3. QCoh *and* IndCoh *on formal completions.* Another set of results we establish concerning QCoh and IndCoh is the following.

In order to prove Theorem 10.1.1 mentioned above, we have to analyze in detail the behavior of the categories QCoh and IndCoh on a DG indscheme obtained as a formal completion $X^\wedge_Y$ of a DG scheme X along a Zariski-closed subset Y.

We show that the category $\mathrm{QCoh}(X^\wedge_Y)$ (resp., $\mathrm{IndCoh}(X^\wedge_Y)$) is equivalent to the localization of $\mathrm{QCoh}(X)$ (resp., $\mathrm{IndCoh}(X)$) with respect to $\mathrm{QCoh}(U)$ (resp., $\mathrm{IndCoh}(U)$), where $U = X - Y$.

This implies some favorable properties of $\mathrm{QCoh}(X^\wedge_Y)$, e.g., that it is compactly generated (something, which is not necessarily true for an arbitrary indscheme). We also endow $\mathrm{QCoh}(X^\wedge_Y)$ with two different t-structures, one compatible with pullbacks from X, and another compatible with push-forwards to X.

In addition, we show that the functors $\Psi, \Xi, \Psi^\vee, \Xi^\vee$ that act between QCoh and IndCoh (see [GL:IndCoh, Sects. 1.1, 1.5, 9.3 and 9.6]) are compatible for $X^\wedge_Y$ and X under the push-forward and pullback functors.

0.4. **Conventions and notation.** Our conventions follow closely those of [GL:IndCoh]. Let us recall the most essential ones.

0.4.1. *The ground field.* Throughout the paper we will be working over a fixed ground field k. We assume that $\mathrm{char}(k) = 0$.

0.4.2. ∞*-categories.* By an ∞-category we always mean an $(\infty, 1)$-category. By a slight abuse of language we will sometimes talk about "categories" when we actually mean ∞-categories. Our usage of ∞-categories is not tied to any particular model, but it is their realization as quasi-categories that we actually have in mind, the basic reference to which is [Lu0].

By ∞-Grpd we denote the ∞-category of ∞-groupoids, which is the same as the category $\mathcal{S}$ of spaces in the notation of [Lu0].

There is a natural functor

$$\infty\text{-Cat} \to \infty\text{-Grpd}$$

which is the right adjoint of the inclusion functor. It sends an ∞-category $\mathbf{C}$ to its maximal subgroupoid, which we will denote by $\mathbf{C}^{\mathrm{grpd}}$. I.e., $\mathbf{C}^{\mathrm{grpd}}$ is obtained from $\mathbf{C}$ by discarding the non-invertible 1-morphisms.

For an ∞-category $\mathbf{C}$, and $x, y \in \mathbf{C}$, we shall denote by $\mathrm{Maps}_{\mathbf{C}}(x, y) \in \infty\text{-Grpd}$ the corresponding mapping space. By $\mathrm{Hom}_{\mathbf{C}}(x, y)$ we denote the set $\pi_0(\mathrm{Maps}_{\mathbf{C}}(x, y))$, i.e., what is denoted $\mathrm{Hom}_{h\mathbf{C}}(x, y)$ in [Lu0].

When working in a fixed ∞-category $\mathbf{C}$, for two objects $x, y \in \mathbf{C}$, we shall call a point of $\mathrm{Maps}_{\mathbf{C}}(x, y)$ an *isomorphism* what is in [Lu0] is called an *equivalence*. I.e., a map that admits a homotopy inverse. We reserve the word "equivalence" to mean a (homotopy) equivalence between ∞-categories.

0.4.3. *Subcategories.* Let $\phi : \mathbf{C}' \to \mathbf{C}$ be a functor between ∞-categoris.

We shall say that ϕ is *0-fully faithful*, or just *fully faithful* if for any $\mathbf{c}'_1, \mathbf{c}'_2 \in \mathbf{C}'$, the map

$$\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}'_1, \mathbf{c}'_2) \to \mathrm{Maps}_{\mathbf{C}}(\phi(\mathbf{c}'_1), \phi(\mathbf{c}'_2)) \tag{0.4}$$

is an isomorphism (=homotopy equivalence) of ∞-groupoids. In this case we shall say that ϕ makes $\mathbf{C}'$ into a *0-full* (or just *full*) subcategory of $\mathbf{C}$.

We also consider two weaker notions:

We shall say that ϕ is *1-fully faithful*, or just *faithful*, if for any $\mathbf{c}'_1, \mathbf{c}'_2 \in \mathbf{C}'$, the map (0.4) is a fully faithful map of ∞-groupoids. Equivalently, the map (0.4) induces an injection on π_0 and a bijection on the homotopy groups π_i, $i \geq 1$ on each connected component of the space $\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}'_1, \mathbf{c}'_2)$.

I.e., 2- and higher morphisms between 1-morphisms in $\mathbf{C}'$ are the same in $\mathbf{C}'$ and $\mathbf{C}$, up to homotopy.

We shall say that ϕ is *faithful and groupoid-full* if it is faithful, and for any $\mathbf{c}'_1, \mathbf{c}'_2 \in \mathbf{C}'$, the map (0.4) is surjective on those connected components of $\mathrm{Maps}_{\mathbf{C}}(\phi(\mathbf{c}'_1), \phi(\mathbf{c}'_2))$ that correspond to isomorphisms. In other words, ϕ is faithful and groupoid-full if it is faithful and the restriction

$$\phi^{\mathrm{grpd}} : \mathbf{C}'^{\,\mathrm{grpd}} \to \mathbf{C}^{\mathrm{grpd}}$$

is fully faithful. In this case, we shall say that ϕ makes $\mathbf{C}'$ into a *1-full* subcategory of $\mathbf{C}$.

0.4.4. *DG categories.* Our conventions regarding DG categories follow [GL:IndCoh, Sects. 0.6.4 and 0.6.5].

In particular, we denote by Vect the DG category of chain complexes of k-vector spaces.

Unless specified otherwise, we will only consider continuous functors between DG categories (i.e., exact functors that commute with direct sums, or equivalently, with all colimits). In other words, we will be working in the category $\mathrm{DGCat}_{\mathrm{cont}}$ in the notation of [GL:DG]. [9]

For a DG category $\mathbf{C}$ and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$ we let

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$$

denote the corresponding object of Vect. We can regard $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ as a not necessarily connective spectrum and thus identify

$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) = \Omega^\infty(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)).$$

For a DG category $\mathbf{C}$ equipped with a t-structure, we denote by $\mathbf{C}^{\leq n}$ (resp., $\mathbf{C}^{\geq m}$, $\mathbf{C}^{\leq n, \geq m}$) the corresponding full subcategories. The inclusion $\mathbf{C}^{\leq n} \hookrightarrow \mathbf{C}$ admits a right adjoint denoted by $\tau^{\leq n}$, and similarly, for the other categories. We let $\mathbf{C}^{\heartsuit}$ denote the heart of the t-structure, and by $H^i : \mathbf{C} \to \mathbf{C}^{\heartsuit}$ the functor of ith cohomology with respect to our t-structure. Note that if $\mathbf{c} \in \mathbf{C}^{\leq n}$ (resp., $\mathbf{C}^{\geq m}$) then $H^i(\mathbf{c}) = 0$ for $i > n$ (resp., $i < m$), but the converse is not true, unless the t-structure is *separated*.

0.4.5. *(Pre)stacks and DG schemes.* Our conventions regarding (pre)stacks and DG schemes follow [GL:Stacks]:

Let $\mathrm{DGSch}^{\mathrm{aff}}$ denote the ∞-category opposite to that of *connective* commutative DG algebras over k.

The category PreStk of prestacks is by definition that of all accessible[10] functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Grpd}\,.$$

The category Stk is a full subcategory in PreStk that consists of those functors that satisfy fppf descent (see [GL:Stacks, Sect. 2.2]). This inclusion admits a left adjoint, denoted L, referred to as the *sheafification functor*.

[9] One can replace $\mathrm{DGCat}_{\mathrm{cont}}$ by (the equivalent) $(\infty,1)$-category of stable presentable ∞-categories tensored over Vect, with colimit-preserving functors.

[10] Recall that an accessible functor is one which commutes with κ-filtered colimits for some regular cardinal κ. This condition ensures that we can avoid set theoretic difficulties when dealing with categories which are not small. See [Lu0] for a discussion of accessible ∞-categories and functors.

We remark that for the purposes of the current paper, the fppf topology can be replaced by the étale, Nisnevich or Zariski topology: all we need is that a non-affine (DG) scheme be isomorphic to the colimit, taken in the category of stacks, of its affine open subschemes.

0.5. **The notion of n-coconnectivity for (pre)stacks.** For the reader's convenience, in this subsection, we briefly review the material of [GL:Stacks] related to the notion of n-connectivity.

0.5.1. Let n be a non-negative integer.

We denote by ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ the full subcategory of $\mathrm{DGSch}^{\mathrm{aff}}$ that consists of affine DG schemes $S = \mathrm{Spec}(A)$, such that $H^{-i}(A) = 0$ for $i > n$. We shall refer to objects of this category as "n-coconnective affine DG schemes." When $n = 0$ we shall also use the terminology "classical affine schemes", and denote this category by $\mathrm{Sch}^{\mathrm{aff}}$.

The inclusion ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$ admits a right adjoint given by cohomological truncation below degree $-n$; we denote this functor by $S \mapsto \tau^{\leq n}(S)$.

0.5.2. *The case of prestacks.* In this paper, we make extensive use of the operation of restricting a prestack $\mathcal{Y}$ to the subcategory ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$. We denote this functor by

$$\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y} : \mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk},$$

where ${}^{\leq n}\mathrm{PreStk}$ is by definition the category of all functors $({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}$.

The above restriction functor admits a (fully faithful) left adjoint, given by left Kan extension along ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$; we denote it by

$$\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}\,.$$

The composition

$$\mathcal{Y} \mapsto \mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{\leq n}\mathcal{Y})$$

is a colocalization functor on PreStk; we denote it by $\mathcal{Y} \mapsto \tau^{\leq n}(\mathcal{Y})$. When $\mathcal{Y}$ is an affine scheme S, this coincides with what was denoted above by $\tau^{\leq n}(S)$.

We shall say that a prestack $\mathcal{Y}$ is *n-coconnective* if it belongs to the essential image of $\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$, or equivalently if the canonical map $\tau^{\leq n}(\mathcal{Y}) \to \mathcal{Y}$ is an isomorphism.

Thus, the functors of restriction and left Kan extension identify ${}^{\leq n}\mathrm{PreStk}$ with the full subcategory of PreStk spanned by *n-coconnective* prestacks.

We shall say that $\mathcal{Y}$ is *eventually coconnective* if it is n-coconnective for some n.

We shall refer to objects of ${}^{\leq 0}\mathrm{PreStk}$ as "classical prestacks"; we shall denote this category also by ${}^{cl}\mathrm{PreStk}$. By the above, the category of classical prestacks is canonically equivalent to that of 0-coconnective prestacks.

0.5.3. *The notion of n-coconnectivity for stacks.* By considering fppf topology on the category ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, we obtain the corresponding full subcategory

$${}^{\leq n}\mathrm{Stk} \subset {}^{\leq n}\mathrm{PreStk}\,.$$

The restriction functor $\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$ sends

$$\mathrm{Stk} \to {}^{\leq n}\mathrm{Stk}, \tag{0.5}$$

but the left adjoint $\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$ does not send ${}^{\leq n}\mathrm{Stk}$ to Stk. The left adjoint to the functor (0.5) is given by the composition

$${}^{\leq n}\mathrm{Stk} \hookrightarrow {}^{\leq n}\mathrm{PreStk} \overset{\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}}{\longrightarrow} \mathrm{PreStk} \overset{L}{\longrightarrow} \mathrm{Stk},$$

and is denoted ${}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$. The functor ${}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$ is fully faithful. The composition of the functor (0.5) with ${}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$ is a colocalization functor on Stk and is denoted $\mathcal{Y}\mapsto {}^L\tau^{\leq n}(\mathcal{Y})$.

We shall say that a stack $\mathcal{Y}\in\mathrm{Stk}$ is n-coconnective *as a stack* if it belongs to the essential image of the functor ${}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$, or equivalently, if the canonical map ${}^L\tau^{\leq n}(\mathcal{Y})\to\mathcal{Y}$ is an isomorphism.

We emphasize, however, that if $\mathcal{Y}$ is n-coconnective as a stack, it is *not* necessarily n-coconnective as a prestack. The corresponding morphism $\tau^{\leq n}(\mathcal{Y})\to\mathcal{Y}$ becomes an isomorphism only after applying the sheafification functor L.

Thus, the functor (0.5) and its left adjoint identify the category ${}^{\leq n}\mathrm{Stk}$ with the full subcategory of Stk spanned by n-coconnective stacks.

We shall say that $\mathcal{Y}$ is *eventually coconnective as a stack* if it is n-coconnective as a stack for some n.

We shall refer to objects of ${}^{\leq 0}\mathrm{Stk}$ as "classical stacks"; we shall also denote this category by ${}^{cl}\mathrm{Stk}$. By the above, the category of classical stacks is canonically equivalent to that of 0-coconnective stacks.

0.5.4. *DG schemes.* The category Stk (resp., ${}^{\leq n}\mathrm{Stk}$) contains the full subcategory DGSch (resp., ${}^{\leq n}\mathrm{DGSch}$), see [GL:Stacks], Sect. 3.2.

The functors of restriction and ${}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$ send the categories DGSch and ${}^{\leq n}\mathrm{DGSch}$ to one another, thereby identifying ${}^{\leq n}\mathrm{DGSch}$ with the subcategory of DGSch that consists of n-coconnective DG schemes, i.e., those DG schemes that are n-coconnective as stacks.

For $n=0$ we shall refer to objects of ${}^{\leq 0}\mathrm{DGSch}$ as "classical schemes", and denote this category also by Sch.

Notational convention: In order to avoid unbearably long formulas, we will sometimes use the following slightly abusive notation: if Z is an object of ${}^{\leq n}\mathrm{DGSch}$, we will use the same symbol Z for the object of DGSch that should properly be denoted

$$ {}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(Z). $$

Similarly, for $n'\geq n$, we shall write Z for the object of ${}^{\leq n'}\mathrm{DGSch}$ that should properly be denoted

$$ {}^{\leq n'}\left({}^L\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(Z)\right). $$

0.5.5. *Convergence.* An object $\mathcal{Y}$ of PreStk (resp., Stk) is said to be convergent if for any $S\in\mathrm{DGSch}^{\mathrm{aff}}$, the natural map

$$ \mathcal{Y}(S)\to \underset{n}{lim}\, \mathcal{Y}(\tau^{\leq n}(S)) $$

is an isomorphism.

Equivalently, $\mathcal{Y}\in\mathrm{PreStk}$ (resp., Stk) is convergent if the map

$$ \mathcal{Y}\to\mathrm{RKE}_{({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}|_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}}) $$

is an isomorphism. Here, ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}$ denotes the full subcategory of $\mathrm{DGSch}^{\mathrm{aff}}$ spanned by eventually coconnective affine DG schemes.

The full subcategory of PreStk (resp., Stk) that consists of convergent objects is denoted ${}^{\mathrm{conv}}\mathrm{PreStk}$ (resp., ${}^{\mathrm{conv}}\mathrm{Stk}$). The embedding

$$ {}^{\mathrm{conv}}\mathrm{PreStk} \hookrightarrow \mathrm{PreStk} $$

admits a left adjoint, called the convergent completion, and denoted $\mathcal{Y} \mapsto {}^{\mathrm{conv}}\mathcal{Y}$. [11] The restriction of this functor to Stk sends

$$ \mathrm{Stk} \to {}^{\mathrm{conv}}\mathrm{Stk}, $$

and is the left adjoint to the embedding ${}^{\mathrm{conv}}\mathrm{Stk} \hookrightarrow \mathrm{Stk}$.

Tautologically, we can describe the functor of convergent completion as the composition

$$ \mathcal{Y} \mapsto \mathrm{RKE}_{({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}|_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}}). $$

I.e, the functor of right Kan extension $\mathrm{RKE}_{({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$ along

$$ ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} $$

identifies the category ${}^{<\infty}\mathrm{PreStk}$ with ${}^{\mathrm{conv}}\mathrm{PreStk}$, and ${}^{<\infty}\mathrm{Stk}$ with ${}^{\mathrm{conv}}\mathrm{Stk}$.

0.5.6. *Weak n-coconnectivity.* For a fixed n, the composite functor

$$ {}^{\mathrm{conv}}\mathrm{PreStk} \hookrightarrow \mathrm{PreStk} \overset{\mathrm{restriction}}{\longrightarrow} {}^{\leq n}\mathrm{PreStk} $$

also admits a left adjoint given by

(0.6)

$$ \mathcal{Y}_n \mapsto {}^{\mathrm{conv}}\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}_n) := {}^{\mathrm{conv}}(\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}_n)). $$

Equivalently, when we identify ${}^{<\infty}\mathrm{PreStk} \simeq {}^{\mathrm{conv}}\mathrm{PreStk}$, the above functor can be described as $\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$.

The composite functor

$$ \mathcal{Y} \mapsto {}^{\mathrm{conv}}\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}) $$

is a colocalization on ${}^{\mathrm{conv}}\mathrm{PreStk}$, and we will denote it by ${}^{\mathrm{conv}}\tau^{\leq n}$.

Similarly, the composite functor

$$ {}^{\mathrm{conv}}\mathrm{Stk} \hookrightarrow \mathrm{Stk} \overset{\mathrm{restriction}}{\longrightarrow} {}^{\leq n}\mathrm{Stk} $$

also admits a left adjoint given by

(0.7)

$$ \mathcal{Y}_n \mapsto {}^{\mathrm{conv},L}\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}_n) := {}^{\mathrm{conv}}({}^{L}\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}_n)). $$

Alternatively, when we identify ${}^{<\infty}\mathrm{Stk} \simeq {}^{\mathrm{conv}}\mathrm{Stk}$, the above functor can be described as

$$ {}^{<\infty L}\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}. $$

The composite functor

$$ \mathcal{Y} \mapsto {}^{\mathrm{conv},L}\mathrm{LKE}_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}) $$

is a colocalization on ${}^{\mathrm{conv}}\mathrm{Stk}$, and we will denote it by ${}^{\mathrm{conv},L}\tau^{\leq n}$.

We shall say that an object $\mathcal{Y}$ of ${}^{\mathrm{conv}}\mathrm{PreStk}$ (resp., ${}^{\mathrm{conv}}\mathrm{Stk}$) is *weakly* n-coconnective if it belongs to the essential image of the functor (0.6) (resp., (0.7)). Equivalently, an object as above is *weakly* n-coconnective if and only if its restriction to ${}^{\leq m}\mathrm{DGSch}^{\mathrm{aff}}$ is n-coconnective for any $m \geq n$.

[11] In [GL:Stacks], this functor was denoted $\mathcal{Y} \mapsto \widehat{\mathcal{Y}}$.

It is clear that if an object is n-coconnective, then it is weakly n-coconnective. However, the converse is false.

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1. DG indschemes

When dealing with usual indschemes, the definition is straightforward: like any "space" in algebraic geometry, an indscheme is a presheaf on the category of affine schemes, and the condition we require is that it should be representable by a filtered family of schemes, where the transition maps are closed embeddings.

The same definition is reasonable in the DG setting as long as we restrict ourselves to n-coconnective DG schemes for some n. However, when dealing with arbitrary DG indschemes, one has to additionally require that the presheaf be *convergent*, see Sect. 0.5.5.

Thus, for reasons of technical convenience we define DG indschemes by requiring the existence of a presentation as a filtered colimit *at the truncated level*. We will later show that a DG indscheme defined in this way itself admit a presentation as a colimit of DG schemes.

In this section we define DG indschemes, first in the n-coconnective setting for some n, and then in general, and study the relationship between these two notions.

As was mentioned in the introduction, the class of (DG) indschemes that we consider in this paper is somewhat smaller than one could in principle consider in general: we will only consider those (DG) indschemes that are ind-quasi compact and ind-quasi separated.

1.1. Definition in the n-coconnective case.

1.1.1. Let us recall the notion of closed embedding in derived algebraic geometry.

Definition 1.1.2. *A map $X_1 \to X_2$ in* DGSch *or* ${}^{\leq n}\mathrm{DGSch}$ *is a closed embedding if the corresponding map of classical schemes ${}^{cl}X_1 \to {}^{cl}X_2$ is.*

Recall that the notation ${}^{cl}X$ means $X|_{{}^{cl}\mathrm{DGSch}^{\mathrm{aff}}}$, i.e., we regard X is a functor on classical affine schemes, and if X was a DG scheme, then ${}^{cl}X$ is a classical scheme (see [GL:Stacks, Sect. 3.2.1]).

Let $(\mathrm{DGSch})_{\mathrm{closed}}$ (resp., $({}^{\leq n}\mathrm{DGSch})_{\mathrm{closed}}$) denote the 1-full subcategory of DGSch (resp., ${}^{\leq n}\mathrm{DGSch}$), where we restrict 1-morphisms to be closed embeddings. Let

$$\mathrm{DGSch}_{\text{qsep-qc}} \subset \mathrm{DGSch},\ (\mathrm{DGSch}_{\text{qsep-qc}})_{\mathrm{closed}} \subset (\mathrm{DGSch})_{\mathrm{closed}},$$

$${}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}} \subset {}^{\leq n}\mathrm{DGSch},\ ({}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}})_{\mathrm{closed}} \subset ({}^{\leq n}\mathrm{DGSch})_{\mathrm{closed}}$$

be the full subcategories corresponding to quasi-separated and quasi-compact DG schemes (by definition, this is a condition on the underlying classical scheme).

1.1.3. We give the following definition:

Definition 1.1.4. *A* $^{\leq n}$DG *indscheme is an object* $\mathcal{X}$ *of* $^{\leq n}$PreStk *that can be represented as a colimit of a functor*

$$\mathsf{A} \to {}^{\leq n}\text{PreStk}$$

which can be factored *as*

$$\mathsf{A} \to ({}^{\leq n}\text{DGSch}_{\text{qsep-qc}})_{\text{closed}} \hookrightarrow {}^{\leq n}\text{PreStk},$$

and where the category A is filtered*.*

I.e., $\mathcal{X} \in$ PreStk is a $^{\leq n}$DG indscheme if it can be written as a filtered colimit in $^{\leq n}$PreStk:

$$\underset{\alpha}{colim}\, X_\alpha, \tag{1.1}$$

where $X_\alpha \in {}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$ and for $\alpha_1 \to \alpha_2$, the corresponding map $i_{\alpha_1,\alpha_2} : X_{\alpha_1} \to X_{\alpha_2}$ is a closed embedding.

Let $^{\leq n}$DGindSch denote the full subcategory of $^{\leq n}$PreStk spanned by $^{\leq n}$DG indschemes. We shall refer to objects of $^{\leq 0}$DGindSch as *classical indschemes*; we shall also use the notation indSch := $^{\leq 0}$DGindSch.

Remark 1.1.5. Note that the quasi-compactness and quasi-separatedness assumption in the definition of $^{\leq n}$DG indschemes means that not every $^{\leq n}$DG scheme X is a $^{\leq n}$DG indscheme. However, a scheme which is an indscheme is not necessarily quasi-separated and quasi-compact: for example, a disjoint union of quasi-separated and quasi-compact $^{\leq n}$DG schemes is a $^{\leq n}$DG indscheme.

1.2. Changing n.

1.2.1. Clearly, for $n' < n$, the functor

$${}^{\leq n}\text{PreStk} \to {}^{\leq n'}\text{PreStk},$$

corresponding to restriction along

$${}^{\leq n'}\text{DGSch}^{\text{aff}} \hookrightarrow {}^{\leq n}\text{DGSch}^{\text{aff}},$$

sends the subcategory $^{\leq n}$DGindSch to $^{\leq n'}$DGindSch.

Indeed, if $\mathcal{X}$ is presented as in (1.1), then $^{\leq n'}\mathcal{X} := \mathcal{X}|_{^{\leq n'}\text{DGSch}^{\text{aff}}}$ can be presented as

$$\underset{\alpha}{colim}\, ({}^{\leq n'}X_\alpha).$$

Thus, restriction defines a functor

$${}^{\leq n'}\text{DGindSch} \leftarrow {}^{\leq n}\text{DGindSch}\,.$$

1.2.2. Vice versa, consider the functor

$$\begin{aligned} &{}^{\leq n}L\text{LKE}_{({}^{\leq n'}\text{DGSch}^{\text{aff}})^{\text{op}} \hookrightarrow ({}^{\leq n}\text{DGSch}^{\text{aff}})^{\text{op}}} := \\ &\qquad {}^{\leq n}L \circ \text{LKE}_{({}^{\leq n'}\text{DGSch}^{\text{aff}})^{\text{op}} \hookrightarrow ({}^{\leq n}\text{DGSch}^{\text{aff}})^{\text{op}}} : {}^{\leq n'}\text{Stk} \to {}^{\leq n}\text{Stk}, \end{aligned} \tag{1.2}$$

left adjoint to the restriction functor. In the above formula $^{\leq n}L : {}^{\leq n}\text{PreStk} \to {}^{\leq n}\text{Stk}$ is the sheafification functor, left adjoint to the embedding $^{\leq n}\text{Stk} \hookrightarrow {}^{\leq n}\text{PreStk}$.

We claim that it sends $^{\leq n'}$DGindSch to $^{\leq n}$DGindSch. Indeed, if $\mathcal{X}' \in {}^{\leq n'}$DGindSch is written as

$$\mathcal{X}' \simeq \underset{\alpha}{colim}\, X_\alpha, \quad X_\alpha \in {}^{\leq n'}\text{DGSch}$$

(the colimit taken in ${}^{\leq n'}\mathrm{PreStk}$), then

$$ {}^{\leq n}L\mathrm{LKE}_{({}^{\leq n'}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow ({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{X}') \simeq \underset{\alpha}{colim}\, X_\alpha, $$

where the colimit is taken in ${}^{\leq n}\mathrm{PreStk}$, and X_α is perceived as an object of ${}^{\leq n}\mathrm{DGSch}$, see the notational convention in Sect. 0.5.4.

1.2.3. We obtain a pair of adjoint functors

$$ {}^{\leq n'}\mathrm{DGindSch} \rightleftarrows {}^{\leq n}\mathrm{DGindSch}, \tag{1.3} $$

with the left adjoint being fully faithful.

An object $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ belongs to the essential image of the left adjoint in (1.3) if and only if it is n'-coconnective as an object of ${}^{\leq n}\mathrm{Stk}$, i.e., if it belongs to the essential image of the left adjoint (1.2).

Moreover, if $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ has this property, it admits a presentation as in (1.1), where the X_α are n'-coconnective.

1.3. Basic properties of ${}^{\leq n}$DG indschemes.

1.3.1. We observe:

Proposition 1.3.2. *Every ${}^{\leq n}$DG indscheme belongs to ${}^{\leq n}\mathrm{Stk}$ i.e., satisfies fppf descent.*

The proof is immediate from the following general assertion:

Lemma 1.3.3. *Let $\alpha \mapsto X_\alpha$ be a filtered diagram in ${}^{\leq n}\mathrm{PreStk}$. Set*

$$ \mathcal{X} := \underset{\alpha}{colim}\, X_\alpha. $$

Then if all X_α belong to ${}^{\leq n}\mathrm{Stk}$ and are k-truncated for some k (see [GL:Stacks, Sect. 1.1.7]*), then $\mathcal{X}$ has the same properties.*

Proof. By assumption,

$$ X_\alpha \text{ and } \mathcal{X} : {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \to \infty\text{-Grpd} $$

take values in the subcategory $(k+n)$-groupoids.

Recall that for a co-simplicial object $\mathbf{c}^\bullet$ in the category of m-groupoids, the totalization $\mathrm{Tot}(\mathbf{c}^\bullet)$ maps isomorphically to $\mathrm{Tot}^{m+1}(\mathbf{c}^\bullet)$, where $\mathrm{Tot}^{m+1}(-)$ denotes the limit taken over the category of finite ordered sets of cardinality $\leq (m+1)$.

Hence, for an fppf cover $S' \to S$ and its Čech nerve $S'^\bullet/S$, for its $(k+n+1)$-truncation $S'^{\bullet \leq k+n+1}/S$, the restriction maps

$$ \mathrm{Tot}(X_\alpha(S'^\bullet/S)) \to \mathrm{Tot}^{\leq (k+n+1)}(X_\alpha(S'^\bullet/S)) \text{ and } \mathrm{Tot}(\mathcal{X}(S'^\bullet/S)) \to \mathrm{Tot}^{\leq (k+n+1)}(\mathcal{X}(S'^\bullet/S)) $$

are isomorphisms.

In particular, it suffices to show that the map $\mathcal{X}(S) \to \mathrm{Tot}^{\leq (k+n+1)}(\mathcal{X}(S'^\bullet/S))$ is an isomorphism.

Consider the commutative diagram

$$ \begin{array}{ccc} \underset{\alpha}{colim}\, X_\alpha(S) & \longrightarrow & \mathrm{Tot}^{\leq (k+n+1)}\left(\underset{\alpha}{colim}\, X_\alpha(S'^\bullet/S)\right) \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow \\ \underset{\alpha}{colim}\, X_\alpha(S) & \longrightarrow & \underset{\alpha}{colim}\left(\mathrm{Tot}^{\leq (k+n+1)}(X_\alpha(S'^\bullet/S))\right). \end{array} $$

The bottom horizontal arrow is an isomorphism, since all X_α satisfy descent. The right vertical arrow is an isomorphism, since filtered colimits commute with *finite* limits. Hence, the top horizontal arrow is also an isomorphism, as desired.

□

1.3.4. We obtain that if $\mathcal{X} \in {}^{\leq n}\mathrm{Stk}$ is written as in (1.1), *but where the colimit is taken in the category* ${}^{\leq n}\mathrm{Stk}$, then $\mathcal{X}$ is a ${}^{\leq n}$DG indscheme.

Indeed, Proposition 1.3.2 implies that the the natural map from the colimit of (1.1) taken in ${}^{\leq n}\mathrm{PreStk}$ to that in ${}^{\leq n}\mathrm{Stk}$ is an isomorphism.

1.3.5. Let Y be an object of ${}^{\leq n}\mathrm{DGSch}$, and let $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ be presented as in (1.1). We have a natural map

$$\underset{\alpha}{colim}\ \mathrm{Maps}(Y, X_\alpha) \to \mathrm{Maps}(Y, \mathcal{X}). \tag{1.4}$$

If Y is affine, the above map is an isomorphism by definition, since colimits in

$${}^{\leq n}\mathrm{PreStk} = \mathrm{Func}({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}, \infty\text{-Grpd})$$

are computed object-wise.

For a general Y, the map (1.4) need not be an isomorphism. However, we have:

Lemma 1.3.6. *If Y is quasi-separated and quasi-compact, then the map* (1.4) *is an isomorphism.*

Proof. This follows from the fact that $\mathcal{X}$ belongs to ${}^{\leq n}\mathrm{Stk}$, and that a quasi-separated and quasi-compact DG scheme can be written as a colimit in ${}^{\leq n}\mathrm{Stk}$ of a *finite* diagram whose terms are in ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, and the fact that filtered colimits in ∞-Grpd commute with finite limits. □

Remark 1.3.7. The reason we ever mention sheafification and work with Stk rather than simply with PreStk is Lemma 1.3.6 above. However, the proof of Lemma 1.3.6 shows that we could equally well work with étale, Nisnevich or Zariski topologies, instead of fppf.

1.4. **General DG indschemes.**

1.4.1. We give the following definition:

Definition 1.4.2. *An object $\mathcal{X} \in$* PreStk *is a DG indscheme if the following two conditions hold:*

1. *As an object of* PreStk*, $\mathcal{X}$ is convergent (see Sect. 0.5.5).*
2. *For every n, ${}^{\leq n}\mathcal{X} := \mathcal{X}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ is a ${}^{\leq n}$*DG *indscheme.*

We shall denote the full subcategory of PreStk spanned by DG indschemes by DGindSch.

1.4.3. We will prove the following (see also Proposition 1.6.4 below for a more precise assertion):

Proposition 1.4.4. *Any DG indscheme $\mathcal{X}$ can be presented as a filtered colimit in* PreStk

$$\underset{\alpha}{colim}\, X_\alpha, \tag{1.5}$$

where $X_\alpha \in \mathrm{DGSch}_{\text{qsep-qc}}$ and for $\alpha_1 \to \alpha_2$, the corresponding map $i_{\alpha_1,\alpha_2} : X_{\alpha_1} \to X_{\alpha_2}$ is a closed embedding.

The above proposition allows us to give the following, in a sense, more straightforward, definition of DG indschemes:

Corollary 1.4.5. *An object* $\mathcal{X} \in \operatorname{PreStk}$ *is a DG indscheme if and only if:*

- *It is convergent;*
- *As an object of* PreStk *it admits a presentation as in* (1.5)*.*

1.4.6. Note that unlike the case of ${}^{\leq n}$DG indschemes, an object of PreStk written as in (1.5) need *not* be a DG indscheme. Indeed, it can fail to be convergent.

However, such a colimit gives rise to a DG indscheme via the following lemma:

Lemma 1.4.7. *For* $\mathcal{X} \in \operatorname{PreStk}$ *given as in* (1.5)*, the object*

$$ {}^{\operatorname{conv}}\mathcal{X} \in \operatorname{PreStk} $$

belongs to DGindSch.

Proof. Indeed, ${}^{\operatorname{conv}}\mathcal{X}$ is convergent by definition, and for any n, we have ${}^{\leq n}({}^{\operatorname{conv}}\mathcal{X}) \simeq {}^{\leq n}\mathcal{X}$. □

1.4.8. If $\mathcal{X}$ is a DG indscheme, then

$$ {}^{\leq n}\mathcal{X} := \mathcal{X}|_{{}^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}} $$

is a ${}^{\leq n}$DG indscheme. In particular, ${}^{cl}\mathcal{X}$ is a classical indscheme. Thus, we obtain a functor

$$ {}^{\leq n}\operatorname{DGindSch} \leftarrow \operatorname{DGindSch}\,. \tag{1.6} $$

Vice versa, if $\mathcal{X}_n$ is a ${}^{\leq n}$DG indscheme, set

$$ \mathcal{X} := {}^{\operatorname{conv},L}\operatorname{LKE}_{({}^{\leq n}\operatorname{DGSch}^{\operatorname{aff}})^{\operatorname{op}} \hookrightarrow (\operatorname{DGSch}^{\operatorname{aff}})^{\operatorname{op}}}(\mathcal{X}_n). $$

Explicitly, if $\mathcal{X}_n$ is given by the colimit as in (1.1), then $\mathcal{X}$ is the convergent completion of the same colimit taken in PreStk, but where X_α are understood as objects of DGSch, see notational convention in Sect. 0.5.4. By Lemma 1.4.7, we obtain that $\mathcal{X}$ is a DG indscheme.

This defines a functor

$$ {}^{\leq n}\operatorname{DGindSch} \to \operatorname{DGindSch}, \tag{1.7} $$

which is left adjoint to the one in (1.6). It is easy to see that the unit map defines an isomorphism from the identity functor to

$$ {}^{\leq n}\operatorname{DGindSch} \to \operatorname{DGindSch} \to {}^{\leq n}\operatorname{DGindSch}\,. $$

I.e., the functor in (1.7) is fully faithful.

1.4.9. In what follows, we shall say that a DG indscheme is *weakly* n-coconnective if it is such as an object of Stk, see Sect. 0.5.6, i.e., if it belongs to the essential image of the functor (1.7).

Thus, the above functor establishes an equivalence between ${}^{\leq n}$DGindSch and the full subcategory of DGindSch spanned by *weakly* n-coconnective DG schemes. In particular, it identifies classical indschemes with weakly 0-coconnective DG indschemes.

We shall say that $\mathcal{X}$ is weakly eventually coconnective if it is weakly n-coconnective for some n.

1.4.10. We shall say that a DG indscheme is n-coconnective if it is n-coconnective as an object of Stk, i.e., if it lies in the essential image of the functor

$$ {}^{L}\operatorname{LKE}_{({}^{\leq n}\operatorname{DGSch}^{\operatorname{aff}})^{\operatorname{op}} \hookrightarrow (\operatorname{DGSch}^{\operatorname{aff}})^{\operatorname{op}}} : {}^{\leq n}\operatorname{Stk} \to \operatorname{Stk}\,. \tag{1.8} $$

We shall say that $\mathcal{X}$ is eventually coconnective if it is n-coconnective for some n.

1.4.11. *The* $\aleph_0$ *condition.* We shall say that $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ is $\aleph_0$ if there exists a presentation as in (1.1) with the category of indices equivalent to the poset $\mathbb{N}$.

We shall say that $\mathcal{X} \in \mathrm{DGindSch}$ is $\aleph_0$ if for it admits a presentation as in Proposition 1.4.4, with the category of indices equivalent to the poset $\mathbb{N}$.

We shall say that $\mathcal{X} \in \mathrm{DGindSch}$ is *weakly* $\aleph_0$ if for every n, the object

$$^{\leq n}\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$$

is $\aleph_0$.

1.5. **Basic properties of DG indschemes.**

1.5.1. We claim:

Proposition 1.5.2. *Every* $\mathcal{X} \in \mathrm{DGindSch}$ *belongs to* Stk*, i.e., satisfies fppf descent.*

Proof. Let $S' \to S$ be an fppf map in $\mathrm{DGSch}^{\mathrm{aff}}$, and let $S'^{\bullet}/S$ be its Čech nerve. We need to show that the map

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Tot}(\mathrm{Maps}(S'^{\bullet}/S, \mathcal{X}))$$

is an isomorphism.

For an integer n, we consider the truncation ${}^{\leq n}S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ of S. Note that since $S' \to S$ is flat, the map ${}^{\leq n}S' \to {}^{\leq n}S$ is flat, and the simplicial object ${}^{\leq n}(S'^{\bullet}/S)$ of ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ is the Čech nerve of ${}^{\leq n}S' \to {}^{\leq n}S$.

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Maps}(S, \mathcal{X}) & \longrightarrow & \mathrm{Tot}(\mathrm{Maps}(S'^{\bullet}/S, \mathcal{X})) \\ \downarrow & & \downarrow \\ \underset{n \in \mathbb{N}^{\mathrm{op}}}{\mathit{lim}}\ \mathrm{Maps}({}^{\leq n}S, {}^{\leq n}\mathcal{X}) & \longrightarrow & \underset{n \in \mathbb{N}^{\mathrm{op}}}{\mathit{lim}}\ \mathrm{Tot}(\mathrm{Maps}({}^{\leq n}(S'^{\bullet}/S), {}^{\leq n}\mathcal{X})) \end{array}.$$

In this diagram the vertical arrows are isomorphisms, since $\mathcal{X}$ is convergent. The bottom horizontal arrow is an isomorphism by Proposition 1.3.2. Hence, the top horizontal arrow is an isomorphism as well, as desired.

□

1.5.3. As in Sect. 1.3.5 we consider maps into a DG indscheme $\mathcal{X}$ from an arbitrary DG scheme Y, and we have the following analog of Lemma 1.3.6 (with the same proof, but relying on Proposition 1.4.4):

Lemma 1.5.4. *For* $\mathcal{X} \in \mathrm{Stk}$ *written as in* (1.5)*, and* $Y \in \mathrm{DGSch}$*, the natural map*

$$\underset{\alpha}{\mathit{colim}}\ \mathrm{Maps}(Y, X_\alpha) \to \mathrm{Maps}(Y, \mathcal{X})$$

is an isomorphism, provided that Y *is quasi-separated and quasi-compact.*

1.6. **The canonical presentation of a DG indscheme.** We shall now formulate a sharper version of Proposition 1.4.4, which will be proved in Sect. 3.

1.6.1. We give the following definition:

Definition 1.6.2. *A map* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *in* PreStk *is said to be a* closed embedding *if the corresponding map* ${}^{cl}\mathcal{Y}_1 \to {}^{cl}\mathcal{Y}_2$ *is a closed embedding (i.e., its base change by an affine scheme yields a closed embedding).*

Note that in the DG setting, being a closed embedding does not imply that a map is schematic[12]. Indeed, a closed embedding of a DG scheme into a DG indscheme is typically not schematic.

It is easy to see that for maps $\mathcal{Y}_1 \to \mathcal{Y}_2 \to \mathcal{Y}_3$ with $\mathcal{Y}_2 \to \mathcal{Y}_3$ being a closed embedding, the map $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a closed embedding if and only if $\mathcal{Y}_1 \to \mathcal{Y}_3$ is.

1.6.3. For a DG indscheme $\mathcal{X}$, let

$$(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \subset (\mathrm{DGSch}_{\text{qsep-qc}})_{/\mathcal{X}}$$

be the full subcategory, consisting of those objects for which the map $Z \to \mathcal{X}$ is a closed embedding in the above sense.

In Sects. 3.2 and 3.5 we will prove:

Proposition 1.6.4. *Let* $\mathcal{X}$ *be a DG scheme.*

(a) *The category* $(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}}$ *is filtered.*

(b) *The natural map*

$$\underset{Z\in(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}}}{colim} Z \to \mathcal{X}, \tag{1.9}$$

where the colimit is taken in PreStk*, is an isomorphism.*

1.6.5. Combined with Lemma 1.5.4, we obtain the following:

Corollary 1.6.6. *Let* $\mathcal{X}$ *be a DG indscheme. The functor*

$$(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \to (\mathrm{DGSch}_{\text{qsep-qc}})_{/\mathcal{X}} \tag{1.10}$$

is cofinal.

Proof. We need to show that for $X \in \mathrm{DGSch}_{\text{qsep-qc}}$ and a map $X \to \mathcal{X}$, the category of its factorizations

$$X \to Z \to \mathcal{X},$$

where $Z \to \mathcal{X}$ is a closed embedding, is contractible. However, the above category of factorizations is the fiber of the map of spaces

$$\underset{Z\in(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}}}{colim} \mathrm{Maps}(X, Z) \to \mathrm{Maps}(X, \mathcal{X})$$

over our given point in $\mathrm{Maps}(X, \mathcal{X})$. □

Finally, we can give the following characterization of DG indschemes among PreStk:

Corollary 1.6.7. *An object* $\mathcal{X} \in {}^{\mathrm{conv}}\mathrm{PreStk}$ *is a DG indscheme if and only if:*

- *The category of closed embeddings* $Z \to \mathcal{X}$*, where* $Z \in \mathrm{DGSch}_{\text{qsep-qc}}$*, is filtered.*
- *The functor* (1.10) *is cofinal.*

---

[12] We recall that a map of prestacks is called *schematic* if its base change by an affine DG scheme yields an affine DG scheme.

1.6.8. Let us also note that Lemma 1.5.4 implies that for any presentation of a DG indscheme as in Proposition 1.4.4, the tautological map

$$\mathsf{A} \to (\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}}$$

is cofinal.

1.7. **The locally almost of finite type condition.**

1.7.1. We shall say that $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ is locally of finite type if it is such as an object of ${}^{\leq n}\mathrm{PreStk}$ (see [GL:Stacks, Sect. 1.3.2]), i.e., it belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ in the terminology of *loc. cit.*

By definition, this means that $\mathcal{X}$, viewed as a functor

$$({}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd},$$

equals the left Kan extension under

$$({}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow ({}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}$$

of its own restriction to $({}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}$, where ${}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \subset {}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}}$ denotes the full subcategory of n-coconnective affine DG schemes *of finite type.* [13]

We shall denote the full subcategory of ${}^{\leq n}\mathrm{DGindSch}$ spanned by ${}^{\leq n}$DG indschemes locally of finite type by ${}^{\leq n}\mathrm{DGindSch}_{\mathrm{lft}}$.

We shall say that $\mathcal{X} \in \mathrm{DGindSch}$ is locally almost of finite type if it is such as an object of PreStk, see [GL:Stacks, Sect. 1.3.9], i.e., if in the notation of *loc.cit.* it belongs to the subcategory $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$. By definition, this means that

$${}^{\leq n}\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$$

must be locally of finite type for every n. We shall denote the full subcategory of DGindSch spanned by DG indschemes locally almost of finite type by $\mathrm{DGindSch}_{\mathrm{laft}}$.

1.7.2. It is natural to wonder whether one can represent objects of $\mathrm{DGindSch}_{\mathrm{laft}}$ as colimits of objects of $\mathrm{DGSch}_{\mathrm{aft}}$ under closed embeddings. (We denote by $\mathrm{DGSch}_{\mathrm{aft}}$ the category of DG schemes almost of finite type, i.e., $\mathrm{DGSch}_{\mathrm{aft}} := \mathrm{DGSch}_{\mathrm{laft}} \cap \mathrm{DGSch}_{\mathrm{qc}}$, see [GL:Stacks, Sect. 3.3.1].)

In fact, there are two senses in which one can ask this question: one may want to have a presentation in a "weak sense", i.e., as in Lemma 1.4.7, or in the "strong" sense, i.e., as in Proposition 1.4.4.

The answer to the "weak" version is affirmative: we will prove the following:

Proposition 1.7.3. *For a DG indscheme $\mathcal{X}$ locally almost of finite type there exists a filtered family*

$$A \to (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} : \alpha \mapsto X_\alpha,$$

such that $\mathcal{X}$ is isomorphic to the convergent completion of

$$\underset{\alpha \in \mathsf{A}}{colim}\, X_\alpha, \tag{1.11}$$

where the colimit is taken in PreStk.

---

[13] We remind that $\mathrm{Spec}(A) \in {}^{\leq n}\,\mathrm{DGSch}^{\mathrm{aff}}$ is said to be *of finite type* if $H^0(A)$ is a finitely generated algebra over k, and each $H^i(A)$ is finitely generated as an $H^0(A)$-module.

1.7.4. Before we answer the "strong question", let us note that it is *not* true that for any $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, the functor

$$(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}} \to (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal. However, if $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$ admitted a presentation as a colimit of objects of $\mathrm{DGSch}_{\mathrm{aft}}$, it would automatically have this property. We have the following general result that will appear in [GR]:

Theorem 1.7.5. *Suppose that* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ admits deformation theory. [14] *Then, the functor*

$$(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}} \to (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal.

This theorem makes it less suprising that that the answer to the "strong" question is also affirmative:

Proposition 1.7.6. *For a DG indscheme* $\mathcal{X}$ *locally almost of finite type there exists a filtered family*

$$A \to (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} : \alpha \mapsto X_\alpha,$$

such that $\mathcal{X}$ *is isomorphic to*

$$\underset{\alpha \in \mathsf{A}}{colim}\, X_\alpha, \tag{1.12}$$

where the colimit is taken in PreStk*.*

1.7.7. In fact, we shall prove a more precise version of the above assertions. Namely, in Sect. 3.5 we will prove:

Proposition 1.7.8. *Let* $\mathcal{X}$ *be an object of* $\mathrm{DGindSch}_{\mathrm{laft}}$*.*

(a) *The category* $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed\ in\ }\mathcal{X}}$ *is filtered.*

(b) *The natural map*

$$\underset{Z \in (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed\ in\ }\mathcal{X}}}{colim} Z \to \mathcal{X},$$

where the colimit is taken in PreStk*, is an isomorphism.*

As a formal consequence, we obtain:

Corollary 1.7.9. *For* $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$ *the following functors are cofinal:*

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed\ in\ }\mathcal{X}} \to (\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{/\mathcal{X}} \tag{1.13}$$

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed\ in\ }\mathcal{X}} \to (\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{\mathrm{closed\ in\ }\mathcal{X}} \tag{1.14}$$

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed\ in\ }\mathcal{X}} \to (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}} \tag{1.15}$$

and

$$({}^{<\infty}\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed\ in\ }\mathcal{X}} \to ({}^{<\infty}\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}. \tag{1.16}$$

[14] A particular case of this notion, namely, what it means for $\mathcal{X}$ to admit connective deformation theory, is reviewed in Sect. 4.

Corollary 1.7.10. *An object* $\mathcal{X} \in \text{DGindSch}_{\text{laft}} \subset \text{PreStk}$ *lies in the essential image of the fully faithful functor*

$$\text{LKE}_{(\text{DGSch}^{\text{aff}}_{\text{aft}})^{\text{op}} \hookrightarrow (\text{DGSch}^{\text{aff}})^{\text{op}}} : \text{Funct}((\text{DGSch}^{\text{aff}}_{\text{aft}})^{\text{op}}, \infty\text{-Grpd}) \to \\ \to \text{Funct}((\text{DGSch}^{\text{aff}})^{\text{op}}, \infty\text{-Grpd}) = \text{PreStk}\,.$$

Equivalently, the functor

$$(\text{DGSch}^{\text{aff}}_{\text{aft}})_{/\mathcal{X}} \to (\text{DGSch}^{\text{aff}})_{/\mathcal{X}}$$

is cofinal.

Corollary 1.7.11. *An object* $\mathcal{X} \in {}^{\text{conv}}\text{PreStk}$ *belongs to* $\text{DGindSch}_{\text{laft}}$ *if and only if:*

- *The category of closed embeddings* $Z \to \mathcal{X}$*, where* $Z \in \text{DGSch}_{\text{aft}}$*, is filtered.*
- *The functor* (1.13) *is cofinal.*

1.7.12. Note that Lemma 1.5.4 implies that for any presentation of $\mathcal{X}$ as in Proposition 1.7.6, the tautological map

$$\mathsf{A} \to (\text{DGSch}_{\text{aft}})_{\text{closed in } \mathcal{X}}$$

is cofinal.

2. Sheaves on DG indschemes

2.1. **Quasi-coherent sheaves on a DG indscheme.**

2.1.1. For any $\mathcal{Y} \in \text{PreStk}$, we have the symmetric monoidal category $\text{QCoh}(\mathcal{Y})$ defined as in [GL:QCoh, Sect. 1.1.3]. Explicitly,

$$\text{QCoh}(\mathcal{Y}) := \underset{S \in ((\text{DGSch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}}}{\mathit{lim}} \text{QCoh}(S).$$

2.1.2. In particular, for $\mathcal{X} \in \text{DGindSch}$ we obtain the symmetric monoidal category $\text{QCoh}(\mathcal{X})$.

If $\mathcal{X} \in \text{DGindSch}$ is written as (1.5), we have:

$$\text{QCoh}(\mathcal{X}) \simeq \underset{\alpha}{\mathit{lim}}\, \text{QCoh}(X_\alpha),$$

where for $\alpha_2 \geq \alpha_1$, the map $\text{QCoh}(X_{\alpha_2}) \to \text{QCoh}(X_{\alpha_1})$ is $i^*_{\alpha_1,\alpha_2}$. This follows from the fact that the functor

$$\text{QCoh}_{\text{PreStk}} : \text{PreStk}^{\text{op}} \to \text{DGCat}_{\text{cont}}$$

takes colimits in PreStk to limits in $\text{DGCat}_{\text{cont}}$.

Since the category $\text{QCoh}(\mathcal{X})$ is given as a limit, it is not at all guaranteed that it will be compactly generated.

2.1.3. We have the following nice feature of the category QCoh on DG indschemes that are locally almost of finite type. Namely, we "only need to know" QCoh on affine DG schemes that are almost of finite type to recover it. More precisely, from Corollary 1.7.10, we obtain:

Corollary 2.1.4. *For* $\mathcal{X} \in \text{DGindSch}_{\text{laft}}$*, the functor*

$$\text{QCoh}(\mathcal{X}) = \underset{S \in ((\text{DGSch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}}}{\mathit{lim}} \text{QCoh}(S) \to \underset{S \in ((\text{DGSch}^{\text{aff}}_{\text{aft}})_{/\mathcal{Y}})^{\text{op}}}{\mathit{lim}} \text{QCoh}(S),$$

given by restriction, is an equivalence.

2.2. **A digression: perfect objects in** QCoh**.**

2.2.1. Recall the notion of a perfect object in $\mathrm{QCoh}(\mathcal{Y})$ for $\mathcal{Y} \in \mathrm{PreStk}$, see, e.g., [GL:QCoh, Sect. 4.1.6].

The subcategory $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ coincides with that of dualizable objects of $\mathrm{QCoh}(\mathcal{Y})$ (see, e.g., [GL:QCoh, Lemma 4.2.2]).

2.2.2. Recall also that if $\mathcal{Y} = X$ is a quasi-separated and quasi-compact scheme, then the category $\mathrm{QCoh}(X)$ is compactly generated and $\mathrm{QCoh}(X)^c = \mathrm{QCoh}(X)^{\mathrm{perf}}$. Moreover, we have the canonical self-duality equivalence

$$\mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X)$$

which can be described in either of the following two equivalent ways:

- The corresponding [15] equivalence $\mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X)} : (\mathrm{QCoh}(X)^c)^{\mathrm{op}} \simeq \mathrm{QCoh}(X)^c$ is the duality functor with respect to the symmetric monoidal structure on $\mathrm{QCoh}(X)$:

$$\mathcal{F} \mapsto \mathcal{F}^\vee : (\mathrm{QCoh}(X)^{\mathrm{perf}})^{\mathrm{op}} \to \mathrm{QCoh}(X)^{\mathrm{perf}}.$$

- The pairing $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{Vect}$ is the composition

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \overset{\otimes}{\to} \mathrm{QCoh}(X) \overset{\Gamma(X,-)}{\longrightarrow} \mathrm{Vect}\,.$$

2.2.3. Note that for an object $\mathcal{Y} \in \mathrm{PreStk}$ (and, in particular, for $\mathcal{X} \in \mathrm{DGindSch}$), the functor $\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$ is not, in general, continuous. Therefore, the functor

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \overset{\otimes}{\to} \mathrm{QCoh}(\mathcal{Y}) \overset{\Gamma(\mathcal{Y},-)}{\longrightarrow} \mathrm{Vect}$$

is not continuous either, and as such cannot serve as a candidate the duality paring.

2.2.4. Let $\mathcal{Y}$ be an arbitrary object of PreStk. We shall say that $\mathcal{Y}$ is quasi-perfect if

(i) The category $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

(ii) The compact objects of $\mathrm{QCoh}(\mathcal{Y})$ are *perfect*, and the duality functor

$$\mathcal{F} \mapsto \mathcal{F}^\vee : (\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}})^{\mathrm{op}} \simeq \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}} \tag{2.1}$$

sends $(\mathrm{QCoh}(\mathcal{Y})^c)^{\mathrm{op}}$ to $\mathrm{QCoh}(\mathcal{Y})^c$.

Note that for $\mathcal{Y}$ quasi-perfect, there exists a canonical equivalence

$$\mathbf{D}_{\mathcal{Y}}^{\mathrm{naive}} : \mathrm{QCoh}(\mathcal{Y})^\vee \simeq \mathrm{QCoh}(\mathcal{Y}),$$

given by the equivalence

$$\mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(\mathcal{Y})} : (\mathrm{QCoh}(\mathcal{Y})^c)^{\mathrm{op}} \simeq \mathrm{QCoh}(\mathcal{Y})^c$$

induced by the duality functor (2.1).

The corresponding pairing $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$ can be described as follows: it is obtained by ind-extending the pairing on compact objects given by

$$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y})^c \mapsto \Gamma(\mathcal{X}, \mathcal{F}_1 \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} \mathcal{F}_2) \in \mathrm{Vect}\,.$$

Indeed, this follows from the fact that for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ and $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{X})$, we have a functorial isomorphism

$$\mathcal{M}aps(\mathcal{F}^\vee, \mathcal{F}') \simeq \Gamma(\mathcal{X}, \mathcal{F} \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{F}').$$

Furthermore, note that $\mathrm{QCoh}(\mathcal{X})^c$ is a monoidal ideal in $\mathrm{QCoh}(X)^{\mathrm{perf}}$.

[15] We recall that for a compactly generated category $\mathbf{C}$ we have a canonical equivalence $(\mathbf{C}^\vee)^c \simeq (\mathbf{C}^c)^{\mathrm{op}}$.

2.2.5. We shall see that certain DG indschemes are quasi-perfect in the above sense (see Sect. 7.2 and Sect. 10.3.1).

2.3. **Ind-coherent sheaves on a DG indscheme.**

2.3.1. Let $\mathcal{Y}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$. Following [GL:IndCoh, Sect. 10.1], we define the category $\mathrm{IndCoh}(\mathcal{Y})$, which is a module category over $\mathrm{QCoh}(\mathcal{Y})$ (see [GL:IndCoh, Sect. 10.3] for the latter piece of structure).

Explicitly,

$$\mathrm{IndCoh}(\mathcal{Y}) = \underset{S\in(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S),$$

where for $(f : S_1 \to S_2) \in ({}^{\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}}$, the functor $\mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$ is $f^!$.

It follows from [GL:IndCoh, Corollaries 10.2.2 and 10.5.5] that in the following commutative diagram all arrows are equivalences:

$$\begin{array}{ccc}
\underset{S\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S) & \longrightarrow & \underset{S\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S) \\
\downarrow & & \downarrow \\
\underset{S\in(({}^{<\infty}\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S) & \longrightarrow & \underset{S\in(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S) =: \mathrm{IndCoh}(\mathcal{Y}).
\end{array}$$

The following is immediate from the definitions:

Lemma 2.3.2. *The functor*

$$\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

takes colimits in $\mathrm{PreStk}_{\mathrm{laft}}$ *to limits in* $\mathrm{DGCat}_{\mathrm{cont}}$.

2.3.3. Let us denote by $\mathrm{IndCoh}^!_{\mathrm{DGindSch}_{\mathrm{laft}}}$ the functor

$$(\mathrm{DGindSch}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained from $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ by restriction along the fully faithful embedding

$$\mathrm{DGindSch}_{\mathrm{laft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}.$$

Thus, for every $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$, we have a well-defined DG category $\mathrm{IndCoh}(\mathcal{X})$, which is a module for $\mathrm{QCoh}(\mathcal{X})$.

We have:

Lemma 2.3.4. *Let* $\mathcal{X} \in \mathrm{DGindSch}$ *be written as in* (1.11)*. Then the natural map*

$$\mathrm{IndCoh}(\mathcal{X}) \to \underset{\alpha}{lim}\, \mathrm{IndCoh}(X_\alpha),$$

is an equivalence.

Proof. Follows from Lemma 2.3.2. □

Remark 2.3.5. We present results using the presentation of a DG indscheme as in (1.11) rather than in (1.12), because many DG schemes that occur in practice come in this form. The possibility of presenting them as in (1.12) is the result of Proposition 1.7.6 and is seldom explicit.

2.4. **Interpretation of** IndCoh **as a colimit and compact generation.**

2.4.1. One of the main advantages of the category $\mathrm{IndCoh}(\mathcal{X})$ over $\mathrm{QCoh}(\mathcal{X})$ for a DG indscheme $\mathcal{X}$ is that the former admits an alternative description as a colimit.

2.4.2. Indeed, recall that for a closed embedding of DG schemes $i : X_1 \to X_2$, the functor

$$i^! : \operatorname{IndCoh}(X_2) \to \operatorname{IndCoh}(X_1)$$

admits a left adjoint, $i_*^{\operatorname{IndCoh}}$, see [GL:IndCoh, Sect. 3.3].

By Lemma 2.3.4 and [GL:DG, Lemma. 1.3.3], we have that for $\mathcal{X}$ as in (1.11),

$$\operatorname{IndCoh}(\mathcal{X}) \simeq \underset{\alpha}{colim}\, \operatorname{IndCoh}(X_\alpha), \tag{2.2}$$

where for $\alpha_2 \geq \alpha_1$, the map $\operatorname{IndCoh}(X_{\alpha_2}) \to \operatorname{IndCoh}(X_{\alpha_1})$ is $(i_{\alpha_1,\alpha_2})_*^{\operatorname{IndCoh}}$.

2.4.3. For $\mathcal{X} \in \operatorname{DGindSch}_{\operatorname{laft}}$, we let $\operatorname{Coh}(\mathcal{X})$ denote the full subcategory of $\operatorname{IndCoh}(\mathcal{X})$ spanned by objects

$$i_*^{\operatorname{IndCoh}}(\mathcal{F}), \quad i : X \to \mathcal{X} \text{ is a closed embedding and } \mathcal{F} \in \operatorname{Coh}(X).$$

By [GL:DG, Sect. 2.2.1], we obtain:

Corollary 2.4.4. *For* $\mathcal{X} \in \operatorname{DGindSch}$*, the category* $\operatorname{IndCoh}(\mathcal{X})$ *is compactly generated by* $\operatorname{Coh}(\mathcal{X})$*.*

2.4.5. We are going to prove:

Proposition 2.4.6.

(a) $\operatorname{Coh}(\mathcal{X})$ *is a (non-cocomplete) DG subcategory of* $\operatorname{IndCoh}(\mathcal{X})$*.*

(b) *The natural functor* $\operatorname{Ind}(\operatorname{Coh}(\mathcal{X})) \to \operatorname{IndCoh}(\mathcal{X}))$ *is an equivalence.*

(c) *Every compact object of* $\operatorname{IndCoh}(\mathcal{X})$ *can be realized as a direct summand of an object of* $\operatorname{Coh}(\mathcal{X})$*.*

2.4.7. For the proof of the above proposition, we will need the following observation:

Let

$$X' \overset{i'}{\to} \mathcal{X} \overset{i''}{\leftarrow} X''$$

be closed embeddings.

We would like to calculate the composition

$$(i')^! \circ (i'')_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X'') \to \operatorname{IndCoh}(X').$$

Let A denote the category $(\operatorname{DGSch}_{\operatorname{aft}})_{\text{closed in } \mathcal{X}}$, so that X' and X'' correspond to indices α and α', respectively. Let B be any category cofinal in

$$A_{\alpha \sqcup \alpha'/} := A_{\alpha/} \underset{A}{\times} A_{\alpha'/}.$$

For $\beta \in B$, let

$$X' = X_\alpha \overset{i_{\alpha,\beta}}{\longrightarrow} X_\beta \overset{i_{\alpha',\beta}}{\longleftarrow} X_{\alpha'} = X''$$

denote the corresponding maps.

The next assertion follows from [GL:DG, Sect. 1.3.5]:

Lemma 2.4.8. *Under the above circumstances, we have a canonican isomorphism*

$$(i')^! \circ (i'')_*^{\operatorname{IndCoh}} \simeq \underset{b \in B}{colim}\, (i_{\alpha,\beta})^! \circ (i_{\alpha',\beta})_*^{\operatorname{IndCoh}}.$$

2.4.9. *Proof of Proposition 2.4.6.* To prove point (a), we only need to show that the category $\operatorname{Coh}(\mathcal{X})$ is preserved by taking cones. I.e., we have to show that in the situation of Sect. 2.4.7, for

$$\mathcal{F}' \in \operatorname{Coh}(X'),\ \mathcal{F}'' \in \operatorname{Coh}(X'')$$

and a map

$$(i')_*^{\operatorname{IndCoh}}(\mathcal{F}') \to (i'')_*^{\operatorname{IndCoh}}(\mathcal{F}'') \in \operatorname{IndCoh}(\mathcal{X}),$$

this map can be realized coming from a map

$$(i_{a',b})_*^{\operatorname{IndCoh}}(\mathcal{F}') \to (i_{a'',b})_*^{\operatorname{IndCoh}}(\mathcal{F}'') \in \operatorname{IndCoh}(X_b)$$

for some $b \in A_{a' \sqcup a''/}$. However, this readily follows from Lemma 2.4.8.

Point (b) follows from point (a) combined with Corollary 2.4.4. Point (c) follows from point (b).

□

2.5. **The t-structure on IndCoh.**

2.5.1. For $\mathcal{X} \in \operatorname{DGindSch}_{\mathrm{laft}}$, we define a t-structure on $\operatorname{IndCoh}(\mathcal{X})$ as follows. An object

$$\mathcal{F} \in \operatorname{IndCoh}(\mathcal{X})$$

belongs to $\operatorname{IndCoh}^{\geq 0}$ if and only if for every closed embedding $i : X \to \mathcal{X}$ with $X \in \operatorname{DGSch}_{\mathrm{aft}}$, the object $i^!(\mathcal{F}) \in \operatorname{IndCoh}(X)$ belongs to $\operatorname{IndCoh}(X)^{\geq 0}$.

By construction, this t-structure is compatible with filtered colimits, i.e., $\operatorname{IndCoh}(\mathcal{X})^{\geq 0}$ is preserved by filtered colimits.

2.5.2. We can describe this t-structure and the category $\operatorname{IndCoh}(\mathcal{X})^{\leq 0}$ more explicitly. Fix a presentation of $\mathcal{X}$ as in (1.11). For each α, let i_α denote the corresponding map $X_\alpha \to \mathcal{X}$. By (2.2), we have a pair of adjoint functors

$$(i_\alpha)_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X_\alpha) \rightleftarrows \operatorname{IndCoh}(\mathcal{X}) : i_\alpha^!.$$

Lemma 2.5.3. *Under the above circumstances we have:*

(a) *An object* $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{X})$ *belongs to* $\operatorname{IndCoh}^{\geq 0}$ *if and only if for every* α*, the object* $i_\alpha^!(\mathcal{F}) \in \operatorname{IndCoh}(X_\alpha)$ *belongs to* $\operatorname{IndCoh}(X_\alpha)^{\geq 0}$.

(b) *The category* $\operatorname{IndCoh}(\mathcal{X})^{\leq 0}$ *is generated under colimits by the essential images of the functors* $(i_\alpha)_*^{\operatorname{IndCoh}}\left(\operatorname{Coh}(X_\alpha)^{\leq 0}\right)$.

Proof. It is easy to see that for a quasi-compact DG scheme X, the category $\operatorname{IndCoh}(X)^{\leq 0}$ is generated under colimits by $\operatorname{Coh}({}^{cl}X)^{\leq 0}$. In particular, by adjunction, an object $\mathcal{F} \in \operatorname{IndCoh}(X)$ is coconnective if and only if its restriction to ${}^{cl}X$ is coconnective.

Hence, in the definition of $\operatorname{IndCoh}(\mathcal{X})^{\geq 0}$, instead of all closed embeddings $X \to \mathcal{X}$, it suffices to consider only those with X a classical scheme.

This implies point (a) of the lemma by Lemma 1.3.6. Point (b) follows formally from point (a).

□

2.5.4. Suppose $i : X \to \mathcal{X}$ is a closed embedding of a DG scheme into a DG indscheme. We then have:

Lemma 2.5.5. *The functor i_*^{IndCoh} is t-exact.*

Proof. Since i_*^{IndCoh} is the left adjoint of $i^!$, it is right t-exact. Thus we need to show that for $\mathcal{F} \in \mathrm{IndCoh}(X)^{\geq 0}$, we have $i_\alpha^! \circ i_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(X_\alpha)^{\geq 0}$ for every closed embedding $i_\alpha : X_\alpha \to \mathcal{X}$. However, this follows from Lemma 2.4.8. □

2.5.6. Recall the full (but not cocomplete) subcategory $\mathrm{Coh}(\mathcal{X}) \subset \mathrm{IndCoh}(\mathcal{X})$, see Sect. 2.4.3 above. From Lemma 2.5.5 we obtain:

Corollary 2.5.7. *The full subcategories*

$$\mathrm{Coh}(\mathcal{X}) \subset \mathrm{IndCoh}(\mathcal{X})^c \subset \mathrm{IndCoh}(\mathcal{X})$$

are preserved by the truncation functors.

Thus, taking into account Proposition 2.4.6, we obtain that the t-structure on $\mathrm{IndCoh}(\mathcal{X})$ can also be described as the ind-extension of the t-structure on $\mathrm{Coh}(\mathcal{X})$:

Corollary 2.5.8. *The category $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ is generated under filtered colimits by $\mathrm{Coh}(\mathcal{X})^{\geq 0}$.*

2.6. **Serre duality on DG indschemes.** We shall now show that the category $\mathrm{IndCoh}(\mathcal{X})$ is canonically self-dual, i.e. there exists a canonical equivalence

$$\mathbf{D}_{\mathcal{X}}^{\mathrm{Serre}} : \mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X}). \tag{2.3}$$

2.6.1. Let us write $\mathcal{X}$ as in (1.11). Combining (2.2) with [GL:DG, Lemma 2.2.2] and [GL:IndCoh, Sect. 9.2.3], we obtain:

Corollary 2.6.2. *Serre duality defines a canonical equivalence:*

$$\mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X}).$$

Note that by Sect. 1.7.12, any other way of writing $\mathcal{X}$ as in (1.12) will give rise to a canonically isomorphic duality functor.

2.6.3. Let us describe the equivalence of Corollary 2.6.2 more explicitly. Namely, we would like to describe the corresponding pairing:

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Vect}\,. \tag{2.4}$$

2.6.4. For a DG scheme X almost of finite type, let

$$\Gamma^{\mathrm{IndCoh}}(X, -) : \mathrm{IndCoh}(X) \to \mathrm{Vect}$$

denote the functor $(p_X)_*^{\mathrm{IndCoh}}$ of [GL:IndCoh], Proposition 3.1.1, where $p_X : X \to \mathrm{pt}$.

For a DG indscheme $\mathcal{X}$, written as in (1.11), we define the functor

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -) : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Vect}$$

to be given by the compatible family of functors $\Gamma^{\mathrm{IndCoh}}(X_\alpha, -) : \mathrm{IndCoh}(X_\alpha) \to \mathrm{Vect}$.

Again, by Sect. 1.7.12, the above definition of $\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -)$ is canonically independent of the choice of the presentation (1.11).

2.6.5. The definition of the functor $\mathbf{D}_{\mathcal{X}}^{\mathrm{Serre}}$ in (2.3) and [GL:IndCoh, Sect. 9.2.2] imply:

Corollary 2.6.6. *The functor* (2.4) *is canonically isomorphic to the composite*

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\boxtimes} \mathrm{IndCoh}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\Delta_{\mathcal{X}}^!} \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -)} \mathrm{Vect}\,.$$

2.7. **Functoriality of** IndCoh **under pushforwards.**

2.7.1. Recall the functor $\operatorname{IndCoh}_{\operatorname{DGSch}_{\operatorname{aft}}} : \operatorname{DGSch}_{\operatorname{aft}} \to \operatorname{DGCat}_{\operatorname{cont}}$ of [GL:IndCoh, Sect. 3.2], which assigns to $X \in \operatorname{DGSch}_{\operatorname{aft}}$ the category $\operatorname{IndCoh}(X)$ and to a map $f : X_1 \to X_2$ the functor

$$f_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X_1) \to \operatorname{IndCoh}(X_2).$$

Let

$$(\operatorname{DGSch}_{\operatorname{aft}})_{\operatorname{closed}} \subset (\operatorname{DGSch}_{\operatorname{aft}})_{\operatorname{proper}} \subset \operatorname{DGSch}_{\operatorname{aft}}$$

be the 1-full subcategories, where we restrict 1-morphisms to be closed embeddings (resp., proper). Let

$$\operatorname{IndCoh}_{(\operatorname{DGSch}_{\operatorname{aft}})_{\operatorname{closed}}} \text{ and } \operatorname{IndCoh}_{(\operatorname{DGSch}_{\operatorname{aft}})_{\operatorname{proper}}}$$

be the restriction of $\operatorname{IndCoh}_{\operatorname{DGSch}_{\operatorname{aft}}}$ to these subcategories.

2.7.2. We shall say that a map of classical indschemes $f : \mathcal{X}_1 \to \mathcal{X}_2$ is an *ind-closed embedding* (resp., *ind-proper*) if the following condition is satisfied:

Whenever $X_i \hookrightarrow \mathcal{X}_i$ are closed embeddings with $X_i \in \operatorname{Sch}_{\operatorname{ft}}$ such that there exists a commutative diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & \mathcal{X}_1 \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X_2 & \longrightarrow & \mathcal{X}_2, \end{array}$$

the map f' (which is automatically unique!), is a closed embedding (resp., proper).

Equivalently, one can reformulate this as follows: if

$$\mathcal{X}_1 := \underset{\alpha \in A}{colim}\, X_{1,\alpha} \text{ and } \mathcal{X}_2 := \underset{\beta \in A}{colim}\, X_{2,\beta},$$

then for every index α, and every/some index β for which $X_{1,\alpha} \to \mathcal{X}_1 \to \mathcal{X}_2$ factors as

$$X_{1,\alpha} \to X_{2,\beta} \to \mathcal{X}_2,$$

the map $X_{1,\alpha} \to X_{2,\beta}$ is a closed embedding (resp., proper).

It is easy to see that if $\mathcal{X}_1 = X_1 \in \operatorname{Sch}_{\text{qsep-qc}}$, then $f : X_1 \to \mathcal{X}_2$ is an ind-closed embedding if and only if it is a closed embedding.

Remark 2.7.3. Note that, in general, "closed embedding" is stronger than "ind-closed embedding." For instance,

$$\operatorname{Spf}(k[\![t]\!]) \to \operatorname{Spec}(k[t])$$

is an an "ind-closed emnedding", but not a closed embedding.

2.7.4. We shall say that a map of DG indschemes $f : \mathcal{X}_1 \to \mathcal{X}_2$ is an *ind-closed embedding* (resp., *ind-proper*) if the induced map of classical indschemes ${}^{cl}\mathcal{X}_1 \to {}^{cl}\mathcal{X}_2$ is an *ind-closed embedding* (resp., *ind-proper*).

Let

$$(\operatorname{DGindSch}_{\operatorname{laft}})_{\text{ind-closed}} \subset (\operatorname{DGindSch}_{\operatorname{laft}})_{\text{ind-proper}}$$

denote the corresponding 1-full subcategories of $\operatorname{DGindSch}_{\operatorname{laft}}$.

Let

$$\operatorname{IndCoh}_{(\operatorname{DGindSch}_{\operatorname{laft}})_{\text{ind-closed}}},\ \operatorname{IndCoh}_{(\operatorname{DGindSch}_{\operatorname{laft}})_{\text{ind-proper}}} \text{ and } \operatorname{IndCoh}_{\operatorname{DGindSch}_{\operatorname{laft}}} \tag{2.5}$$

denote the left Kan extensions of the functors

$$\operatorname{IndCoh}_{(\operatorname{DGSch}_{\operatorname{aft}})_{\operatorname{closed}}},\ \operatorname{IndCoh}_{(\operatorname{DGSch}_{\operatorname{aft}})_{\operatorname{proper}}} \text{ and } \operatorname{IndCoh}_{\operatorname{DGSch}_{\operatorname{aft}}}$$

along the fully faithful embeddings

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-closed}},\ (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}}$$

and

$$\mathrm{DGSch}_{\mathrm{aft}} \hookrightarrow \mathrm{DGindSch}_{\mathrm{laft}},$$

respectively.

From (2.2) and Sect. 1.7.12 we obtain:

Corollary 2.7.5. *For $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$, the value of the functor* $\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-closed}}}$ *on $\mathcal{X}$ is canonically equivalent to* $\mathrm{IndCoh}(\mathcal{X})$*.*

2.7.6. By construction, we have the natural transformations

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}}} \to \mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}}|_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}}} \text{ and}$$
$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-closed}}} \to \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}}|_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-closed}}}. \tag{2.6}$$

Proposition 2.7.7. *The natural transformations* (2.6) *are equivalences.*

Proof. For a given $\mathcal{X} \in \mathrm{DGindSch}$, the value of the functors (2.5) on it are given by

$$\underset{X\in(\mathrm{DGSch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}}}{colim} \mathrm{IndCoh}(X),\quad \underset{X\in(\mathrm{DGSch}_{\mathrm{aft}})_{\text{proper over } \mathcal{X}}}{colim} \mathrm{IndCoh}(X)$$

and

$$\underset{X\in(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}}{colim} \mathrm{IndCoh}(X),$$

respectively.

Hence, to prove the proposition, it suffices to show that the functors

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}} \to (\mathrm{DGSch}_{\mathrm{aft}})_{\text{proper over } \mathcal{X}} \to (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}$$

are cofinal. Since both arrows are fully faithful embeddings, it suffices to show that the functor

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}} \to (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}$$

is cofinal, but the latter is given by Corollary 1.7.9.

□

2.7.8. Thus, from Proposition 2.7.7 we obtain that for a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ we have a well-defined functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2).$$

Concretely, the functor f_*^{IndCoh} can be described as follows. By (2.2), objects of $\mathrm{IndCoh}(X)$ are colimits of objects of the form $(i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)$, where X_1 is a DG scheme almost of finite type equipped with a closed embedding $X_1 \overset{i_1}{\to} \mathcal{X}_1$. By continuity, the functor f_*^{IndCoh} is completely determined by its values on such objects.

By Corollary 1.7.9, we can factor the map

$$X_1 \overset{i_1}{\to} \mathcal{X}_1 \overset{f}{\to} \mathcal{X}_2$$

as

$$X_1 \overset{g}{\to} X_2 \overset{i_2}{\to} \mathcal{X}_2,$$

where $X_2 \in \mathrm{DGSch}_{\mathrm{aft}}$ and i_2 being a closed embedding. We set

$$f_*^{\mathrm{IndCoh}}((i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)) := (i_2)_*^{\mathrm{IndCoh}}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_1)).$$

The content of Proposition 2.7.7 is that this construction extends to a well-defined functor $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$.

Note that the functor $\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -)$ of Sect. 2.6.4 is a particular instance of this construction for $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = \mathrm{pt}$.

2.7.9. It follows from the definition of the self-duality functors

$$\mathbf{D}^{\mathrm{Serre}}_{\mathcal{X}_i} : \mathrm{IndCoh}(\mathcal{X}_i)^\vee \to \mathrm{IndCoh}(\mathcal{X}_i), \quad i = 1, 2$$

that the dual of the functor f_*^{IndCoh} identifies canonically with $f^!$.

2.7.10. For a morphism of DG indschemes, the pushforward functor on IndCoh interacts with the t-structure in the usual way:

Lemma 2.7.11. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map of indschemes. Then the functor f_*^{IndCoh} is left t-exact. Furthermore, if f is a closed embedding, then it is t-exact.*

Proof. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\geq 0}$. We wish to show that $f_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathcal{X}_2)^{\geq 0}$. By Corollary 2.5.8, we can assume that $\mathcal{F} = (i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)^{\geq 0}$ where $i_1 : X_1 \to \mathcal{X}_1$ is a closed embedding.

Let now

$$X_1 \overset{g}{\to} X_2 \overset{i_2}{\to} \mathcal{X}_2$$

be a factorization of $f \circ i_1$, where i_2 is a closed embedding. We have:

$$f_*^{\mathrm{IndCoh}}(\mathcal{F}) \simeq f_*^{\mathrm{IndCoh}}((i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)) = (i_2)_*^{\mathrm{IndCoh}}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_1)).$$

By Lemma 2.5.5, $(i_2)_*^{\mathrm{IndCoh}}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_1)) \in \mathrm{IndCoh}(\mathcal{X}_2)^{\geq 0}$.

Suppose now that f is a closed embedding. In this case, we wish to show that f_*^{IndCoh} is also right t-exact. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\leq 0}$. By Lemma 2.5.3(b), we can assume that $\mathcal{F} = (i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)^{\leq 0}$ where $i_1 : X_1 \to \mathcal{X}_1$ is a closed embedding. The result now follows from the fact that the composed map

$$X_1 \to \mathcal{X}_1 \to \mathcal{X}_2$$

is a closed embedding and Lemma 2.5.5. □

2.8. **Adjunction for proper maps.**

2.8.1. Consider the functor

$$\mathrm{IndCoh}^!_{\mathrm{DGindSch}_{\mathrm{laft}}} : \mathrm{DGindSch}^{\mathrm{op}}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and let

$$\mathrm{IndCoh}^!_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} \text{ and } \mathrm{IndCoh}^!_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}$$

be the restrictions of $\mathrm{IndCoh}^!_{\mathrm{DGindSch}_{\mathrm{laft}}}$ to the corresponding 1-full subcategories.

In addition, consider the corresponding functors

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}},\ \mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}} \text{ and } \mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}}$$

for $\mathrm{DGSch}_{\mathrm{aft}}$ instead of $\mathrm{DGindSch}_{\mathrm{laft}}$.

As in Proposition 2.7.7, we have:

Lemma 2.8.2. *The natural maps*

$$\mathrm{IndCoh}^!_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}}} \to \mathrm{RKE}_{(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}}_{\mathrm{proper}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})^{\mathrm{op}}_{\text{ind-proper}}}(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}})$$

and

$$\mathrm{IndCoh}^!_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-closed}}} \to \mathrm{RKE}_{(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}}_{\mathrm{closed}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})^{\mathrm{op}}_{\text{ind-closed}}}(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}})$$

are isomorphisms.

We shall now deduce the following:

Corollary 2.8.3. *The functor*

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}}} : (\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is obtained from the functor

$$\mathrm{IndCoh}^!_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\text{ind-proper}}} : (\mathrm{DGindSch}_{\mathrm{laft}})^{\mathrm{op}}_{\text{ind-proper}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

by passing to left adjoints.

This corollary means that for a proper map $f : \mathcal{X}_1 \to \mathcal{X}_2$ in $\mathrm{DGindSch}_{\mathrm{laft}}$, the functor

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$$

is the left adjoint of $f^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_1)$ in a way compatible with compositions, and that this data is homotopy-coherent.

Proof. This follows from the corresponding fact for the functors $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}$ and $\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}$ (see [GL:IndCoh, Theorem 5.2.2(a)]), and the following general assertion:

Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between ∞-categories. Let $\Phi_1 : \mathbf{C}_1 \to \mathrm{DGCat}_{\mathrm{cont}}$ be a functor such that for every $\mathbf{c}'_1 \to \mathbf{c}''_1$, the corresponding functor

$$\Phi_1(\mathbf{c}'_1) \to \Phi_1(\mathbf{c}''_1)$$

admits a continuous right adjoint. Let $\Psi_1 : \mathbf{C}_1^{\mathrm{op}} \to \mathrm{DGCat}$ be the resulting functor given by taking the right adjoints.

Let Φ_2 and Ψ_2 be the left (resp., right) Kan extension of Φ_1 (resp., Ψ_1) along F (resp., F^{op}).

The following is a version of [GL:DG, Lemma 1.3.3]:

Lemma 2.8.4. *Under the above circumstances, the functor Ψ_2 is obtained from Φ_2 by taking right adjoints.*

□

2.9. Proper base change.

2.9.1. Let

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f_X} \\ \mathcal{Y}_2 & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

be a Cartesian diagram of DG indschemes, with the maps f_X and f_Y ind-proper. From the isomorphism of functors

$$g_1^! \circ f_X^! \simeq f_Y^! \circ g_2^!,$$

by adjunction, we obtain a natural transformation

$$(f_Y)^{\mathrm{IndCoh}}_* \circ g_1^! \to g_2^! \circ (f_X)^{\mathrm{IndCoh}}_*. \tag{2.7}$$

Proposition 2.9.2. *The natural transformation* (2.7) *is an isomorphism.*

The proof of this proposition will occupy the next few subsections.

2.9.3. *Proof of Proposition 2.9.2, Step 1.* The assertion readily reduces to the case when $\mathcal{Y}_2$ is a DG scheme, denote it Y_2. Next, we are going to show that we can assume $\mathcal{X}_2$ is also a DG scheme.

2.9.4. *Interlude.* Consider the following general paradigm. Let $G : \mathbf{C}_2 \to \mathbf{C}_1$ be a functor between ∞-categories. Let A be a category of indices, and suppose we are given an A-family of commutative diagrams

$$\begin{array}{ccc} \mathbf{C}_{1,\alpha} & \xleftarrow{i_{1,\alpha}} & \mathbf{C}_1 \\ {\scriptstyle G_\alpha}\uparrow & & \uparrow{\scriptstyle G} \\ \mathbf{C}_{2,\alpha} & \xleftarrow{i_{2,\alpha}} & \mathbf{C}_2. \end{array}$$

Assume that for each $\alpha \in A$, the functor G_α admits a left adjoint F_α. Furthermore, assume that for each map $\alpha' \to \alpha''$ in A, the natural transformation in the diagram

(2.8)
$$\begin{array}{ccc} \mathbf{C}_{1,\alpha''} & \xleftarrow{i_{1,\alpha',\alpha''}} & \mathbf{C}_{1,\alpha'} \\ {\scriptstyle F_{\alpha''}}\downarrow & \searrow\!\!\!\searrow & \downarrow{\scriptstyle F_{\alpha'}} \\ \mathbf{C}_{2,\alpha''} & \xleftarrow[i_{2,\alpha',\alpha''}]{} & \mathbf{C}_{2,\alpha'} \end{array}$$

is an isomorphism.

Finally, assume that the functors

$$\mathbf{C}_1 \to \underset{\alpha \in A}{lim}\, \mathbf{C}_{1,\alpha} \text{ and } \mathbf{C}_2 \to \underset{\alpha \in A}{lim}\, \mathbf{C}_{2,\alpha}$$

are equivalences.

Under the above circumstances we have:

Lemma 2.9.5. *The functor G admits a left adjoint, denoted F, and for every $\alpha \in A$, the natural transformation in the diagram*

$$\begin{array}{ccc} \mathbf{C}_{1,\alpha} & \xleftarrow{i_{1,\alpha}} & \mathbf{C}_1 \\ {\scriptstyle F_\alpha}\downarrow & \searrow\!\!\!\searrow & \downarrow{\scriptstyle F} \\ \mathbf{C}_{2,\alpha} & \xleftarrow[i_{2,\alpha}]{} & \mathbf{C}_2 \end{array}$$

is an isomorphism.

2.9.6. *Proof of Proposition 2.9.2, Step 2.* Write

$$\mathcal{X}_2 \simeq \underset{\alpha \in A}{colim}\, X_{2,\alpha}$$

where A is the category $(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}_2}$. Set

$$\mathcal{X}_{1,\alpha} := X_{2,\alpha} \underset{\mathcal{X}_2}{\times} \mathcal{X}_1.$$

It is clear that

$$\mathcal{X}_1 \simeq \underset{\alpha \in A}{colim}\, \mathcal{X}_{1,\alpha},$$

where the colimit is taken in $\mathrm{PreStk}_{\mathrm{laft}}$.

Hence, by Lemma 2.3.2,

$$\mathrm{IndCoh}(\mathcal{X}_1) \simeq \underset{\alpha \in A^{\mathrm{op}}}{lim}\, \mathrm{IndCoh}(\mathcal{X}_{1,\alpha}).$$

Set

$$\mathbf{C}_2 = \mathrm{IndCoh}(\mathcal{X}_2),\ \mathbf{C}_1 = \mathrm{IndCoh}(\mathcal{X}_1),\ \mathbf{C}_{2,\alpha} = \mathrm{IndCoh}(X_{2,\alpha}),\ \mathbf{C}_{1,\alpha} = \mathrm{IndCoh}(\mathcal{X}_{1,\alpha}).$$

The condition of Lemma 2.9.5 is equivalent to the assertion of Proposition 2.9.2 when instead of $\mathcal{X}_2 \in \mathrm{DGindSch}_{\mathrm{laft}}$ we take $X_{2,\alpha} \in \mathrm{DGSch}_{\mathrm{laft}}$.

Thus, the assertion of Lemma 2.9.5 reduces the assertion of Proposition 2.9.2 to the case when both $\mathcal{Y}_2 = Y_2$ and $\mathcal{X}_2 = X_2$ are DG schemes.

2.9.7. *Proof of Proposition 2.9.2, Step 3.* Write

$$\mathcal{X}_1 \simeq \underset{\beta \in B}{colim}\, X_{1,\beta},$$

where $X_{1,\beta} \in \mathrm{DGSch}_{\mathrm{aft}}$ and $i_{X,\beta} : X_{1,\beta} \to \mathcal{X}_1$ are closed embeddings.

Set

$$Y_{1,\beta} := Y_2 \underset{X_2}{\times} X_{1,\beta}.$$

We have:

$$\mathcal{Y}_1 \simeq \underset{\beta \in B}{colim}\, Y_{1,\beta},$$

Let $i_{Y,\beta}$ denote the correspoding closed embedding $Y_{1,\beta} \to \mathcal{Y}_1$, and let g_β denote the map $Y_{1,\beta} \to X_{1,\beta}$. Note that the maps $f_X \circ i_{X,\beta} : X_{1,\beta} \to X_2$ and $f_Y \circ i_{Y,\beta} : Y_{1,\beta} \to Y_2$ are proper, by assumption.

By (2.2), we have:

$$\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1)} \simeq \underset{\beta \in B}{colim}\, (i_{X,\beta})_*^{\mathrm{IndCoh}} \circ (i_{X,\beta})^! \text{ and } \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y}_1)} \simeq \underset{\beta \in B}{colim}\, (i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ (i_{Y,\beta})^!$$

Hence, we can rewrite the functor $(f_Y)_*^{\mathrm{IndCoh}} \circ g_1^!$ as

$$\underset{\beta \in B}{colim}\, (f_Y)_*^{\mathrm{IndCoh}} \circ (i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ (i_{Y,\beta})^! \circ g_1^!,$$

and the functor $g_2^! \circ (f_X)_*^{\mathrm{IndCoh}}$ as

$$\underset{\beta \in B}{colim}\, g_2^! \circ (f_X)_*^{\mathrm{IndCoh}} \circ (i_{X,\beta})_*^{\mathrm{IndCoh}} \circ (i_{X,\beta})^!.$$

It follows from the construction that the map in (2.7) is given by a compatible system of maps for each $\beta \in B$

$$(f_Y)_*^{\mathrm{IndCoh}} \circ (i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ (i_{Y,\beta})^! \circ g_1^! \simeq (f_Y \circ i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ (g_1 \circ i_{Y,\beta})^! \simeq$$
$$(f_Y \circ i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ (i_{X,\beta} \circ g_\beta)^! \simeq (f_Y \circ i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ g_\beta^! \circ i_{X,\beta}^! \to$$
$$\to g_2^! \circ (f_X \circ i_{X,\beta})_*^{\mathrm{IndCoh}} \circ i_{X,\beta}^! \simeq g_2^! \circ (f_X)_*^{\mathrm{IndCoh}} \circ (i_{X,\beta})_*^{\mathrm{IndCoh}} \circ i_{X,\beta}^!,$$

where the arrow

$$(f_Y \circ i_{Y,\beta})_*^{\mathrm{IndCoh}} \circ g_\beta^! \to g_2^! \circ (f_X \circ i_{X,\beta})_*^{\mathrm{IndCoh}}$$

is base change for the Cartesian square

$$\begin{array}{ccc} Y_{1,\beta} & \xrightarrow{g_\beta} & X_{1,\beta} \\ {\scriptstyle f_Y \circ i_{Y,\beta}} \downarrow & & \downarrow {\scriptstyle f_X \circ i_{X,\beta}} \\ Y_2 & \xrightarrow{g_2} & X_2. \end{array}$$

Hence, the required isomorphism follows from proper base change in the case of DG schemes, see [GL:IndCoh, Proposition 3.4.2].

□

2.9.8. Let

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f_Y} \downarrow & & \downarrow {\scriptstyle f_X} \\ \mathcal{Y}_2 & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

now be a Cartesian diagram of DG indschemes, where the maps g_1 and g_2 are ind-proper. From the isomorphism of functors

$$(g_2)_*^{\mathrm{IndCoh}} \circ (f_Y)_*^{\mathrm{IndCoh}} \simeq (f_X)_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}}$$

by adjunction, we obtain a natural transformation

$$(f_Y)_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ (f_X)_*^{\mathrm{IndCoh}}. \tag{2.9}$$

Proposition 2.9.9. *The natural transformation* (2.9) *is an isomorphism.*

Remark 2.9.10. It is easy to see from Corollary 2.8.3 that when both pairs of morphisms (i.e., (f_X, f_Y) and (g_1, g_2)) are ind-proper, then the natural transformations (2.7) and (2.9) are canonically isomorphic.

Proof. By (2.2), we can assime that $\mathcal{X}_1 = X_1 \in \mathrm{DGSch}_{\mathrm{aft}}$. Factor the map $f : X_1 \to \mathcal{X}_2$ as a composition

$$X_1 \to X_2 \to \mathcal{X}_2,$$

where $X_2 \in \mathrm{DGSch}_{\mathrm{aft}}$ and $X_2 \to \mathcal{X}_2$ is a closed embedding. Such a factorization is possible by Corollary 1.7.9.

This reduces the assertion of the proposition to the analyses of the following two cases: (1) when the morphism f is a closed embedding (and, in particular, proper); and (2) when both $\mathcal{X}_1 = X_1$ and $\mathcal{X}_2 = X_2$ are DG schemes.

Now, the assertion in case (1) follows from Proposition 2.9.2. The assertion in case (2) follows by repeating the argument of Step 3 in the proof of Proposition 2.9.2.

□

Remark 2.9.11. The isomorphisms as in (2.7) and (2.9) can be defined for all Cartesian diagrams of DG indschemes, i.e., we one does not need to require that either pair of maps be ind-proper. However, the construction is more involved as there is no a priori map in either direction.

For an individual diagram, such an isomorphism is easy to deduce from [GL:IndCoh, Sect. 5], where the corresponding natural transformations were constructed in the case of DG schemes.

A functorial construction of these natural transformations for indschemes compatible with composition requires additional work and will be carried out in [GR]. Furthermore, as in [GL:IndCoh, Sect. 10.6] one can combine the functors

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$\mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}} : \mathrm{DGindSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to a functor

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:ind\text{-}sch;all}}} : (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:ind\text{-}sch;all}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:ind\text{-}sch;all}}$ is the category of correspondences, whose objects are prestacks locally almost of finite type $\mathcal{Y}$, and whose morphisms are correspondences

$$\begin{array}{ccc} \mathcal{Y}_{1,2} & \xrightarrow{g} & \mathcal{Y}_1 \\ {\scriptstyle f}\downarrow & & \\ \mathcal{Y}_2, & & \end{array}$$

where the morphism g is arbitrary, and the morphism f is ind-schematc (i.e., a morphism such that its base change by an affine DG scheme yields a DG indscheme).

2.10. **Groupoids in** DGindSch.

2.10.1. Let $\mathcal{X}^\bullet$ be a simplicial object in DGindSch, arising from a groupoid object

$$p_s, p_t : \mathcal{X}^1 \rightrightarrows \mathcal{X}^0 \tag{2.10}$$

(see [Lu0], Definition 6.1.2.7).

Suppose that the face maps in the above simplicial DG indscheme are ind-proper (equivalently, the maps p_s, p_t in (2.10) are ind-proper).

In this case, the forgetful functor

$$\mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)) \to \mathrm{IndCoh}(\mathcal{X}^0)$$

admits a left adjoint; moreover, the resulting monad on $\mathrm{IndCoh}(\mathcal{X}^0)$, when viewed as a plain endo-functor of $\mathrm{IndCoh}(\mathcal{X}^0)$, is naturally isomorphic to

$$(p_t)^{\mathrm{IndCoh}}_* \circ (p_s)^!.$$

The proof is the same as that of [GL:IndCoh, Proposition 8.2.3].

2.10.2. Assume that in the situation of Sect. 2.10.1, the groupoid arises as the Čech nerve of a morphism $f : \mathcal{X} \to \mathcal{Y}$, which is ind-proper and surjective.[16] Let $\mathcal{X}^\bullet/\mathcal{Y}$ denote the resulting simplicial object.

In this case, the augmentation

$$\mathcal{X}^\bullet/\mathcal{Y} \to \mathcal{Y}$$

gives rise to a functor

$$\text{IndCoh}(\mathcal{Y}) \to \text{Tot}(\text{IndCoh}(\mathcal{X}^\bullet/\mathcal{Y})). \tag{2.11}$$

As in [GL:IndCoh, Proposition 8.2.3] we have:

Lemma 2.10.3. *Under the above circumstances, the functor* (2.11) *is an equivalence.*

Note that the composition

$$\text{IndCoh}(\mathcal{Y}) \to \text{Tot}(\text{IndCoh}(\mathcal{X}^\bullet/\mathcal{Y})) \to \text{IndCoh}(\mathcal{X})$$

is the functor $f^!$, and hence its left adjoint is f_*^{IndCoh}.

3. Closed embeddings into a DG indscheme and push-outs

Let X be a scheme, and Z_1 and Z_2 be two closed subschemes. In this case, we can consider the subscheme given by the union of Z_1 and Z_2; in fact, this is the coproduct in the category of closed subschemes of X (locally, the ideal of the union is the intersection of the ideals of Z_1 and Z_2). The same operation is well-defined when X is no longer a scheme, but an indscheme: indeed the union of Z_1 and Z_2 in X is the same as their union in X', if X' is another closed subscheme of X which contains Z_1 and Z_2.

However, one might be suspicious of the operation of union in the DG setting since closed DG subschemes are no longer in bijection with "ideals."

The goal of this section is to show that in this case, the operation of union behaves as well as for schemes.

In addition, we will consider a particular situation in which push-outs in the category of DG schemes exist and are well-behaved. This will allow us, in particular, to show that DG indschemes contain "many" closed subschemes.

3.1. Closed embeddings into a DG scheme.

3.1.1. For a morphism $f : Y \to X$ in $\text{DGSch}_{\text{qsep-qc}}$ consider the category

$$(\text{DGSch}_{\text{qsep-qc}})_{Y//X}$$

of factorizations of f; i.e. objects are given by

$$Y \to Z \overset{\phi}{\to} X$$

and morphisms are commutative diagrams

$$\begin{array}{ccccc} & & Z_1 & & \\ & \nearrow & \downarrow & \overset{\phi_1}{\searrow} & \\ Y & & & & X. \\ & \searrow & \downarrow & \underset{\phi_2}{\nearrow} & \\ & & Z_2 & & \end{array} \tag{3.1}$$

[16] I.e., the base change of f by an object of $\text{DGSch}^{\text{aff}}_{\text{aft}}$ yields a morphism surjective on geometric points.

Let

$$\mathrm{DGSch}_{Y/,\text{closed in } X} \subset (\mathrm{DGSch}_{\text{qsep-qc}})_{Y//X}$$

be the full subcategory, spanned by those objects $Y \to Z \overset{\phi}{\to} X$, for which the map ϕ is a closed embedding.

3.1.2. We shall prove:

Proposition 3.1.3.

(a) *The category* $\mathrm{DGSch}_{Y/,\text{closed in } X}$ *contains finite colimits (and, in particular, an initial object).*

(b) *The formation of colimits in* $\mathrm{DGSch}_{Y/,\text{closed in } X}$ *is compatible with Zariski localization on* X.

Proof.

Step 1. Assume first that X is affine, given by $X = \mathrm{Spec}(A)$. Let

$$i \rightsquigarrow (Y \to Z_i \overset{\phi_i}{\to} X), \tag{3.2}$$

be a finite diagram in $\mathrm{DGSch}_{Y/,\text{ closed in } X}$.

Set $B := \Gamma(Y, \mathcal{O}_Y)$. This is a (not necessarily connective) commutative k-algebra. Set also $Z_i = \mathrm{Spec}(C_i)$. Consider the corresponding diagram

$$i \rightsquigarrow (A \to C_i \to B) \tag{3.3}$$

in $\mathrm{ComAlg}_{A//B}$.

Set

$$(\widetilde{C} \to B) := \underset{i}{lim}\, (C_i \to B),$$

where the limit taken in $\mathrm{ComAlg}_{/B}$. Note that we have a canonical map $A \to \widetilde{C}$, and

$$(A \to \widetilde{C} \to B) \in \mathrm{ComAlg}_{A//B}$$

maps isomorphically to the limit of (3.3) taken in, $\mathrm{ComAlg}_{A//B}$.

Set

$$C := \tau^{\leq 0}(\widetilde{C}) \underset{H^0(\widetilde{C})}{\times} \mathrm{Im}\left(H^0(A) \to H^0(\widetilde{C})\right),$$

where the fiber product is taken in the category of *connective* commutative algebras (i.e., it is $\tau^{\leq 0}$ of the fiber product taken in the category of all commutative algebras).

We still have canonical maps

$$A \to C \to B,$$

and it is easy to see that for $Z := \mathrm{Spec}(C)$, the object

$$(X \to Z \to Y) \in \mathrm{DGSch}_{X/,\text{ closed in } Y}$$

is the colimit of (3.2).

Step 2. To treat the general case it suffices to show that the formation of colimits in the affine case commutes with Zariski localization. I.e., that if X is affine, $\overset{\circ}{X} \subset X$ is a basic open, then for $\overset{\circ}{Y} := f^{-1}(\overset{\circ}{X})$, $\overset{\circ}{Z}_i := \phi_i^{-1}(\overset{\circ}{X})$, $\overset{\circ}{Z} := \phi^{-1}(\overset{\circ}{X})$, the map

$$\underset{i}{colim}\, \overset{\circ}{Z}_i \to \overset{\circ}{Z},$$

is an isomorphism, where the colimit is taken in $\text{DGSch}_{\overset{\circ}{Y}/,\,\text{closed}\ \overset{\circ}{X}}$.

However, the required isomorphism follows from the description of the colimit in Step 1. □

3.1.4. As before, let

$$i \rightsquigarrow (Y \to Z_i \overset{\phi_i}{\to} X),$$

be a finite diagram in $\text{DGSch}_{Y/,\,\text{closed in}\ X}$. In this case, note the following property of colimits.

Let $g : X \to X'$ be a closed embedding. Set

$$(Y \to Z \to X) = \underset{i}{colim}\,(Y \to Z_i \to X) \text{ and } (Y \to Z' \to X') = \underset{i}{colim}\,(Y \to Z_i \to X'),$$

where the colimits are taken in $\text{DGSch}_{Y/,\,\text{closed in}\ X}$ and $\text{DGSch}_{Y/,\,\text{closed}\ X'}$, respectively.

Consider the composition

$$Y \to Z \to X \to X',$$

and the corresponding object

$$(Y \to Z \to X') \in \text{DGSch}_{Y/,\,\text{closed}\ X'}.$$

It is endowed with a compatible family of maps in $\text{DGSch}_{Y/,\,\text{closed}\ X'}$:

$$(Y \to Z_i \to X') \to (Y \to Z \to X').$$

Hence, by the universal property of $(Y \to Z' \to X') \in \text{DGSch}_{Y/,\,\text{closed}\ X'}$, we obtain a canonically defined map

$$Z' \to Z. \tag{3.4}$$

We claim:

Lemma 3.1.5. *The map* (3.4) *is an isomorphism.*

Proof. We construct the inverse map as follows. We note that by the universal property of $(Y \to Z' \to X') \in \text{DGSch}_{Y/,\,\text{closed}\ X'}$, we have a canonical map

$$(Y \to Z' \to X') \to (Y \to X \to X'),$$

and hence a compatible family of maps

$$(Y \to Z_i \to X') \to (Y \to Z' \to X') \to (Y \to X \to X').$$

The latter gives rise to a compatible family of maps in $\text{DGSch}_{Y/,\,\text{closed}\ X}$

$$(Y \to Z_i \to X) \to (Y \to Z' \to X),$$

and hence, by the universal property of $(Y \to Z \to X) \in \text{DGSch}_{Y/,\,\text{closed}\ X}$, the desired map

$$Z \to Z'.$$

□

3.1.6. *The closure of the image.* For $f : X \to Y$ a morphism in in $\text{DGSch}_{\text{qsep-qc}}$, let

$$\overline{\text{Im}(f)} \in \text{DGSch}_{Y/,\,\text{closed}\ X}$$

denote the initial object of this category. We will refer to it as *the closure of the image of f*.

3.1.7. We have the following properties of the formation of colimits in $\mathrm{DGSch}_{Y/,\,\mathrm{closed}\;X}$:

Lemma 3.1.8. *Let* $i \mapsto (Y \to Z_i \to X)$ *be a finite diagram in* $\mathrm{DGSch}_{Y/,\,\mathrm{closed}\;X}$*, and let*

$$Y \to Z \to X$$

be its colimit.

(a) *Suppose that the DG schemes* Z_i *are* n*-coconnective. Then so is* Z*.*

(b) *Suppose that* $f : Y \to X$ *is affine (resp., of cohomological amplitude* k *for the functor* $f_* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$*). For an integer* m*, consider the diagram*

$$^{\leq m}Y \to {}^{\leq m}Z_i \to X,$$

and let

$$^{\leq m}Y \to Z' \to X$$

be its colimit in $\mathrm{DGSch}_{^{\leq m}Y/,\,\mathrm{closed}\;X}$*. Then the natural map*

$$^{\leq n}Z' \to {}^{\leq n}Z$$

is an isomorphism whenever $m \geq n+1$ *(resp.,* $m \geq n+1+k$*).*

Proof. Both assertions follow from the explicit construction of colimits in Step 1 in the proof of Proposition 3.1.3.

□

3.2. The case of DG indschemes.

3.2.1. For $\mathfrak{X} \in \mathrm{DGindSch}$, $Y \in \mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$ and a morphism $Y \to \mathfrak{X}$, we consider the category

$$(\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{Y/\,/\mathfrak{X}}$$

and the corresponding full subcategory

$$\mathrm{DGSch}_{Y/,\mathrm{closed\;in}\;\mathfrak{X}}\,.$$

Proposition 3.2.2. *The category* $\mathrm{DGSch}_{Y/,\mathrm{closed\;in}\;\mathfrak{X}}$ *contains finite colimits.*

As in the case of DG schemes, for a given map $f : Y \to \mathfrak{X}$, we let $\overline{\mathrm{Im}(f)}$ denote the initial object of the category $\mathrm{DGSch}_{Y/,\mathrm{closed\;in}\;\mathfrak{X}}$.

Remark 3.2.3. As Proposition 3.2.2 will be used in the proof of Proposition 1.4.4, we will not be able to use the existence of a presentation as in (1.5). If we could assume such a presentation, the proof would be immediate.

Proof of Proposition 3.2.2. Assume first that Y, Z_1 and Z_2 are eventually coconnective, i.e., n-coconnective for some n. Then we can work in the categories $^{\leq n}\mathrm{DGSch}$ and $^{\leq n}\mathrm{DGindSch}$. We replace $\mathfrak{X}$ by $^{\leq n}\mathfrak{X}$, and representing it as in (1.1), we obtain that the statement follows from Lemma 3.1.5.

Writing $^{cl}\mathfrak{X}$ as in (1.1), let $\alpha \in A$ be an index such that the map $^{cl}Y \to {}^{cl}\mathfrak{X}$ factors via a map

$$^{cl}f_\alpha : {}^{cl}Y \to X_\alpha \to {}^{cl}\mathfrak{X}.$$

Let k denote the cohomological amplitude of the functor

$$(^{cl}f_\alpha)_* : \mathrm{QCoh}(^{cl}Y) \to \mathrm{QCoh}(X_\alpha).$$

Let

$$i \mapsto (Y \to Z_i \to \mathfrak{X}) \tag{3.5}$$

be a finite diagram in $\mathrm{DGSch}_{Y/,\,\mathrm{closed}\,\mathcal{X}}$. For an integer m, consider the corresponding diagram

$$^{\leq m}Y \to {}^{\leq m}Z_i \to X.$$

Let

$$^{\leq m}Y \to \widetilde{Z}^m \to \mathcal{X}$$

denote its colimit in $\mathrm{DGSch}_{^{\leq m}Y/,\,\mathrm{closed}\,\mathcal{X}}$.

For an integer n set

$$Z^n = {}^{\leq n}\widetilde{Z}^m$$

for any $m \geq n+1+k$. Note that this is independent of the choice of m by Corollary 3.1.8(b). For the same reason, for $n_1 \leq n_2$ we have

$$Z^{n_1} \simeq {}^{\leq n_1}Z^{n_2}.$$

The sought-for colimit of (3.5) is $Y \to Z \to \mathcal{X}$, where $Z \in \mathrm{DGSch}$ is such that

$$^{\leq n}Z = Z^n.$$

□

3.2.4. As a corollary of Proposition 3.2.2, we obtain:

Corollary 3.2.5. *For* $\mathcal{X} \in \mathrm{DGindSch}$*, the category of closed embeddings* $Z \to \mathcal{X}$*, where* $Z \in \mathrm{DGSch}_{\text{qsep-qc}}$*, is filtered.*

Note that the assertion of Corollary 3.2.5 coincides with that of Proposition 1.6.4(a).

3.3. **A digression on push-outs.** Let

$$\begin{array}{ccc} & & Y_1 \\ & \overset{f_1}{\nearrow} & \\ Y & & \\ & \underset{f_2}{\searrow} & \\ & & Y_2 \end{array} \tag{3.6}$$

be a diagram in DGSch.

We wish to consider the push-out of this diagram in DGSch. Note that push-outs of (DG) schemes are not among the standard practices in algebraic geometry; this operation is in general quite ill-behaved unless we impose some particular conditions on morphisms under which we are taking push-outs. In what follows we will consider three rather special situations where push-outs are manageable.

3.3.1. *Push-outs in the category of affine schemes.* Let

$$i \mapsto Y_i, \quad i \in I$$

be an I-diagram in $\mathrm{DGSch}^{\mathrm{aff}}$ for some $I \in \infty\text{-Cat}$.

Let $\widetilde{Y}$ denote its colimit in the category $\mathrm{DGSch}^{\mathrm{aff}}$. I.e., if $Y_i = \mathrm{Spec}(A_i)$, then $\widetilde{Y} = \mathrm{Spec}(\widetilde{A})$, where

$$\widetilde{A} = \underset{i}{lim}\, A_i,$$

where the limits is taken in the category of connective k-algebras.

3.3.2. In particular, consider a diagram $Y_1 \leftarrow Y \rightarrow Y_2$ in $\mathrm{DGSch}^{\mathrm{aff}}$ and set $\widetilde{Y} := Y_1 \underset{Y}{\sqcup} Y_2$, where the push-out is taken in $\mathrm{DGSch}^{\mathrm{aff}}$. I.e., if $Y_i = \mathrm{Spec}(A_i)$ and $Y = \mathrm{Spec}(A)$, then $\widetilde{Y} = \mathrm{Spec}(\widetilde{A})$, where

$$\widetilde{A} := A_1 \underset{A}{\times} A_2,$$

where the fiber product is taken in the category of *connective* k-algebras.

Note that if $Y \rightarrow Y_1$ is a closed embedding, then so is the map $Y_2 \rightarrow \widetilde{Y}$.

3.3.3. *The case of closed embeddings.* We observe the following:

Lemma 3.3.4. *Suppose that in the setting of Sect. 3.3.2, both maps $Y \rightarrow Y_i$ are closed embeddings. Then:*

(a) *The Zariski topology on $\widetilde{Y}$ is induced by that on $Y_1 \sqcup Y_2$.*

(b) *For open affine DG subschemes $\overset{\circ}{Y}_i \subset Y_i$ such that $\overset{\circ}{Y}_1 \cap Y = \overset{\circ}{Y}_2 \cap Y =: \overset{\circ}{Y}$, and the corresponding open DG subscheme $\widetilde{\overset{\circ}{Y}} \subset \widetilde{Y}$, the map*

$$\overset{\circ}{Y}_1 \underset{\overset{\circ}{Y}}{\sqcup} \overset{\circ}{Y}_2 \rightarrow \widetilde{\overset{\circ}{Y}}$$

is an isomorphism.

(c) *The diagram*

$$\begin{array}{ccc} Y & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & \widetilde{Y} \end{array}$$

is a push-out diagram in DGSch.

3.3.5. From here we obtain:

Corollary 3.3.6. *Let $Y_1 \leftarrow Y \rightarrow Y_2$ be a diagram in* DGSch*, where both maps $Y_i \rightarrow Y$ are closed embeddings. Then:*

(a) *The push-out $\widetilde{Y} := Y_1 \underset{Y}{\sqcup} Y_2$ in* DGSch *exists.*

(b) *The Zariski topology on $\widetilde{Y}$ is induced by that on $Y_1 \sqcup Y_2$.*

(c) *For open DG subschemes $\overset{\circ}{Y}_i \subset Y_i$ such that $\overset{\circ}{Y}_1 \cap Y = \overset{\circ}{Y}_2 \cap Y =: \overset{\circ}{Y}$, and the corresponding open DG subscheme $\widetilde{\overset{\circ}{Y}} \subset \widetilde{Y}$, the map*

$$\overset{\circ}{Y}_1 \underset{\overset{\circ}{Y}}{\sqcup} \overset{\circ}{Y}_2 \rightarrow \widetilde{\overset{\circ}{Y}}$$

is an isomorphism.

Remark 3.3.7. Note that if one of the maps f_i fails to be a closed embedding, it is no longer true that the push-out in the category of affine DG schemes is a push-out in the category of schemes. A counter-example is

$$\mathbb{A}^1 \times (\mathbb{A}^1 - 0) \hookleftarrow \{0\} \times (\mathbb{A}^1 - 0) \hookrightarrow \{0\} \times \mathbb{A}^1.$$

3.3.8. We give the following definition:

Definition 3.3.9. *A map $f : X_1 \to X_2$ in* DGSch *is said to be a* nil-immersion *if it induces an isomorphism*

$$^{cl,red}X_1 \to {}^{cl,red}X_2,$$

where for a DG scheme X, we let $^{cl,red}X$ denote the underlying classical reduced scheme. If f is in addition a closed embedding, then it is said to be a closed nil-immersion*.*

3.3.10. *Push-outs with respect to nil-immersions.* Consider the following situation. Let $i \mapsto Y_i$ and $\widetilde{Y}$ be as in Sect. 3.3.1.

Assume that the maps $Y_i \to \widetilde{Y}$ are nil-immersions. In particular, the transition maps

$$Y_{i_1} \to Y_{i_2}$$

are nil-immersions as well. In this case we have:

Lemma 3.3.11. *Assume that the maps $Y_i \to \widetilde{Y}$ are nil-immersions.*

(a) *For an open affine DG subscheme $\widetilde{\overset{\circ}{Y}} \subset \widetilde{Y}$, and the corresponding open DG subschemes $\overset{\circ}{Y}_i \subset Y_i$, the map*

$$\underset{i}{colim}\, \overset{\circ}{Y}_i \to \widetilde{\overset{\circ}{Y}}$$

is an isomorphism, where the colimit is taken in $\mathrm{DGSch}^{\mathrm{aff}}$*.*
(b) *The diagram*

$$i \mapsto (Y_i \to \widetilde{Y})$$

is also a colimit diagram in DGSch*.*

3.3.12. From Lemma 3.3.11 we obtain:

Corollary 3.3.13. *Let $Y_1 \leftarrow Y \to Y_2$ be a diagram in* DGSch *where the maps $Y \to Y_i$ are nil-immersions. Then:*
(a) *The push-out $\widetilde{Y} := Y_1 \underset{Y}{\sqcup} Y_2$ in* DGSch *exists, and the maps $Y_i \to \widetilde{Y}$ are nil-immersions.*

(b) *For an open DG subscheme $\widetilde{\overset{\circ}{Y}} \subset \widetilde{Y}$, and the corresponding open DG subschemes $\overset{\circ}{Y}_i \subset Y_i$, $\overset{\circ}{Y} \subset Y$, the map*

$$\overset{\circ}{Y}_1 \underset{\overset{\circ}{Y}}{\sqcup} \overset{\circ}{Y}_2 \to \widetilde{\overset{\circ}{Y}}$$

is an isomorphism.

3.3.14. *The push-out of a closed nil-immersion.* Finally, we will consider the following situation. Let

$$Y_1 \to Y_1'$$

be a closed nil-immersion of affine schemes, and let $f : Y_1 \to Y_2$ be a map, where $Y_2 \in \mathrm{DGSch}^{\mathrm{aff}}$.

Let $Y_2' = Y_1' \underset{Y_1}{\sqcup} Y_2$, where the colimit is taken in $\mathrm{DGSch}^{\mathrm{aff}}$. Note that the map

$$Y_2 \to Y_2'$$

is a closed nil-immersion.

We claim:

Lemma 3.3.15. (a) *For an open affine DG subscheme* $\overset{\circ}{Y}_2 \subset Y_2$, $f^{-1}(\overset{\circ}{Y}_2) =: \overset{\circ}{Y}_1 \subset Y_1$, *and the corresponding open affine DG subscheme* $\overset{\circ}{Y}{}'_i \subset Y'_i$, *the map*

$$\overset{\circ}{Y}{}'_1 \underset{\overset{\circ}{Y}_1}{\sqcup} \overset{\circ}{Y}_2 \to \overset{\circ}{Y}{}'_2$$

is an isomorphism, where the push-out is taken in $\mathrm{DGSch}^{\mathrm{aff}}$.
(b) *The diagram*

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_2 \\ \downarrow & & \downarrow \\ Y'_1 & \longrightarrow & Y'_2 \end{array}$$

is also a push-out diagram in DGSch.

3.3.16. As a corollary we obtain:

Corollary 3.3.17. *Let* $Y_1 \to Y'_1$ *be a closed nil-immersion, and* $f : Y_1 \to Y_2$ *be a quasi-separated quasi-compact map between schemes. Then:*
(a) *The push-out* $Y'_2 := Y'_1 \underset{Y_1}{\sqcup} Y_2$ *exists, and the map* $Y_2 \to Y'_2$ *is a nil-immersion.*
(b) *For an open affine DG subscheme* $\overset{\circ}{Y}_2 \subset Y_2$, $f^{-1}(\overset{\circ}{Y}_2) =: \overset{\circ}{Y}_1 \subset Y_1$, *and the corresponding open affine DG subscheme* $\overset{\circ}{Y}{}'_i \subset Y'_i$, *the map*

$$\overset{\circ}{Y}{}'_1 \underset{\overset{\circ}{Y}_1}{\sqcup} \overset{\circ}{Y}_2 \to \overset{\circ}{Y}{}'_2$$

is an isomorphism, where the push-out is taken in DGSch.
(c) *If* f *is an open embedding, then so is the map* $Y'_1 \to Y'_2$.

Proof. We observe that it suffices to prove the corollary when Y_2 is affine. Let us write Y_1 as $\underset{i}{colim}\, U_i$, where U_i are affine and open in Y_1. In this case,

$$Y'_1 \simeq \underset{i}{colim}\, U'_i,$$

where U'_i are the corresponding open DG subschemes in Y'_1.

We construct $Y'_1 \underset{Y_1}{\sqcup} Y_2$ as

$$\underset{i}{colim}\, (U'_i \underset{U_i}{\sqcup} Y_2).$$

This implies points (a) and (b) of the corollary via Lemma 3.3.15. Point (c) follows formally from point (b).

□

3.3.18. We will use the following additional properties of push-outs:

Lemma 3.3.19. *Let* Y_1, Y'_1, Y_2, Y'_2 *be as in Corollary 3.3.17. Suppose that the map* $f : Y_1 \to Y_2$ *is such that the cohomological amplitude of the functor* $f_* : \mathrm{QCoh}(Y_1) \to \mathrm{QCoh}(Y_2)$ *is bounded by* k. *Then the map*

$${}^{\leq m}Y'_1 \underset{{}^{\leq m}Y_1}{\sqcup} {}^{\leq m}Y_2 \to {}^{\leq m}Y'_2$$

defines an isomorphism of the n*-coconnective truncations whenever* $m \geq n + k$.

3.4. DG indschemes and push-outs.

3.4.1. Let us observe the following property enjoyed by ind-schemes:

Proposition 3.4.2. *Let*

$$\begin{array}{ccc} Y & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & \widetilde{Y} \end{array}$$

be a push-out diagram in $\mathrm{DGSch}_{\text{qsep-qc}}$*, where* Y, Y_1, Y_2 *are eventually coconnective. Then for* $\mathcal{X} \in \mathrm{DGindSch}$*, the natural map*

$$\mathrm{Maps}(\widetilde{Y}, \mathcal{X}) \to \mathrm{Maps}(Y_1, \mathcal{X}) \underset{\mathrm{Maps}(Y,\mathcal{X})}{\times} \mathrm{Maps}(Y_2, \mathcal{X})$$

is an isomorphism.

Proof. Suppose that Y, Y_1, Y_2 are n-coconnective. By adjunction, we obtain that $\widetilde{Y}$ is n-coconnective as well.

The assertion of the proposition now follows from Lemma 1.3.6 and the fact that fiber products commute with filtered colimits.

□

Remark 3.4.3. In the above proposition we had to make the eventual coconnectivity assumption, because it will be used for the proof of Proposition 1.4.4. However, assuming this proposition, and hence, Lemma 1.5.4, we will be able to prove the same assertion for any $Y, Y_1, Y \in \mathrm{DGSch}_{\text{qsep-qc}}$. The next corollary, which will be also used in the proof of Proposition 1.4.4, gives a partial result along these lines.

Corollary 3.4.4. *Let*

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_2 \\ \downarrow & & \downarrow \\ Y_1' & \longrightarrow & Y_2' \end{array}$$

be a push-out diagram as in Lemma 3.3.17, where $Y_1, Y_2 \in \mathrm{DGSch}_{\text{qsep-qc}}$*. Then the natural map*

$$\mathrm{Maps}(Y_2', \mathcal{X}) \to \mathrm{Maps}(Y_1', \mathcal{X}) \underset{\mathrm{Maps}(Y_1,\mathcal{X})}{\times} \mathrm{Maps}(Y_2, \mathcal{X})$$

is an isomorphism.

Proof. Consider the following two inverse families of objects of $\mathrm{DGSch}_{\text{qsep-qc}}$:

$$n \mapsto {}^{\leq n}Y_2' \text{ and } n \mapsto {}^{\leq n}Y_1' \underset{{}^{\leq n}Y_1}{\sqcup} {}^{\leq n}Y_2.$$

There is a natural map $\leftarrow$. By Lemma 3.3.19, this map induces an isomorphism of m-coconnective truncations whenever $n \gg m$.

Therefore, for any $\mathcal{X} \in \mathrm{DGindSch}$ (and, indeed, any $\mathcal{X} \in {}^{\mathrm{conv}}\mathrm{PreStk}$), the induced map

$$\underset{n}{\mathit{lim}}\, \mathrm{Maps}({}^{\leq n}Y_2', \mathcal{X}) \to \underset{n}{\mathit{lim}}\, \mathrm{Maps}\left({}^{\leq n}Y_1' \underset{{}^{\leq n}Y_1}{\sqcup} {}^{\leq n}Y_2, \mathcal{X}\right)$$

is an isomorphism.

Consider the composite map

$$\mathrm{Maps}(Y_2',\mathcal{X})\to \mathrm{Maps}(Y_1',\mathcal{X}) \underset{\mathrm{Maps}(Y_1,\mathcal{X})}{\times} \mathrm{Maps}(Y_2,\mathcal{X}) \overset{\sim}{\to}$$
$$\overset{\sim}{\to} \underset{n}{lim}\, \mathrm{Maps}({}^{\leq n}Y_1',\mathcal{X}) \underset{\underset{n}{lim}\, \mathrm{Maps}({}^{\leq n}Y_1,\mathcal{X})}{\times} \underset{n}{lim}\, \mathrm{Maps}({}^{\leq n}Y_2',\mathcal{X}) \overset{\sim}{\to}$$
$$\overset{\sim}{\to} \underset{n}{lim}\left(\mathrm{Maps}({}^{\leq n}Y_1',\mathcal{X}) \underset{\mathrm{Maps}({}^{\leq n}Y_1,\mathcal{X})}{\times} \mathrm{Maps}({}^{\leq n}Y_2,\mathcal{X})\right).$$

It equals the map

$$\mathrm{Maps}(Y_2',\mathcal{X})\to \underset{n}{lim}\, \mathrm{Maps}({}^{\leq n}Y_2',\mathcal{X}) \overset{\sim}{\to}$$
$$\overset{\sim}{\to} \underset{n}{lim}\, \mathrm{Maps}\left({}^{\leq n}Y_1' \underset{{}^{\leq n}Y_1}{\sqcup} {}^{\leq n}Y_2,\mathcal{X}\right) \overset{\sim}{\to}$$
$$\overset{\sim}{\to} \underset{n}{lim}\left(\mathrm{Maps}({}^{\leq n}Y_1',\mathcal{X}) \underset{\mathrm{Maps}({}^{\leq n}Y_1,\mathcal{X})}{\times} \mathrm{Maps}({}^{\leq n}Y_2,\mathcal{X})\right),$$

where the last arrow is an isomorphism by Proposition 3.4.2 above. This shows that

$$\mathrm{Maps}(Y_2',\mathcal{X})\to \mathrm{Maps}(Y_1',\mathcal{X}) \underset{\mathrm{Maps}(Y_1,\mathcal{X})}{\times} \mathrm{Maps}(Y_2,\mathcal{X})$$

is an isomorphism as well.

□

3.5. **Presentation of indschemes.**

3.5.1. We shall now prove point (b) of Proposition 1.6.4. In fact, we will prove a slightly stronger (but, in fact, equivalent) statement; namely, we will prove Corollary 1.6.6.

Proof. We have to show that for $Y\in \mathrm{DGSch}_{\text{qsep-qc}}$ and a map $f:Y\to\mathcal{X}$, the category of its factorizations

$$Y\to Z\to\mathcal{X},$$

where $Z\in\mathrm{DGSch}_{\text{qsep-qc}}$, and $Z\to\mathcal{X}$ is a closed embedding, is contractible.

By Proposition 3.1.3, the category in question admits coproducts. Hence, to prove that it is contractible, it remains to show that it is non-empty.

Consider the map ${}^{cl}f:{}^{cl}Y\to{}^{cl}\mathcal{X}$. Since ${}^{cl}\mathcal{X}$ is a classical indscheme, there exists a factorization

$${}^{cl}Y\overset{h_{cl}}{\longrightarrow} Z_{cl}\overset{g_{cl}}{\longrightarrow}{}^{cl}\mathcal{X},$$

where $Z_{cl}\in\mathrm{Sch}_{\text{qsep-qc}}$ and g_{cl} is a closed embedding.

Let k be the cohomological amplitude of the functor $(h_{cl})_*:\mathrm{QCoh}({}^{cl}Y)\to\mathrm{QCoh}(Z_{cl})$, and let n be an integer $>k$.

Consider the truncation ${}^{\leq n}Y$ and its map ${}^{\leq n}f$ to ${}^{\leq n}\mathcal{X}$. Since ${}^{\leq n}\mathcal{X}$ is a ${}^{\leq n}$DG indscheme, the map ${}^{\leq n}f$ can be factored as

$${}^{\leq n}Y\overset{h_n}{\longrightarrow} Z_n\overset{g_n}{\longrightarrow}{}^{\leq n}\mathcal{X},$$

where $Z_n \in {}^{\leq n}\operatorname{DGSch}_{\text{qsep-qc}}$ and g_n is a closed embedding. Moreover, without loss of generality, we can assume that we have a commutative square

$$\begin{array}{ccc} {}^{cl}Y & \xrightarrow{h_{cl}} & Z_{cl} \\ \sim\downarrow & & \downarrow \\ {}^{cl}({}^{\leq n}Y) & \xrightarrow{{}^{cl}h_n} & {}^{cl}Z_n, \end{array}$$

where the right vertical map is automatically a closed embedding. In particular, we obtain that the cohomological amplitude of the functor $({}^{cl}h_n)_*$ also equals k. Therefore, the same is true for the functor

$$(h_n)_* : \operatorname{QCoh}({}^{\leq n}Y) \to \operatorname{QCoh}(Z_n).$$

Thus, Lemma 3.3.19 applies to h_n. Let

$$Z := Y \underset{{}^{\leq n}Y}{\sqcup} Z_n \in \operatorname{DGSch}.$$

By Corollary 3.4.4, we have a canonical map $g : Z \to \mathcal{X}$, which is a closed embedding since at the classical level this map is the same as g_n. Thus

$$Y \to Z \to \mathcal{X}$$

is the required factorization of f.

□

3.5.2. Let us now prove Proposition 1.7.8. Our proof will rely on the notion of square-zero extension, which will be reviewed in Sect. 4.5.1.

We begin with the following observation:

Lemma 3.5.3. *Let $\mathbf{C}$ be an ∞-category and $i : \mathbf{C}_1 \to \mathbf{C}$ a fully faithful functor. Assume that $\mathbf{C}$ is filtered. Then i is cofinal if and only if every object of $\mathbf{C}$ admits a map to an object in $\mathbf{C}_1$. In this case $\mathbf{C}_1$ is also filtered.*

We take $\mathbf{C} := (\operatorname{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}}$ and $\mathbf{C}_1 = (\operatorname{DGSch}_{\text{aft}})_{\text{closed in } \mathcal{X}}$. Having proved Corollary 1.6.6, it remains to show that every closed embedding

$$f : Y \to X$$

admits a factorization

$$Y \to Z \xrightarrow{g} \mathcal{X},$$

where $Z \in \operatorname{DGSch}_{\text{aft}}$ and g is also a closed embedding.

Step 1. Consider a factorization of ${}^{cl}f$

$${}^{cl}Y \xrightarrow{h_{cl}} Z_{cl} \xrightarrow{g_{cl}} {}^{cl}\mathcal{X},$$

where g_{cl} is a closed embedding. We claim that the "locally almost of finite type" assumption on $\mathcal{X}$ implies that the classical scheme Z_{cl} is automatically of finite type.

This follows from the next lemma:

Lemma 3.5.4. *If $\mathcal{X}_{cl}$ is a classical indscheme locally of finite type, and $X_{cl} \to \mathcal{X}_{cl}$ a closed embedding, where $X_{cl} \in \operatorname{Sch}$, then $X_{cl} \in \operatorname{Sch}_{\text{ft}}$.*

Proof of Lemma 3.5.4. Note that a classical scheme X_{cl} is of finite type if and only if for any classical k-algebra A and a filtered family $i \mapsto A_i$ of *subalegbras* such that $A = \underset{i}{\cup} A_i$, the map

$$\underset{i}{colim}\ \mathrm{Maps}(\mathrm{Spec}(A_i), X_{cl}) \to \mathrm{Maps}(\mathrm{Spec}(A), X_{cl})$$

is an isomorphism.

Note that since $A_i \to A$ are injective, the diagram

$$\begin{array}{ccc} \underset{i}{colim}\ \mathrm{Maps}(\mathrm{Spec}(A_i), X_{cl}) & \longrightarrow & \mathrm{Maps}(\mathrm{Spec}(A), X_{cl}) \\ \downarrow & & \downarrow \\ \underset{i}{colim}\ \mathrm{Maps}(\mathrm{Spec}(A_i), \mathcal{X}_{cl}) & \longrightarrow & \mathrm{Maps}(\mathrm{Spec}(A), \mathcal{X}_{cl}) \end{array}$$

is Cartesian. However, the bottom horizontal arrow is an isomorphism since $\mathcal{X} \in {}^{cl}\mathrm{PreStk}_{\mathrm{lft}}$. □

Step 2. We shall construct the required factorization of f by induction on $n \geq 0$. Namely, we shall construct a sequence of factorizations of ${}^{\leq n}f : {}^{\leq n}Y \to {}^{\leq n}\mathcal{X}$ as

$${}^{\leq n}Y \xrightarrow{h_n} Z_n \xrightarrow{g_n} {}^{\leq n}\mathcal{X},$$

with $Z_n \in {}^{\leq n}\mathrm{DGSch}_{\mathrm{ft}}$, g_n a closed embedding, and such that for $n \geq n'$, we have a commutative diagram

$$\begin{array}{ccc} {}^{\leq n'}Z_n & \xrightarrow{{}^{\leq n'}g_n} & {}^{\leq n'}\mathcal{X} \\ \sim\uparrow & & \uparrow\mathrm{id} \\ {}^{\leq n'}Z_{n'} & \xrightarrow{{}^{\leq n'}g_{n'}} & {}^{\leq n'}\mathcal{X}. \end{array}$$

Setting

$$Z := \underset{n}{colim}\ Z_n$$

(where the colimit is taken in DGSch) we will then obtain the desired factorization of f.

Step 3. Suppose (Z_{n-1}, g_{n-1}) have been constructed. Note that the maps

$$h_{n-1} : {}^{\leq n-1}Y \to Z_{n-1} \text{ and } {}^{\leq n-1}Y \to {}^{\leq n}Y$$

satisfy the conditions of Corollary 3.3.17. Set

$$Z_n' := Z_{n-1} \underset{{}^{\leq n-1}Y}{\sqcup} {}^{\leq n}Y.$$

We have ${}^{\leq n-1}Z_n' \simeq {}^{\leq n-1}Z_{n-1}$, and by Proposition 3.4.2 we obtain a natural map $g_n' : Z_n' \to {}^{\leq n}\mathcal{X}$.

To find the sought-for pair (Z_n, g_n), it suffices to find a factorization of g_n' as

$$Z_n' \to Z_n \xrightarrow{g_n} {}^{\leq n}\mathcal{X},$$

so that $Z_n \in {}^{\leq n}\mathrm{DGSch}_{\mathrm{ft}}$, and ${}^{\leq n-1}Z_n' \to {}^{\leq n-1}Z_n$ is an isomorphism.

Step 4. Note that the closed embedding

$${}^{\leq n-1}Y \to {}^{\leq n}Y$$

has a natural structure of a *square-zero extension*, see Corollary 4.5.8, by an ideal

$$\mathcal{I} \in \mathrm{QCoh}\left({}^{\leq n-1}Y\right)^{\heartsuit}[n].$$

Hence, the closed embedding $Z_{n-1} \to Z'_n$ also has a structure of a square-zero extension by

$$\mathcal{J} := (h_{n-1})_*(\mathcal{I}) \in \mathrm{QCoh}(Z_{n-1})^{\heartsuit}[n].$$

Step 5. Write $\mathcal{J}$ as a filtered colimit $\underset{\alpha}{colim}\, \mathcal{J}_\alpha$, where

$$\mathcal{J}_\alpha \in \mathrm{Coh}(Z_{n-1})^{\heartsuit}[n].$$

The category $\mathrm{Coh}(Z_{n-1})$ is well-defined since Z_{n-1} is almost of finite type.

By Sect. 4.5.1, we obtain a family $\alpha \mapsto Z_{n,\alpha}$ of objects of ${}^{\leq n}\mathrm{DGSch}$, for all of which ${}^{\leq n-1}Z_{n,\alpha} \simeq {}^{\leq n-1}Z_{n-1}$; moreover, we have isomorphisms

$$Z'_n \simeq \underset{\alpha}{lim}\, Z_{n,\alpha}$$

as objects of ${}^{\leq n}\mathrm{DGSch}$.

Now, since $\mathcal{X}$ is locally almost of finite type as an object of ${}^{\leq n}\mathrm{PreStk}$, the map

$$\underset{\alpha}{colim}\ \mathrm{Maps}(Z_{n,\alpha}, {}^{\leq n}\mathcal{X}) \to \mathrm{Maps}(Z'_n, {}^{\leq n}\mathcal{X})$$

is an isomorphism. In particular, the map g'_n factors through some $g_{n,\alpha} : Z_{n,\alpha} \to {}^{\leq n}\mathcal{X}$.

Now, the DG schemes $Z_{n,\alpha}$ all belong to ${}^{\leq n}\mathrm{DGSch}_{\mathrm{ft}}$, by construction. This gives the required factorization.

□

4. Deformation theory: recollections

This section is preparation for Sect. 5. Our goal is the following: given $\mathcal{X} \in \mathrm{PreStk}$ such that ${}^{cl}\mathcal{X}$ is a classical indscheme, we would like to give necessary and sufficient conditions for $\mathcal{X}$ to be a DG indscheme. In this section we shall discuss what will be called Conditions (A), (B) and (C) that are satisfied by every DG indscheme. In Sect. 5 we will show that these conditions are also sufficient.

Conditions (A), (B) and (C) say that $\mathcal{X}$ has a reasonable deformation theory. We will encode this by the property of sending certain push-outs (in $\mathrm{DGSch}^{\mathrm{aff}}$) to fiber products (in ∞-Grpd).

4.1. **Split square-zero extensions and Condition (A).**

4.1.1. *Split square-zero extensions.*

For $Z \in {}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}}$. We define the category ${}^{\leq n}\mathrm{SplitSqZExt}(Z)$ of *split square-zero extensions of Z* to be the opposite of $\mathrm{QCoh}(Z)^{\geq -n, \leq 0}$.

There is a natural forgetful functor

$${}^{\leq n}\mathrm{SplitSqZExt}(Z) \to ({}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}})_{Z/}, \quad \mathcal{F} \mapsto Z_{\mathcal{F}}.$$

Explicitly, locally in the Zariski topology if $Z = S = \mathrm{Spec}(A)$, and $\mathcal{M} := \Gamma(S, \mathcal{F})$,

$$S_{\mathcal{F}} := \mathrm{Spec}(A \oplus \mathcal{M}),$$

where the multiplication on $\mathcal{M}$ is zero.

4.1.2. The category

$$^{\leq n}\text{SplitSqZExt}(Z) = (\text{QCoh}(Z)^{\geq -n, \leq 0})^{\text{op}}$$

has push-outs: for $\mathcal{F}_1, \mathcal{F}_2 \to \mathcal{F} \in \text{QCoh}(Z)^{\geq -n, \leq 0}$ the sought-for push-out is given by

$$\mathcal{F}' := \mathcal{F}_1 \underset{\mathcal{F}}{\times} \mathcal{F}_2,$$

where the fiber product is taken in $\text{QCoh}(Z)^{\geq -n, \leq 0}$, i.e.,

$$\mathcal{F}' \simeq \tau^{\leq 0}\left(\mathcal{F}_1 \underset{\mathcal{F}}{\times} \mathcal{F}_2\right).$$

By Corollary 3.3.17, the forgetful functor

$$^{\leq n}\text{SplitSqZExt}(Z) \to (^{\leq n}\text{DGSch}_{\text{qsep-qc}})_{Z/} \to {}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$$

commutes with push-outs. I.e., for $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}, \mathcal{F}'$ as above, the map

$$Z_{\mathcal{F}_1} \underset{Z_{\mathcal{F}}}{\sqcup} Z_{\mathcal{F}_2} \to Z_{\mathcal{F}'}$$

is an isomorphism, where the latter push-out is taken in the category $^{\leq n}\text{DGSch}_{\text{qsep-qc}}$. Moreover, if Z is affine, the above push-out agrees with the push-out in the category $^{\leq n}\text{DGSch}^{\text{aff}}$.

4.1.3. Let $\mathcal{X}$ be an object of $^{\leq n}\text{PreStk}$. For $S \in {}^{\leq n}\text{DGSch}^{\text{aff}}$ and a map $x : S \to \mathcal{X}$, consider the category $^{\leq n}\text{SplitSqZExt}(S, x)$ consisting of triples

$$\{\mathcal{F} \in \text{QCoh}(S)^{\geq -n, \leq 0},\ x' : S_{\mathcal{F}} \to \mathcal{X},\ x'|_S \simeq x\}.$$

I.e.,

$$^{\leq n}\text{SplitSqZExt}(S, x) := {}^{\leq n}\text{SplitSqZExt}(S) \underset{(^{\leq n}\text{DGSch}^{\text{aff}})_{S/}}{\times} (^{\leq n}\text{DGSch}^{\text{aff}})_{S//\mathcal{X}}.$$

Definition 4.1.4. *We shall say that $\mathcal{X}$ satisfies indscheme-like Condition (A) if for any S and x as above, the category $^{\leq n}\text{SplitSqZExt}(S, x)$ is filtered.*

We can reformulate the above condition in more familiar terms. Another familiar reformulation is described in Sect. 4.1.7 below.

4.1.5. Consider the functor

$$^{\geq -n}(T^*_x\mathcal{X}) : \text{QCoh}(S)^{\geq -n, \leq 0} \to \infty\text{-Grpd}$$

defined by

$$^{\geq -n}(T^*_x\mathcal{X})(\mathcal{F}) := \{x' : S_{\mathcal{F}} \to \mathcal{X},\ x'|_S \simeq x\}. \tag{4.1}$$

I.e.,

$$\mathcal{F} \mapsto \{S_{\mathcal{F}}\} \underset{^{\leq n}\text{SplitSqZExt}(S)}{\times} {}^{\leq n}\text{SplitSqZExt}(S, x) = \{S_{\mathcal{F}}\} \underset{(^{\leq n}\text{DGSch}^{\text{aff}})_{S/}}{\times} (^{\leq n}\text{DGSch}^{\text{aff}})_{S//\mathcal{X}}.$$

The following results from [Lu0, Prop. 5.3.2.9]:

Lemma 4.1.6. *The prestack $\mathcal{X}$ satisfies Condition (A) if and only if the functor $^{\geq -n}(T^*_x\mathcal{X})$ preserves fiber products.*

4.1.7. *The pro-cotangent space.* Recall ([Lu0, Cor. 5.3.5.4]) that for an arbitrary ∞-category $\mathbf{C}$ that has fiber products, and a functor $F : \mathbf{C} \to \infty\text{-Grpd}$, the condition that F preserve fiber products is equivalent to the condition that F be pro-representable.

Thus, we obtain:

Corollary 4.1.8. *A prestack $\mathcal{X}$ satisfies Condition (A) if and only if for every*

$$(S, x : S \to \mathcal{X}) \in ({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{X}},$$

the functor

$${}^{\geq -n}(T^*_x\mathcal{X}) : \mathrm{QCoh}(S)^{\geq -n, \leq 0} \to \infty\text{-Grpd}$$

is pro-representable[17].

Henceforth, whenever $\mathcal{X}$ satisfies Condition (A), we shall denote by ${}^{\geq -n}(T^*_x\mathcal{X})$ the corresponding object of $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$. We shall refer to ${}^{\geq -n}(T^*_x\mathcal{X})$ as "the pro-cotangent space to $\mathcal{X}$ at $x : S \to \mathcal{X}$."

Thus, an alternative terminology for Condition (A) is that the prestack $\mathcal{X}$ *admits connective pro-cotangent spaces.* [18]

4.1.9. Since fiber products in $\mathrm{QCoh}(S)^{\geq -n, \leq 0}$ correspond to push-outs in ${}^{\leq n}\mathrm{SplitSqZExt}(S)$, from Lemma 4.1.6 we obtain that Condition (A) is equivalent to requiring that the functor

$${}^{\leq n}\mathrm{SplitSqZExt}(S) \to \infty\text{-Grpd}$$

given by

$$S_{\mathcal{F}} \mapsto \{x' : S_{\mathcal{F}} \to \mathcal{X},\ x'|_S \simeq x\} = \{S_{\mathcal{F}}\} \underset{{}^{\leq n}\mathrm{SplitSqZExt}(S)}{\times} {}^{\leq n}\mathrm{SplitSqZExt}(S, x) \tag{4.2}$$

take push-outs to fiber products.

Since the forgetful functor

$${}^{\leq n}\mathrm{SplitSqZExt}(S) \to {}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}}$$

preserves push-outs, from Proposition 3.4.2, we obtain:

Corollary 4.1.10. *Any $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ satisfies Condition (A).*

4.1.11. Going back to a general prestack $\mathcal{X}$, assume that $\mathcal{X}$ satisfies Zariski descent. This allows us to extend $\mathcal{X}$ to a functor

$$({}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}})^{\mathrm{op}} \to \infty\text{-Grpd}$$

by

$$Z \mapsto \underset{S \in Zar(Z)}{lim} \mathrm{Maps}(S, \mathcal{X}),$$

where $Zar(Z)$ is the category of affine schemes endowed with an open embedding into X.

The following is straightforward:

---

[17]Since $\mathcal{X}$ is an accessible functor, so is ${}^{\geq -n}(T^*_x\mathcal{X})$.

[18]Note that k-Artin stacks for $k > 0$ viewed as objects of PreStk typically do not satisfy the above condition, as their (pro)-cotangent spaces belong to $\mathrm{QCoh}(S)^{\geq -n, \leq k}$ but not to $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$; i.e., they do not satisfy the connectivity condition.

Lemma 4.1.12. *If $\mathcal{X}$ satisfies Condition (A), $Z \in {}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}}$ and $x : Z \to \mathcal{X}$ is a map, then the functor*

$$ {}^{\geq -n}(T^*_x\mathcal{X}) : \mathrm{QCoh}(Z)^{\geq -n, \leq 0} \to \infty\text{-Grpd}, \quad \mathcal{F} \mapsto \{Z_{\mathcal{F}}\} \underset{({}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}})_{Z/}}{\times} ({}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}})_{S/\,/\mathcal{X}} $$

preserves fiber products.

In particular, we obtain that ${}^{\geq -n}(T^*_x\mathcal{X})$ is given by an object of $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n, \leq 0})$.

4.1.13. *The relative situation.* The functor ${}^{\geq -n}(T^*_x\mathcal{X})$ can be defined in a relative situation, i.e., when we are dealing with a map of prestacks $\phi : \mathcal{X} \to \mathcal{Y}$. Namely, for $x : S \to \mathcal{X}$ as above, we set $T^*_x\mathcal{X}/\mathcal{Y}$ to be the functor

$$ \mathrm{QCoh}(S)^{\geq -n, \leq 0} \to \infty\text{-Grpd} $$

defined by

$$ \mathcal{F} \mapsto \{S_{\mathcal{F}}\} \underset{{}^{\leq n}\mathrm{SplitSqZExt}(S, \phi\circ x)}{\times} {}^{\leq n}\mathrm{SplitSqZExt}(S, x). $$

where $S_{\mathcal{F}}$ defines the point of ${}^{\leq n}\mathrm{SplitSqZExt}(S, \phi \circ x)$ equal to the composite

$$ S_{\mathcal{F}} \overset{\pi}{\to} S \overset{\phi\circ x}{\longrightarrow} \mathcal{Y}, $$

and where $\pi : S_{\mathcal{F}} \to S$ is the canonical projection.

Note that if both $\mathcal{X}$ and $\mathcal{Y}$ admit connective pro-cotangent spaces, $T^*_x\mathcal{X}/\mathcal{Y}$, as an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$, is given by

$$ \tau^{\geq -n}\left(\mathrm{Cone}(T^*_{\phi\circ x}\mathcal{Y} \to T^*_x\mathcal{X})\right). $$

4.2. **A digression: pro-objects in** QCoh**.**

4.2.1. Let $\mathbf{C}$ be an ∞-category. We consider the category $\mathrm{Pro}(\mathbf{C})$, which is, by definition, the full subcategory of $\mathrm{Funct}(\mathbf{C}, \infty\text{-Grpd})$ that consists of accessible functors

$$ F : \mathbf{C} \to \infty\text{-Grpd} $$

that can be written as *filtered* colimits of co-representable functors.

Let $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between ∞-categories. Then the functor

$$ \mathrm{LKE}_\Phi : \mathrm{Funct}(\mathbf{C}_1, \infty\text{-Grpd}) \to \mathrm{Funct}(\mathbf{C}_2, \infty\text{-Grpd}) $$

sends $\mathrm{Pro}(\mathbf{C}_1)$ to $\mathrm{Pro}(\mathbf{C}_2)$; we shall denote by

$$ \mathrm{Pro}(\Phi) : \mathrm{Pro}(\mathbf{C}_1) \to \mathrm{Pro}(\mathbf{C}_2) $$

the resulting functor.

Note that if Φ admits a right adjoint, denoted Ψ, then $\mathrm{Pro}(\Phi)$ can be computed as

$$ (\mathrm{Pro}(\Phi)(F))(\mathbf{c}_2) = F(\Psi(\mathbf{c}_2)), \quad F \in \mathrm{Pro}(\mathbf{C}_1), \mathbf{c}_2 \in \mathbf{C}_2. \tag{4.3} $$

4.2.2. Let $\mathbf{C}$ be a stable ∞-category. In this case, the category $\mathrm{Pro}(\mathbf{C})$ is also stable. [19]

If $\mathbf{C}_1$ and $\mathbf{C}_2$ is a pair of stable categories and $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$ is an exact functor, then $\mathrm{Pro}(\Phi)$ is also exact.

[19] Note, however, that even if $\mathbf{C}$ is presentable, the category $\mathrm{Pro}(\mathbf{C})$ is not, so caution is required when applying such results as the adjoint functor theorem.

4.2.3. Let $\mathbf{C}$ be a stable ∞-category and F an object of $\operatorname{Pro}(\mathbf{C})$. Then F gives rise to an exact functor

$$F^{\mathrm{Sp}}:\mathbf{C}\to \mathrm{Spectra},$$

such that

$$F\simeq \Omega^\infty\circ F^{\mathrm{Sp}}.$$

If $\mathbf{C}$ arises from a DG category (or, equivalently, is tensored over Vect), then the functor F^{Sp} can be further upgraded to a functor

$$F^{\mathrm{Vect}}:\mathbf{C}\to \mathrm{Vect}\,.$$

4.2.4. Suppose that $\mathbf{C}$ is endowed with a t-structure. In this case, $\operatorname{Pro}(\mathbf{C})$ also inherits a t-structure: its connective objects are those $F\in \operatorname{Pro}(\mathbf{C})$ such that $F(x)=0$ for $x\in \mathbf{C}^{>0}$.

Restriction of functors defines a map

$$\operatorname{Pro}(\mathbf{C})^{\leq 0}\to \operatorname{Pro}(\mathbf{C}^{\leq 0}),$$

which is easily seen to be an equivalence. Similarly, for any $n\geq 0$, the natural functor

$$\operatorname{Pro}(\mathbf{C})^{\geq -n,\leq 0}\to \operatorname{Pro}(\mathbf{C}^{\geq -n,\leq 0})$$

is an equivalence.

4.2.5. Now consider the following situation specific to QCoh. Let Z be a DG scheme. We have the following two categories

$$\operatorname{Pro}(\operatorname{QCoh}(Z)) \text{ and } \underset{S\in Zar(Z)}{lim} \operatorname{Pro}(\operatorname{QCoh}(S)).$$

Left Kan extension along

$$\mathcal{F}\mapsto \mathcal{F}|_S:\operatorname{QCoh}(Z)\to \operatorname{QCoh}(S)$$

defines a functor

$$\operatorname{Pro}(\operatorname{QCoh}(Z))\to \underset{S\in Zar(Z)}{lim} \operatorname{Pro}(\operatorname{QCoh}(S)). \tag{4.4}$$

This functor admits a right adjoint, which is tautologically described as follows. To

$$\{S\mapsto (F_S\in \operatorname{Pro}(\operatorname{QCoh}(S)))\}\in \underset{S\in Zar(Z)}{lim} \operatorname{Pro}(\operatorname{QCoh}(S))$$

it assigns $F\in \operatorname{Pro}(\operatorname{QCoh}(Z))$ given by

$$F(\mathcal{F}):=\underset{S\in Zar(Z)}{lim} F_S(\mathcal{F}|_S).$$

We claim:

Lemma 4.2.6. *Assume that Z is quasi-separated and quasi-compact. Then the above two functors*

$$\operatorname{Pro}(\operatorname{QCoh}(Z))\rightleftarrows \underset{S\in Zar(Z)}{lim} \operatorname{Pro}(\operatorname{QCoh}(S)) \tag{4.5}$$

are mutually inverse.

Proof. A standard argument shows that instead of $Zar(S)$ we can consider a *finite* limit corresponding to a Zariski hypercovering.

Note that by (4.3), the left Kan extension $\text{Pro}(\text{QCoh}(Z)) \to \text{Pro}(\text{QCoh}(S))$ can be also expressed as the functor

$$F \mapsto F \circ (j_S)_*,$$

where j_S denotes the open embedding $S \hookrightarrow Z$.

Then the fact that the two adjunction maps are isomorphisms follows from the fact that

$$\text{Id}_{\text{QCoh}(S)} \to \underset{S}{lim}\ (j_S)_* \circ j_S^*$$

is an isomorphism and the functors F and F_S involved commute with finite limits. □

Note that the lemma (with the same proof) also applies when we replace the category $\text{Pro}(\text{QCoh}(Z))$ by $\text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$ for any $n \geq 0$.

4.3. **Functoriality of split square-zero extensions and Condition (B).**

4.3.1. Let $\phi : Z_1 \to Z_2$ be an map between objects of ${}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$. Direct image ϕ_* composed with the truncation $\tau^{\leq 0}$ defines a functor

$${}^{\leq 0}\phi_* : \text{QCoh}(Z_1)^{\geq -n, \leq 0} \to \text{QCoh}(Z_2)^{\geq -n, \leq 0},$$

i.e., a functor

$${}^{\leq n}\text{SplitSqZExt}(Z_1) \to {}^{\leq n}\text{SplitSqZExt}(Z_2).$$

It follows from Corollary 3.3.17 that the following diagram is commutative

$$\begin{array}{ccc} {}^{\leq n}\text{SplitSqZExt}(Z_1) & \longrightarrow & {}^{\leq n}\text{SplitSqZExt}(Z_2) \\ \downarrow & & \downarrow \\ (\text{DGSch}_{\text{qsep-qc}})_{Z_1/} & \longrightarrow & (\text{DGSch}_{\text{qsep-qc}})_{Z_2/}, \end{array} \tag{4.6}$$

where the bottom horizontal arrow is the push-out functor

$$Z_1' \mapsto Z_1' \underset{Z_1}{\sqcup} Z_2.$$

4.3.2. Assume now that $Z_1 = S_1$ and $Z_2 = S_2$ are affine. Let $\mathcal{X}$ be an object of ${}^{\leq n}\text{PreStk}$, and x_2 an S_2-point of $\mathcal{X}$. Set $x_1 := x_2 \circ \phi : S_1 \to \mathcal{X}$. Composition defines a map

$${}^{\leq n}\text{SplitSqZExt}(S_1) \underset{{}^{\leq n}\text{SplitSqZExt}(S_2)}{\times} {}^{\leq n}\text{SplitSqZExt}(S_2, x_2) \to {}^{\leq n}\text{SplitSqZExt}(S_1, x_1). \tag{4.7}$$

Definition 4.3.3. *We shall say that* $\mathcal{X} \in {}^{\leq n}\text{PreStk}$ *satisfies indscheme-like Condition (B) if the above functor is an equivalence for any* (S_1, S_2, ϕ).

4.3.4. Using (4.6), we can reformulate Condition (B) as saying that the presheaf $\mathcal{X}$ should take push-outs in ${}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$ of the form $(S_1)_{\mathcal{F}_1} \underset{S_1}{\sqcup} S_2$ to fiber products, where $S_1, S_2 \in \text{DGSch}^{\text{aff}}$.

By Proposition 3.4.2, we obtain:

Corollary 4.3.5. *Any* $\mathcal{X} \in {}^{\leq n}\text{DGindSch}$ *satisfies Condition (B).*

4.3.6. Let us assume that $\mathcal{X}$ satisfies Condition (A). In this case, by (4.3), the map (4.7) can be interpreted as a map in $\text{Pro}(\text{QCoh}(S_1)^{\geq -n, \leq 0})$:

$$ {}^{\geq -n}(T^*_{x_1}\mathcal{X}) \to \text{Pro}({}^{\geq -n}\phi^*)\left({}^{\geq -n}(T^*_{x_2}\mathcal{X})\right). \tag{4.8} $$

We obtain:

Lemma 4.3.7. *As object $\mathcal{X} \in {}^{\leq n-1}\text{PreStk}$, satisfying condition (A), satisfies Condition (B) if and only if the map* (4.8) *be an isomorphism.*

4.3.8. We shall use the following terminology:

Definition 4.3.9. *We shall say that $\mathcal{X} \in {}^{\leq n}\text{PreStk}$* admits a connective pro-cotangent complex *if it satisfies both Conditions (A) and (B).*

In other words, $\mathcal{X}$ admits a connective pro-cotangent complex if it admits connective pro-cotangent spaces, whose formation is compatible with pullbacks under morphisms of affine DG schemes.

4.3.10. Let us now assume that $\mathcal{X}$ satisfies Zariski descent, as well as Conditions (A) and (B).

Thus, for $Z \in {}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$ and $x : Z \to \mathcal{X}$, we have a well-defined object

$$ {}^{\geq -n}(T^*_x\mathcal{X}) \in \text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0}). $$

We wish to compare the restriction of ${}^{\geq -n}(T^*_x\mathcal{X})$ to a given affine Zariski open $S \subset Z$ with

$$ {}^{\geq -n}(T^*_{x|_S}\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^{\geq -n, \leq 0}). $$

As in (4.8), we have a natural map

$$ {}^{\geq -n}(T^*_{x|_S}\mathcal{X}) \to {}^{\geq -n}(T^*_x\mathcal{X})|_S. \tag{4.9} $$

We claim:

Lemma 4.3.11. *The map* (4.9) *is an isomorphism.*

Proof. This follows from the description of $\text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$ given by Lemma 4.2.6.

□

4.4. **The cotangent complex of a DG scheme.**

4.4.1. Assume for a moment that $\mathcal{X} = X \in {}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$. It is well-known that in this case the object ${}^{\geq -n}(T^*_x X) \in \text{Pro}\left(\text{QCoh}(S)^{\geq -n, \leq 0}\right)$ actually belongs to $\text{QCoh}(S)^{\geq -n, \leq 0}$:

Proof. It is easy to readuce the assertion to the case when X is affine. It is enough to show that the functor ${}^{\geq -n}(T^*_x X)$ commutes with filtered limits. But filtered limits in $\text{QCoh}(S)^{\geq -n, \leq 0}$ map to filtered colimits in ${}^{\leq n}\text{DGSch}^{\text{aff}}$, and the assertion follows. □

4.4.2. We obtain that for any $X \in {}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$ we have a well-defined object ${}^{\geq -n}(T^*X) \in \text{QCoh}(X)^{\geq -n, \leq 0}$, such that for any affine S with a map $x : S \to X$, we have

$$ {}^{\geq -n}(T^*_x X) \simeq {}^{\geq -n}x^*({}^{\geq -n}(T^*X)). \tag{4.10} $$

Moreover, as schemes are sheaves in the Zariski topology, the isomorphism (4.10) remains valid when $S \in {}^{\leq n}\text{DGSch}^{\text{aff}}$ is replaced by an arbitrary object $Z \in {}^{\leq n}\text{DGSch}_{\text{qsep-qc}}$.

4.4.3. In particular, taking $Z = X$ and x to be the identity map, we obtain that the identity map on ${}^{\geq -n}(T^*Z)$ defines a canonical map

$$ \mathfrak{d}_{can} : Z_{{}^{\geq -n}(T^*Z)} \to Z. $$

4.4.4. Assume now that $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$, and is written as in (1.1) for some index set A, and let $Z \in {}^{\leq n}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$.

Let $x : Z \to \mathcal{X}$ be a map that factors through a map $x_{\alpha_0} : Z \to X_{\alpha_0}$. We obtain that ${}^{\geq -n}(T^*_x\mathcal{X})$ can be explicitly presented as a pro-object of $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n, \leq 0})$. Namely, we have:

$$ {}^{\geq -n}(T^*_x\mathcal{X}) \simeq \underset{\alpha \in \mathsf{A}_{\alpha_0/}}{\text{“}lim\text{”}} \, {}^{\geq -n}(T^*_{x_\alpha} X_\alpha), \tag{4.11} $$

where x_α denotes the composition $Z \xrightarrow{x_{\alpha_0}} X_{\alpha_0} \to X_\alpha$.

4.4.5. Let $\mathcal{X}$ again be an arbitrary object of ${}^{\leq n}\mathrm{PreStk}$, satisfying Condition (A), $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ and $x : S \to \mathcal{X}$ a point. We claim that there exists a canonical map in $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$

$$ (dx)^* : {}^{\geq -n}(T^*_x\mathcal{X}) \to {}^{\geq -n}(T^*S). \tag{4.12} $$

Indeed, it corresponds to the map $S_{{}^{\geq -n}(T^*S)} \to \mathcal{X}$ given by the composite

$$ S_{{}^{\geq -n}(T^*S)} \overset{\mathfrak{d}_{can}}{\to} S \overset{x}{\to} \mathcal{X}. $$

The same remains true with $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ replaced by $Z \in {}^{\leq n}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$, whenever $\mathcal{X}$ satisfies Zariski descent.

4.5. **General square-zero extensions.**

4.5.1. Let Z be an object of ${}^{\leq n-1}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$. The category ${}^{\leq n-1}\mathrm{SqZExt}(Z)$ of *square-zero extensions of* Z is defined to be the opposite of

$$ \left((\mathrm{QCoh}(Z)^{\geq -n+1, \leq 0})_{{}^{\geq -n}(T^*Z)[-1]/} \right)^{\mathrm{op}}. $$

4.5.2. We have a natural forgetful functor

$$ {}^{\leq n-1}\mathrm{SqZExt}(Z) \to ({}^{\leq n-1}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{Z/}, $$

defined as follows.

For $\mathcal{I} \in \mathrm{QCoh}(Z)^{\geq -n+1, \leq 0}$ and a map $\gamma : {}^{\geq -n}(T^*Z) \to \mathcal{I}[1]$, we construct the corresponding scheme Z' as the push-out in ${}^{\leq n}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$

$$ Z \underset{Z_{\mathcal{I}[1]}}{\sqcup} Z, \tag{4.13} $$

where the first map $Z_{\mathcal{I}[1]} \to Z$ is the projection, and the second map corresponds to γ via the universal property of ${}^{\geq -n}(T^*Z)$.

We note that when Z is affine, by Corollary 3.3.17, the push-out in (4.13) is isomorphic to the corresponding push-out taken in $\mathrm{DGSch}^{\mathrm{aff}}$.

4.5.3. Let us denote by i the resulting closed embedding

$$ Z \to Z \underset{Z_{\mathcal{I}[1]}}{\sqcup} Z $$

corresponding to the canonical map of the first factor.

We have an exact triangle in $\mathrm{QCoh}(Z')$:

$$ i_*(\mathcal{I}) \to \mathcal{O}_{Z'} \to i_*(\mathrm{QCoh}(Z)). $$

Remark 4.5.4. Informally, we can think of the data of $i_*(\mathcal{I}) \in \mathrm{QCoh}(Z)^{\geq -n+1,\leq 0}$ for

$$(\mathcal{I},\gamma) \in (\mathrm{QCoh}(Z)^{\geq -n+1,\leq 0})_{\geq -n(T^*Z)[-1]/}$$

as the "ideal" of Z inside Z'. The fact that this "ideal" comes as the direct image of an object in $\mathrm{QCoh}(Z)$ reflects the fact that its square is zero. This explains the terminology of "square-zero extensions."

Remark 4.5.5. Let us emphasize that, unlike the situation of classical schemes, the forgetful functor

$${}^{\leq n-1}\mathrm{SqZExt}(Z) \to ({}^{\leq n-1}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{Z/}$$

is *not* fully faithful. I.e., being a square-zero extension is not a property, but is extra structure.

4.5.6. However, we have the following:

Lemma 4.5.7. *For $Z \in {}^{\leq n-1}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$, the forgetful functor*

$${}^{\leq n-1}\mathrm{SqZExt}(Z) \to ({}^{\leq n-1}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{Z/}$$

induces an equivalence between the full subcategories of both sides corresponding to $Z \hookrightarrow Z'$ for which ${}^{\leq n-2}Z \to {}^{\leq n-2}Z'$ is an isomorphism.

Corollary 4.5.8. *For $Z' \in ({}^{\leq n}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})$, the canonical map ${}^{\leq n-1}Z \to Z$ has a canonical structure of an object of ${}^{\leq n}\mathrm{SqZExt}(Z')$.*

In addition, we have:

Lemma 4.5.9. *For $Z \in \mathrm{Sch}_{\mathrm{qsep\text{-}qc}}$, the forgetful unctor*

$${}^{\leq 0}\mathrm{SqZExt}(Z) \to ({}^{\leq 0}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}})_{Z/}$$

is fully faithful and its essential image consists of closed embeddings $Z \hookrightarrow Z'$, such that the ideal $\mathcal{I}$ of Z in Z' satisfies $\mathcal{I}^2 = 0$.

4.5.10. Let $\phi : Z_1 \to Z_2$ be an *affine* map between objects of ${}^{\leq n-1}\mathrm{DGSch}_{\mathrm{qsep\text{-}qc}}$. There is a canonically defined functor

$$(4.14)\qquad {}^{\leq n-1}\mathrm{SqZExt}(Z_1) \to {}^{\leq n-1}\mathrm{SqZExt}(Z_2),$$

which it sends

$$(\mathcal{I}_1,\gamma_1) \in (\mathrm{QCoh}(Z_1)^{\geq -n+1,\leq 0})_{\geq -n(T^*Z_1)[-1]/}$$

to

$$(\mathcal{I}_2,\gamma_2) \in (\mathrm{QCoh}(Z_2)^{\geq -n+1,\leq 0})_{\geq -n(T^*Z_2)[-1]/},$$

where

$$\mathcal{I}_2 := \phi_*(\mathcal{I}_1),$$

and γ_2 is obtained by the (ϕ^*,ϕ_*) adjunction from the map

$${}^{\geq -n}\phi^*({}^{\geq -n}(T^*Z_2)) \overset{(d\phi)^*}{\longrightarrow} {}^{\geq -n}(T^*Z_1) \overset{\gamma_1}{\longrightarrow} \mathcal{I}_1.$$

4.5.11. The following assertion results from the construction:

Lemma 4.5.12. *The following diagram commutes*

$$\begin{array}{ccc} {}^{\leq n-1}\mathrm{SqZExt}(Z_1) & \longrightarrow & {}^{\leq n-1}\mathrm{SqZExt}(Z_2) \\ \downarrow & & \downarrow \\ ({}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}})_{Z_1/} & \longrightarrow & ({}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}})_{Z_2/}, \end{array}$$

where the bottom horizontal arrow is the push-out functor

$$Z_1' \mapsto Z_1' \underset{Z_1}{\sqcup} Z_2.$$

4.6. **Infinitesimal cohesiveness and Condition (C).**

4.6.1. Let $S \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$ and let $(\mathcal{I}, \gamma)$ be an object of ${}^{\leq n-1}\mathrm{SqZExt}(S)$. Let

$$S' := S \underset{S_{\mathcal{I}[1]}}{\sqcup} S$$

be as in (4.13).

For $\mathcal{X} \in {}^{\leq n}\mathrm{PreStk}$, consider the resulting map

$$\mathrm{Maps}(S', \mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X}) \underset{\mathrm{Maps}(S_{\mathcal{I}[1]}, \mathcal{X})}{\times} \mathrm{Maps}(S, \mathcal{X}). \tag{4.15}$$

Definition 4.6.2. *We shall say that $\mathcal{X}$ satisfies indscheme-like Condition (C) if the map* (4.15) *is an isomorphism for any $(S, \mathcal{I}, \gamma)$ as above.*

An alternative terminology for prestacks satisfying Condition (C) is *infinitesimally cohesive*.

4.6.3. Note that from Proposition 3.4.2 we obtain:

Corollary 4.6.4. *Any $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ satisfies Condition (C).*

4.6.5. For $S \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$, let ${}^{\leq n-1}\mathrm{SqZExt}(S, x)$ be the category of triples

$$\{S \hookrightarrow S',\ x' : S' \to \mathcal{X},\ x'|_S \simeq x\},$$

where $S \hookrightarrow S'$ is a square-zero extension with $S' \in {}^{\leq n-1}\mathrm{DGSch}$. I.e.,

$${}^{\leq n-1}\mathrm{SqZExt}(S, x) := {}^{\leq n-1}\mathrm{SqZExt}(S) \underset{{}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}_{S/}}{\times} {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}_{S//\mathcal{X}}.$$

Suppose now that $\mathcal{X}$ satisfies Condition (A). For $S \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$, recall the map in $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$

$$(dx)^* : {}^{\geq -n}(T^*_x\mathcal{X}) \to {}^{\geq -n}(T^*S).$$

Consider the object

$$\mathrm{Cone}((dx)^*)[-1] \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n+1, \leq 1}).$$

Hence, we obtain:

Lemma 4.6.6. *An object $\mathcal{X} \in {}^{\leq n}\mathrm{PreStk}$, satisfying Condition (A), satisfies condition (C) if and only if the naturally defined functor*

$${}^{\leq n-1}\mathrm{SqZExt}(S, x) \to \left(\left(\mathrm{QCoh}(S)^{\geq -n+1, \leq 0} \right)_{\mathrm{Cone}((dx)^*)[-1]/} \right)^{\mathrm{op}}$$

is an equivaence.

4.6.7. Assume now that $\mathcal{X}$ satisfies Zariski descent as well as Conditions (A) and (C). We obtain that for $Z \in {}^{\leq n-1}\mathrm{DGSch}$ and a given $x : Z \to \mathcal{X}$, the description of the category ${}^{\leq n-1}\mathrm{SqZExt}(Z,x)$ as

$$\left(\left(\mathrm{QCoh}(Z)^{\geq -n+1,\leq 0}\right)_{\mathrm{Cone}((dx)^*)[-1]/}\right)^{\mathrm{op}}$$

remains valid.

Moreover, we have the following:

Lemma 4.6.8. *Under the above circumstances the following are equivalent:*

(a) *The category* ${}^{\leq n-1}\mathrm{SqZExt}(Z,x)$ *is filtered.*

(b) $T^*_x Z/\mathcal{X}[-1] \simeq \mathrm{Cone}((dx)^*)[-1]$ *belongs to* $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n+1,\leq 0})$.

(c) *The map*

$$H^0((dx)^*) : H^0\left({}^{\geq -n}(T^*_x\mathcal{X})\right) \to H^0\left({}^{\geq -n}(T^*Z)\right)$$

is surjective.

4.6.9. We note that condition (c) in Lemma 4.6.8 is satisfied when $\mathcal{X}$ is an indscheme, and the map $x : Z \to \mathcal{X}$ is a closed embedding.

4.6.10. Let $\phi : S_1 \to S_2$ be a map in ${}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$. For $x_2 : S_2 \to \mathcal{X}$, composition defines a map

$$(4.16) \qquad {}^{\leq n-1}\mathrm{SqZExt}(S_2,x_2) \underset{{}^{\leq n-1}\mathrm{SqZExt}(S_2)}{\times} {}^{\leq n-1}\mathrm{SqZExt}(S_1) \to {}^{\leq n-1}\mathrm{SqZExt}(S_1,x_1).$$

using the functor (4.14).

From the definitions, we obtain:

Lemma 4.6.11. *If $\mathcal{X}$ satisfies Conditions (B) and (C), then the map* (4.16) *is an isomorphism.*

4.6.12. If $\mathcal{X}$ satisfies Zariski descent, then the same continues to be true for S_1 and S_2 replaced by arbitrary objects $Z_1, Z_2 \in {}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}}$, but keeping the assumption that $f : Z_1 \to Z_2$ be affine.

In other words, the map (4.16) is an isomorphism, where

$${}^{\leq n-1}\mathrm{SqZExt}(Z_1) \to {}^{\leq n-1}\mathrm{SqZExt}(Z_2)$$

is the functor defined in (4.14).

4.6.13. Now suppose that $Z_1, Z_2 \in {}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}}$ are as above, but the map f is not necessarily affine. Assume that $\mathcal{X}$ satisfies Zariski descent, and let $x_2 : Z_2 \to \mathcal{X}$ be a map satisfying the equivalent conditions of Lemma 4.6.8. Let $x_1 = x_2 \circ f$. In this situation, the Sect. 4.6.12 still applies. Namely, we have:

Lemma 4.6.14. *In the above situation, if $\mathcal{X}$ satisfies conditions (A), (B) and (C), there is a canonically defined functor*

$${}^{\leq n-1}\mathrm{SqZExt}(Z_1,x_1) \to {}^{\leq n-1}\mathrm{SqZExt}(Z_2,x_2),$$

such that the diagram

$$\begin{array}{ccc} {}^{\leq n-1}\mathrm{SqZExt}(Z_1,x_1) & \longrightarrow & {}^{\leq n-1}\mathrm{SqZExt}(Z_2,x_2) \\ \downarrow & & \downarrow \\ ({}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}})_{Z_1/} & \longrightarrow & ({}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}})_{Z_2/} \end{array}$$

commutes, where the bottom horizontal arrow is the push-out functor

$$Z_1' \mapsto Z_1' \underset{Z_1}{\sqcup} Z_2.$$

Proof. By definition, an object of ${}^{\leq n-1}\mathrm{SqZExt}(Z_1, x_1)$ is given by a map $T^*_{x_1} Z_1/\mathcal{Y} \to \mathcal{I}[1]$ for $\mathcal{I} \in \mathrm{QCoh}(Z_1)^{\geq -n+1, \leq 0}$. This gives a map $T^*_{x_2} Z_2/\mathcal{Y} \to f_*(\mathcal{I}[1])$. By assumption on x_2, this map canonically factors through $\tau^{\leq -1} f_*(\mathcal{I}[1]) = \tau^{\leq 0}(f_*\mathcal{I})[1]$. This gives the desired functor

$${}^{\leq n-1}\mathrm{SqZExt}(Z_1, x_1) \to {}^{\leq n-1}\mathrm{SqZExt}(Z_2, x_2).$$

Let Z_1' be the square zero extension of Z_1, and Z_2' the corresponding square zero extension of Z_2; i.e.,

$$Z_2' = Z_2 \underset{(Z_2)_{\tau^{\leq 0}(f_*\mathcal{I})[1]}}{\sqcup} Z_2.$$

It follows from Corollary 3.3.17 that

$$Z_1' \underset{Z_1}{\sqcup} Z_2 \simeq Z_2 \underset{(Z_2)_{\tau^{\leq 0} f_*(\mathcal{I}[1])}}{\sqcup} Z_2.$$

Furthermore, by the above discussion, both maps $(Z_2)_{\tau^{\leq 0} f_*(\mathcal{I}[1])} \to Z_2$ canonically factor through $(Z_2)_{\tau^{\leq 0}(f_*\mathcal{I})[1]}$ (compatibly with the map to $\mathcal{X}$). This gives the comparison map

$$Z_1' \underset{Z_1}{\sqcup} Z_2 \to Z_2',$$

and it remains to show that it is an isomorphism. By Corollary 3.3.17, it suffices to show it is an isomorphism when Z_2 is affine. In this case, by Lemma 3.3.15, the pushouts can be taken in $\mathrm{DGSch}^{\mathrm{aff}}$, in which case the statement is evident. □

4.7. **Dropping n-coconnectivity.** Finally, note that the above considerations are valid for an object $\mathcal{X} \in \mathrm{PreStk}$, simply by omitting the n-coconnectivity conditions.

Definition 4.7.1. *We shall say that $\mathcal{X} \in \mathrm{PreStk}$ admits* connective deformation theory *if it is convergent, and satisfies Conditions (A), (B) and (C).*

5. A characterization of DG indschemes via deformation theory

5.1. **The statement.** Let $\mathcal{X}$ be an object of ${}^{\leq n}\mathrm{PreStk}$, such that ${}^{cl}\mathcal{X}$ is a classical indscheme. We would like to give a criterion for when $\mathcal{X}$ belongs to ${}^{\leq n}\mathrm{DGindSch}$.

Theorem 5.1.1. *Under the above circumstances, $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$ if and only if $\mathcal{X}$ admits an extension to an object $\mathcal{X}_{n+1} \in {}^{\leq n+1}\mathrm{PreStk}$, which satisfies indscheme-like Conditions (A), (B) and (C).*

The rest of this subsection is devoted to the proof of this theorem [20]. The "only if" direction is clear: if $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$, the extension

$$\mathcal{X}_{n+1} := {}^{\leq n+1}L\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow {}^{\leq n+1}\mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{X})$$

belongs to ${}^{\leq n+1}\mathrm{DGindSch}$, and hence satisfies Conditions (A), (B) and (C).

For the opposite implication, we will argue by induction on n, assuming that the statement is true for $n' < n$. In particular, we can assume that ${}^{\leq n-1}\mathcal{X} := \mathcal{X}|_{{}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}}$ belongs to ${}^{\leq n-1}\mathrm{DGindSch}$.

[20] A more streamlined proof will be given in [GR].

5.1.2. *Step 0: initial remarks.*

First, we note that by Corollary 4.5.8, the induction hypothesis combined with Condition (C) implies that the prestack $\mathcal{X}$ satisfies Zariski descent. Hence, deformation theory of maps into it from objects of ${}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}}$, described in the previous section applies.

Thus, for $X \in {}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}}$ and a map $f : X \to {}^{\leq n-1}\mathcal{X}$, we have a well-defined object

$$^{\geq -n-1}(T^*_f \mathcal{X}_{n+1}) \in \mathrm{Pro}(\mathrm{QCoh}(X)^{\geq -n-1, \leq 0}),$$

whose formation is compatible with pull-backs.

Moreover, we have:

$$^{\leq n}\mathrm{SqZExt}(X, f) \simeq \left((\mathrm{QCoh}(X)^{\geq -n, \leq 0})_{\mathrm{Cone}(^{\geq -n-1}(T^*_f \mathcal{X}_{n+1}) \to ^{\geq -n-1}(T^*X))[-1]/} \right)^{\mathrm{op}}.$$

Let ${}^{\leq n-1}\mathrm{DGSch}_{\text{closed in } \mathcal{X}}$ denote the full subcategory of $({}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}})_{/\mathcal{X}}$ that consists of those $f : X \to \mathcal{X}$, for which f is a closed embedding. In particular, a map

$$(X_1, f_1) \to (X_2, f_2)$$

in this category is given by

$$(\phi : X_1 \to X_2,\ f_1 \simeq \phi \circ f_2),$$

where the underlying map $\phi : X_1 \to X_2$ is also a closed embedding, and in particular, affine.

We obtain that push-out makes the assignment

$$(X, f) \mapsto {}^{\leq n}\mathrm{SqZExt}(X, f)$$

into a category co-fibered over ${}^{\leq n-1}\mathrm{DGSch}_{\text{closed in } \mathcal{X}}$. We denote it by

$$^{\leq n}\mathrm{SqZExt}(^{\leq n-1}\mathrm{DGSch}_{\text{closed in } \mathcal{X}}).$$

By Sect. 4.6.9, we have that for $(X, f) \in {}^{\leq n-1}\mathrm{DGSch}_{\text{closed in } \mathcal{X}}$,

$$\mathrm{Cone}(^{\geq -n-1}(T^*_f \mathcal{X}_{n+1}) \to {}^{\geq -n-1}(T^*X))[-1] \in \mathrm{Pro}(\mathrm{QCoh}(X)^{\geq -n, \leq 0}). \tag{5.1}$$

Hence, by Lemma 4.6.8, the category ${}^{\leq n}\mathrm{SqZExt}(X, f)$ is filtered.

5.1.3. *Step 1: creating closed embeddings.*

It is of course not true that for any $(X, f) \in {}^{\leq n-1}\mathrm{DGSch}_{\text{closed in } \mathcal{X}}$ and

$$(i : X \hookrightarrow X',\ f' : X' \to \mathcal{X}) \in {}^{\leq n}\mathrm{SqZExt}(X, f), \tag{5.2}$$

the map f' is also a closed embedding.

Let

$$^{\leq n}\mathrm{SqZExt}(X, f)_{\text{closed in } \mathcal{X}} \subset {}^{\leq n}\mathrm{SqZExt}(X, f) \tag{5.3}$$

denote the full subcategory spanned by objects for which the map f' is a closed embedding. We claim that the functor (5.3) admits a left adjoint.

Indeed, for an object (5.2), given by a pair

$$\mathrm{Cone}(^{\geq -n-1}(T^*_f \mathcal{X}_{n+1}) \to {}^{\geq -n-1}(T^*X))[-1] \to \mathcal{I}, \quad \mathcal{I} \in \mathrm{QCoh}(X)^{\geq -n, \leq 0},$$

the image of the map

$$H^0\left(\mathrm{Cone}(^{\geq -n-1}(T^*_f \mathcal{X}_{n+1}) \to {}^{\geq -n-1}(T^*X))[-1]\right) \to H^0(\mathcal{I})$$

is a well-defined object $\mathcal{J} \in \mathrm{QCoh}(X)^{\heartsuit}$, by (4.11).

The value of the sough-for left adjoint on the above object of ${}^{\leq n}\mathrm{SqZExt}(X,f)$ is given by

$$\mathrm{Cone}({}^{\geq -n-1}(T^*_f\mathcal{X}_{n+1}) \to {}^{\geq -n-1}(T^*X))[-1] \to \widetilde{\mathcal{I}},$$

where $\widetilde{\mathcal{I}} \in \mathrm{QCoh}(X)^{\geq -n,\leq 0}$ fits into the exact triangle

$$\widetilde{\mathcal{I}} \to \mathcal{I} \to \mathcal{J}.$$

In particular, we obtain that the embedding (5.3) is cofinal. We also obtain that the category ${}^{\leq n}\mathrm{SqZExt}(X,f)_{\text{closed in }\mathcal{X}}$ is also filtered.

Let

$${}^{\leq n}\mathrm{SqZExt}({}^{\leq n-1}\mathrm{DGSch}_{/\mathcal{X}})_{\text{closed in }\mathcal{X}}$$

denote the corresponding full subcategory of ${}^{\leq n}\mathrm{SqZExt}({}^{\leq n-1}\mathrm{DGSch}_{\text{closed in }\mathcal{X}})$. The existence of the left adjoint to (5.3) implies that the forgetful functor

$${}^{\leq n}\mathrm{SqZExt}({}^{\leq n-1}\mathrm{DGSch}_{/\mathcal{X}})_{\text{closed in }\mathcal{X}} \to {}^{\leq n-1}\mathrm{DGSch}_{\text{closed in }\mathcal{X}}$$

is also a co-Cartesian fibration.

5.1.4. *Step 2: construction of the inductive system.* Let

$${}^{\leq n-1}\mathcal{X} \simeq \underset{\alpha\in\mathsf{A}}{colim}\, X_\alpha$$

be a presentation as in (1.1) with $X_\alpha \in {}^{\leq n-1}\mathrm{DGSch}_{\text{qsep-qc}}$. For every $\alpha \in \mathsf{A}$, let f_α denote the corresponding map $X_\alpha \to {}^{\leq n-1}\mathcal{X}$. For an arrow $\alpha_1 \to \alpha_2$, let f_{α_1,α_2} denote the corresponding map $X_{\alpha_1} \to X_{\alpha_2}$.

For each α, let B_α denote the category

$${}^{\leq n}\mathrm{SqZExt}(X_\alpha, f_\alpha)_{\text{closed in }\mathcal{X}}.$$

For β an object of B_α, we will denote by X_β the corresponding ${}^{\leq n}$DG scheme X'_α, and by f_β the closed embedding f'_α. Let i_β denote the closed embedding $X_\alpha \to X_\beta$. We have an evident functor from B_α to the category of ${}^{\leq n}$DG schemes endowed with a closed embedding into $\mathcal{X}$.

The above construction makes the assignment

$$\alpha \mapsto \mathsf{B}_\alpha$$

into a category co-fibered over A. Let ϕ denote the tautological map $\mathsf{B} \to \mathsf{A}$. Since A is filtered and all B_α are filtered, the category B is also filtered.

It is also clear that the assignment

$$(\beta \in \mathsf{B}) \mapsto (X_\beta \xrightarrow{f_\beta} \mathcal{X})$$

is a functor from B to the category of ${}^{\leq n}$DG schemes equipped with a closed embedding into $\mathcal{X}$. For an arrow $(\beta_1 \to \beta_2) \in \mathsf{B}$, let f_{β_1,β_2} denote the corresponding closed embedding $X_{\beta_1} \to X_{\beta_2}$.

Thus, we obtain a map

$$\underset{\beta\in\mathsf{B}}{colim}\, X_\beta \to \mathcal{X}, \tag{5.4}$$

and we claim that it is an isomorphism.

In other words, we have to show that for $S' \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, the maps f_β induce an isomorphism:

$$\underset{\beta\in\mathsf{B}}{colim}\, \mathrm{Maps}(S', X_\beta) \simeq \mathrm{Maps}(S', \mathcal{X}). \tag{5.5}$$

5.1.5. *Step 3: a map in the opposite direction.* Let us construct a map that we shall eventually prove to be the inverse of (5.5):

$$\text{Maps}(S', \mathcal{X}) \to \underset{\beta \in \mathsf{B}}{colim}\ \text{Maps}(S', X_\beta). \tag{5.6}$$

For $S' \in {}^{\leq n}\text{DGSch}^{\text{aff}}$, set $S := {}^{\leq n-1}S'$. Tautologically, we have:

$$\text{Maps}(S', \mathcal{X}) \simeq \underset{\alpha \in \mathsf{A}}{colim}\, \{x : S \to X_\alpha,\, x' : S' \to \mathcal{X},\, x'|_S \simeq f_\alpha \circ x\}. \tag{5.7}$$

We can view S' as a square-zero extension of S by the object

$$H^{-n}(\mathcal{O}_{S'}) \in \text{QCoh}(S)^\heartsuit[n] \subset \text{QCoh}(S)^{\geq -n, \leq 0},$$

see Corollary 4.5.8.

By Lemma 4.6.14, we obtain an isomorphism in ∞-Grpd:

$$\{x : S \to X_\alpha,\, x' : S' \to \mathcal{X},\, x'|_S \simeq f_\alpha \circ x\} \simeq$$
$$\simeq \underset{(X_\alpha \hookrightarrow X'_\alpha) \in {}^{\leq n}\text{SqZExt}(X_\alpha, f_\alpha)}{colim} \{x : S \to X_\alpha,\, S' \underset{S}{\sqcup} X_\alpha \simeq X'_\alpha\}.$$

Taking into account that

$$\mathsf{B}_\alpha := {}^{\leq n}\text{SqZExt}(X_\alpha, f_\alpha)_{\text{closed in } \mathcal{X}} \hookrightarrow {}^{\leq n}\text{SqZExt}(X_\alpha, f_\alpha)$$

is cofinal, we have an isomorphism in ∞-Grpd:

$$\{x : S \to X_\alpha,\, x' : S' \to \mathcal{X},\, x'|_S \simeq f_\alpha \circ x\} \simeq \underset{\beta \in \mathsf{B}_\alpha}{colim}\, \{x : S \to X_\alpha,\, S' \underset{S}{\sqcup} X_\alpha \simeq X_\beta\}.$$

Combining this with (5.7), we obtain a canonical isomorphism in ∞-Grpd:

$$\text{Maps}(S', \mathcal{X}) \simeq \underset{\alpha \in \mathsf{A}}{colim} \left(\underset{\beta \in \mathsf{B}_\alpha}{colim}\, \{x : S \to X_\alpha,\, S' \underset{S}{\sqcup} X_\alpha \simeq X_\beta\} \right).$$

We have a canonical forgetful map

$$\underset{\alpha \in \mathsf{A}}{colim} \left(\underset{\beta \in \mathsf{B}_\alpha}{colim}\, \{x : S \to X_\alpha,\, S' \underset{S}{\sqcup} X_\alpha \simeq X_\beta\} \right) \to \underset{\alpha \in \mathsf{A}}{colim} \left(\underset{\beta \in \mathsf{B}_\alpha}{colim}\, \{x' : S' \to X_\beta\} \right) \simeq$$
$$\simeq \underset{\beta \in \mathsf{B}}{colim}\, \{x' : S' \to X_\beta\}.$$

Thus, we obtain the desired map

$$\text{Maps}(S', \mathcal{X}) \to \underset{\beta \in \mathsf{B}}{colim}\, \{x' : S' \to X_\beta\}$$

of (5.6).

It is immediate from the construction, the composite arrow

$$\text{Maps}(S', \mathcal{X}) \overset{(5.6)}{\longrightarrow} \underset{\beta \in \mathsf{B}}{colim}\ \text{Hom}(S', X_\beta) \overset{(5.5)}{\longrightarrow} \text{Hom}(S', \mathcal{X})$$

is the identity map.

5.1.6. *Step 4: computation of the other composition.* It remains to show that the composition

$$\underset{\beta\in\mathsf{B}}{colim}\ \mathrm{Maps}(S',X_\beta)\overset{(5.5)}{\longrightarrow}\mathrm{Maps}(S',\mathcal{X})\overset{(5.6)}{\longrightarrow}\underset{\beta\in\mathsf{B}}{colim}\ \mathrm{Maps}(S',X_\beta) \tag{5.8}$$

is isomorphic to the identity map.

To do this, we introduce yet another category, denoted Γ. An object of Γ is given by the following data.

- An arrow $(\beta\to\beta_1)\in\mathsf{B}$, which projects by means of ϕ to an arrow $(\alpha\to\alpha_1)\in\mathsf{A}$,
- A map $g_{\beta,\alpha_1}: {}^{\leq n-1}X_\beta\to X_{\alpha_1}$,
- A commutative diagram of square-zero extensions compatible with maps to $\mathcal{X}$

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_{\beta,\beta_1}} & X_{\beta_1} \\ {\scriptstyle j_\beta}\uparrow & & \uparrow{\scriptstyle i_{\beta_1}} \\ {}^{\leq n-1}X_\beta & \xrightarrow{g_{\beta,\alpha_1}} & X_{\alpha_1}, \end{array}$$

  where j_β is the canonical map, corresponding to the truncation (see Corollary 4.5.8),
- An identification of the composition

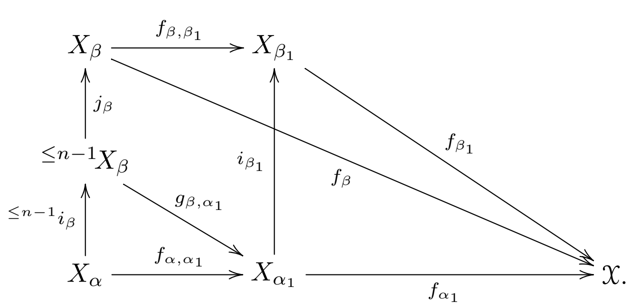
$$X_\alpha\overset{{}^{\leq n-1}i_\beta}{\longrightarrow}{}^{\leq n-1}X_\beta\overset{g_{\beta,\alpha_1}}{\longrightarrow}X_{\alpha_1}\text{ with }f_{\alpha,\alpha_1},$$

- A homotopy between the resulting two identifications, making the following diagram commutative:

$$\begin{array}{ccc} f_{\alpha_1}\circ g_{\beta,\alpha_1}\circ{}^{\leq n-1}i_\beta & \overset{\sim}{\longrightarrow} & f_\beta\circ j_\beta\circ{}^{\leq n-1}i_\beta\simeq f_\beta\circ i_\beta \\ \sim\downarrow & & \sim\downarrow \\ f_{\alpha_1}\circ f_{\alpha,\alpha_1} & \longrightarrow & f_\alpha. \end{array}$$

We can depict this data in a diagram:

(5.9)



Morphisms in Γ are defined naturally (so that the corresponding diagrams of DG schemes commute).

There are tautological maps $\psi,\psi_1:\Gamma\to\mathsf{B}$ that remember the data of β and β_1, respectively.

The colimit

$$\underset{\gamma\in\Gamma}{colim}\ \mathrm{Maps}(S',X_{\psi(\gamma)}),$$

admits a tautological map

$$r:\underset{\gamma\in\Gamma}{colim}\ \mathrm{Maps}(S',X_{\psi(\gamma)})\to\underset{\beta\in\mathsf{B}}{colim}\ \mathrm{Maps}(S',X_\beta). \tag{5.10}$$

Note, however, that we have another map

$$r_1 : \underset{\gamma\in\Gamma}{colim}\ \mathrm{Maps}(S', X_{\psi(\gamma)}) \to \underset{\beta\in\mathsf{B}}{colim}\ \mathrm{Maps}(S', X_\beta), \tag{5.11}$$

which for $\gamma \in \Gamma$, sends $\mathrm{Maps}(S', X_{\psi(\gamma)})$ to $\mathrm{Maps}(S', X_{\psi_1(\gamma})$ by means of $f_{\psi(\gamma),\psi_1(\gamma)}$. However, the same edge $f_{\psi(\gamma),\psi_1(\gamma)}$ provides a homotopy between these two maps of colimits.

It follows from the construction that the composite

$$\underset{\gamma\in\Gamma}{colim}\ \mathrm{Maps}(S, X_{\psi(\gamma)}) \xrightarrow{r} \underset{\beta\in\mathsf{B}}{colim}\ \mathrm{Maps}(S', X_\beta) \xrightarrow{(5.5)} \mathrm{Maps}(S', \mathcal{X}) \xrightarrow{(5.6)} \underset{\beta\in\mathsf{B}}{colim}\ \mathrm{Maps}(S', X_\beta)$$

coincides with the map r_1.

Therefore, to prove that the composition in (5.8) is isomorphic to the identity map, it suffices to show that the map r is an isomorphism in ∞-Grpd. To do this, we will repeatedly use the following observation:

Lemma 5.1.7. *Let $F : \mathbf{C}' \to \mathbf{C}$ be a functor between ∞-categories.*

(a) *Suppose that F is a Cartesian fibration. Then F is cofinal if and only if it has contractible fibers.*

(b) *Suppose that F is a co-Cartesian fibration, and that F has contractible fibers. Then it is cofinal.*

It is easy to see that the functor ψ is a Cartesian fibration. Applying Lemma 5.1.7, we obtain that it is sufficient to show that the fibers of ψ are contractible.

5.1.8. *Step 5: contractibility of the fibers of ψ.* For $\beta \in \mathsf{B}$, let Γ_β denote the fiber of ψ. Explicitly, Γ_β consists of the data of

- An object $\alpha_1 \in \mathsf{A}$, and an arrow $\phi(\beta) =: \alpha \to \alpha_1$.
- A map $g_{\beta,\alpha_1} : {}^{\leq n-1}X_\beta \to X_{\alpha_1}$,
- An identification of the composition

$$X_\alpha \overset{{}^{\leq n-1}i_\beta}{\longrightarrow} {}^{\leq n-1}X_\beta \overset{g_{\beta,\alpha_1}}{\longrightarrow} X_{\alpha_1} \text{ with } f_{\alpha,\alpha_1},$$

- A homotopy between the resulting two identifications, making the following diagram commutative:

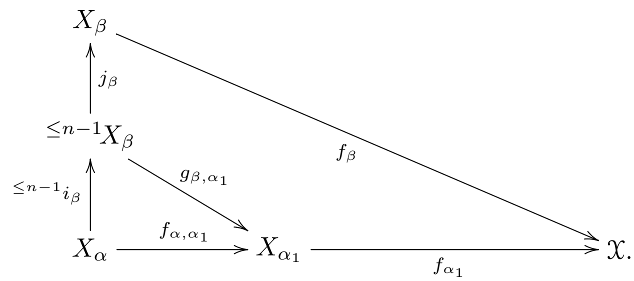
$$\begin{array}{ccc} f_{\alpha_1} \circ g_{\beta,\alpha_1} \circ {}^{\leq n-1}i_\beta & \xrightarrow{\sim} & f_\beta \circ j_\beta \circ {}^{\leq n-1}i_\beta \simeq f_\beta \circ i_\beta \\ \sim\big\downarrow & & \sim\big\downarrow \\ f_{\alpha_1} \circ f_{\alpha,\alpha_1} & \longrightarrow & f_\alpha. \end{array}$$

- A lift of $\alpha \to \alpha_1$ to an arrow $\beta \to \beta_1$.
- A commutative diagram of square-zero extensions compatible with maps to $\mathcal{X}$

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_{\beta,\beta_1}} & X_{\beta_1} \\ {\scriptstyle j_\beta}\big\uparrow & & \big\uparrow{\scriptstyle i_{\beta_1}} \\ {}^{\leq n-1}X_\beta & \xrightarrow{g_{\beta,\alpha_1}} & X_{\alpha_1}. \end{array}$$

We introduce the category Δ_β to consist of the first four out of six of the pieces of data in the description of Γ_β given above. I.e., an object of Δ_β corresponds to a diagram

(5.12)



We have a natural forgetful map $\Gamma_\beta \to \Delta_\beta$. It is easy to see that this functor is a co-Cartesian fibration. Hence, by Lemma 5.1.7, it is enough to show that Δ_β is contractible, and that the fibers of Γ_β over Δ_β are contractible.

5.1.9. *Step 6: contractibility of* Δ_β. By construction, we have a left fibration $\Delta_\beta \to \mathsf{A}_{\alpha/}$, and hence the homotopy type of Δ_β is

$$\underset{\alpha_1\in \mathsf{A}_{\alpha/}}{colim}\left(\operatorname{Maps}({}^{\leq n-1}X_\beta, X_{\alpha_1}) \underset{\operatorname{Maps}({}^{\leq n-1}X_\beta,\mathcal{X}) \underset{\operatorname{Maps}(X_\alpha,\mathcal{X})}{\times} \operatorname{Maps}(X_\alpha,X_{\alpha_1})}{\times} \mathrm{pt}\right),$$

where $\mathrm{pt} \to \operatorname{Maps}({}^{\leq n-1}(X_\beta),\mathcal{X})$ is the map $f_\beta\circ j_\beta$ and $\mathrm{pt}\to \operatorname{Maps}(X_\alpha,X_{\alpha_1})$ is f_{α,α_1}.

Since the category $A_{\alpha/}$ of objects $\alpha_1\in\mathsf{A}$ *under* α is filtered, we can commute the colimit and the Caretesian products, and we obtain that the homotopy type of Δ_β is

$$\left(\underset{\alpha_1\in \mathsf{A}_{\alpha/}}{colim}\operatorname{Maps}({}^{\leq n-1}X_\beta, X_{\alpha_1})\right) \underset{\operatorname{Maps}({}^{\leq n-1}X_\beta,\mathcal{X}) \underset{\operatorname{Maps}(X_\alpha,\mathcal{X})}{\times} \left(\underset{\alpha_1\in \mathsf{A}_{\alpha/}}{colim}\operatorname{Maps}(X_\alpha,X_{\alpha_1})\right)}{\times} \mathrm{pt}\,.$$

Since the DG schemes and X_α and X_β are quasi-separated and quasi-compact, the maps

$$\underset{\alpha_1\in \mathsf{A}_{\alpha/}}{colim}\operatorname{Maps}({}^{\leq n-1}X_\beta, X_{\alpha_1}) \simeq \underset{\alpha_1\in \mathsf{A}}{colim}\operatorname{Maps}({}^{\leq n-1}X_\beta, X_{\alpha_1}) \to \operatorname{Maps}({}^{\leq n-1}X_\beta, {}^{\leq n-1}\mathcal{X})$$

and

$$\underset{\alpha_1\in \mathsf{A}_{\alpha/}}{colim}\operatorname{Maps}(X_\alpha, X_{\alpha_1}) \simeq \underset{\alpha_1\in \mathsf{A}}{colim}\operatorname{Maps}(X_\alpha, X_{\alpha_1}) \to \operatorname{Maps}(X_\alpha, {}^{\leq n-1}\mathcal{X}) \simeq \operatorname{Maps}(X_\alpha,\mathcal{X})$$

are isomorphisms.

We obtain that the homotopy type of Δ_β is

$$\operatorname{Maps}({}^{\leq n-1}X_\beta,\mathcal{X}) \underset{\operatorname{Maps}({}^{\leq n-1}X_\beta,\mathcal{X}) \underset{\operatorname{Maps}(X_\alpha,\mathcal{X})}{\times} \operatorname{Maps}(X_\alpha,\mathcal{X})}{\times} \mathrm{pt} \simeq$$

$$\simeq \operatorname{Maps}({}^{\leq n-1}X_\beta,\mathcal{X}) \underset{\operatorname{Maps}({}^{\leq n-1}X_\beta,\mathcal{X})}{\times} \mathrm{pt} \simeq \mathrm{pt}\,.$$

5.1.10. *Step 7: contractibility of the fibers $\Gamma_\beta \to \Delta_\beta$.* For an object $\delta_\beta \in \Delta_\beta$ as above, the fiber of Γ_β over it is the category of

- $\beta_1 \in \mathsf{B}_{\alpha_1}$,
- A map of square-zero extensions:

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_{\beta,\beta_1}} & X_{\beta_1} \\ {\scriptstyle j_\beta}\uparrow & & \uparrow{\scriptstyle i_{\beta_1}} \\ {}^{\leq n-1}X_\beta & \xrightarrow{g_{\beta,\alpha_1}} & X_{\alpha_1}, \end{array}$$

compatible with the maps to $\mathcal{X}$.

Let $j : Z \hookrightarrow Z'$ be any square-zero extension in ${}^{\leq n}\mathrm{DGSch}$, and let $x_1 : Z \to X_{\alpha_1}$, $x' : Z' \to \mathcal{X}$ be fixed maps equipped with an identification

$$f_{\alpha_1} \circ x_1 \simeq x' \circ j.$$

(In our case, we are going to take $Z = {}^{\leq n-1}X_\beta$ and $Z' = X_\beta$.) Consider the category of pairs:

- $\beta_1 \in \mathsf{B}_{\alpha_1}$,
- A map of square-zero extensions

$$\begin{array}{ccc} Z' & \xrightarrow{x'_1} & X_{\beta_1} \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i_{\beta_1}} \\ Z & \xrightarrow{x_1} & X_{\alpha_1}, \end{array}$$

compatible with the maps to $\mathcal{X}$

We claim that this category is contractible. Indeed, if we omit the condition of compatibility with the given map $x' : Z' \to \mathcal{X}$, we obtain the category whose homotopy type is

$$\underset{\beta_1 \in \mathsf{B}_{\alpha_1}}{\mathit{colim}} \{\text{maps of square-zero extensions as above}\},$$

which, by the definition of B_{α_1}, is homotopy equivalent to

$$\mathrm{Maps}(Z', \mathcal{X}) \underset{\mathrm{Maps}(Z,\mathcal{X})}{\times} \mathrm{pt},$$

where the map $\mathrm{pt} \to \mathrm{Maps}(Z, \mathcal{X})$ is given by $f_{\alpha_1} \circ x_1 = x' \circ j$.

Reinstating the compatibility condition results in taking the fiber product

$$\underset{\beta_1 \in \mathsf{B}_{\alpha_1}}{\mathit{colim}} \left(\{\text{maps of square-zero extensions}\} \underset{\mathrm{Maps}(Z',\mathcal{X}) \underset{\mathrm{Maps}(Z,\mathcal{X})}{\times} \mathrm{pt}}{\times} \mathrm{pt} \right).$$

Since B_{α_1} is filtered, the above colimit can be rewritten as

$$\underset{\beta_1 \in \mathsf{B}_{\alpha_1}}{\mathit{colim}} \{\text{maps of square-zero extensions}\} \underset{\mathrm{Maps}(Z',\mathcal{X}) \underset{\mathrm{Maps}(Z,\mathcal{X})}{\times} \mathrm{pt}}{\times} \mathrm{pt} \simeq$$

$$\simeq \left(\mathrm{Maps}(Z', \mathcal{X}) \underset{\mathrm{Maps}(Z,\mathcal{X})}{\times} \mathrm{pt} \right) \underset{\mathrm{Maps}(Z',\mathcal{X}) \underset{\mathrm{Maps}(Z,\mathcal{X})}{\times} \mathrm{pt}}{\times} \mathrm{pt} \simeq \mathrm{pt}\,.$$

□

5.2. **The $\aleph_0$ condition.** In this subsection we will give a characterization of the $\aleph_0$ property in terms of pro-cotangent spaces.

5.2.1. Let $\mathbf{C}$ be a category. We shall say that an object of $\text{Pro}(\mathbf{C})$ is $\aleph_0$ if it can be presented as an inverse limit over a category equivalent to $\mathbb{N}$ as a poset.

5.2.2. Let $\mathcal{X}$ be an object of ${}^{\leq n}\text{DGindSch}$. We shall denote by $\mathcal{X}_{n+1}$ its canonical extension to an object of ${}^{\leq n+1}\text{DGindSch}$, i.e.,

$$\mathcal{X}_{n+1} := {}^{\leq n+1}L\text{LKE}(\mathcal{X}).$$

Proposition 5.2.3. *An object* $\mathcal{X} \in {}^{\leq n}\text{DGindSch}$ *is* $\aleph_0$ *if and only if the following two conditions hold:*

(a) *The classical indscheme* ${}^{cl}\mathcal{X}$ *is* $\aleph_0$*.*

(b) *The following equivalent conditions hold:*

(i) *There exists a cofinal family of closed embeddings* $x : Z \to {}^{cl}\mathcal{X}$*, where* $Z \in \text{Sch}_{\text{qsep-qc}}$*, such that the object* ${}^{\geq -n-1}(T^*_x\mathcal{X}_{n+1}) \in \text{Pro}(\text{QCoh}(Z)^{\geq -n-1, \leq 0})$ *is* $\aleph_0$*.*

(ii) *Same as* (i) *but for any map* $x : Z \to {}^{cl}\mathcal{X}$ *(i.e., not necessarily a closed embedding).*

(iii) *Same as* (ii)*, but with* Z *required to be affine.*

5.2.4. *Proof of the equivalence of (i), (ii) and (iii).* The implication (ii) $\Rightarrow$ (i) is tautological. The implication (i) $\Rightarrow$ (ii) follows from the fact that the formation of ${}^{\geq -n-1}(T^*_x\mathcal{X}_{n+1})$ is compatible with pull-backs, i.e., Condition (B). The implication (ii) $\Rightarrow$ (iii) is again tautological. The implication (iii) $\Rightarrow$ (ii) follows from the next lemma:

Lemma 5.2.5. *The equivalence of Lemma 4.2.6 for* $\text{Pro}(\text{QCoh}(-)^{\geq -n, \leq 0})$ *preserves the corresponding* $\aleph_0$ *subcategories.*

Proof. It is easy to see that it is enough to prove the lemma for $\text{Pro}(\text{QCoh}(-))$ instead of $\text{Pro}(\text{QCoh}(-)^{\geq -n, \leq 0})$.

By induction, the assertion reduces to the following statement: let $Z = Z_1 \cup Z_2$ be a covering of Z be two Zariski open subsets. Let $F \in \text{Pro}(\text{QCoh}(Z))$ be an object such that

$$\text{Pro}(j_1^*)(F) \in \text{Pro}(\text{QCoh}(Z_1)) \text{ and } \text{Pro}(j_2^*)(F) \in \text{Pro}(\text{QCoh}(Z_2))$$

are $\aleph_0$. Then F is $\aleph_0$.

It is easy to see that if $F' \to F'' \to F'''$ is an exact triangle in $\text{Pro}(\text{QCoh}(S))$, then the condition of being $\aleph_0$ has the "2 out of 3" property. Considering the exact triangle

$$F \to \text{Pro}(j_{1*}) \circ \text{Pro}(j_1{}^*)(F) \to \text{Cone}\left(F \to \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F)\right),$$

we obtain that it is sufficient to show that $\text{Cone}\left(F \to \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F)\right)$ is $\aleph_0$.

However, $\text{Cone}\left(F \to \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F)\right)$ is supported on a Zariski-closed subset contained in Z_2 and isomorphic to

$$\text{Cone}\left(\text{Pro}(j_2^*)(F) \to \text{Pro}(j_{12,2*}) \circ \text{Pro}(j_{12,2}^*)(F)\right),$$

(where $j_{12,2}$ denotes the open embedding $Z_1 \cap Z_2 \hookrightarrow Z_2$), which is $\aleph_0$ by the "2 out of 3" property.

□

This finishes the proof of the equivalence of properties (i), (ii) and (iii) in Proposition 5.2.3.

□

5.2.6. *Proof of "the only if" direction.* Suppose $\mathcal{X}$ is $\aleph_0$. Fix its presentation as in (1.1), where the index set A is equivalent to $\mathbb{N}$. For $\alpha \in \mathsf{A}$ (resp., for an arrow $\alpha_1 \to \alpha_2$) let f_α (resp., f_{α_1,α_2}) denote the corresponding closed embedding $f_\alpha : X_\alpha \to \mathcal{X}$ (resp., $X_{\alpha_1} \to X_{\alpha_2}$).

For a quasi-separated and quasi-compact $Z \in {}^{\leq n}\mathrm{DGSch}$ equipped with a map $x : Z \to \mathcal{X}$, let α_0 be an index such that x factors through a map $x_{\alpha_0} : Z \to X_{\alpha_0}$. By (4.11), we have:

$$^{\geq -n-1}(T^*_x \mathcal{X}_{n+1}) \simeq \underset{\alpha \in \mathsf{A}_{\alpha_0/}}{\text{“}lim\text{”}} \, ^{\geq -n-1}(T^*_{x_{\alpha_0} \circ f_{\alpha_0,\alpha}} X_\alpha),$$

and the category of indices is explicitly equivalent to $\mathbb{N}$.

□

5.2.7. *Proof of "if" direction.* First, we observe that the "2 out of 3" property of an object of $\mathrm{QCoh}(Z)^{\geq -n-1,\leq 0}$ of being $\aleph_0$ implies that if conditions (i), (ii) or (iii) hold for $Z \in \mathrm{Sch}_{\text{qsep-qc}}$ equipped with a map to ${}^{cl}\mathcal{X}$, then the same will be true for any $Z \in {}^{\leq n}\mathrm{DGSch}_{\text{qsep-qc}}$ equipped with a map to ${}^{\leq n}\mathcal{X}$.

By induction, we may assume that the truncation

$$^{\leq n-1}\mathcal{X} := \mathcal{X}|_{^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}}$$

is $\aleph_0$.

Fix a presentation of ${}^{\leq n-1}\mathcal{X}$ as in (1.1), where the category A is equivalent to the poset $\mathbb{N}$. Consider the corresponding category B (see Step 3 in the proof of Theorem 5.1.1), mapping to A by means of ϕ. We shall use the following lemma:

Lemma 5.2.8. *Let $\phi : \mathsf{B} \to \mathsf{A}$ be a co-Cartesian fibration of categories, where A is equivalent to $\mathbb{N}$, and every fiber admits a cofinal functor from $\mathbb{N}$. Then B also admits a cofinal functor from $\mathbb{N}$.*

Hence, by the lemma, it suffices to show that for each $\alpha \in \mathsf{A}$, the corresponding category B_α admits a cofinal functor from $\mathbb{N}$. By construction, the category

$$\mathsf{B}_\alpha = {}^{\leq n}\mathrm{SqZExt}(X_\alpha, f_\alpha)_{\text{closed in } \mathcal{X}}$$

is cofinal in ${}^{\leq n}\mathrm{SqZExt}(X_\alpha, f_\alpha)$, and the embedding admits a left adjoint. Therefore, it is enough to show that the latter admits a cofinal map from $\mathbb{N}$.

We have

$$^{\leq n}\mathrm{SqZExt}(X_\alpha, f_\alpha) \simeq \left((\mathrm{QCoh}(X_\alpha)^{\geq -n,\leq 0})_{\mathrm{Cone}((df_\alpha)^*)[-1]}\right)^{\mathrm{op}},$$

where $(df_\alpha)^*$ is the canonical map in $\mathrm{Pro}(\mathrm{QCoh}(X_\alpha)^{\geq -n-1})$:

$$^{\geq -n-1}(T^*_{f_\alpha} \mathcal{X}_{n+1}) \to {}^{\geq -n-1}(T^* X_\alpha).$$

The assertion now follows from the assumption that ${}^{\geq -n-1}(T^*_{f_\alpha} \mathcal{X}_{n+1})$ is $\aleph_0$.

□

5.3. **The "locally almost of finite type" condition.**

5.3.1. We shall characterize ${}^{\leq n}$DG indschemes locally of finite type in terms of their pro-cotangent spaces.

As before, let $\mathcal{X}$ be an object of ${}^{\leq n}$DGindSch, and set

$$\mathcal{X}_{n+1} := {}^{\leq n+1}{}^{L}\mathrm{LKE}(\mathcal{X}).$$

Proposition 5.3.2. *The following conditions are equivalent:*

(a) $\mathcal{X}$ *is locally of finite type.*

(b) ${}^{cl}\mathcal{X}$ *is locally of finite type and the following equivalent conditions hold:*

(i) *There exists a cofinal family of closed embeddings* $x : Z \to {}^{cl}\mathcal{X}$, *where* $Z \in \mathrm{Sch}_{\mathrm{ft}}$, *such that the object*

$$^{\geq -n-1}(T^*_x\mathcal{X}_{n+1}) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n-1,\leq 0})$$

belongs to $\mathrm{Pro}(\mathrm{Coh}(Z)^{\geq -n-1,\leq 0})$.

(ii) *Same as* (i) *but for any map* $x : Z \to {}^{cl}\mathcal{X}$ *(i.e., not necessarily a closed embedding).*

(iii) *Same as* (ii), *but with* Z *required to be affine.*

5.3.3. *Proof of the equivalence of (i), (ii), and (iii).* This is similar to Sect. 5.2.4, using the following interpretation of

$$\mathrm{Pro}(\mathrm{Coh}(Z)^{\geq -n,\leq 0}) \subset \mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n,\leq 0})$$

for $Z \in \mathrm{DGSch}_{\mathrm{aft}}$:

Lemma 5.3.4. *For a Noetherian scheme* Z, *an object* $F \in \mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n,\leq 0})$ *belongs to* $\mathrm{Pro}(\mathrm{Coh}(Z)^{\geq -n,\leq 0})$ *if and only if, when viewed as a functor*

$$F : \mathrm{QCoh}(Z)^{\geq -n,\leq 0} \to \infty\text{-Grpd},$$

it commutes with filtered colimits.

5.3.5. *Proof of Proposition 5.3.2.* The implication (a) $\Rightarrow$ (b) follows using Lemma 5.3.4 from the fact that an object of ${}^{\leq n+1}$PreStk, which is locally of finite type, takes limits in ${}^{\leq n+1}\mathrm{DGSch}^{\mathrm{aff}}$ to colimits, see [GL:Stacks, Corollary 1.3.8].

Let us show that (b) implies (a). By induction, we can assume that ${}^{\leq n-1}\mathcal{X} := \mathcal{X}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ is locally of finite type.

We claim now that conditions (i), (ii) and (iii) of Proposition 5.3.2(b) hold for any $Z \in {}^{\leq n}\mathrm{DGSch}_{\mathrm{ft}}$ mapping to ${}^{\leq n}\mathcal{X}$ (and not just classical schemes). This follows from the next observation:

Lemma 5.3.6. *Let* Z *be an object of* $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, *and* $F \in \mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n,\leq 0})$. *Then* F *belongs to* $\mathrm{Pro}(\mathrm{Coh}(Z)^{\geq -n,\leq 0})$ *if and only if its restriction to* ${}^{cl}Z$ *does.*

Remark 5.3.7. The assertion of Lemma 5.3.6 is valid, with the same proof, when we replace the categories $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n,\leq 0})$ and $\mathrm{Pro}(\mathrm{Coh}(Z)^{\geq -n,\leq 0})$ by the categories $\mathrm{QCoh}(Z)^{\geq -n,\leq 0}$ and $\mathrm{Coh}(Z)^{\geq -n,\leq 0}$, respectively.

Proof. The property of commutation with filtered colimits is enough to check on $\mathrm{QCoh}(Z)^{\heartsuit}$, and direct image defines an equivalence $\mathrm{QCoh}({}^{cl}Z)^{\heartsuit} \simeq \mathrm{QCoh}(Z)^{\heartsuit}$.

□

The rest of the proof of Proposition 5.3.2 is the same as that of Proposition 1.7.6 in Sect. 3.5.2.

6. Formal completions

6.1. The setting.

6.1.1. In this section we will study the following situation. Let $\mathcal{X}$ be an object of PreStk, and let $\mathcal{Y}$ be an object of

$$ {}^{cl,red}\mathrm{PreStk} := \mathrm{Funct}(({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \infty\text{-Grpd}), $$

where ${}^{red}\mathrm{Sch}^{\mathrm{aff}}$ denotes the category of classical reduced affine schemes. Let $\mathcal{Y} \to {}^{cl,red}\mathcal{X}$ be a map, where ${}^{cl,red}\mathcal{X} := \mathcal{X}|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}$.

Definition 6.1.2. *By the formal completion of* $\mathcal{X}$ *along* $\mathcal{Y}$*, denoted* $\mathcal{X}^\wedge_\mathcal{Y}$*, we shall mean the object of* PreStk *equal to the fiber product*

$$ \begin{array}{ccc} \mathcal{X}^\wedge_\mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{RKE}_{({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}) & \longrightarrow & \mathrm{RKE}_{({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl,red}\mathcal{X}). \end{array} $$

In plain terms for $S \in \mathrm{DGSch}^{\mathrm{aff}}$, we set $\mathrm{Maps}(S, \mathcal{X}^\wedge_\mathcal{Y})$ to be the groupoid consisting of pairs (x, y), where $x : S \to \mathcal{X}$, and y is a lift of the map

$$ x|_{{}^{cl,red}S} : {}^{cl,red}S \to {}^{cl,red}\mathcal{X} $$

to a map $y : {}^{cl,red}S \to \mathcal{Y}$.

6.1.3. Several remarks are in order:

(i) If $\mathcal{X}$ is convergent, then so is $\mathcal{X}^\wedge_\mathcal{Y}$.

(ii) $\mathcal{X}$ and $\mathcal{X}^\wedge_\mathcal{Y}$ have "the same deformation theory." In particular, if $\mathcal{X}$ satisfies Conditions (A), (B) or (C), then so does $\mathcal{X}^\wedge_\mathcal{Y}$, and for any $x : S \to \mathcal{X}^\wedge_\mathcal{Y}$, the map

$$ T^*_x\mathcal{X} \to T^*_x\mathcal{X}^\wedge_\mathcal{Y} $$

is an isomorphism of functors out of $\mathrm{QCoh}(S)^{\leq 0}$.

(iii) The formation of $\mathcal{X}^\wedge_\mathcal{Y}$ is compatible with filtered colimits in the sense that for a filtered category A and functors

$$ \mathsf{A} \to \mathrm{PreStk} : \alpha \mapsto \mathcal{X}_\alpha \text{ and } \mathsf{A} \to {}^{cl,red}\mathrm{PreStk} : \alpha \mapsto \mathcal{Y}_\alpha, $$

and a natural transformation $\mathcal{Y}_\alpha \to {}^{cl,red}\mathcal{X}_\alpha$, the resulting map

$$ \underset{\alpha \in \mathsf{A}}{colim}\, (\mathcal{X}_\alpha)^\wedge_{\mathcal{Y}_\alpha} \to \mathcal{X}^\wedge_\mathcal{Y} $$

is an isomorphism, where

$$ \mathcal{X} := \underset{\alpha \in \mathsf{A}}{colim}\, \mathcal{X}_\alpha \text{ and } \mathcal{Y} := \underset{\alpha \in \mathsf{A}}{colim}\, \mathcal{Y}_\alpha. $$

(iv) For a map $\mathcal{X}' \to \mathcal{X}$ in PreStk, let $\mathcal{Y}' := \mathcal{Y} \underset{{}^{cl,red}\mathcal{X}}{\times} {}^{cl,red}\mathcal{X}'$. Then

$$ \mathcal{X}'^\wedge_{\mathcal{Y}'} \simeq \mathcal{X}^\wedge_\mathcal{Y} \underset{\mathcal{X}}{\times} \mathcal{X}'. $$

6.1.4. When defining formal completions, we can take $\mathcal{Y} \to \mathcal{X}|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}$ to be an arbitrary map.

For example, taking $\mathcal{X} = \mathrm{pt}$, we obtain an object of PreStk isomorphic to

$$ \mathrm{RKE}_{({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Y}). $$

The latter object is otherwise known as the "de Rham space of $\mathcal{Y}$" and is denoted $\mathcal{Y}_{\mathrm{dR}}$.

6.1.5. Given a map of prestacks $\mathcal{Y} \to \mathcal{X}$, let $\mathcal{X}^\wedge_\mathcal{Y}$ denote the formal completion of $\mathcal{X}$ along ${}^{cl,red}\mathcal{Y} \to {}^{cl,red}\mathcal{X}$. We can express $\mathcal{X}^\wedge_\mathcal{Y}$ in terms of the de Rham spaces of $\mathcal{X}$ and $\mathcal{Y}$; namely,

$$\mathcal{X}^\wedge_\mathcal{Y} \simeq \mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{Y}_{\mathrm{dR}}.$$

6.2. Formal completions along monomorphisms.

6.2.1. Let us now assume that the map $\mathcal{Y} \to {}^{cl,red}\mathcal{X}$ is a monomorphism. I.e., for $S \in {}^{red}\mathrm{Sch}^{\mathrm{aff}}$ and a map $S \to {}^{cl,red}\mathcal{X}$, if there exists a lifting $S \to \mathcal{Y}$, then the space of such liftings is contractible.

Note that in this case the map $\mathcal{X}^\wedge_\mathcal{Y} \to \mathcal{X}$ is also a monomorphism. In particular, if $\mathcal{Z}_i \to \mathcal{X}$, $i = 1, 2$ are maps in PreStk that factor through $\mathcal{X}^\wedge_\mathcal{Y}$, then the map

$$\mathcal{Z}_1 \underset{\mathcal{X}^\wedge_\mathcal{Y}}{\times} \mathcal{Z}_2 \to \mathcal{Z}_1 \underset{\mathcal{X}}{\times} \mathcal{Z}_2 \tag{6.1}$$

is an isomorphism.

6.2.2. The above observation implies that if $f : \mathcal{Z} \to \mathcal{X}^\wedge_\mathcal{Y}$ is a map such that the composition $\mathcal{Z} \to \mathcal{X}^\wedge_\mathcal{Y} \to \mathcal{X}$ is a closed embedding, then the original map f is a closed embedding.

Remark 6.2.3. Note that the converse to the above statement is not true: consider $\mathcal{X} := \mathbb{A}^1_{\mathrm{dR}}$, and $\mathcal{Y} = \mathrm{pt}$. We have ${}^{cl,red}\mathcal{X} = \mathbb{A}^1|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}$, and we let $\mathcal{Y} \to {}^{cl,red}\mathcal{X}$ be the map corresponding to $\{0\} \in \mathbb{A}^1$. Then $\mathcal{X}^\wedge_\mathcal{Y} = \mathrm{pt}$. The tautological map $\mathcal{X}^\wedge_\mathcal{Y} \to \mathcal{X}$ is now

$$\mathrm{pt} \to \mathbb{A}^1_{\mathrm{dR}},$$

and it is not a closed embedding: indeed, its base change with respect to $\mathbb{A}^1 \to \mathbb{A}^1_{\mathrm{dR}}$ yields $(\mathbb{A}^1)^\wedge_{\{0\}}$ which is not a closed subscheme of $\mathbb{A}^1$.

6.2.4. We would like to consider descent for $\mathcal{X}^\wedge_\mathcal{Y}$. This is not completely straightforward since the restriction of the fppf topology to ${}^{red}\mathrm{Sch}^{\mathrm{aff}}$ does not make much sense.

Let

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & {}^{cl,red}\mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & {}^{cl,red}(L(\mathcal{X})) \end{array} \tag{6.2}$$

be a Cartesian diagram in ${}^{cl,red}\mathrm{PreStk}$ in which the horizontal arrows are monomorphisms.

Lemma 6.2.5. *Under the above circumstances, the natural map*

$$L(\mathcal{X}^\wedge_\mathcal{Y}) \to (L(\mathcal{X}))^\wedge_{\mathcal{Y}'}$$

is an isomorphism.

Proof. Recall that the sheafification functor L maps monomorphisms into monomorphisms. Therefore both maps

$$L(\mathcal{X}^\wedge_\mathcal{Y}) \to L(\mathcal{X}) \text{ and } (L(\mathcal{X}))^\wedge_{\mathcal{Y}'} \to L(\mathcal{X})$$

are monomorphisms. Hence, the map in the lemma is a monomorphism as well. It requires to see that it is essentially surjective.

Thus, let $x : S \to L(\mathcal{X})$ be a map that factors through $(L(\mathcal{X}))^\wedge_{\mathcal{Y}'}$. We wish to show that it factors through $L(\mathcal{X}^\wedge_{\mathcal{Y}})$ as well. Let $\pi : \widetilde{S} \to S$ be an fppf cover, such that $x \circ \pi$ lifts to a map $\widetilde{x} : \widetilde{S} \to \mathcal{X}$. It suffices to show that $\widetilde{x}|_{^{cl,red}\widetilde{S}}$ factors through $\mathcal{Y}$. However, $\widetilde{x}|_{^{cl,red}\widetilde{S}}$ factors through

$$\mathcal{Y}' \underset{^{cl,red}(L(\mathcal{X}))}{\times} {}^{cl,red}\mathcal{X},$$

by construction, and the required factorization follows from the fact that the diagam (6.2) is Cartesian.

□

Corollary 6.2.6. *If $\mathcal{X}$ is a stack, then so is $\mathcal{X}^\wedge_{\mathcal{Y}}$.*

6.2.7. From now on, we will assume that the map $\mathcal{Y} \to {}^{cl,red}\mathcal{X}$ is a closed embedding. I.e., for $S \in {}^{red}\mathrm{Sch}^{\mathrm{aff}}$ and a map $S \to {}^{cl,red}\mathcal{X}$, the fiber product

$$S \underset{^{cl,red}\mathcal{X}}{\times} \mathcal{Y},$$

taken in ${}^{cl,red}\mathrm{PreStk}$, is representable by a (reduced) closed subscheme of S.

6.3. **Formal completions of DG indshemes.** The next proposition shows that the procedure of formal completion is a way of generating DG indschemes:

Proposition 6.3.1. *Suppose that in the setting of Sect. 6.2.7, $\mathcal{X}$ is a DG indscheme. Then the formal completion $\mathcal{X}^\wedge_{\mathcal{Y}}$ is also a DG indscheme.*

We shall give two proofs.

Proof. (an overkill)

We shall prove the proposition by applying Theorem 5.1.1. We note that Conditions (A), (B) and (C) hold for $\mathcal{X}^\wedge_{\mathcal{Y}}$ because they do for $\mathcal{X}$, see Sect. 6.1.3(ii) above. Hence, it remains to show that ${}^{cl}(\mathcal{X}^\wedge_{\mathcal{Y}})$ is a classical indscheme. However, this is obvious, as the latter is the colimit

$$\underset{Z_{cl} \to {}^{cl}\mathcal{X}}{colim}\, Z_{cl},$$

taken over the (filtered!) category of closed embeddings that at the reduced level factor through $\mathcal{Y}$.

□

Note that using Proposition 5.3.2 and Sect. 6.1.3(ii), the above argument also gives:

Corollary 6.3.2. *If $\mathcal{X}$ is locally almost of finite type, then so is $\mathcal{X}^\wedge_{\mathcal{Y}}$.*

6.3.3. The second proof of Proposition 6.3.1 comes along with an explicit description of $\mathcal{X}^\wedge_{\mathcal{Y}}$ as a colimit of DG schemes:

For $\mathcal{X} \in \mathrm{DGindSch}$, consider the full subcategory

$$(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \underset{\mathrm{DGSch}_{/\mathcal{X}}}{\times} \mathrm{DGSch}_{/\mathcal{X}^\wedge_{\mathcal{Y}}}.$$

I.e., it consits of those closed embedding $Z \to \mathcal{X}$, which factor through $\mathcal{X}^\wedge_{\mathcal{Y}}$. Note that by Sect. 6.2.2, for any

$$Z \in (\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \underset{\mathrm{DGSch}_{/\mathcal{X}}}{\times} \mathrm{DGSch}_{/\mathcal{X}^\wedge_{\mathcal{Y}}},$$

the resulting map $Z \to \mathcal{X}^\wedge_{\mathcal{Y}}$ is a closed embedding.

Proposition 6.3.4. *The category* $(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \underset{\mathrm{DGSch}_{/\mathcal{X}}}{\times} \mathrm{DGSch}_{/\mathcal{X}^\wedge_\mathcal{Y}}$ *is filtered, and the map*

$$\underset{Z \in (\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \underset{\mathrm{DGSch}_{/\mathcal{X}}}{\times} \mathrm{DGSch}_{/\mathcal{X}^\wedge_\mathcal{Y}}}{colim} Z \to \mathcal{X}^\wedge_\mathcal{Y},$$

is an isomorphism, where the colimit is taken in PreStk*.*

Proof. It suffices to show that for $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and a map $S \to \mathcal{X}$ that factors through $\mathcal{Y}$ at the reduced classical level, the full subcategory of

$$\mathrm{DGSch}_{S/,\text{closed in } \mathcal{X}}$$

consisting of

$$S \to Z \to \mathcal{X}, \quad Z \in (\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \underset{\mathrm{DGSch}_{/\mathcal{X}}}{\times} \mathrm{DGSch}_{/\mathcal{X}^\wedge_\mathcal{Y}}$$

contains finite colimits.

The proof follows from the description of finite colimits in $(\mathrm{DGSch})_{S/,\text{closed in } \mathcal{X}}$, given in the proof of Proposition 3.2.2.

□

Remark 6.3.5. It is not difficult to see that the category

$$(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}} \underset{\mathrm{DGSch}_{/\mathcal{X}}}{\times} \mathrm{DGSch}_{/\mathcal{X}^\wedge_\mathcal{Y}}$$

used in the above proof is the same as $(\mathrm{DGSch}_{\text{qsep-qc}})_{\text{closed in } \mathcal{X}^\wedge_\mathcal{Y}}$, i.e., the assertion on Sect. 6.2.2 is "if and only if" for $\mathcal{X}$ a DG indscheme and $\mathcal{Z} = Z \in \mathrm{DGSch}_{\text{qsep-qc}}$.

Indeed, let $\overline{Z}$ denote the closure of the image of Z in $\mathcal{X}$. It is enough to show that the map $Z \underset{\mathcal{X}}{\times} \overline{Z} \to \overline{Z}$ is a closed embedding. However, since $\overline{Z} \to \mathcal{X}$ also factors through $\mathcal{X}^\wedge_\mathcal{Y}$, the map

$$Z \underset{\mathcal{X}^\wedge_\mathcal{Y}}{\times} \overline{Z} \to Z \underset{\mathcal{X}}{\times} \overline{Z}$$

is an isomorphism, and the map

$$Z \underset{\mathcal{X}^\wedge_\mathcal{Y}}{\times} \overline{Z} \to \overline{Z},$$

being a base change of $Z \to \mathcal{X}^\wedge_\mathcal{Y}$, is a closed embedding, by assumption.

6.3.6. Note also that if $\mathcal{X}$ is written as in (1.5), then if we set $Y_\alpha := \mathcal{Y} \cap {}^{cl,red}X_\alpha$, by Sect. 6.1.3 (iii) and (iv), we have:

$$\mathcal{X}^\wedge_\mathcal{Y} \simeq \underset{\alpha}{colim}\, (X_\alpha)^\wedge_{Y_\alpha},$$

where the colimit is taken in PreStk.

6.4. **Formal (DG) schemes.**

Let us recall the following definition:

Definition 6.4.1. *A classical indscheme $\mathcal{X}_{cl}$ is called a formal scheme if ${}^{red}(\mathcal{X}_{cl})$ is a scheme.* [21]

In the derived setting, we give the following one:

Definition 6.4.2. *A DG indscheme $\mathcal{X}$ is called a formal DG scheme if the underlying classical indscheme ${}^{cl}\mathcal{X}$ is formal.*

[21] Recall that we denote by $\mathcal{Y} \mapsto {}^{red}\mathcal{Y}$ the functor ${}^{cl}\mathrm{PreStk} \to {}^{cl,red}\mathrm{PreStk}$ corresponding to restriction along ${}^{red}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Sch}^{\mathrm{aff}}$.

We have, tautologically:

Lemma 6.4.3. *In the situation of Proposition 6.3.1, if $\mathcal{Y}$ is a scheme, then $\mathcal{X}^\wedge_{\mathcal{Y}}$ is a formal DG scheme.*

6.5. **Formal completions of DG schemes.** For the rest of this section we will take $\mathcal{X}$ to be a DG scheme X, and $\mathcal{Y}$ to be a Zariski closed subset Y of ${}^{cl,red}X$. Consider the corresponding formal completion $X^\wedge_Y$.

In this situation, we shall always assume Y is quasi-compact and quasi-separated, in order for $X^\wedge_Y$ to be a DG indscheme according to our definition.

6.5.1. First, we have:

Proposition 6.5.2. *$X^\wedge_Y$ is a DG indscheme.*

We note that Proposition 6.5.2 is not, strictly speaking a consequence of Proposition 6.3.1, since if X fails to be quasi-separated and quasi-compact, then it is not a DG indscheme in our definition. However, it is easy to see that either of the first two proofs of Proposition 6.3.1 applies in this case as well.

We also note that ${}^{cl,red}(X^\wedge_Y) \simeq Y$. Hence, we obtain:

Corollary 6.5.3. *$X^\wedge_Y$ is a formal DG scheme.*

6.5.4. In the present situation, we can slightly improve the presentation of $X^\wedge_Y$ given by Proposition 6.3.4:

Proposition 6.5.5. *As an object of* PreStk*, $X^\wedge_Y$ is isomorphic to*

$$\underset{Y' \to X}{colim}\, Y',$$

where the colimit is taken over the category of closed embeddings whose set-theoretic image is Y.

Proof. By Corollary 1.6.6, we know that $X^\wedge_Y$ is isomorphic to

$$\underset{Z \to X}{colim}\, Z,$$

where the colimit is taken over the category of closed embeddings $Z \to X$ whose image is set-theoretically contained in Y.

By Lemma 3.5.3, is suffices to show that any such $Z \to X$ can be factored as $Z \to Y' \to Z$, where $Y' \to Z$ is a closed embedding whose set-theoretic image is exactly Y.

Let Y'_{can} be the reduced closed subscheme of X corresponding to Y.

Consider the map ${}^{cl,red}Z \to {}^{cl}X$. The latter canonically factors as ${}^{cl,red}Z \to Y'_{can} \to {}^{cl}X$. The required Y' is then given by

$$Z \underset{{}^{cl,red}Z}{\sqcup} Y'_{can}.$$

∎

6.5.6. Note, however, that in general $X^\wedge_Y$ will fail to be weakly $\aleph_0$, even at the classical level. E.g., we take $X = \mathbb{A}^\infty := \operatorname{Spec}(k[t_1, t_2, ...])$ and $Y = \{0\} \subset \mathbb{A}^\infty$.

However, $X^\wedge_Y$ is weakly $\aleph_0$ under the following additional condition:

Proposition 6.5.7. *Assume that Y can be represented by a subscheme Y' of ${}^{cl}X$, whose ideal is locally finitely generated. Then $X^\wedge_Y$ is weakly $\aleph_0$ as a DG indscheme.*

Remark 6.5.8. We expect that $X^\wedge_Y$ is actually $\aleph_0$ (see Sect. 1.4.11 for the distinction between the two notions), but we cannot prove it at this time. However, we will prove this when X is affine, and for general X, "up to sheafification", see Proposition 6.7.7.

Proof. We shall deduce the assertion of the proposition from Proposition 5.2.3.

We note that condition (b) of Proposition 5.2.3 follows from Sect. 6.1.3(ii), as it is satisfied for X.

It remains to show that the classical indscheme underlying $X^\wedge_Y$ is $\aleph_0$. However, the quasi-compactness hypothesis in Y and the assumption that the ideal $\mathcal{I}$ of Y' is locally finitely generated imply that the subschemes Y'_n given by $\mathcal{I}^n$ are cofinal among all subschemes of X whose underlying set is Y. □

6.6. **Formal completion of the affine line at a point.**

6.6.1. We continue to study formal completions of the form $X^\wedge_Y$, where X is a DG scheme, and Y is a Zariski closed subset of ${}^{cl}X$, which is quasi-separated and quasi-compact.

We will impose the assumption made in Proposition 6.5.7. Namely, will assume that Y can be represented by a subscheme Y' of ${}^{cl}X$, whose ideal is locally finitely generated.

We will show that in this case, the behavior of $X^\wedge_Y$ exhibits some additional favorable features.

6.6.2. First, we shall calculate the most basic example: the formal completion of $\mathbb{A}^1$ at the point $\{0\}$. Namely, we have:

Proposition 6.6.3. *The natural map*

$$\underset{n}{colim}\ \operatorname{Spec}(k[t]/t^n) \to (\mathbb{A}^1)^\wedge_{\{0\}},$$

where the colimit is taken in PreStk*, is an isomorphism.*

The statement of the proposition is obvious at the level of the underlying classical prestacks, i.e., when we evaluate both sides on $\operatorname{Sch}^{\mathrm{aff}} \subset \operatorname{DGSch}^{\mathrm{aff}}$. However, some care is needed in the derived setting.

The rest of this subsection is devoted to the proof of this proposition.

6.6.4. *Proof of Proposition 6.6.3, Step 1.*

Both sides of the proposition are a priori functors $(\operatorname{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}$. However, we claim that they both, along with the map between them, naturally upgrade to functors

$$(\operatorname{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd},$$

where ∞-PicGrpd is the category of ∞-Picard groupoids, i.e., connective spectra.

Consider first the functor $\operatorname{Maps}_{\operatorname{PreStk}}(-, \mathbb{A}^1) : (\operatorname{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}$ represented by $\mathbb{A}^1$. We claim that it naturally upgrades to a functor

$$\mathcal{M}aps_{\operatorname{PreStk}}(-, \mathbb{A}^1) : (\operatorname{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd}\,.$$

This comes from the structure on $\mathbb{A}^1$ of abelian group object in

$$\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{DGSch}^{\mathrm{aff}} \subset \mathrm{PreStk}\,.$$

Consider now the object

$$\underset{n}{colim}\ \mathrm{Spec}(k[t]/t^n) \in {}^{cl}\mathrm{PreStk},$$

where the colimit in the above formula is taken in ${}^{cl}\mathrm{PreStk}$. The binomial formula endows the above object with a structure of abelian group object in ${}^{cl}\mathrm{PreStk}$.

Consider the object

$$\mathrm{nilp} := \mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\underset{n}{colim}\ \mathrm{Spec}(k[t]/t^n)) \in \mathrm{PreStk}\,.$$

It equals

$$\underset{n}{colim}\ \mathrm{nilp}_n,$$

where the colimit is now taken in PreStk, and where

$$\mathrm{nilp}_n(S) = \mathrm{Maps}_{\mathrm{DGSch}^{\mathrm{aff}}}(S, \mathrm{Spec}(k[t]/t^n)), \quad S \in \mathrm{DGSch}^{\mathrm{aff}}\,.$$

By the functoriality of $\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}$, and since the forgetful functor

$$\infty\text{-PicGrpd} \to \infty\text{-Grpd}$$

commutes with filtered colimits, we obtain that nilp canonically lifts to a functor

$$\mathcal{N}ilp : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd}\,.$$

The same construction shows that the map of functors

$$\mathrm{nilp} \to \mathrm{Maps}_{\mathrm{PreStk}}(-, \mathbb{A}^1)$$

naturally upgrades to a map of functors with values in ∞-PicGrpd

$$\mathcal{N}ilp \to \mathcal{M}aps_{\mathrm{PreStk}}(-, \mathbb{A}^1).$$

Consider now the functor $\mathrm{Maps}_{\mathrm{PreStk}}\left(-, (\mathbb{A}^1)^\wedge_{\{0\}}\right)$. Since $(\mathbb{A}^1)^\wedge_{\{0\}} \hookrightarrow \mathbb{A}^1$ is a monomorphism and gives rise to subgroups at the level of π_0, we obtain that this functor also naturally upgrades to a functor

$$\mathcal{M}aps_{\mathrm{PreStk}}\left(-, (\mathbb{A}^1)^\wedge_{\{0\}}\right) : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd},$$

and the natural transformation $\mathcal{N}ilp \to \mathcal{M}aps_{\mathrm{DGSch}^{\mathrm{aff}}}(-, \mathbb{A}^1)$ factors canonically as

$$\mathcal{N}ilp \to \mathcal{M}aps_{\mathrm{PreStk}}\left(-, (\mathbb{A}^1)^\wedge_{\{0\}}\right) \to \mathcal{M}aps_{\mathrm{DGSch}^{\mathrm{aff}}}(-, \mathbb{A}^1).$$

6.6.5. *Step 2.*

To prove the proposition, we need to show that for $S = \text{Spec}(A) \in \text{DGSch}^{\text{aff}}$, the map in ∞-PicGrpd

$$\mathcal{N}ilp(A) \to A$$

is an isomorphism onto those connected components of A that correspond to nilpotent elements in $\pi_0(A) = {}^{cl}A$. In the above formula, we view a connective algebra A as a connective spectrum, i.e., object of ∞-PicGrpd.

Hence, it suffices to show that for a connective commutative DG algebra A, the map

$$\pi_i(\mathcal{N}ilp(A)) \to \pi_i(A)$$

is an isomorphism for $i \geq 1$, and that $\pi_0(\mathcal{N}ilp(A))$ maps isomorphically to the set of nilpotent elements in $\pi_0(A) = {}^{cl}A$. Here by π_i for $i \geq 1$ we mean the ith homotopy group of the corresponding space based at the point 0.

6.6.6. *Step 3.* We first consider the case $i \geq 1$.

We regard each $\text{nilp}_n(A)$ as a pointed object of ∞-Grpd. Hence, from the isomorphism

$$\Omega^\infty(\mathcal{N}ilp(A)) = \text{nilp}(A) \simeq \underset{n}{colim}\ \text{nilp}_n(A)$$

in $\infty\text{-Grpd}_{*/}$, for each $i \geq 1$, we have an isomorphism of (ordinary) groups:

$$\pi_i(\mathcal{N}ilp(A)) \simeq \underset{n}{colim}\ \pi_i(\text{nilp}_n(A)).$$

Hence, it suffices to show that the map

$$\underset{n}{colim}\ \pi_i(\text{nilp}_n(A)) \to \pi_i(\Omega^\infty(A)) \tag{6.3}$$

is an isomorphism.

We have a Cartesian square in DGSch:

$$\begin{array}{ccc} \text{Spec}(k[t]/t^n) & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \text{power } n \\ \{0\} & \longrightarrow & \mathbb{A}^1, \end{array}$$

and the corresponding Cartesian square in ∞-Grpd:

$$\begin{array}{ccc} \text{nilp}_n(A) & \longrightarrow & \Omega^\infty(A) \\ \downarrow & & \downarrow \text{power } n \\ * & \longrightarrow & \Omega^\infty(A). \end{array}$$

Hence, we obtain a long exact sequence of homotopy groups

$$...\pi_{i+1}(\Omega^\infty(A)) \overset{\text{power } n}{\longrightarrow} \pi_{i+1}(\Omega^\infty(A)) \to \pi_i(\text{nilp}_n(A)) \to \pi_i(\Omega^\infty(A)) \overset{\text{power } n}{\longrightarrow} \pi_i(\Omega^\infty(A))...$$

However, for $i \geq 1$ and $n > 1$, the map $\pi_i(\Omega^\infty(A)) \overset{\text{power } n}{\longrightarrow} \pi_i(\Omega^\infty(A))$ is zero. Indeed, this follows from the fact for any two connective algebras A_1 and A_2, the canonical map

$$\Omega^\infty(A_1) \times \Omega^\infty(A_2) \to \Omega^\infty(A_1 \otimes A_2)$$

induces a zero map

$$\pi_i(\Omega^\infty(A_1)) \oplus \pi_i(\Omega^\infty(A_2)) \to \pi_i(\Omega^\infty(A_1 \otimes A_2))$$

for $i \geq 1$.

Hence, every n we have a short exact sequence

$$0 \to \pi_{i+1}(\Omega^\infty(A)) \to \pi_i(\mathrm{nilp}_n(A)) \to \pi_i(\Omega^\infty(A)) \to 0.$$

Moreover, for $n'' \geq n'$, in the diagram

$$\begin{array}{ccccc} \pi_{i+1}(\Omega^\infty(A)) & \longrightarrow & \pi_i(\mathrm{nilp}_{n'}(A)) & \longrightarrow & \pi_i(\Omega^\infty(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{i+1}(\Omega^\infty(A)) & \longrightarrow & \pi_i(\mathrm{nilp}_{n''}(A)) & \longrightarrow & \pi_i(\Omega^\infty(A)) \end{array}$$

the right vertical map is the identity, whereas the left vertical map corresponds to the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by raising to the power $n'' - n'$, and so vanishes for $n'' > n'$.

This shows that (6.3) is an isomorphism.

6.6.7. *Step 4.* The fact that

$$\pi_0(\mathcal{N}ilp(A)) \to {}^{cl}A$$

is an isomorphism onto the set of nilpotent elements is proved similarly.

□(Proposition 6.6.3)

6.7. **Formal completions along subschemes of finite codimension.** We now return to the case of a general X and Y satisfying the assumption of Sect. 6.6.1.

6.7.1. Assume that the DG scheme X is eventually coconnective. It is natural to ask whether the same will be true for the DG indscheme $X^\wedge_Y$.

Note, however, that asking for a DG indscheme to be eventually coconnective (i.e., eventually coconnective as a stack) is a strong requirement, since it is difficult to satisfy it together with convergence, see [GL:Stacks, Sect. 1.2.6].

However, the answer to the above question turns out to be affirmative:

Proposition 6.7.2. *If X is eventually coconnective, then $X^\wedge_Y$ is eventually coconnective as a DG indscheme.*

6.7.3. In order to prove Proposition 6.7.2, we will give a more explicit description of the formal completion $X^\wedge_Y$ in the situation of Sect. 6.6.1 when X is affine. This description will be handy for the proof of several other assertions in this paper.

Let $X = \mathrm{Spec}(A)$, and let Y' be a closed subscheme of ${}^{cl}X$ whose ideal is generated by elements $\overline{f_1},...,\overline{f_m}$ in

$${}^{cl}A = H^0(A) = \pi_0(\Omega^\infty(\mathsf{Sp}(A))).$$

Let $f_1, ..., f_m$ be points of $\Omega^\infty(A)$ that project to the $\overline{f_1},...,\overline{f_m}$.

For an integer n, set $A_n := A[t_{n,1}, ..., t_{n,m}]$, where the generators $t_{n,i}$ are in degree -1, and $d(t_{n,i}) = f_i^n$.

For $n' \leq n''$ we have a natural map $A_{n''} \to A_{n'}$ which is identity on A, and which sends $t_{n'',i} \mapsto f_i^{n''-n'} \cdot t_{n',i}$. We will prove:

Proposition 6.7.4. *The natural map*

$$\underset{n}{colim}\ \mathrm{Spec}(A_n) \to X^\wedge_Y, \tag{6.4}$$

where the colimit is taken in PreStk*, is an isomorphism.*

Proof. The functions $f_1, ..., f_m$ define a map

$$\mathrm{Spec}(A) \to \mathbb{A}^m,$$

and by definition, $X^\wedge_Y$ maps isomorphically to the fiber product

$$(\mathbb{A}^m)^\wedge_{\{0\}} \underset{\mathbb{A}^m}{\times} \mathrm{Spec}(A).$$

Since fiber products commute with filtered colimits, from Proposition 6.6.3, we obtain that

$$(\mathbb{A}^m)^\wedge_{\{0\}} \underset{\mathbb{A}^m}{\times} \mathrm{Spec}(A)$$

is isomorphic to the colimit over n of

$$(\{0\} \underset{\mathbb{A}^m}{\times} \mathbb{A}^m) \underset{\mathbb{A}^m}{\times} \mathrm{Spec}(A), \tag{6.5}$$

where the map $\mathbb{A}^m \to \mathbb{A}^m$ is given by raising to the power n along each coordinate. Now, by definition, the DG scheme in (6.5) is isomorphic to $\mathrm{Spec}(A_n)$, as required.

□

6.7.5. Let us show how Proposition 6.7.4 implies Proposition 6.7.2:

Proof of Proposition 6.7.2. First, note that the assertion is local in the Zariski topology on X. Thus, we can assume that $X = \mathrm{Spec}(A)$ is affine.

Now, the assertion follows from the fact that if A is l-coconnective, then each of the algebras A_n is $(m + l)$-coconnective, by construction.

□

6.7.6. Here is another corollary of Proposition 6.7.4:

Proposition 6.7.7. *The DG indscheme $X^\wedge_Y$ can be written as a colimit in* Stk

$$\underset{\alpha \in \mathsf{A}}{colim}\, Y'_\alpha,$$

where $Y'_\alpha \to X$ are closed embeddings with set-theoretic image is equal to Y, and where the category A of indices is equivalent to the poset $\mathbb{N}$.

Remark 6.7.8. This proposition does not prove that $X^\wedge_Y$ is $\aleph_0$, because the colimit is taken in Stk and not PreStk.

Proof. First, note that Proposition 6.7.4 gives such a presentation if X is affine (moreover, in this case, the colimit can be taken in PreStk). I.e., in this case, $X^\wedge_Y$ is $\aleph_0$ as a DG indscheme.

Let S_i be a (finite) collection of affine open DG subschemes of X that covers Y. For each i, let A_i be the corresponding index set (isomorphic to $\mathbb{N}$) for the formal completion $(S_i)^\wedge_{S_i \cap Y}$.

For $\alpha_i \in \mathsf{A}_i$ let Y'_{i,α_i} be the corresponding DG scheme equipped with a closed embedding into S_i. Let $\overline{Y}'_{i,\alpha_i}$ be the closure of its image in X, see Sect. 3.1.6.

For $\alpha := \{i \mapsto (\alpha_i \in \mathsf{A}_i)\}$ set Y'_α be the coproduct of $\overline{Y}'_{i,\alpha_i}$ in $(\mathrm{DGSch}_{\text{qs-qs}})_{\text{closed in } X}$.

We claim that the family $\alpha \mapsto Y'_\alpha$ has the desired property. Indeed, it is sufficient to show that for every i, the colimit of the family

$$\alpha \mapsto Y'_\alpha \underset{X}{\times} S_i$$

is isomorphic to $(S_i)^\wedge_{S_i \cap Y}$. However, this is clear since this colimit is also given by the colimit of the family $i \mapsto Y'_{i,\alpha_i}$.

□

6.8. **Classical vs. derived formal completions.** We shall now show how Proposition 6.7.4 helps answer another natural question regarding the behavior of $X^\wedge_Y$.

6.8.1. Let X a DG scheme, which is 0-coconnective (=classical), i.e., the sheafification of a left Kan extension of a classical scheme.

One can ask whether the DG indscheme $X^\wedge_Y$ is also 0-coconnective. That is, we consider the classical indscheme ${}^{cl}(X^\wedge_Y)$, and let

$$\mathcal{X} := {}^{L}\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}} \left({}^{cl}(X^\wedge_Y)\right).$$

By adjunction, we obtain a map

$$\mathcal{X} \to X^\wedge_Y, \tag{6.6}$$

and we wish to know whether it is an isomorphism.

Again, we emphasize that it is a rather strong property for a DG indscheme (or any convergent stack) to be 0-coconnective (rather than weakly 0-coconnective), see [GL:Stacks, Remark 1.2.6].

However, the answer to the above question turns out to be affirmative, under an additional assumption that X be Noetherian (see [GL:IndCoh, Sect. 0.6.9] for the notion of Noetherianness in the DG setting):

Proposition 6.8.2. *If X is Noetherian, the map* (6.6) *is an isomorphism.*

6.8.3. *Proof of Proposition 6.8.2, Step 1.* The assertion readily reduces to the case when X is affine; $X = \mathrm{Spec}(A)$, where A is a classical k-algebra. Let $f_1, ..., f_m \in A$ be the generators of the ideal of some subscheme $Y' \subset X$ whose underlying Zariski-closed subset is Y. Let A_n be the algebras as in Proposition 6.7.4.

For each n, let A'_n be the classical algebra $H^0(A_n)$, so that

$$\mathcal{X} \simeq \underset{n}{colim}\ \mathrm{Spec}(A'_n),$$

where the colimit is taken in PreStk. We will show that inverse systems $\{A_n\}$ and $\{A'_n\}$ are equivalent, i.e., that the natural map

$$\underset{n\in\mathbb{N}}{colim}\ \mathrm{Spec}(A'_n) \to \underset{n\in\mathbb{N}}{colim}\ \mathrm{Spec}(A_n) \tag{6.7}$$

is an isomorphism in PreStk.

6.8.4. *Proof of Proposition 6.8.2, Step 2.* We will prove:

Lemma 6.8.5. *For every n there exists $N \geq n$ such that the map $A_N \to A_n$ can be factored as*

$$A_N \to A'_N \to A_n.$$

Let us show how Lemma 6.8.5 implies that (6.7) is an isomorphism. We construct the sequence $i_1, i_2, ..., \subset \mathbb{N}$ inductively, starting with $i_1 = 1$. Suppose i_k has been constructed. We take $n := i_k$, and we let i_{k+1} be the integer N given by Lemma 6.8.5.

We obtain the maps

$$\underset{k\in\mathbb{N}}{colim}\ \mathrm{Spec}(A'_{i_k}) \to \underset{k\in\mathbb{N}}{colim}\ \mathrm{Spec}(A_{i_k})$$

and

$$\underset{k\in\mathbb{N}}{colim}\ \mathrm{Spec}(A_{i_k}) \to \underset{k\in\mathbb{N}}{colim}\ \mathrm{Spec}(A'_{i_{k+1}})$$

that induce mutually inverse maps in (6.7).

6.8.6. *Proof of Proposition 6.8.2, Step 3.* We will deduce the assertion of Lemma 6.8.5 from the following version of the Artin-Rees lemma:

Lemma 6.8.7. *Let $B \to A$ be a map of (classical) Noetherian rings; let $I \subset B$ be an ideal , and let M be a finitely generated A-module. Then for every $i > 0$, the inverse system*

$$n \mapsto \operatorname{Tor}_i^B(M, B/I^n)$$

is equivalent to zero, i.e., for every n there exists an $N \geq n$, such that the map

$$\operatorname{Tor}_i^B(M, B/I^N) \to \operatorname{Tor}_i^B(M, B/I^n)$$

is zero.

Proof of Lemma 6.8.5. It is easy to see by induction that a map of connective commutative DG algebras $C_1 \to C_2$ can be factored as

$$C_1 \to H^0(C_1) \to C_2$$

if and only if the maps

$$H^{-i}(C_1) \to H^{-i}(C_2)$$

are zero for $i > 0$.

Hence, we need to show that for every n, we can find $N \geq n$ such that the maps

$$H^{-i}(A_N) \to H^{-i}(A_n)$$

are zero for $i > 0$.

Let us apply Lemma 6.8.7 to A being our algebra A, $B = k[t_1, ..., t_m]$, and $B \to A$ being given by $f_1, ..., f_m$. Let $I \subset B$ be the ideal generated by $t_1, ..., t_m$. Let $'I^n$ be the ideal generated by $t_1^n, ..., t_m^n$. Note that

$$H^{-i}(A_n) \simeq \operatorname{Tor}_i^B(A, B/'I^n).$$

Finally, the system of ideals $n \mapsto I^n$ is cofinal with $n \mapsto {}'I^n$.

□

6.8.8. *Exponential map.* Let $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ be the formal completions of $\mathbb{G}_a$ and $\mathbb{G}_m$ at 0 and 1, respectively. These are both formal group schemes. By Proposition 6.8.2, we have that $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ are both 0-coconnective as prestacks. Hence, the exponential map in ${}^{cl}\mathrm{PreStk}$

$${}^{cl}\widehat{\mathbb{G}}_a \to {}^{cl}\widehat{\mathbb{G}}_m,$$

defined by the usual fomula, gives rise to a canonical isomorphism in PreStk.

$$\exp : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_m.$$

Furthermore, exp is an isomorphism of $\mathbb{E}_\infty$-group objects in PreStk, i.e., as functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd}\,.$$

6.8.9. For a connective k-akgebra A, let $\mathcal{N}ilp(A)$ denote the connective spectrum (i.e., ∞-Picard groupoid)

$$\ker\left(A \to {}^{cl,red}A\right).$$

Note that by Proposition 6.6.3, the above definition of $\mathcal{N}ilp(A)$ agrees with one in Sect. 6.6.4.

Let $A^\times$ denote the connective spectrum of invertible elements in the $\mathbb{E}_\infty$-ring spectrum A, and similarly for ${}^{cl,red}A$. Set

$$\mathcal{U}nip(A) := \ker\left(A^\times \to {}^{cl,red}A^\times\right).$$

We obtain that the exponential map defines a functorial isomorphism

$$\exp : \mathcal{N}ilp(A) \to \mathcal{U}nip(A)$$

of functors $(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty$-PicGrpd.

7. Quasi-coherent and ind-coherent sheaves on formal completions

7.1. **Quasi-coherent sheaves on a formal completion.** Let X be a DG scheme, and $Y \to X$ a Zariski closed subset. We shall assume that Y is quasi-separated and quasi-compact. Let U be the open DG subscheme of X equal to the complement of Y; let j denote the corresponding open embedding.

7.1.1. We have a pair of mutually adjoint functors

$$j^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(U) : j_*,$$

which realizes $\mathrm{QCoh}(U)$ as a localization of $\mathrm{QCoh}(X)$. Note, however, that the functor j_* is not a priori continuous, since j is not necessarily quasi-compact.

Let $\mathrm{QCoh}(X)_Y$ denote the full subcategory of $\mathrm{QCoh}(X)$ equal to

$$\ker(j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)).$$

Let $\widehat{i}$ denote the canonical map $X^\wedge_Y \to X$, and consider the corresponding functor

$$\widehat{i}^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^\wedge_Y).$$

We can ask the following questions:

(i) Is the composition $\widehat{i}^* \circ j_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(X^\wedge_Y)$ zero?

(ii) Does the functor $\widehat{i}^*$ induce an equivalence $\mathrm{QCoh}(X)_Y \to \mathrm{QCoh}(X^\wedge_Y)$?

We will answer these questions in the affirmative under an additional hypothesis on the pair X and Y. We learned the corresponding assertion from J. Lurie.

7.1.2. We will impose the assumption of Sect. 6.6.1, i.e., that Y can be represented by a closed subscheme Y' of ${}^{cl}X$, whose ideal is locally finitely generated.

In this case, the morphism j is quasi-compact (being an open embedding, it is automatically quasi-separated). In particular, by [GL:QCoh, Proposition 2.1.1], the functor j_* is continuous and satisfies the base change formula, which immediately implies that the composition

$$\widehat{i}^* \circ j_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(X^\wedge_Y)$$

vanishes.

Proposition 7.1.3. *Under the above hypothesis, the composite functor*

$$\widehat{i}^* : \mathrm{QCoh}(X)_Y \hookrightarrow \mathrm{QCoh}(X) \xrightarrow{\widehat{i}^*} \mathrm{QCoh}(X^\wedge_Y)$$

is an equivalence.

7.1.4. *Proof of Proposition 7.1.3.* The assertion is Zarski-local, so we can assume that $X = \mathrm{Spec}(A)$ is affine. Let $f_1, ..., f_m$ and A_n be as in the proof of Proposition 6.7.4.

Consider the functor

$$\mathrm{QCoh}(X^\wedge_Y) \to \mathrm{QCoh}(X)$$

given by direct image $\widehat{i}_*$ with respect to the morphism $\widehat{i}$, i.e., the *right* adjoint of the functor $\widehat{i}^*$. [22]

Warning: The functor $\widehat{i}_*$ is not continuous and does not commute with Zariski localization.

We obtain that $\widehat{i}_*$ and $\widehat{i}^*$ induce a pair of mutually adjoint functors

$$(\mathrm{QCoh}(X))_{\mathrm{QCoh}(U)} \rightleftarrows \mathrm{QCoh}(X^\wedge_Y), \tag{7.1}$$

where $(\mathrm{QCoh}(X))_{\mathrm{QCoh}(U)}$ denotes the localization of $\mathrm{QCoh}(X)$ with respect to $\mathrm{QCoh}(U)$, and the latter is mapped in by means of j_*. To prove the proposition it suffices to show that:

(a) The functor $\leftarrow$ in (7.1) is fully faithful, and

(b) The functor $\rightarrow$ in (7.1) is conservative.

Assertion (a) is equivalent to the functor $\widehat{i}_*$ being fully faithful. I.e., we need to show that the adjunction map $\widehat{i}^* \circ \widehat{i}_* \to \mathrm{Id}$ is an isomorphism.

Fix an object of $\mathrm{QCoh}(X^\wedge_Y)$, thought of as a compatible system of A_n-modules $\{\mathcal{F}_n\}$; let $\mathcal{F}$ be its direct image on X. By definition,

$$\mathcal{F} \simeq \underset{n}{lim}\, \mathcal{F}_n,$$

where in the right-hand side, the $\mathcal{F}_n$'s are regarded as A-modules.

We need to show that for every n_0, the map

$$A_{n_0} \underset{A}{\otimes} \mathcal{F} \to \mathcal{F}_{n_0}$$

is an isomorphism.

Since A_{n_0} is compact as an A-module, we can rewrite the left-hand side as $\underset{n}{lim} \left(A_{n_0} \underset{A}{\otimes} \mathcal{F}_n \right)$, and further as

$$\underset{n}{lim} \left((A_{n_0} \underset{A}{\otimes} A_n) \underset{A_n}{\otimes} \mathcal{F}_n \right).$$

For $n \geq n_0$ consider the canonical map $A_{n_0} \underset{A}{\otimes} A_n \to A_0$, and let K_n denote its kernel. The required assertion follows from the next claim:

Lemma 7.1.5. *For every n there exists $N \geq n$, such that the map $K_N \to K_n$ is zero as a map of A_N-modules.*

[22] Recall that direct image g_*, although in general non-continuous, is defined for any morphism $g : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk, by the adjoint functor theorem.

Proof. Let B denote the polynomial algebra $k[t_1, ..., t_m]$. We have $A_n \simeq A \underset{B}{\otimes} B_n$, as commutative k-algebras. With respect to this identification, we have

$$K_n \simeq A \underset{B}{\otimes} K_n^B,$$

as A_n-modules, where K_B^n denotes the corresponding object for the algebra B. Hence, it is enough to prove the assertion for A replaced by B.

To simplify notation, we will only consider the case when $m = 1$, i.e., $B = k[t]$. In this case

$$B_n \underset{B}{\otimes} B_{n_0} \simeq \operatorname{Cone}(t^n : k[t]/t^{n_0} \to k[t]/t^{n_0}).$$

When $n \geq n_0$, the map $t^n : k[t]/t^{n_0} \to k[t]/t^{n_0}$ is zero, so $K_n \simeq k[t]/t^{n_0}[1]$. For $n' \geq n$, the corresponding map $K_{n'} \to K_n$ is given by multiplication by $t^{n'-n}$. Hence, we can take $N = 2n$. $\square$

Let us now prove point (b). Recall the elements $f_1, ..., f_m$ of A. Let Y_k be the closed DG subscheme of X cut out (in the derived sense) by the equations $f_1, ..., f_k$; i.e.,

$$Y_k = \operatorname{Spec}(A[t_1, ..., t_k],\, d(t_i) = f_i);$$

let $i_k : Y_k \to X$ denote the corresponding closed embedding. In particular ${}^{cl,red}Y_m = Y$.

It suffices to show that if for $\mathcal{F} \in \operatorname{QCoh}(X)$ we have $i_m^*(\mathcal{F}) = 0$, then $\mathcal{F}$ belongs to the essential image of j_*. Taking the cone we can assume that $j^*(\mathcal{F}) = 0$ as well, and we need to show that $\mathcal{F} = 0$.

By induction on k, we may assume that $m = 1$. The assumption that $i_1^*(\mathcal{F}) = 0$ means that $f_1 : \mathcal{F} \to \mathcal{F}$ acts invertibly, i.e., $\mathcal{F} \to (\mathcal{F})_{f_1}$ is an isomorphism, where $(\mathcal{F})_{f_1}$ denotes the localization of $\mathcal{F}$ with respect to f_1. However, $j^*(\mathcal{F}) = 0$ implies $(\mathcal{F})_{f_1} = 0$.

$\square$(Proposition 7.1.3)

7.1.6. Let us denote by $\mathbf{e}^{\mathrm{QCoh}}$ the tautological embedding

$$\operatorname{QCoh}(X)_Y \to \operatorname{QCoh}(X).$$

We note that it admits a (continuous) right adjoint, denoted $\mathbf{r}^{\mathrm{QCoh}}$, given by

$$\mathcal{F} \mapsto \operatorname{Cone}(\mathcal{F} \to j_* \circ j^*(\mathcal{F}))[-1].$$

The adjoint pair $(\mathbf{e}^{\mathrm{QCoh}}, \mathbf{r}^{\mathrm{QCoh}})$ realizes $\operatorname{QCoh}(X)_Y$ as a *co-localization* of $\operatorname{QCoh}(X)$.

By construction, we have a commutative diagram

(7.2)
$$\begin{array}{ccc}
\operatorname{QCoh}(X^\wedge_Y) & \xleftarrow{\widehat{i}^*} & \operatorname{QCoh}(X) \\
\Big\uparrow{\scriptstyle \widehat{i}^*} & & \Big\uparrow{\scriptstyle \mathrm{Id}} \\
\operatorname{QCoh}(X)_Y & \xleftarrow{\mathbf{r}^{\mathrm{QCoh}}} & \operatorname{QCoh}(X),
\end{array}$$

where the left vertical arrow is the functor from Proposition 7.1.3.

Hence, we obtain that the functor $\widehat{i}^* : \operatorname{QCoh}(X) \to \operatorname{QCoh}(X^\wedge_Y)$, in addition to having a non-continuous right adjoint $\widehat{i}_*$, admits a left adjoint, which we denote by $\widehat{i}_?$.

Thus, we can think of $\operatorname{QCoh}(X^\wedge_Y)$ as both a localization and a co-localization of $\operatorname{QCoh}(X)$ with respect to the essential image of $\operatorname{QCoh}(U)$.

Note that under such circumstances, we have a canonical natural transformation

(7.3)
$$\widehat{i}_? \to \widehat{i}_*.$$

7.1.7. Consider now the *non-continuous* functor

$$\mathrm{QCoh}(X) \xrightarrow{\widehat{i}^*} \mathrm{QCoh}(X^\wedge_Y) \xrightarrow{\widehat{i}_*} \mathrm{QCoh}(X),$$

i.e., the localization functor on $\mathrm{QCoh}(X)$ with respect to the essential image of $\mathrm{QCoh}(U)$.

This functor is called *the functor of formal completion* of a quasi-coherent sheaf along Y. Its essential image (i.e., the essential image of $\widehat{i}_*$) is referred to as objects of $\mathrm{QCoh}(X)$ that are *adically-complete* with respect to Y.

7.2. **Compact generation and duality.** Assume now that the scheme X is quasi-separated and quasi-compact. It is well-known that if Y is locally given by a finitely generated ideal, then the category $\mathrm{QCoh}(X)_Y$ is compactly generated by $\mathrm{QCoh}(X)_Y \cap \mathrm{QCoh}(X)^{\mathrm{perf}}$.

Combining this with Proposition 7.1.3 and (7.2) we obtain:

Corollary 7.2.1. *The category* $\mathrm{QCoh}(X^\wedge_Y)$ *is compactly generated. The compact objects are obtained as images under* $\widehat{i}^*$ *of compact objects of* $\mathrm{QCoh}(X)$ *that are set-theoretically supported on* Y.

Recall now the notion of quasi-perfectness, see Sect. 2.2.4. We obtain:

Corollary 7.2.2. *For* X *and* Y *as above, the DG indscheme* $\mathrm{QCoh}(X^\wedge_Y)$ *is quasi-perfect.*

Let us recall that being quasi-perfect means by definition that the category $\mathrm{QCoh}(X^\wedge_Y)$ is compactly generated, and that its compact objects belong to $\mathrm{QCoh}(X^\wedge_Y)^{\mathrm{perf}}$.

As was shown in Sect. 2.2.4, the above property implies that there exists a canonical equivalence:

$$\mathbf{D}^{\mathrm{naive}}_{X^\wedge_Y} : \mathrm{QCoh}(X^\wedge_Y)^\vee \simeq \mathrm{QCoh}(X^\wedge_Y), \tag{7.4}$$

characterized by either of the following two properties:

- The canonical anti self-equivalence $\mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X^\wedge_Y)} : (\mathrm{QCoh}(X^\wedge_Y)^c)^{\mathrm{op}} \to (\mathrm{QCoh}(X^\wedge_Y)^c$ is given by the restriction of the functor $\mathcal{F} \mapsto \mathcal{F}^\vee : ((\mathrm{QCoh}(X^\wedge_Y)^{\mathrm{perf}})^{\mathrm{op}} \to \mathrm{QCoh}(X^\wedge_Y)^{\mathrm{perf}}$.
- The pairing

$$\mathrm{QCoh}(X^\wedge_Y) \otimes \mathrm{QCoh}(X^\wedge_Y) \to \mathrm{Vect} \tag{7.5}$$

is given by ind-extension of the pairing

$$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(X^\wedge_Y)^c \mapsto \Gamma(X^\wedge_Y, \mathcal{F}_1 \underset{\mathcal{O}_{X^\wedge_Y}}{\otimes} \mathcal{F}_2) \in \mathrm{Vect}\,.$$

7.2.3. Note that although the natural transformation (7.3) is not an isomorphism, we have the following:

Lemma 7.2.4. *The natural transformation* (7.3) *it induces an isomorphism when restricted to compact objects of* $\mathrm{QCoh}(X^\wedge_Y)$.

Proof. Follows from the fact that compact objects of $\mathrm{QCoh}(X)$ with set-theoretic support on Y are both left and right orthogonal to the essential image of $\mathrm{QCoh}(U)$. □

7.2.5. Recall that the category $\mathrm{QCoh}(X)$ is also self-dual. From the description of the functor $\mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X^\wedge_Y)}$ we obtain that there exists a canonical isomorphism

$$\mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X)} \circ \widehat{i}_? \simeq \widehat{i}_? \circ \mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X^\wedge_Y)} : (\mathrm{QCoh}(X^\wedge_Y)^c)^{\mathrm{op}} \to \mathrm{QCoh}(X)^c.$$

By [GL:DG, Lemma 2.3.3], this implies:

Corollary 7.2.6. *Under the identifications*

$$\mathbf{D}^{\mathrm{naive}}_X : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathbf{D}^{\mathrm{naive}}_{X^\wedge_Y} : \mathrm{QCoh}(X^\wedge_Y)^\vee \simeq \mathrm{QCoh}(X^\wedge_Y),$$

the dual of the functor $\widehat{i}^*$ identifies with $\widehat{i}_?$.

7.2.7. Note under the identifications

$$\mathbf{D}^{\mathrm{naive}}_X : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathbf{D}^{\mathrm{naive}}_U : \mathrm{QCoh}(U)^\vee \simeq \mathrm{QCoh}(U)$$

we have $(j_*)^\vee \simeq j^*$. This implies that the category $\mathrm{QCoh}(X)_Y$ is also naturally self-dual, such that the dual of the natural embedding

$$\mathbf{e}^{\mathrm{QCoh}} : \mathrm{QCoh}(X)_Y \to \mathrm{QCoh}(X)$$

is the functor $\mathbf{r}^{\mathrm{QCoh}}$.

By [GL:DG, Lemma 2.3.3], this implies:

$$\mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X)} \circ \mathbf{e}^{\mathrm{QCoh}} \simeq \mathbf{r}^{\mathrm{QCoh}} \circ \mathbb{D}^{\mathrm{naive}}_{\mathrm{QCoh}(X)_Y} : (\mathrm{QCoh}(X)^c_Y)^{\mathrm{op}} \to \mathrm{QCoh}(X)^c.$$

It follows that:

Corollary 7.2.8. *The above self-duality of $\mathrm{QCoh}(X)_Y$ is compactible with the self-duality $\mathbf{D}^{\mathrm{naive}}_{X^\wedge_Y}$ of $\mathrm{QCoh}(X^\wedge_Y)$ via the equivalence of Proposition 7.1.3.*

7.3. **t-structures on $\mathrm{QCoh}(X^\wedge_Y)$.** In this subsection we will show that the category

$$\mathrm{QCoh}(X^\wedge_Y) \simeq \mathrm{QCoh}(X)_Y$$

possesses two natural t-structures: for one of them the functor $\mathbf{e}^{\mathrm{QCoh}}$ (i.e., the *left* adjoint of $\mathbf{r}^{\mathrm{QCoh}} \simeq \widehat{i}^*$) is t-exact, and for the other, the functor $\widehat{i}_*$ (i.e., the *right* adjoint of $\widehat{i}^* \simeq \mathbf{r}^{\mathrm{QCoh}}$) is t-exact.

7.3.1. Let us recall the following general paradigm: let $\mathbf{C}$ be a DG category equipped with a t-structure. Let $F : \mathbf{C}_1 \hookrightarrow \mathbf{C}$ be a fully faithful functor. Assume that F admits a left (resp., right) adjoint, denoted F^L (resp., F^R). We have:

Lemma 7.3.2.

(*a*) *If the composition $F \circ F^L$ (resp., $F \circ F^R$) is right (resp., left) t-exact, then $\mathbf{C}_1$ has a unique t-structure such that F is t-exact. With respect to this t-structure, the functor F^L (resp., F^R) is right (resp., left) t-exact.*

(*b*) *If the composition $F \circ F^L$ (resp., $F \circ F^R$) is left (resp., right) t-exact, then $\mathbf{C}_1$ has a unique t-structure such that F^L (resp., F^R) is t-exact. With respect to this t-structure, the functor F is left (resp., right) t-exact.*

7.3.3. We will apply point (a) of the lemma (with right adjoints) to $\mathbf{C} = \mathrm{QCoh}(X)$ and $\mathbf{C}_1 = \mathrm{QCoh}(X)_Y$.

Let us first take $F := \mathbf{e}^{\mathrm{QCoh}}$ and $F^R := \mathbf{r}^{\mathrm{QCoh}}$. We obtain that $\mathrm{QCoh}(X)_Y$ admits a t-structure, compatible with its embedding into $\mathrm{QCoh}(X)$. This t-structure is compatible with filtered colimits (i.e., truncation functors commute with filtered colimits).

We shall refer to this t-structure on $\mathrm{QCoh}(X^\wedge_Y)$ as the "inductive t-structure."

7.3.4. We shall now introduce another t-structure on $\mathrm{QCoh}(X^\wedge_Y)$.

Recall (see [GL:QCoh, Sect. 1.2.3]) that for any $\mathcal{Z} \in \mathrm{PreStk}$, the category $\mathrm{QCoh}(\mathcal{Z})$ has a canonical t-structure defined by the following requirement: an object $\mathcal{F}$ belongs to $\mathrm{QCoh}(\mathcal{Z})^{\leq 0}$ if and only if for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and $\phi : S \to \mathcal{Z}$, we have $\phi^*(\mathcal{F}) \in \mathrm{QCoh}(S)^{\leq 0}$. Let us call it "the canonical t-structure on $\mathrm{QCoh}(\mathcal{Z})$."

Proposition 7.3.5. *The functor*

$$\widehat{i}_* : \mathrm{QCoh}(X^\wedge_Y) \to \mathrm{QCoh}(X)$$

is t-exact for the canonical t-structure on $\mathrm{QCoh}(X^\wedge_Y)$.

A few remarks are in order:

(i) Since the functor $\widehat{i}^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^\wedge_Y)$ is right t-exact, we obtain that the proposition implies that the localization functor

$$\widehat{i}_* \circ \widehat{i}^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$$

is also right t-exact. Thus, the canonical t-structure on $\mathrm{QCoh}(\mathcal{Z})$ falls into the paradigm of Lemma 7.3.2(a) with left adjoints.

(ii) The canonical t-structure on $\mathrm{QCoh}(X^\wedge_Y)$ is different from the one of Sect. 7.3.3: for the former the functor $\widehat{i}^*$ is right t-exact, and for the latter it is left t-exact.

(iii) Let $\mathcal{F}$ be an object of $\mathrm{QCoh}(X)^\heartsuit$ which is *scheme-theoretically* supported on some subscheme $Y' \subset X$ whose underlying set is Y. Then it is easy to see that $\mathcal{F}$, regarded as an object of $\mathrm{QCoh}(X)_Y$, lies in the heart of both t-structures.

(iv) The canonical t-structure on $\mathrm{QCoh}(X^\wedge_Y)$ is typically not compatible with colimits, as can be seen in the example of $X = \mathbb{A}^1$ and $Y = \mathrm{pt}$.

Proof of Proposition 7.3.5. The functor $\widehat{i}_*$ is left t-exact, being the right adjoint of a right t-exact functor, namely, $\widehat{i}^*$. Hence, we need to show that $\widehat{i}_*$ is right t-exact.

Let $\mathcal{Y}$ be an object of Stk, and let $f : \mathcal{Y} \to X$ be a morphism, where $X \in \mathrm{DGSch}$. Assume that $\mathcal{Y}$ is written as a colimit in Stk

$$\underset{g_\alpha : Y'_\alpha \to \mathcal{Y}}{colim} Y'_\alpha,$$

where $Y'_\alpha \in \mathrm{DGSch}$. In this case, the functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \underset{\alpha}{lim}\, \mathrm{QCoh}(Y'_\alpha)$$

is an equivalence (this follows from [GL:QCoh, Corollary 1.3.7], and the fact that the functor $\mathrm{QCoh}(-)$ takes colimits in PreStk to limits in DGCat).

This implies that the (non-continuous) functor $f_* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(X)$ can be calculated as follows: for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, given as a compatible family $\mathcal{F}_\alpha := g_\alpha^*(\mathcal{F}) \in \mathrm{QCoh}(Y'_\alpha)$,

$$f_*(\mathcal{F}) \simeq \underset{\alpha}{lim}\, (f \circ g_\alpha)_*(\mathcal{F}_\alpha).$$

We apply this to $\mathcal{Y} := X^\wedge_Y$ written as a colimit as in Proposition 6.7.7. Thus, in order to show that $\widehat{i}_*$ is right t-exact, we need to check that if $\mathcal{F}_\alpha \in \mathrm{QCoh}(Y'_\alpha)^{\leq 0}$ for all $\alpha \in \mathsf{A}$, then

$$\underset{\alpha \in \mathsf{A}}{lim}\, (i_\alpha)_*(\mathcal{F}_\alpha) \in \mathrm{QCoh}(X)^{\leq 0}$$

where i_α denotes the map $Y'_\alpha \to X$.

Since the index category is A is $\mathbb{N}$, we need to show that the functor lim^1 applied to the family $\alpha \mapsto H^0((i_\alpha)_*(\mathcal{F}_\alpha))$ vanishes. However, this is the case, since the maps in this family are surjective.

□

7.4. **Ind-coherent sheaves on formal completions.** Let X be a DG scheme almost of finite type; in particular, it is quasi-compact and quasi-separated.

7.4.1. Recall (see [GL:IndCoh, Sect. 4.1]) that we have a pair of adjoint functors

$$j^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{IndCoh}(U) : j_*^{\mathrm{IndCoh}}$$

that realize $\mathrm{IndCoh}(U)$ as a localization of $\mathrm{IndCoh}(X)$. Let $\mathrm{IndCoh}(X)_Y \subset \mathrm{IndCoh}(X)$ be the full subcategory equal to

$$\ker(j^{\mathrm{IndCoh},*}) : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(U).$$

We let $\mathbf{e}^{\mathrm{IndCoh}}$ denote the tautological embedding

$$\mathrm{IndCoh}(X)_Y \hookrightarrow \mathrm{IndCoh}(X).$$

This functor admits a right adjoint, denoted $\mathbf{r}^{\mathrm{IndCoh}}$ given by

$$\mathcal{F} \mapsto \mathrm{Cone}\left(\mathcal{F} \to j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*}(\mathcal{F})\right)[-1].$$

7.4.2. As was shown in Corollary 6.3.2, for a Zariski-closed subset Y, the DG indscheme $X^\wedge_Y$ is locally almost of finite type, so $\mathrm{IndCoh}(X^\wedge_Y)$ is well-defined.

Consider the functor [23]

$$\widehat{i}^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X^\wedge_Y),$$

i.e., the !-pullback functor with respect to the morphism $\widehat{i} : X^\wedge_Y \to X$. It is easy to see that this functor annihilates the essential image of $\mathrm{IndCoh}(U)$ under j_*^{IndCoh}.

7.4.3. We now claim that the functor $\widehat{i}^!$ admits a left adjoint, to be denoted by $\widehat{i}_*^{\mathrm{IndCoh}}$.

Indeed, by Sect. 2.4.2, we have:

$$\mathrm{IndCoh}(X^\wedge_Y) \simeq \underset{\alpha}{colim}\, \mathrm{IndCoh}(Y_\alpha), \tag{7.6}$$

where Y_α run over a family of closed DG subschemes of X with the underlying set contained in Y. If we denote by i_α the closed embedding $Y_\alpha \hookrightarrow X$, the functor $\widehat{i}_*^{\mathrm{IndCoh}}$, left adjoint to $\widehat{i}^!$, is given by the compatible family of functors

$$(i_\alpha)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Y_\alpha) \to \mathrm{IndCoh}(X).$$

[23]The usage of notation $\widehat{i}^!$ here is different from [GL:IndCoh, Corollary 4.1.5]. Nevertheless, this notation is consistent as will follow from Proposition 7.4.5.

7.4.4. By construction, the essential image of the functor $\widehat{i}_*^{\mathrm{IndCoh}}$ belongs to

$$\mathrm{IndCoh}(X)_Y \subset \mathrm{IndCoh}(X).$$

(Or, equivalently, the right adjoint $\widehat{i}^!$ of $\widehat{i}_*^{\mathrm{IndCoh}}$ factors through the co-localization functor $\mathbf{r}^{\mathrm{IndCoh}}$.)

Let

$$'\widehat{i}_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_Y^\wedge) \rightleftarrows \mathrm{IndCoh}(X)_Y : {}'\widehat{i}^! \tag{7.7}$$

denote the resulting pair of adjoint functors.

We will show:

Proposition 7.4.5. *The adjoint functors of* (7.7) *are equivalences.*

Proof. As in the proof of Proposition 7.1.3, we need to show two things:

(a) The functor $\widehat{i}_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_Y^\wedge) \to \mathrm{IndCoh}(X)$ is fully faithful.

(b) The essential image of the functor $'\widehat{i}_*^{\mathrm{IndCoh}}$ generates $\mathrm{IndCoh}(X)_Y$.

We note that (b) follows from [GL:IndCoh, Proposition 4.1.7(a)]. It remains to prove (a). The assertion is Zariski-local, so we can assume $X = \mathrm{Spec}(A)$. Let A_n be as in the proof of Proposition 7.1.3. Set $Y_n := \mathrm{Spec}(A_n)$.

For $n' \leq n''$, let $i_{n',n''}$ denote the closed embedding $Y_{n'} \to Y_{n''}$, and i_n the closed embedding $Y_n \to X$. To prove (a), we need to show that for an index n_0 and $\mathcal{F} \in \mathrm{IndCoh}(Y_{n_0})$, the map

$$\underset{n \geq n_0}{\mathit{colim}}\, i^!_{n_0,n} \circ (i_{n_0,n})_*^{\mathrm{IndCoh}}(\mathcal{F}) \to i^!_{n_0} \circ (i_{n_0})_*^{\mathrm{IndCoh}}(\mathcal{F}) \tag{7.8}$$

is an isomorphism.

Both sides in (7.8) commute with colimits in the $\mathcal{F}$ variable. So, we can take $\mathcal{F} \in \mathrm{Coh}(Y_{n_0})$. In this case both sides of (7.8) belong to $\mathrm{IndCoh}(Y_{n_0})^+$. Hence, by [GL:IndCoh, Proposition 1.2.4], it suffices to show that the map in (7.8) induces an isomorphisms by applying the functor $\Psi_{Y_{n_0}} : \mathrm{IndCoh}(Y_{n_0}) \to \mathrm{QCoh}(Y_{n_0})$. Since Y_0 is affine, we can furthermore test whether a map is an isomorphism by taking global sections.

Hence, we obtain that it suffices to show that

$$\underset{n \geq n_0}{\mathit{colim}}\, \mathrm{Maps}_{A_n\text{-mod}}(A_{n_0}, \mathcal{F}) \to \mathrm{Maps}_{A\text{-mod}}(A_{n_0}, \mathcal{F}) \tag{7.9}$$

is an isomorphism. The map (7.9) can be rewritten as

$$\underset{n \geq n_0}{\mathit{colim}}\, \mathrm{Maps}_{A_{n_0}\text{-mod}}((A_{n_0} \underset{A_n}{\otimes} A) \underset{A}{\otimes} A_{n_0}, \mathcal{F}) \to \mathrm{Maps}_{A_{n_0}\text{-mod}}(A_{n_0} \underset{A}{\otimes} A_{n_0}, \mathcal{F}).$$

Hence, the required assertion follows from Lemma 7.1.5.

□

7.4.6. Proposition 7.4.5 implies the commutativity of the following diagram, analogous to (7.2):

$$\begin{array}{ccc} \mathrm{IndCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^!} & \mathrm{IndCoh}(X) \\ {}^{'\widehat{i}^!}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{Id}} \\ \mathrm{IndCoh}(X)_Y & \xleftarrow{\mathbf{r}^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(X), \end{array} \tag{7.10}$$

7.4.7. *Compatibility with the t-structure.* Recall from Sect. 2.5, that the category $\mathrm{IndCoh}(X^\wedge_Y)$ has a natural t-structure. Note that the category $\mathrm{IndCoh}(X)_Y$ also has a natural t-structure for which the functor $\mathbf{e}^{\mathrm{IndCoh}}$ is t-exact. Indeed, this follows by Lemma 7.3.2(a) from the fact that the functor

$$\mathbf{e}^{\mathrm{IndCoh}}\circ\mathbf{r}^{\mathrm{IndCoh}}:\mathrm{IndCoh}(X)\to\mathrm{IndCoh}(X),\quad \mathcal{F}\mapsto \mathrm{Cone}\left(\mathcal{F}\to j_*^{\mathrm{IndCoh}}\circ j^{\mathrm{IndCoh},*}(\mathcal{F})\right)[-1]$$

is left t-exact.

We claim:

Lemma 7.4.8. *The equivalence in* (7.7) *is t-exact.*

Proof. By Lemma 7.3.2(a) we only have to show that the functor

$$\widehat{i}_*^{\mathrm{IndCoh}}:\mathrm{IndCoh}(X^\wedge_Y)\to\mathrm{IndCoh}(X)$$

is t-exact. However, this follows from the description of this functor given in Sect. 7.4.3. ☐

7.5. **Comparison of** QCoh **and** IndCoh **on a formal completion.**

7.5.1. Recall ([GL:IndCoh, Sect. 10.3.3], Sect. 9.3.2) that for any $\mathcal{Y}\in\mathrm{PreStk}_{\mathrm{laft}}$ we have a canonical functor

$$\Upsilon_{\mathcal{Y}}:\mathrm{QCoh}(\mathcal{Y})\to\mathrm{IndCoh}(\mathcal{Y}),$$

given by tensoring with the dualizing object $\omega_{\mathcal{Y}}\in\mathrm{IndCoh}(\mathcal{Y})$.

By construction, the following diagram of functors commutes:

$$\begin{array}{ccc}\mathrm{QCoh}(X^\wedge_Y) & \xleftarrow{\widehat{i}^*} & \mathrm{QCoh}(X)\\ \Upsilon_{X^\wedge_Y}\big\downarrow & & \big\downarrow\Upsilon_X\\ \mathrm{IndCoh}(X^\wedge_Y) & \xleftarrow{\widehat{i}^!} & \mathrm{IndCoh}(X)\end{array}\tag{7.11}$$

7.5.2. Recall that if Z is a DG scheme, then the category $\mathrm{IndCoh}(Z)$ is self-dual, and the functor Υ_Z identifies with the functor $\Psi_Z^\vee$, the dual of the naturally defined functor

$$\Psi_Z:\mathrm{IndCoh}(Z)\to\mathrm{QCoh}(Z),$$

see [GL:IndCoh, Proposition 9.3.3]. However, the functor $\Psi_{\mathcal{Z}}$ is *not* intrinsically defined for $\mathcal{Z}\in\mathrm{PreStk}_{\mathrm{laft}}$.

Nevertheless, for $\mathcal{X}\in\mathrm{DGindSch}$, we still have a canonical self-duality

$$\mathbf{D}^{\mathrm{Serre}}_{\mathcal{X}}:\mathrm{IndCoh}(\mathcal{X})^\vee\simeq\mathrm{IndCoh}(\mathcal{X})$$

(see Corollary 2.6.2), and if $\mathcal{X}$ is quasi-perfect (see Sect. 2.2.4), then we also have a self-duality

$$\mathbf{D}^{\mathrm{naive}}_{\mathcal{X}}:\mathrm{QCoh}(\mathcal{X})^\vee\simeq\mathrm{QCoh}(\mathcal{X}).$$

So, in this case, we can consider the functor $\Upsilon^\vee_{\mathcal{X}}:\mathrm{IndCoh}(\mathcal{X})\to\mathrm{QCoh}(\mathcal{X})$, dual to $\Upsilon_{\mathcal{X}}$.

Consider the resulting functor

$$\mathrm{QCoh}(\mathcal{X})\otimes\mathrm{IndCoh}(\mathcal{X})\overset{\mathrm{Id}\otimes\Upsilon^\vee_{\mathcal{X}}}{\longrightarrow}\mathrm{QCoh}(\mathcal{X})\otimes\mathrm{QCoh}(\mathcal{X})\to\mathrm{Vect},\tag{7.12}$$

where the last arrow is the pairing corresponding to the self-duality $\mathbf{D}^{\mathrm{naive}}_{\mathcal{X}}$ of $\mathrm{QCoh}(\mathcal{X})$:

By construction and Corollary 2.6.6, it is isomorphic to the composite

$$\mathrm{QCoh}(\mathcal{X})\otimes\mathrm{IndCoh}(\mathcal{X})\to\mathrm{IndCoh}(\mathcal{X})\overset{\Gamma^{\mathrm{IndCoh}}(\mathcal{X},-)}{\longrightarrow}\mathrm{Vect},$$

where the first arrow is the canonical action of $\mathrm{QCoh}(-)$ on $\mathrm{IndCoh}(-)$.

7.5.3. The discussion in Sect. 7.5.2 applies in particular to $\mathcal{X} = X^\wedge_Y$.

Passing to dual functors in (7.11), and using Corollary 7.2.6 we obtain another commutative diagram:

$$
\begin{CD}
\operatorname{QCoh}(X^\wedge_Y) @>{\widehat{i}_?}>> \operatorname{QCoh}(X) \\
@A{\Upsilon^\vee_{X^\wedge}}AA @AA{\Psi_X}A \\
\operatorname{IndCoh}(X^\wedge_Y) @>{\widehat{i}^{\operatorname{IndCoh}}_*}>> \operatorname{IndCoh}(X).
\end{CD}
\tag{7.13}
$$

Lemma 7.5.4. *The functor $\Upsilon^\vee_{X^\wedge_Y}$ is t-exact, when we consider the t-structure on* $\operatorname{IndCoh}(X^\wedge_Y)$ *of Sect. 2.5 and the inductive t-structure on* $\operatorname{QCoh}(X^\wedge_Y)$ *of Sect. 7.3.3.*

Proof. The assertion follows from the fact that the functors $\widehat{i}^{\operatorname{IndCoh}}_*$ and $\widehat{i}_?$ are t-exact and conservative, and the fact that Ψ_X is t-exact. □

7.5.5. Consider now the functors

$$
\widehat{i}_? \circ \widehat{i}^* : \operatorname{QCoh}(X) \to \operatorname{QCoh}(X) \text{ and } \widehat{i}^{\operatorname{IndCoh}}_* \circ \widehat{i}^! : \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(X).
\tag{7.14}
$$

Lemma 7.5.6. *The functor* $\Psi_X : \operatorname{IndCoh}(X) \to \operatorname{QCoh}(X)$ *intertwines the functors of* (7.14).

Proof. The functors of (7.14) are isomorphic to

$$
\operatorname{Cone}(\operatorname{Id} \to j_* \circ j^*)[-1] \text{ and } \operatorname{Cone}(\operatorname{Id} \to j^{\operatorname{IndCoh}}_* \circ j^{\operatorname{IndCoh},*})[-1],
$$

respectively. So, it is enough to show that the functor Ψ_X intertwines the functors $j_* \circ j^*$ and $j^{\operatorname{IndCoh}}_* \circ j^{\operatorname{IndCoh},*}$. However, the latter follows from [GL:IndCoh, Propositions 3.1.1 and 3.5.4]. □

Combining this with the fact that the horizontal arrows in (7.13) are conservative (in fact, fully faithful), we obtain:

Corollary 7.5.7. *The diagram of functors*

$$
\begin{CD}
\operatorname{QCoh}(X^\wedge_Y) @<{\widehat{i}^*}<< \operatorname{QCoh}(X) \\
@A{\Upsilon^\vee_{X^\wedge_Y}}AA @AA{\Psi_X}A \\
\operatorname{IndCoh}(X^\wedge_Y) @<{\widehat{i}^!}<< \operatorname{IndCoh}(X),
\end{CD}
\tag{7.15}
$$

obtained from (7.13) *by passing to right adjoint functors along the horizontal arrows, and which a priori commutes up to a natural transformation, is commutative.*

Passing to dual functors in (7.15), we obtain yet another commutative diagram of functors:

$$
\begin{CD}
\operatorname{QCoh}(X^\wedge_Y) @>{\widehat{i}_?}>> \operatorname{QCoh}(X) \\
@V{\Upsilon_{X^\wedge_Y}}VV @VV{\Psi^\vee_X}V \\
\operatorname{IndCoh}(X^\wedge_Y) @>{\widehat{i}^{\operatorname{IndCoh}}_*}>> \operatorname{IndCoh}(X).
\end{CD}
\tag{7.16}
$$

The diagram (7.16) can be alternatively obtained by passing to left adjoint functors along the horizontal arrows in (7.11). Thus, the resulting diagram, which a priori commutes up to a natural transformation, is actually commutative.

7.6. QCoh **and** IndCoh **in the eventually coconnective case.** In this subsection we will assume that X is eventually coconnective. By Proposition 6.7.2, the ind-scheme $X^{\wedge}_Y$ is also eventually coconnective.

7.6.1. Recall (see [GL:IndCoh, Proposition 1.5.3]) that for $X \in \mathrm{DGSch}_{\mathrm{aft}}$ eventually coconnective, the functor $\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ admits a left adjoint, denoted Ξ_X. It is characterized by the property that it sends $\mathrm{QCoh}(X)^{\mathrm{perf}} \simeq \mathrm{QCoh}(X)^c$ to $\mathrm{Coh}(X) \simeq \mathrm{IndCoh}(X)^c$ via the tautological map

$$\mathrm{QCoh}(X)^{\mathrm{perf}} \to \mathrm{Coh}(X),$$

which is well-defined because X is eventually coconnective.

Also, recall that the functor $\Xi^\vee_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$, dual to Ξ_S, is the right adjoint of $\Psi^\vee_X$, and it can be described as

$$\Xi^\vee_X \simeq \underline{\mathrm{Hom}}_{\mathrm{QCoh}(X)}(\omega_X, -),$$

(see [GL:IndCoh, Lemma 9.6.7]).

We emphasize that the functors Ξ and $\Xi^\vee$ are defined specifically for DG schemes, and not arbitrary eventually coconnective objects of $\mathrm{PreStk}_{\mathrm{laft}}$.

However, for any object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ we can still ask whether the right adjoint $\Xi^\vee_{\mathcal{Y}}$ of $\Upsilon_{\mathcal{Y}}$ is continuous.

If $\mathcal{Y}$ is equipped with self-duality data for $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$, in which case the functor $\Upsilon^\vee_{\mathcal{Y}}$ is well-defined, we can ask whether the left adjoint $\Xi_{\mathcal{Y}}$ of $\Upsilon^\vee_{\mathcal{Y}}$ exists.

7.6.2. By passing to right (resp., left) adjoint functors is Diagrams (7.16) and (7.15), respectively, we obtain two more commutative diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(X^{\wedge}_Y) & \xleftarrow{\widehat{i}^*} & \mathrm{QCoh}(X) \\ {\scriptstyle \Xi^\vee_{X^{\wedge}_Y}}\uparrow & & \uparrow{\scriptstyle \Xi^\vee_X} \\ \mathrm{IndCoh}(X^{\wedge}_Y) & \xleftarrow{\widehat{i}^!} & \mathrm{IndCoh}(X), \end{array} \tag{7.17}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(X^{\wedge}_Y) & \xrightarrow{\widehat{i}_?} & \mathrm{QCoh}(X) \\ {\scriptstyle \Xi_{X^{\wedge}_Y}}\downarrow & & \downarrow{\scriptstyle \Xi_X} \\ \mathrm{IndCoh}(X^{\wedge}_Y) & \xrightarrow{\widehat{i}^{\mathrm{IndCoh}}_*} & \mathrm{IndCoh}(X). \end{array} \tag{7.18}$$

In particular, we obtain that the functor $\Xi^\vee_{X^{\wedge}_Y}$ is continuous, and $\Xi_{X^{\wedge}_Y}$ is defined, for the DG indscheme $X^{\wedge}_Y$.

7.6.3. We now claim the following:

Proposition 7.6.4. *The diagrams of functors*

$$\begin{array}{ccc} \mathrm{QCoh}(X^{\wedge}_Y) & \xleftarrow{\widehat{i}^*} & \mathrm{QCoh}(X) \\ {\scriptstyle \Xi_{X^{\wedge}_Y}}\downarrow & & \downarrow{\scriptstyle \Xi_X} \\ \mathrm{IndCoh}(X^{\wedge}_Y) & \xleftarrow{\widehat{i}^!} & \mathrm{IndCoh}(X) \end{array} \tag{7.19}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}(X^\wedge_Y) & \xrightarrow{\widehat{i}_?} & \mathrm{QCoh}(X) \\ {\scriptstyle \Xi^\vee_{X^\wedge_Y}}\Big\uparrow & & \Big\uparrow{\scriptstyle \Xi^\vee_X} \\ \mathrm{IndCoh}(X^\wedge_Y) & \xrightarrow{\widehat{i}^{\mathrm{IndCoh}}_*} & \mathrm{IndCoh}(X), \end{array} \tag{7.20}$$

obtained from the diagrams (7.17) *and* (7.18)*, respectively, by passing to adjoint functors along the vertical arrows, and which a priori commute up to natural transformations, are commutative.*

Proof. The two diagrams are obtained from one another by passing to dual functors. Therefore, it is sufficient to show that (7.20) is commutative. Taking into account (7.17) and the fact that in the latter diagram the horizontal arrows are co-localizations, it suffices to show that the functor $\Xi^\vee_X$ intertwines the functors

$$\widehat{i}_? \circ \widehat{i}^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \text{ and } \widehat{i}^{\mathrm{IndCoh}}_* \circ \widehat{i}^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X).$$

As in the proof of Lemma 7.5.6, it suffices to show that $\Xi^\vee_X$ intertwines the functors

$$j_* \circ j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \text{ and } j^{\mathrm{IndCoh}}_* \circ j^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X).$$

It is clear that

$$j^* \circ \Xi^\vee_X \simeq \Xi^\vee_U \circ j^{\mathrm{IndCoh},*}.$$

So, we have to show that the natural map

$$\Xi^\vee_X \circ j^{\mathrm{IndCoh}}_* \to j_* \circ \Xi^\vee_U$$

is an isomorphism. Let $\mathcal{F}_X \in \mathrm{QCoh}(X)$ and $\mathcal{F}_U \in \mathrm{IndCoh}(U)$ be two objects. We have

$$\mathrm{Maps}_{\mathrm{QCoh}(X)}(\mathcal{F}_X, \Xi^\vee_X \circ j^{\mathrm{IndCoh}}_*(\mathcal{F}_U)) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{F}_X \otimes \omega_X, j^{\mathrm{IndCoh}}_*(\mathcal{F}_U)) \simeq$$
$$\simeq \mathrm{Maps}_{\mathrm{IndCoh}(U)}(j^{\mathrm{IndCoh},*}(\mathcal{F}_X \otimes \omega_X), \mathcal{F}_U) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(U)}(j^*(\mathcal{F}_X) \otimes j^{\mathrm{IndCoh},*}(\omega_X), \mathcal{F}_U) \simeq$$
$$\simeq \mathrm{Maps}_{\mathrm{IndCoh}(U)}(j^*(\mathcal{F}_X) \otimes \omega_U, \mathcal{F}_U) \simeq \mathrm{Maps}_{\mathrm{QCoh}(U)}(j^*(\mathcal{F}_X), \Xi^\vee_U(\mathcal{F}_U)).$$

□

8. Formally smooth DG indschemes

8.1. **The notion of formal smoothness.** Let $\mathcal{X}_{cl}$ be an object of ${}^{cl}\mathrm{PreStk}$.

Definition 8.1.1. *We say that $\mathcal{X}_{cl}$ is formally smooth if for every closed embedding*

$$S \hookrightarrow S'$$

of classical affine schemes, such that the ideal of S inside S' is nilpotent*, the map of sets*

$$\pi_0(\mathcal{X}_{cl}(S')) \to \pi_0(\mathcal{X}_{cl}(S))$$

is surjective.

Clearly, in order to test formal smoothness, it is sufficient to consider closed embeddings of classical affine schemes

$$S \hookrightarrow S',$$

such that the ideal $\mathcal{I}$ of S inside S' satisfies $\mathcal{I}^2 = 0$.

8.1.2. Let $\mathcal{X}$ be an object of PreStk.

Definition 8.1.3. *We say that $\mathcal{X}$ is formally smooth if:*

(1) *The classical prestack ${}^{cl}\mathcal{X} := \mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}}$ is formally smooth in the sense of Definition 8.1.1.*
(2) *For every n and $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the map $\mathcal{X}(S) \to \mathcal{X}({}^{\leq n}S)$ induces an isomorphism on π_n.*

We can reformulate Definition 8.1.3 as follows.

Lemma 8.1.4. *Let $\mathcal{X} \in \mathrm{PreStk}$ be such that ${}^{cl}\mathcal{X}$ is formally smooth. Then $\mathcal{X}$ is formally smooth if and only if $\mathcal{X}$ is convergent and for any integers $i \geq j$ and $S \in {}^{\leq i}\mathrm{DGSch}^{\mathrm{aff}}$ the map*

$$\mathcal{X}(S) \to \mathcal{X}({}^{\leq j}S)$$

induces an isomorphism on π_j.

Remark 8.1.5. As was alluded to in the introduction, the property of formal smoothness, in both the classical and derived contexts, has a substantial drawback of being non-local in the Zariski topology. For example, we could have given a different definition by requiring the corresponding properties to hold after Zariski localization with respect to the test affine scheme S. We will see a manifestation of this phenomenon in Sect. 8.2.9 for 0-truncated prestacks that admit connective deformation theory.

However, it will turn out that in the latter case the difference between the two definitions disappears if we restrict ourselves to prestacks locally almost of finite type (see Sect. 8.3), which will be the main case of interest in the rest of this paper.

8.1.6. All the examples of prestacks that we consider in this paper are 0-truncated in the sense of [GL:Stacks, Sect. 1.1.7]. I.e., we consider prestacks $\mathcal{Y}$ such that for all n and $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$,

$$\mathcal{Y}(S) \in n\text{-}\mathrm{Grpd} \subset \infty\text{-}\mathrm{Grpd}\,.$$

In this case, we have the following reformulation of the Definition 8.1.3.

Lemma 8.1.7. *Let $\mathcal{X} \in \mathrm{PreStk}$ be a 0-truncated prestack such that ${}^{cl}\mathcal{X}$ is formally smooth as a classical prestack. Then $\mathcal{X}$ is formally smooth if and only if for every n and $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the map*

$$\mathcal{X}(S) \to \mathcal{X}({}^{\leq n}S)$$

identifies the right-hand side with the n-truncation of (the Postnikov tower of) $\mathcal{X}(S)$.

8.1.8. Let $\mathcal{X}$ be a DG indscheme, and let Y be a reduced classical scheme, equipped with a closed embedding $Y \hookrightarrow {}^{cl,red}\mathcal{X}$. Consider the formal completion $\mathcal{X}^\wedge_Y$.

Proposition 8.1.9. *The following conditions are equivalent:*

(a) *$\mathcal{X}$ is formally smooth.*

(b) *For every $Y \hookrightarrow {}^{cl,red}\mathcal{X}$ as above, the formal completion $\mathcal{X}^\wedge_Y$ is formally smooth.*

Proof. Since $\mathcal{X}^\wedge_Y(S)$ is a connected component of $\mathcal{X}(S)$, condition (a) implies condition (b). For the opposite implication, write ${}^{cl}\mathcal{X}$ as $\underset{\alpha}{colim}\, X_\alpha$. We claim that it is enough to show that each $\mathcal{X}^\wedge_{X_\alpha}$ is formally smooth. Indeed, both conditions of formal smoothness can be checked separately over each point of ${}^{cl,red}S \to {}^{cl,red}\mathcal{X}$, and every such point factors through some X_α. □

8.2. **Formal smoothness via deformation theory.**

8.2.1. Let $\mathcal{X} \in \text{PreStk}$ admit connective deformation theory (see Definition 4.7.1).

Proposition 8.2.2. *Suppose that* ${}^{cl}\mathcal{X}$ *is* 0*-truncated. Then* $\mathcal{X}$ *is formally smooth if and only if the following equivalent conditions hold:*

(a) *For every* $S \in \text{DGSch}^{\text{aff}}$ *and* $x : S \to \mathcal{X}$*, the object*

$$T^*_x\mathcal{X} \in \text{Pro}(\text{QCoh}(S)^{\leq 0})$$

has the property that

$$\text{Hom}(T^*_x\mathcal{X}, \mathcal{F}[i]) = 0, \ \forall \mathcal{F} \in \text{QCoh}(S)^{\heartsuit} \text{ and } i > 0.$$

(b) *Same as* (a)*, but for* S *a classical affine scheme.*

(c) *Under an additional assumption that* $\mathcal{X}$ *is locally almost of finite type, the same as* (b)*, but for* S *reduced.*

Proof. It is clear that if $\mathcal{X}$ is formally smooth, then it satisfies (a): indeed, consider the split square-zero extension of S corresponding to $\mathcal{F}[i]$. The converse implication follows from deformation theory using Lemmas 4.5.7, 4.5.9 and Lemma 8.1.4.

Condition (a) implies condition (b) tautologically. The converse implication follows from the fact that every object of $\text{QCoh}(S)^{\heartsuit}$ is the direct image under the canonical map ${}^{cl}S \to S$. Indeed, for a point $x : S \to \mathcal{X}$, the pull-back of $T^*_x\mathcal{X}$ under ${}^{cl}S \to S$ identifies with $T^*_{{}^{cl}x}\mathcal{X}$, where ${}^{cl}x$ is the composition ${}^{cl}S \to S \xrightarrow{x} \mathcal{X}$.

Condition (b) implies condition (c) tautologically. For the converse implication, we note that under the assumption that $\mathcal{X}$ is locally of finite type, by Lemma 5.3.4, the functor $T^*_x\mathcal{X}$ commutes with colimits in $\text{QCoh}(S)^{\heartsuit}$. This allows to replace any $\mathcal{F} \in \text{QCoh}(S)^{\heartsuit}$ by one obtained as a direct image from ${}^{red}S$.

□

8.2.3. The following definition will be convenient in the sequel. Let S be an affine DG scheme, and let F be an object of $\text{Pro}(\text{QCoh}(S)^{\leq 0})$.

We shall say that F is *convergent* if for every $\mathcal{F} \in \text{QCoh}(S)^{\leq 0}$, the natural map

$$F(\mathcal{F}) \to \underset{n \in \mathbb{N}^{\text{op}}}{lim} F(\tau^{\geq -n}(\mathcal{F})) \tag{8.1}$$

is an isomorphism in ∞-Grpd.

We have:

Lemma 8.2.4. *Let* $\mathcal{X} \in \text{PreStk}$ *admit connective deformation theory, and let* $x : S \to \mathcal{X}$ *be a map. Then* $T^*_x\mathcal{X} \in \text{Pro}(\text{QCoh}(S)^{\leq 0})$ *is convergent.*

Proof. Follows from the fact that the condition of admitting connective deformation theory includes convergence. □

8.2.5. Let S be an affine classical scheme. Let us characterize those objects

$$F \in \text{Pro}(\text{QCoh}(S)^{\leq 0})$$

that satisfy property (a) of Proposition 8.2.2.

We have:

Lemma 8.2.6. *For $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $F \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$ the following are equivalent:*

(a) *F is convergent and $\pi_0\left(F(\mathcal{F}[i])\right) = 0$ for all $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$ and $i > 0$.*

(a') *$\pi_0\left(F(\mathcal{F})\right) = 0$ for all $\mathcal{F} \in \mathrm{QCoh}(S)^{<0}$.*

(b) *F belongs to the full subcategory*

$$\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit,\mathrm{proj}}) \subset \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$$

where $\mathrm{QCoh}(S)^{\heartsuit,\mathrm{proj}}$ is the full subcategory of projective objects in $\mathrm{QCoh}(S)^{\heartsuit}$.

(b') *F is convergent, belongs to the full subcategory*

$$\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit}) \subset \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0}),$$

and the functor

$$\mathcal{F} \mapsto \pi_0 \circ F(\mathcal{F}), \quad \mathrm{QCoh}(S)^{\heartsuit} \to \mathrm{Sets}$$

is right exact.

Remark 8.2.7. This lemma is not specific to $\mathrm{QCoh}(S)$; it is applicable to any stable ∞-category equipped with a t-structure, whose heart has enough projectives and injectives.

Proof. The equivalence of (a) and (a') is immediate. It is also clear that (b) implies (a).

Suppose that F satisfies (a'), and let us deduce (b). Consider the category

$$\{P \in \mathrm{QCoh}(S)^{\heartsuit,\mathrm{proj}}, f_P \in H^0(F(P))\}.$$

The assumption implies that this category is cofiltered, and it is easy to see that the resulting map in $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$

$$F \to \underset{(P,f_P)}{\text{“}lim\text{”}}\, P$$

is an isomorphism.

The implication (b) $\Rightarrow$ (b') is also immediate. Let us show that (b') implies (a). By assumption, F is given as an object

$$\underset{\alpha \in \mathsf{A}}{\text{“}lim\text{”}}\, \mathcal{F}_\alpha \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit}),$$

where the category of indices A is filtered. By definition,

$$\pi_0(F(\mathcal{F})) \simeq \underset{\alpha \in \mathsf{A}}{colim}\, \mathrm{Hom}(\mathcal{F}_\alpha, \mathcal{F}).$$

Hence, if $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$ is injective, then $\pi_0(F(\mathcal{F}[i])) = 0$ for $i > 0$. The exactness of F on the abelian category implies that $\pi_0(F(\mathcal{F}[1])) = 0$ for any $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$ by the long exact cohomology sequence. The assertion that $\pi_0(F(\mathcal{F}[i])) = 0$ for $n > i > 1$ and any $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$ follows by induction on i, again by the long exact cohomology sequence.

$\square$

8.2.8. In what follows, for $S \in \mathrm{Sch}^{\mathrm{aff}}$, we shall refer to objects of $F \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$ satisfying the equivalent conditions of Lemma 8.2.6 as *pro-projective.*

8.2.9. We can now better explain the non-locality of the definition of formal smoothness mentioned in Remark 8.1.5:

Let S be an affine classical scheme, and let F be an object of $\text{Pro}(\text{QCoh}(S)^\heartsuit)$. It is a natural question to ask whether the property of F to be pro-projective is local in the Zariski topology.

Namely, if S_i is an open cover of S by affine subchemes and $F|_{S_i} \in \text{Pro}(\text{QCoh}(S_i)^{\heartsuit,\text{proj}})$, will it be true that F itself belongs to $\text{Pro}(\text{QCoh}(S)^{\heartsuit,\text{proj}})$?

Unfortunately, we do not know the answer to this question, but we think that it is probably negative.

Remark 8.2.10. Note, however, if we ask the same question for F being an object of $\text{QCoh}(S)^\heartsuit$, rather than $\text{Pro}(\text{QCoh}(S)^\heartsuit)$, the answer will be affirmative, due to a non-trivial theorem of Raynaud-Gruson, [RG].

8.3. **Formal smoothness for prestacks locally of finite type.**

8.3.1. Let S be an affine DG scheme, and let F be an object of $\text{Pro}(\text{QCoh}(S)^{\geq -n, \leq 0})$.

We shall say that F is *pro-coherent* if, when viewed as a functor

$$\text{QCoh}(S)^{\leq 0} \to \infty\text{-Grpd},$$

it commutes with filtered colimits.

Note that this condition is satisfied for F arising as ${}^{\geq -n}(T^*_x \mathcal{X})$ for $x : S \to \mathcal{X}$, where $\mathcal{X}$ admits connective deformation theory and belongs to $\text{PreStk}_{\text{laft}}$.

Also note that when S is Noetherian, by Lemma 5.3.4, pro-coherence is equivalent to F belonging to $\text{Pro}(\text{Coh}(S)^{\geq -n, \leq 0})$.

In general, F is pro-coherent if and only if it can be represented by a complex

$$P^{-n-1} \to P^{-n} \to ... \to P^{-1} \to P^0$$

in $\text{Pro}(\text{QCoh}(S)^\heartsuit)$, whose terms belong to $\text{Pro}(\text{QCoh}(S)^{\heartsuit,\text{proj,f.g.}})$, where

$$\text{QCoh}(S)^{\heartsuit,\text{proj,f.g.}} \subset \text{QCoh}(S)^\heartsuit$$

denotes the category of projective finitely generated quasi-coherent sheaves.

8.3.2. We have:

Lemma 8.3.3. *Let S be a classical affine scheme and let $F \in \text{Pro}(\text{QCoh}(S)^\heartsuit)$ be pro-coherent. Then its property of being pro-projective is local in the Zariski topology.*

Proof. We will check the locality of condition (b') of Lemma 8.2.6.

First, it is easy to see that the property for an object of $\text{Pro}(\text{QCoh}(S)^{\leq 0})$ to be convergent is Zariski-local.

Hence, it remains to check that the property of the functor

$$\mathcal{F} \mapsto \pi_0(F(\mathcal{F})), \quad \text{QCoh}(S)^\heartsuit \to \text{Sets}$$

to be right exact is also Zariski-local, under the assumption that F is pro-coherent. We will show that this property is in fact fpqc-local.

Thus, let $f : S' \to S$ be an fpqc map, where $S = \text{Spec}(A)$ and $S' = \text{Spec}(B)$. We assume that the functor

$$F' := \text{Pro}(f^*)(F) : B\text{-mod} \to \infty\text{-Grpd}$$

is such that

$$F'^{\heartsuit} := \pi_0 \circ F' : (B\text{-mod})^{\heartsuit} \to \mathrm{Sets}$$

is right exact, and we wish to deduce the same for

$$F^{\heartsuit} := \pi_0 \circ F : (A\text{-mod})^{\heartsuit} \to \mathrm{Sets}\,.$$

Note that by adjunction, $F'^{\heartsuit}(\mathcal{M}) = F^{\heartsuit}(f_*(\mathcal{M}))$ for $\mathcal{M} \in (B\text{-mod})^{\heartsuit}$.

Consider the object $F^{\heartsuit}(A) \in \mathrm{Sets}$. The action of A on itself as an A-module defines on $F^{\heartsuit}(A)$ a structure of an A-module. There is a natural map of functors $(A\text{-mod})^{\heartsuit} \to \mathrm{Sets}$

$$\mathcal{N} \underset{A}{\otimes} F^{\heartsuit}(A) \to F^{\heartsuit}(\mathcal{N}), \tag{8.2}$$

where in the above formula we are using the *non-derived* tensor product.

Note the map in (8.2) is an isomorphism whenever F is pro-coherent and $F^{\heartsuit}$ is right exact. Indeed, both functors are right exact and commute with filtered colimits, so the isomorphism for any $\mathcal{N}$ follows from the case $\mathcal{N} = A$.

And vice versa, if (8.2) is an isomorphism then $F^{\heartsuit}$ is right exact. Indeed, the left-hand side is a right exact, and the right-hand side is left exact, so if the map in question is an isomorphism, and both functors are actually exact.

Also note that (8.2) is an isomorphism for $F^{\heartsuit}$ pro-coherent whenever $\mathcal{N}$ is A-flat, by Lazard's lemma.

In order to show that (8.2) is an isomorphism under our assumptions, let us tensor both sides with B, and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{N} \underset{A}{\otimes} F^{\heartsuit}(A) \underset{A}{\otimes} B & \longrightarrow & F^{\heartsuit}(\mathcal{N}) \underset{A}{\otimes} B \\ \downarrow & & \downarrow \\ \mathcal{N} \underset{A}{\otimes} F^{\heartsuit}(B) & \longrightarrow & F^{\heartsuit}(\mathcal{N} \underset{A}{\otimes} B). \end{array}$$

Since B is faithfully flat over A, it is enough to show that the upper horizontal arrow is an isomorphism.

In the above diagram the vertical arrows are isomorphisms since B is A-flat. However, the lower horizontal arrow identifies with

$$\mathcal{N} \underset{A}{\otimes} F'^{\heartsuit}(B) \simeq (\mathcal{N} \underset{A}{\otimes} B) \underset{B}{\otimes} F'^{\heartsuit}(B) \to F'^{\heartsuit}(\mathcal{N} \underset{A}{\otimes} B),$$

which is an isomorphism by (8.2) applied to $F'^{\heartsuit}$.

□

8.3.4. In view of Proposition 8.2.2, the above lemma implies that for 0-truncated prestacks that admit connective deformation theory and are locally almost of finite type, the definition of formal smoothness is reasonable, in the sense that it is Zariski-local.

As a manifestation of this, we have the following assertion that will be useful in the sequel.

8.3.5. Let $\mathcal{X}$ be a formal DG scheme with the underlying reduced classical scheme X. Denote

$$T^*\mathcal{X}|_X := T^*_x\mathcal{X}$$

where $x : X \to \mathcal{X}$ is the tautological point.

Corollary 8.3.6. *Suppose that $\mathcal{X}$ is locally almost of finite type, and that $T^*\mathcal{X}|_X$ is Zariski-locally pro-projective. Then $\mathcal{X}$ is formally smooth.*

Proof. We will check that the conditions of Proposition 8.2.2(c). Note that every map $S \to \mathcal{X}$, where S is a reduced classical affine scheme, factors through a map $f : S \to X$. Thus, we need to show that for every such f, the object

$$\text{Pro}(f^*)(T^*\mathcal{X}|_X) \in \text{Pro}(\text{QCoh}(S)^{\leq 0})$$

is pro-projective.

First, the Zariski-locality of the t-structure on $\text{Pro}(\text{QCoh}(X)^{\leq 0})$ implies that $T^*\mathcal{X}|_X$ belongs to the full subcategory

$$\text{Pro}(\text{QCoh}(X)^{\heartsuit}) \subset \text{Pro}(\text{QCoh}(X)^{\leq 0}).$$

Now, since $\mathcal{X}$ is locally almost of finite type, the classical scheme X is of finite type. Hence, Proposition 5.3.2 implies that $T^*\mathcal{X}|_X$ belongs to

$$\text{Pro}(\text{Coh}(X)^{\heartsuit}) \subset \text{Pro}(\text{QCoh}(X)^{\heartsuit}) \subset \text{Pro}(\text{QCoh}(X)^{\leq 0}).$$

Finally, by Lemma 8.3.3, we obtain that $T^*\mathcal{X}|_X$ belongs to

$$\text{Pro}(\text{Coh}(X)^{\heartsuit,\text{proj}}) \subset \text{Pro}(\text{Coh}(X)^{\heartsuit}) \subset \text{Pro}(\text{QCoh}(X)^{\heartsuit}) \subset \text{Pro}(\text{QCoh}(X)^{\leq 0}),$$

and in particular to

$$\text{Pro}(\text{QCoh}(X)^{\heartsuit,\text{proj}}) \subset \text{Pro}(\text{QCoh}(X)^{\leq 0}).$$

However, it is clear that for any $f : S \to X$ with $S \in \text{Sch}^{\text{aff}}$, the functor $\text{Pro}(f^*)$ sends pro-projective objects to pro-projective objects, as required.

□

8.4. **Examples of formally smooth DG indschemes.** In this subsection we will give three examples of formally smooth DG indschemes.

8.4.1. The first example is the most basic one: we claim that the affine space $\mathbb{A}^n$, considered as an object of PreStk, is formally smooth. Indeed, the definition of formal smoothness is satisfied on the nose as

$$\text{Maps}(\text{Spec}(A), \mathbb{A}^n) \simeq \Omega^\infty(\mathsf{Sp}(A))^{\times n}.$$

8.4.2. Let X be a classical smooth scheme of finite type over k. We claim that X, considered as an object of PreStk (i.e., ${}^L\text{LKE}_{(\text{Sch}^{\text{aff}})^{\text{op}} \hookrightarrow (\text{DGSch}^{\text{aff}})^{\text{op}}}(X)$), is formally smooth.

Indeed, it suffices to show that the conditions of Proposition 8.2.2 are satisfied. In fact, we claim that T^*X, is an object of $\text{Coh}(X)^{\heartsuit}$, and is locally projective.

The question is local on X, so we can assume that X fits into a Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & \mathbb{A}^m, \end{array} \tag{8.3}$$

where the map f is smooth, and where the fiber product *is taken in the category of classical schemes.*

Since f is flat, the above square is also Cartesian in the category of DG schemes. Hence, T^*X can be calculated as

$$\mathrm{Cone}(f^*(T^*\mathbb{A}^m)|_X \to T^*\mathbb{A}^n|_X),$$

and the smoothness hypothesis on f implies the required properties of T^*X.

Corollary 8.4.3. *Let X be a smooth classical scheme locally of finite type, and let $Y \subset X$ be a Zariski-closed subset. Then the formal completion $X^\wedge_Y$ is formally smooth as an object of* PreStk.

Proof. This follows from Proposition 8.2.2 as $\widehat{i} : X^\wedge_Y \to X$ induces an isomorphism on pro-cotangent complexes. □

Also, note that by Proposition 6.8.2, the DG indscheme $X^\wedge_Y$ is 0-coconnective, i.e., is a left Kan extension of a classical indscheme.

8.4.4. The following example will be needed for the proof of Theorem 9.1.2. Consider the formal DG scheme $\mathbb{A}^{n,m} := \mathrm{Spf}\,(k[x_1, ..., x_n][\![y_1, ..., y_m]\!])$, i.e., the formal completion of $\mathbb{A}^{n+m}$ along the subscheme $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$ embedded along the first n coordinates.

Let $f_1, ..., f_k$ be elements of $k[x_1, ..., x_n][\![y_1, ..., y_m]\!]$, and let $\bar{f}_1, ...\bar{f}_k$ be their images under

$$k[x_1, ..., x_n][\![y_1, ..., y_m]\!] \twoheadrightarrow k[x_1, ..., x_n].$$

Set

$$\mathcal{X} := 0 \underset{\mathbb{A}^k}{\times} \mathbb{A}^{n,m} \text{ and } X := 0 \underset{\mathbb{A}^k}{\times} \mathbb{A}^n.$$

Suppose that the Jacobi matrix of $f_1, ..., f_k$ is non-degenerate when restricted to X. I.e., the matrix $k \times (m + n)$-matrix $\partial_i(f_j)|_X$, viewed as a map

$$\mathcal{O}_X^{\oplus n+m} \to \mathcal{O}_X^{\oplus k},$$

is a surjective map of vector bundles when restricted to X.

From Corollary 8.3.6 and Corollary 6.3.2, we obtain:

Corollary 8.4.5. *Under the above circumstances, the DG indscheme $\mathcal{X}$ is formally smooth.*

We now claim:

Proposition 8.4.6. *The DG indscheme $\mathcal{X}$ is* 0*-coconnective.*

Proof. Consider the *scheme*

$$^\sim\mathbb{A}^{n,m} := \mathrm{Spec}\,(k[x_1, ..., x_n][\![y_1, ..., y_m]\!])$$

and its map to $\mathbb{A}^k$ given by $f_1, ..., f_k$. The assumption on the Jacobi matrix implies that this map is flat on a Zariski neighborhood U of $X \subset {}^\sim\mathbb{A}^{n,m}$. Therefore, the Cartesian product *taken in the category of DG schemes*

$$^\sim\mathcal{X} \simeq 0 \underset{\mathbb{A}^k}{\times} U$$

is 0-coconnective as a DG scheme.

The formal DG scheme $\mathcal{X}$ is obtained from $^\sim\mathcal{X}$ as a formal completion along X. Since all the schemes involved are Noetherian, the assertion follows from Proposition 6.8.2. □

8.4.7. In what follows we shall refer to formal DG schemes $\mathcal{X}$ of the type described in Sect. 8.4.4 as *elementary.*

We shall say that a classical formal scheme is elementary if it is of the form ${}^{cl}\mathcal{X}$ for $\mathcal{X}$ an elementary formal DG scheme.

9. Classical vs. derived formal smoothness

The focus of this section is the relation between the notions of formal smoothness in the classical and derived contexts when $\mathcal{X}$ is a DG indscheme. Namely, we would like to know under what circumstances a formally smooth DG indscheme $\mathcal{X}$ is 0-coconnective, i.e., arises as a left Kan extension from a classical indscheme. The reader may have observed that this was the case in all the examples that we considered in Sect. 8.4.

And vice versa, we would like to know when, for a classical formally smooth indscheme $\mathcal{X}_{cl}$, the object

$$\mathcal{X} := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{X}_{cl}) \in \mathrm{Stk}$$

is a formally smooth DG indscheme. (Note that it is not clear that $\mathcal{X}$ defined as above is a DG indscheme, since the convergence condition is not a priori guaranteed.)

Unfortunately, we do not have a general answer for this question even in the case of schemes: we do not even know that the DG scheme

$$X := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(X_{cl})$$

is smooth when X_{cl} is a smooth classical scheme, except when X_{cl} is locally of finite type.

9.1. **The main result.** The main result of this section and the first of the two main results of this paper is a partial answer to the above questions, under the assumption that our (DG) indschemes are locally (almost) of finite type.

9.1.1. Let $\mathcal{X}_{cl}$ be a classical formally smooth $\aleph_0$ indscheme. Assume that $\mathcal{X}_{cl}$ is locally of finite type. Set

$$\mathcal{X} := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{X}_{cl}) \in \mathrm{PreStk}\,.$$

We will prove:

Theorem 9.1.2. *Under the above circumstances, $\mathcal{X}$ is a formally smooth DG indscheme.*

This theorem gives a partial answer to the second of the two questions above. We shall presently show that it also gives a partial answer to the first question.

9.1.3. We have the following observation:

Proposition 9.1.4. *If $\mathcal{X}$ is a formally smooth DG indscheme such that*

$${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl}\mathcal{X}) \in \mathrm{Stk}$$

is also a formally smooth DG indscheme, then the natural map

$${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl}\mathcal{X}) \to \mathcal{X}$$

is an isomorphism. In particular, $\mathcal{X}$ is 0*-coconnective.*

Proof. By assumption, both sides in

$$\mathcal{X}' := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl}\mathcal{X}) \to \mathcal{X} \tag{9.1}$$

are formally smooth DG indschemes, and the above map induces an isomorphism of the underlying classical indschemes.

By deformation theory, it suffices to show that for every affine DG scheme S and a map $x' : S \to \mathcal{X}'$, the map

$$T^*_x\mathcal{X} \to T^*_{x'}\mathcal{X}'$$

is an isomorphism, where x is the composition of x' and the map (9.1).

Using Proposition 8.2.2(a), we obtain that it suffices to check that the map

$$T^*_{x'}\mathcal{X}'(\mathcal{F}) \to T^*_x\mathcal{X}(\mathcal{F}) \tag{9.2}$$

is an isomorphism for every $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$. Since any such $\mathcal{F}$ comes as a direct image from ${}^{cl}S$, this reduces the assertion to the case when S is classical.

We have

$$T^*_{x'}\mathcal{X}'(\mathcal{F}) \simeq \mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X}') \underset{\mathrm{Maps}(S,\mathcal{X}')}{\times} x',$$

and similarly for $T^*_x\mathcal{X}(\mathcal{F})$. When S is classical and $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$, the DG scheme $S_{\mathcal{F}}$ is also classical. So, both sides of (9.2) only depend on the restrictions of $\mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}}$ and $\mathcal{X}'|_{\mathrm{Sch}^{\mathrm{aff}}}$, respectively, and, hence are isomorphic by construction.

□

9.1.5. Combining Proposition 9.1.4 and Theorem 9.1.2, we obtain:

Theorem 9.1.6. *Let $\mathcal{X}$ be a formally smooth DG indscheme, such that ${}^{cl}\mathcal{X} := \mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}}$ is locally of finite type and $\aleph_0$. Then $\mathcal{X}$ is 0-coconnective, i.e., the natural map*

$${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl}\mathcal{X}) \to \mathcal{X}$$

is an isomorphism. Moreover, $\mathcal{X}$ is locally almost of finite type and weakly $\aleph_0$.

Proof. The first assertion is immediate.

Writing ${}^{cl}\mathcal{X}$ as a colimit in ${}^{cl}\mathrm{PreStk}$ of X_α, with X_α being classical schemes closed in ${}^{cl}\mathcal{X}$ and hence of finite type, we obtain that

$$\mathcal{X} \simeq \underset{\alpha}{colim}\, X_\alpha,$$

where the colimit is taken in PreStk, and X_α are now understood as objects of $\mathrm{DGSch}_{\mathrm{aft}}$. Hence, $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

The fact that $\mathcal{X}$ is weakly $\aleph_0$ follows from Proposition 5.2.3. □

Thus, we obtain:

Corollary 9.1.7. *There exists an equivalence of categories between the category of classical formally smooth $\aleph_0$ indschemes locally of finite type and that of formally smooth weakly $\aleph_0$ DG indschemes locally almost of finite type.*

9.1.8. Prior to proving Theorem 9.1.2, let us see some of its corollaries in concrete geometric situations.

9.2. **Loop spaces.**

9.2.1. Let Z be an object of PreStk. We define the objects $Z[t]/t^k$, $Z[\![t]\!]$ and $Z(\!(t)\!)$ of PreStk as follows: for $S = \operatorname{Spec}(A) \in \operatorname{DGSch}^{\operatorname{aff}}$,

$$\operatorname{Maps}(S, Z[t]/t^k) := \operatorname{Maps}(\operatorname{Spec}(A[t]/t^k), Z),\ \operatorname{Maps}(S, Z[\![t]\!]) := \operatorname{Maps}(\operatorname{Spec}(A[\![t]\!]), Z)$$

and

$$\operatorname{Maps}(S, Z(\!(t)\!)) := \operatorname{Maps}(\operatorname{Spec}(A(\!(t)\!)), Z).$$

Note that by definition,

$$Z[\![t]\!] \simeq \underset{k}{lim}\, Z[t]/t^k,$$

as objects of PreStk.

Lemma 9.2.2. *Assume that Z is formally smooth as an object of* PreStk*. Then so are $Z[t]/t^k$, $Z[\![t]\!]$ and $Z(\!(t)\!)$.*

Proof. This is immediate from the fact that for a DG algebra A, the maps

$$\tau^{\leq n}(A[t]/t^k) \to (\tau^{\leq n}(A))[t]/t^k,\ \tau^{\leq n}(A[\![t]\!]) \to (\tau^{\leq n}A)[\![t]\!] \text{ and } \tau^{\leq n}(A(\!(t)\!)) \to (\tau^{\leq n}A)(\!(t)\!)$$

are isomorphisms, and that for a surjection of classical algebras $A_1 \twoheadrightarrow A_2$ with a nilpotent kernel, the corresponding maps

$$A_1[t]/t^k \to A_2[t]/t^k,\ A_1[\![t]\!] \to A_2[\![t]\!] \text{ and } A_1(\!(t)\!) \to A_2(\!(t)\!)$$

have the same property. □

9.2.3. From now on we are going to consider the case when $Z \in \operatorname{DGSch}_{\operatorname{aft}}$. We have:

Proposition 9.2.4. *Under the above circumstances, we have:*

(a) $Z[t]/t^k \in \operatorname{DGSch}_{\operatorname{aft}}$*, and is affine if Z is affine.*

(b) $Z[\![t]\!] \in \operatorname{DGSch}$*, and it is affine if Z is affine.*

(c) *If Z is affine, then $Z(\!(t)\!)$ is a DG indscheme.*

Proof. For all three statements, it is enough to assume that Z is affine. Note that $Z[t]/t^k$, $Z[\![t]\!]$ and $Z(\!(t)\!)$, considered as objects of PreStk are convergent. Hence, it is sufficient to show that

$$^{\leq n}(Z[t]/t^k) := Z[t]/t^k|_{^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}},\ ^{\leq n}(Z[\![t]\!]) := Z[\![t]\!]|_{^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}} \text{ and}$$

$$^{\leq n}(Z(\!(t)\!)) := Z(\!(t)\!)|_{^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}}$$

are representable by objects from $^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}$ (for the first two) and $^{\leq n}\operatorname{DGindSch}$, respectively. Note that the above objects only depend on the truncation $^{\leq n}Z$. The assertions of the proposition result from combining the following observations:

(i) The assignments $Z \mapsto Z[t]/t^k$, $Z \mapsto Z[\![t]\!]$ and $Z \mapsto Z(\!(t)\!)$ commute with limits.

(ii) Every object of $^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{ft}}$ can be obtained as the totalization of a truncated cosimplicial object whose terms are isomorphic to affine spaces $\mathbb{A}^n$.

(iii) The subcategories

$$^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{ft}} \subset {}^{\leq n}\operatorname{DGSch}^{\operatorname{aff}} \subset {}^{\leq n}\operatorname{PreStk} \text{ and } {}^{\leq n}\operatorname{DGindSch} \subset {}^{\leq n}\operatorname{PreStk}$$

are stable under finite products.

(iv) For $Z = \mathbb{A}^n$, both assertions of the proposition are manifest.

□

9.2.5. Suppose now that Z is a classical scheme which is smooth over k (and in particular, locally of finite type). We have:

Proposition 9.2.6. *The DG schemes $Z[t]/t^k$ and $Z[\![t]\!]$ are 0-coconnective.*

Proof. To prove that $Z[t]/t^k$ is 0-coconnective, by Proposition 9.1.4 and Sect. 8.4.2, it is sufficient to show that the classical scheme

$$^{cl}(Z[t]/t^k) := Z[t]/t^k|_{\mathrm{Sch}^{\mathrm{aff}}}$$

is smooth. By Lemma 9.2.2, $^{cl}(Z[t]/t^k)$ is formally smooth as a classical scheme, which implies that it is smooth, since $^{cl}(Z[t]/t^k)$ is locally of finite type by Proposition 9.2.4(a).

To treat the case of $Z[\![t]\!]$, we will have to go back to the proof of Proposition 9.2.4. We can assume that Z is affine and that it fits into a Cartesian square (8.3). Hence, we have a Cartesian square

$$\begin{array}{ccc} Z[\![t]\!] & \longrightarrow & \mathbb{A}^n[\![t]\!] \\ \downarrow & & \downarrow {\scriptstyle f[\![t]\!]} \\ 0 & \longrightarrow & \mathbb{A}^m[\![t]\!]. \end{array}$$

Since the affine schemes $\mathbb{A}^n[\![t]\!]$ and $\mathbb{A}^m[\![t]\!]$ are 0-coconnective, to show that $Z[\![t]\!]$ is also 0-coconnective, it suffices to show that the map $f[\![t]\!]$ is flat. The latter is the limit of the maps $f[t]/t^k : \mathbb{A}^n[t]/t^k \to \mathbb{A}^m[t]/t^k$, and smoothness of f implies that each of these maps is flat. Hence, $f[t]/t^k$ is flat as well.

□

9.2.7. *Question.* What are the conditions on a classical scheme of finite type Z (viewed as a 0-coconnective DG scheme), that will guarantee that $Z[\![t]\!]$ will also be 0-coconnective?

It is easy to see that this is not always the case: for instance, consider $Z = \mathrm{Spec}(k[t]/t^2)$. However, the smoothness condition on Z is not necessary, as can be seen from the following example:

Let $\mathfrak{g}$ be a semi-simple Lie algebra, and let $\mathcal{N} \subset \mathfrak{g}$ be its nilpotent cone. We have:

Corollary 9.2.8. *The DG scheme $\mathcal{N}[\![t]\!]$ is 0-coconnective.*

Proof. By definition, $\mathcal{N}$ fits into a Cartesian square

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow {\scriptstyle \varpi} \\ 0 & \longrightarrow & \mathfrak{g}/\!/G, \end{array}$$

taken in the category of classical schemes, where $\mathfrak{g}/\!/G$ is the GIT quotient of $\mathfrak{g}$ by the adjoint action of G, i.e., $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^*)^G)$, and ϖ is the Chevalley map.

However, by Kostant's theorem, the map ϖ is flat, so the above square is also Cartesian in the category of DG schemes. Hence, we have a Cartesian square

$$\begin{array}{ccc} \mathcal{N}[\![t]\!] & \longrightarrow & \mathfrak{g}[\![t]\!] \\ \downarrow & & \downarrow {\scriptstyle \varpi[\![t]\!]} \\ 0 & \longrightarrow & \mathfrak{g}/\!/G[\![t]\!]. \end{array}$$

Since $\mathfrak{g}$ and $\mathfrak{g}//G$ are smooth schemes of finite type (in fact, isomorphic to affine spaces), the DG schemes $\mathfrak{g}[\![t]\!]$ and $\mathfrak{g}//G[\![t]\!]$ are 0-coconnective. Hence, to show that $\mathcal{N}[\![t]\!]$ is 0-coconnective, it suffices to know that the map $\varpi[\![t]\!]$ is flat. However, the latter is Theorem A.4 in [EF].

□

9.2.9. *The case of loops.* Let Z be an affine smooth scheme of finite type over the ground field. We propose:

Conjecture 9.2.10. *The DG indscheme $Z(\!(t)\!)$ is 0-coconnective.*

In this next subsection we will prove this conjecture in a particular case when Z is an algebraic group G.

9.3. **Loop groups and the affine Grassmannian.**

9.3.1. Let G be an algebraic group. We define the affine Grassmannian Gr_G as an object of PreStk as follows:

$\mathrm{Maps}(\mathrm{Spec}(A), \mathrm{Gr}_G)$ is the ∞-groupoid of principal G-bundles on $\mathrm{Spec}(A[\![t]\!])$ equipped with a trivialization over $\mathrm{Spec}(A(\!(t)\!))$.

It is easy to show that Gr_G is convergent and that it belongs to Stk (i.e., it satisfies fppf descent). We have a naturally defined map $G(\!(t)\!) \to \mathrm{Gr}_G$, which identifies Gr_G with the quotient of $G(\!(t)\!)$ by $G[\![t]\!]$ in the fppf and the étale topology (indeed, it is easy to see that any G-bundle on $\mathrm{Spec}(A[\![t]\!])$ admits a trivialization after an étale localization with respect to $\mathrm{Spec}(A)$).

It is well-known that the underlying object ${}^{cl}\mathrm{Gr}_G \in {}^{cl}\mathrm{Stk}$ is a classical indscheme, which is $\aleph_0$ and locally of finite type.

Proposition 9.3.2. Gr_G *is a DG indscheme. Moreover, it is formally smooth.*

Proof. To prove that Gr_G is a DG indscheme, we will apply Theorem 5.1.1. For $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and a point $g : S \to \mathrm{Gr}_G$ we need to study the category of extensions of g to a point $g' : S' \to \mathrm{Gr}_G$ for square-zero extensions $S \hookrightarrow S'$. The question is local in the étale topology on S, so we can assume that the point g admits a lift to a point $\tilde{g} : S \to G(\!(t)\!)$. Multiplication by $\tilde{g}$ defines a map

$$\mathrm{SplitSqZExt}(S, 1_{G[\![t]\!]}) \to \mathrm{SplitSqZExt}(S, \tilde{g}),$$

where $1_{G[\![t]\!]} : S \to G[\![t]\!]$ is the constant map to the unit point of $G[\![t]\!]$. Consider the corresponding map

$$\alpha : T^*_{\tilde{g}} G(\!(t)\!) \to T^*_{1_{G[\![t]\!]}} G[\![t]\!].$$

We claim that $\mathrm{Cone}(\alpha)[-1]$ represents $T^*_g \mathrm{Gr}_G$. This follows from the fact that any extension $g' : S' \to \mathrm{Gr}_G$ also admits a lift to $\tilde{g}' : S' \to G(\!(t)\!)$ and if $\mathcal{F} \in \mathrm{QCoh}(S)$ is the ideal of S inside S', the ambiguity for such lift is given by the fiber of $\mathrm{SplitSqZExt}(S, 1_{G[\![t]\!]})$ over $S_{\mathcal{F}} \in \mathrm{SplitSqZExt}(S)$. This shows that Gr_G satisfies scheme-like Conditions (A) and (C), while Condition (B) follows from the construction.

To show that Gr_G is formally smooth, it suffices to show that $\mathrm{Cone}(\alpha)[-1]$ satisfies property (b) of Proposition 8.2.2. Multiplication by the inverse $\tilde{g}$ defines an isomorphism between $\mathrm{Cone}(\alpha)$ and the situation when $\tilde{g} = 1_{G(\!(t)\!)}$. In the latter case, $\mathrm{Cone}(\alpha)$ isomorphic to the object of $\mathrm{Pro}(\mathrm{Coh}(S)^{\heartsuit})$ equal to "$\underset{\alpha}{lim}$" $\mathcal{O}_S \otimes V^*_\alpha$, where $\alpha \mapsto V_\alpha$ is the filtered family of finite-dimensional k-vector spaces, such that

$$\underset{\alpha}{colim}\, V_\alpha \simeq \mathfrak{g}(\!(t)\!)/\mathfrak{g}[\![t]\!].$$

□

9.3.3. Let us now observe the following corollary of Theorem 9.1.6:

Theorem 9.3.4. Gr_G *is* 0*-coconnective. Moreover, it is weakly* $\aleph_0$*, and locally almost of finite type.*

We shall now use Theorem 9.3.4 to prove the following:

Theorem 9.3.5. *The indscheme* $G((t))$ *is* 0*-coconnective.*

Proof. Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in Stk such that $\mathcal{X}_2$ is 0-coconnective, and for any $S \in \mathrm{Sch}^{\mathrm{aff}}$ and a map $S \to \mathcal{X}_2$, the fiber product $S \underset{\mathcal{X}_1}{\times} \mathcal{X}_2 \in \mathrm{Stk}$ is also 0-coconnective.

Lemma 9.3.6. *Under the above circumstances,* $\mathcal{X}_2$ *is also* 0*-coconnective.*

We apply this lemma to $\mathcal{X}_1 = G((t))$ and $\mathcal{X}_2 = \mathrm{Gr}_G$. It remains to verify that for a classical affine scheme S and a map $g : S \to \mathrm{Gr}_G$, the fiber product $S \underset{\mathrm{Gr}_G}{\times} G((t))$ is 0-coconnective. The question is local in the étale topology on S. Hence, we can assume that g admits a lift to an S-point of $G((t))$. However, this left defines an isomorphism

$$S \underset{\mathrm{Gr}_G}{\times} G((t)) \simeq S \times G[[t]],$$

and the assertion follows from Theorem 9.3.4 and Proposition 9.2.6.

□

9.4. **The (pro)-cotangent complex of a classical formally smooth (ind)scheme.** For the proof of Theorem 9.1.2 we will need to establish several facts concerning the pro-cotangent complex of classical formally smooth indschemes.

9.4.1. Let $\mathcal{X}_{cl}$ be a classical indscheme; set

$$\mathcal{X} := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{X}_{cl}) \in \mathrm{PreStk}\,.$$

Proposition 9.4.2. *The indscheme* $\mathcal{X}_{cl}$ *is classically formally smooth if and only if for every* $S \in \mathrm{Sch}^{\mathrm{aff}}$ *and* $x : S \to \mathcal{X}_{cl}$*, the object*

$$^{\geq -1}(T^*_x\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -1, \leq 0})$$

is pro-projective.

Proof. Let $^{\geq -1}(T^*_x\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -1, \leq 0})$ be pro-projective. By Lemma 4.5.9, it suffices to show that

$$\pi_0\left(\mathrm{Maps}(^{\geq -1}(T^*_x\mathcal{X}), \mathcal{F}[1])\right) = 0$$

for $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$. However, the latter is given by condition (a) Lemma 8.2.6.

For the opposite implication, let us assume that $\mathcal{X}_{cl}$ is formally smooth. We will check that $^{\geq -1}(T^*_x\mathcal{X})$ satisfies condition (b') of Lemma 8.2.6. Let x be a map $S \to \mathcal{X}$, where $S \in \mathrm{Sch}^{\mathrm{aff}}$.

The fact that the functor

$$\mathcal{F} \mapsto \pi_0\left(\mathrm{Maps}(^{\geq -1}(T^*_x\mathcal{X}), \mathcal{F})\right), \quad \mathrm{QCoh}(S)^{\heartsuit} \to \mathrm{Sets}$$

is right exact follows from the assumption on $\mathcal{X}_{cl}$ and the definition of $T^*_x\mathcal{X}$ in terms of split square-zero extensions in Sect. 4.1.3. Hence, it remains to show that $H^{-1}(T^*_x\mathcal{X}) = 0$.

Let

$$\mathcal{X}_{cl} \simeq \underset{\alpha \in \mathsf{A}}{colim}\, X_\alpha,$$

where $X_\alpha \in \mathrm{Sch}_{\mathrm{qsep\text{-}qc}}$. Let α_0 be an index such that x factors through a map $x_{\alpha_0} : S \to X_{\alpha_0}$.

Since the t-structure on Pro(QCoh(S)) is Zariski-local, we can assume that the map x_{α_0} factors as

$$S \to U_{\alpha_0} \overset{j}{\hookrightarrow} X_{\alpha_0},$$

where U_{α_0} is an open affine inside X_{α_0}.

Let $\iota_{\alpha_0} : X_{\alpha_0} \to \mathcal{X}_{cl}$ denote the tautological map. For $(\alpha_0 \to \alpha) \in \mathsf{A}$, let $\iota_{\alpha_0,\alpha}$ denote the corresponding closed embedding $X_{\alpha_0} \to X_\alpha$.

It is easy to see that it is sufficient to show that

$$H^{-1}\left(T^*_{\iota_{\alpha_0}\circ j}\mathcal{X}\right) = 0$$

as an object of Pro(QCoh$(U_{\alpha_0})^\heartsuit$).

By (4.11), we have

$$H^{-1}\left(T^*_{\iota_{\alpha_0}\circ j}\mathcal{X}\right) \simeq \underset{\alpha\in\mathsf{A}_{\alpha_0/}}{\text{“}lim\text{”}}\, H^{-1}\left(T^*_{\iota_{\alpha_0,\alpha}\circ j}X_\beta\right).$$

So, we need to show that for a given $\mathcal{M} \in \mathrm{QCoh}(U_{\alpha_0})^\heartsuit$, $\alpha_0 \to \alpha$ and

$$\phi : H^{-1}(T^*_{\iota_{\alpha_0,\alpha}\circ j}X_\alpha) \to \mathcal{M},$$

there exists $\alpha \to \beta$, such that the composition

$$H^{-1}(T^*_{\iota_{\alpha_0,\beta}\circ j}X_\beta) \to H^{-1}(T^*_{\iota_{\alpha_0,\alpha}\circ j}X_\alpha) \overset{\phi}{\to} \mathcal{M}$$

vanishes.

Since $\iota_{\alpha_0,\alpha}$ is a closed embedding, U_{α_0} is the pre-image of an open affine U_α in X_α. Replacing $\mathcal{M}$ by its direct image under $U_{\alpha_0} \to U_\alpha$, we can assume that $\alpha = \alpha_0$. Further, embedding $\mathcal{M}$ into an injective sheaf, we can assume that the map ϕ extends to a map $\psi : T^*U_\alpha \to \mathcal{M}[1]$. We wish to find an index $\beta \in \mathsf{A}_{\alpha/}$ such that the composition

$$T^*_{\iota_{\alpha,\beta}\circ j}X_\beta \to T^*U_\alpha \overset{\psi}{\to} \mathcal{M}[1] \tag{9.3}$$

vanishes.

However, the data of ψ as above is equivalent to that of a square-zero extension U'_α of U_α. And the data of a splitting of (9.3) is equivalent to that of an extension of the map $\iota_{\alpha,\beta} \circ j : U_\alpha \to X_\beta$ to a map $U'_\alpha \to X_\beta$. Thus, giving such an index β is equivalent to extending the map $U_\alpha \to \mathcal{X}_{cl}$ to a map $U'_\alpha \to \mathcal{X}_{cl}$. The existence of such an extension follows from the classical formal smoothness of $\mathcal{X}_{cl}$.

□

Corollary 9.4.3. *Let X_{cl} be a classical scheme, and consider it as a DG scheme. Then X_{cl} is classically formally smooth if and only if T^*X_{cl} satisfies:*

(a) $H^{-1}(T^*X_{cl}) = 0$.

(b) $H^0(T^*X_{cl})$ *is projective over every affine subscheme of* X_{cl}.

Proof. We only need to show that if S is an affine scheme mapping to X_{cl}, then the pull-back of T^*X_{cl} to it is projective. By assumption, we know this locally in the Zariski topology on S. The assertion now follows from the theorem of Raynaud-Gruson mentioned earlier that projectivity of a module is a Zariski-local property.

□

9.4.4. The following somewhat technical assertion will be needed in the sequel:

Let $\mathcal{X}_{cl}$ be a classical formal scheme with the underlying reduced scheme X, and set

$$\mathcal{X} :={}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{X}_{cl}).$$

Corollary 9.4.5. *Suppose that $\mathcal{X}_{cl}$ is locally of finite type, and that ${}^{\geq -1}(T^*\mathcal{X}|_X)$ is locally pro-projective. Then $\mathcal{X}_{cl}$ is classically formally smooth.*

This follows from Proposition 9.4.2 in the same way as Corollary 8.3.6 follows from Proposition 8.2.2.

9.5. **Classical formally smooth indschemes locally of finite type case.** In this subsection we will reduce Theorem 9.1.2 to a key proposition (Proposition 9.5.2) that describes the general shape of formal classical schemes locally of finite type.

9.5.1. Let $\mathcal{X}_{cl}$ be as in Theorem 9.1.2. Let Y be a reduced classical scheme and let $Y \subset {}^{red}(\mathcal{X}_{cl})$ be a closed embedding. (Note that such a Y is automatically locally of finite type.)

By Proposition 8.1.9, in order to prove Theorem 9.1.2, it suffices to show that the formal completion $\mathcal{X}^\wedge_Y$ is formally smooth. Moreover, by Proposition 8.2.2 and Lemma 8.3.3, we can assume that Y is affine.

We will prove that $\mathcal{X}^\wedge_Y$ is formally smooth by quoting/reproving the following result (see [BD, Proposition 7.12.22]). This proposition will also be useful to us in the sequel.

Proposition 9.5.2. *Let $\mathcal{Z}_{cl}$ be a classical formal scheme. Assume that:*

- *As a classical indscheme, $\mathcal{Z}_{cl}$ is locally of finite type and $\aleph_0$.*
- *The classical scheme ${}^{red}(\mathcal{Z}_{cl})$ is affine.*
- *$\mathcal{Z}_{cl}$ is classically formally smooth.*

Then $\mathcal{Z}_{cl}$ is isomorphic to retract of a filtered colimit, taken in ${}^{cl}\mathrm{PreStk}$, of classical formal schemes each of which is elementary *(see Sect. 8.4.7).*

9.5.3. Let us deduce Theorem 9.1.2 from Proposition 9.5.2. This will be done via a series of lemmas. First, we have:

Lemma 9.5.4. *For any $\mathcal{X}_{cl} \in {}^{\leq 0}\mathrm{DGindSch}_{\mathrm{lft}}$, a reduced classical scheme Y and a closed embedding $Y \subset {}^{red}(\mathcal{X}_{cl})$, the canonical map*

$${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl}(\mathcal{X}^\wedge_Y)) \to \mathcal{X}^\wedge_Y,$$

where $\mathcal{X} := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{X}_{cl})$, is an isomorphism.

Proof. Let $\mathcal{X}_{cl} \simeq \underset{\alpha}{colim}\, X_\alpha$, where the colimit is taken in ${}^{cl}\mathrm{PreStk}$. Then

$$\mathcal{X} \simeq \underset{\alpha}{colim}\, X_\alpha,$$

where the colimit is taken in Stk.

Without loss of generality, we can assume that Y is contained in each $X_{\alpha,cl}$. Then

$${}^{cl}(\mathcal{X}^\wedge_Y) \simeq \underset{\alpha}{colim}\, {}^{cl}((X_\alpha)^\wedge_Y),$$

and the left-hand side in the lemma is

$$\underset{\alpha}{colim}\, {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}\, {}^{cl}((X_\alpha)^\wedge_Y),$$

where the colimit is taken in Stk.

We now claim that the map

$$L\left(\underset{\alpha}{colim}\,((X_\alpha)^\wedge_Y)\right) \to \left(L(\underset{\alpha}{colim}\, X_\alpha)\right)^\wedge_Y = \mathcal{X}^\wedge_Y,$$

where both colimits are taken in PreStk, is an isomorphism. This follows from Lemma 6.2.5 and Sect. 6.1.3(iii).

Thus, the right-hand side in the lemma identifies with $\underset{\alpha}{colim}\,(X_\alpha)^\wedge_Y$, where the colimit is taken in Stk.

Hence, to prove the lemma, it suffices to show that for every α, the canonical map

$${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}\,{}^{cl}((X_\alpha)^\wedge_Y) \to (X_\alpha)^\wedge_Y$$

is an isomorphism. However, this is the content of Proposition 6.8.2, which is applicable since X_α is of finite type, and in particular, Noetherian.

□

Lemma 9.5.5. *Let $\mathcal{Z}_{cl}$ be as in Proposition 9.5.2. Then*

$${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Z}_{cl})$$

is a formally smooth DG indscheme.

Let us assume this lemma and finish the proof of Theorem 9.1.2.

Proof of Theorem 9.1.2. We need to show that $\mathcal{X}^\wedge_Y$ is formally smooth. By Lemma 9.5.4, this is equivalent to ${}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}({}^{cl}(\mathcal{X}^\wedge_Y))$ being formally smooth. The required assertion follows Lemma 9.5.5 applied to $\mathcal{Z}_{cl} := {}^{cl}(\mathcal{X}^\wedge_Y)$.

□

9.5.6. *Proof of Lemma 9.5.5.* Since the notion of formal smoothness is stable under taking retracts, we can assume that

$$\mathcal{Z}_{cl} \simeq \underset{\alpha}{colim}\, \mathcal{Z}_{\alpha,cl},$$

(colimit taken in ^{cl}PreStk), where each $\mathcal{Z}_{\alpha,cl}$ is elementary.

Hence, $\mathcal{Z} := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Z}_{cl})$ is isomorphic to

$$\underset{\alpha}{colim}\, \mathcal{Z}_\alpha,$$

where the colimit is taken in Stk, where

$$\mathcal{Z}_\alpha := {}^L\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\hookrightarrow(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Z}_{\alpha,cl}).$$

By Proposition 8.4.6 combined with Corollary 8.4.5, each $\mathcal{Z}_\alpha$ is a formally smooth DG indscheme. Thus, it remains to prove the following:

Lemma 9.5.7. *Let $\alpha \mapsto \mathcal{Z}_\alpha$ be filtered family of objects if* Stk*, each of which is formally smooth as an object of* PreStk*. Assume that for every n, all $\mathcal{Z}_\alpha|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ are k-truncated for some k. Then $\mathcal{Z} := \underset{\alpha}{colim}\, \mathcal{Z}_\alpha$ is also formally smooth as an object of* PreStk*, where the colimit is taken in* Stk*.*

Proof. Consider the object $\mathcal{Z}' := \underset{\alpha}{colim}\, \mathcal{Z}_\alpha$, where the colimit is taken in PreStk. Since homotopy groups commute with filtered colimits, we obtain that $\mathcal{Z}'$ is formally smooth as an object of PreStk. In particular, it is convergent. It remains to show that the canonical map $\mathcal{Z}' \to \mathcal{Z}$ is an isomorphism. To show this, it suffices to show that $\mathcal{Z}'$ satisfies descent. By convergence, it is enough to check the descent condition on ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$. But the latter follows from the truncatedness assumption by Lemma 1.3.3. □

□(Lemma 9.5.5)

9.6. **Proof of the key proposition.** In this subsection we will prove Proposition 9.5.2, reproducing a slightly modified argument from [BD], pages 328-331. [24]

Remark 9.6.1. Note that the statement of [BD, Proposition 7.12.22] is slightly stronger: it asserts that, under the (innocuous) additional assumption that ${}^{red}(\mathcal{Z}_{cl})$ is connected, we have an isomorphism

$$\mathcal{Z}_{cl} \simeq \mathcal{Z}_{0,cl} \times {}^{cl}(\mathrm{Spf}\,(k[\![z_1, z_2, \ldots]\!]))\,,$$

where $\mathcal{Z}_{0,cl}$ is elementary. The reason we choose the formulation given in Proposition 9.5.2 is that it makes it more amenable for generalization in the non-finite type situation.

9.6.2. *Step 0: initial remarks.* Denote $Z := {}^{red}(\mathcal{Z}_{cl})$ and $\mathcal{Z} := {}^{L}\mathrm{LKE}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}}(\mathcal{Z}_{cl})$.

By Proposition 5.3.2, the finite type condition implies that the object $T^*\mathcal{Z}|_Z$ belongs to $\mathrm{Pro}(\mathrm{Coh}(Z)^{\leq 0})$, where

$$T^*\mathcal{Z}|_Z := T^*_z\mathcal{Z},$$

where $z : Z \to \mathcal{Z}$ is the tautological map. Proposition 5.2.3 implies that $T^*\mathcal{Z}|_Z$ is $\aleph_0$ as an object of $\mathrm{Pro}(\mathrm{Coh}(Z)^{\leq 0})$.

By Proposition 9.4.2 and [BD, Proposition 7.12.6(iii)], $T^*\mathcal{Z}|_Z$ is the dual of a Mittag-Leffler quasi-coherent sheaf $\mathcal{M}$ on Z. By [BD, Theorem 7.12.8], the $\aleph_0$ condition implies that $\mathcal{M}$ is actually projective.

Multiplying $\mathcal{Z}$ by a suitable formally smooth classical indscheme as in [BD, Proposition 7.12.14], we can assume that $\mathcal{M}$ is a free countably generated $\mathcal{O}_Z$-module. [25]

Thus, we obtain that we can assume that

$$H^0(T^*\mathcal{Z}|_Z) \in \mathrm{Pro}(\mathrm{Coh}(Z)^{\heartsuit}) \tag{9.4}$$

can be represented as $P := \text{``}\underset{k\in\mathbb{N}}{lim}\text{''}\, P_k$, where P_k are locally free (in fact, free) sheaves on Z of finite rank, and the maps $P_{k+1} \to P_k$ are surjective.

Let us write $\mathcal{Z}_{cl} \simeq \underset{n\in\mathbb{N}}{colim}\, Z_n$ with $Z = Z_0$. Let $\mathcal{I}_n$ denote the sheaf of ideals of Z in Z_n. The finite type hypothesis implies that $\mathcal{I}_n \in \mathrm{Coh}(Z_n)^{\heartsuit}$. Consider $\mathcal{I}_n|_Z \simeq \mathcal{I}_n/\mathcal{I}_n^2 \in \mathrm{Coh}(Z)^{\heartsuit}$ and denote

$$\mathcal{I}|_Z := \text{``}\underset{n}{lim}\text{''}\, \mathcal{I}_n|_Z \in \mathrm{Pro}(\mathrm{Coh}(Z)^{\heartsuit}).$$

By Proposition 9.4.2, the long exact cohomology sequence for the map $Z \hookrightarrow \mathcal{Z}$ gives rise to a 4-term exact sequence in $\mathrm{Pro}(\mathrm{Coh}(Z)^{\heartsuit})$:

$$0 \to H^{-1}(T^*Z) \to \mathcal{I}|_Z \to H^0(T^*\mathcal{Z}|_Z) \to H^0(T^*Z) \to 0. \tag{9.5}$$

9.6.3. *Step 1: "the finite-dimensional case".* Let us first assume that $H^0(T^*\mathcal{Z}|_Z)$ is an object of $\mathrm{Coh}(Z)$. In this case we will prove that ${}^{cl}\mathcal{Z}$ is elementary.

By (9.4), $H^0(T^*\mathcal{Z}|_Z)$ is locally free of finite rank over Z.

As in [BD], top of page 329, it suffices to show that the system of coherent sheaves $n \mapsto \mathcal{I}_n|_Z$ stabilizes. However, (9.5) implies that $\mathcal{I}|_Z$ is in fact an object of $\mathrm{Coh}(Z)^{\heartsuit}$. Since the maps $\mathcal{I}_{n+1}|_Z \to \mathcal{I}_n|_Z$ are surjective, this implies the stabilization statement.

[24] The reason that we include the proof instead of just quoting the result from [BD] is that it seems that the considerations that involve derived pro-cotangent spaces that we introduce help to make the argument of *loc. cit.* more conceptual.

[25] This last procedure is the reason the word "retract" appears in the formulation of Proposition 9.5.2.

9.6.4. *Step 2: choosing generators for the ideal.* For a general $\mathcal{Z}_{cl}$ we will construct an object $Q \in \text{Pro}(\text{Coh}(Z)^{\heartsuit})$ of the form

$$Q \simeq \text{``}\underset{m \in \mathbb{N}}{lim}\text{''}\, Q_m,$$

where Q_m are locally free sheaves on Z of finite rank, and the maps $Q_{m+1} \to Q_m$ are surjective, and a map $f : Q \to \mathcal{I}|_Z$, such that the composition

$$Q \to \mathcal{I}|_Z \to H^0(T^*\mathcal{Z}|_Z) =: P$$

in injective and has the property that $\text{coker}(Q \to P)$ belongs to $\text{Coh}(Z)$ and is locally free.

Consider again (9.5). Let k be an index such that the map $P \to H^0(T^*Z)$ factors through a map $P_k \twoheadrightarrow H^0(T^*Z)$, and let $Q := \ker(P \to P_k)$. Set

$$R := \mathcal{I}|_Z \underset{P}{\times} Q.$$

By construction, the map $R \to Q$ is surjective, i.e., we have the following short exact sequence in $\text{Pro}(\text{Coh}(Z)^{\heartsuit})$:

$$0 \to H^1(T^*Z) \to R \to Q \to 0.$$

Since Q is pro-projective and the category of indices is $\mathbb{N}$, the map $R \to Q$ admits a right inverse, which gives rise to the desired map $f : Q \to R \to \mathcal{I}|_Z$.

9.6.5. *Step 3.* We shall now use the above pair $(Q, f : Q \to \mathcal{I}|_Z)$ to construct the desired family of sub-indschemes of $\mathcal{Z}_{cl}$, each being as in Step 1.

For every n consider the object

$$\mathcal{I}|_{Z_n} := \text{``}\underset{n' \geq n}{lim}\text{''}\, \mathcal{I}_{n'}|_{Z_n} \in \text{Pro}(\text{Coh}(Z_n)^{\heartsuit}).$$

We can extend the locally free sheaves Q_m to a compatible family of locally free finite rank coherent sheaves $n \mapsto Q_m|_{Z_n}$ and a compatible family of maps

$$f|_{Z_n} : Q|_{Z_n} := \text{``}\underset{m}{lim}\text{''}\, Q_m|_{Z_n} \to \mathcal{I}|_{Z_n}.$$

Let $Q^m|_{Z_n}$ be the kernel of the map $Q|_{Z_n} \to Q_m|_{Z_n}$. For each m we define the closed sub-scheme Z_n^m of Z_n to be given by the ideal $\mathcal{J}_n^m$ equal to the image of

$$Q^m|_{Z_n} \hookrightarrow Q|_{Z_n} \xrightarrow{f|_{Z_n}} \mathcal{I}|_{Z_n} \to \mathcal{I}_n.$$

We set

$$\mathcal{Z}_{cl}^m := \underset{n}{colim}\, Z_n^m.$$

It is clear from the construction that

$$\mathcal{Z}_{cl} \simeq \underset{m}{colim}\, \mathcal{Z}_{cl}^m.$$

9.6.6. *Step 4.* It remains to show that each $\mathcal{Z}_{cl}^m$ is a classical indscheme satisfying the assumptions of Step 1. Let

$$\mathcal{Z}^m := {}^L\text{LKE}_{(\text{Sch}^{\text{aff}})^{\text{op}} \hookrightarrow (\text{DGSch}^{\text{aff}})^{\text{op}}}(\mathcal{Z}_{cl}^m).$$

Using Corollary 9.4.5, it suffices to show that $H^0(T^*\mathcal{Z}^m|_Z) \in \text{Pro}(\text{Coh}(Z)^{\heartsuit})$ is locally free of finite rank and that $H^{-1}(T^*\mathcal{Z}^m|_Z) = 0$.

For that it suffices to show that the map

$$\text{``}\underset{n}{lim}\text{''}\, \mathcal{J}_n^m|_Z \to H^0(T^*\mathcal{Z}|_Z)$$

is injective and that the quotient belongs to $\mathrm{Coh}(Z)^{\heartsuit}$ and is locally free of finite rank. However, by construction, we have a surjective map

$$Q^m := \ker(Q \to Q_m) \twoheadrightarrow \text{“}\underset{n}{lim}\text{”}\, \mathcal{J}_n^m|_Z,$$

and the required properties follow from the corresponding properties of the map f.

□(Proposition 9.5.2)

10. QCoh and IndCoh on formally smooth indschemes

10.1. **The main result.** The goal of this section is to prove the following result, originally established by J. Lurie using a different method:

Theorem 10.1.1. *Let $\mathcal{X}$ be a formally smooth DG indscheme, which is weakly $\aleph_0$ and locally almost of finite type. Then the functor*

$$\Upsilon_{\mathcal{X}} := - \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \omega_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{IndCoh}(X)$$

is an equivalence.

Note that by Theorem 9.1.2, the DG indscheme $\mathcal{X}$ is 0-coconnected, so $\mathrm{QCoh}(\mathcal{X})$ is equivalent to

$$\mathrm{QCoh}({}^L\tau^{cl}(\mathcal{X})) \simeq \mathrm{QCoh}(\tau^{cl}(\mathcal{X})),$$

where the latter equivalence is because of [GL:QCoh, Corollary 1.3.7].

We also note the following corollary of Theorem 10.1.1 and Corollary 2.4.4:

Corollary 10.1.2. *Let $\mathcal{X}$ be a formally smooth DG indscheme, which is weakly $\aleph_0$ and locally almost of finite type. Then the category $\mathrm{QCoh}(\mathcal{X})$ is compactly generated.*

The rest of this section is devoted to the proof of Theorem 10.1.1.

10.2. **Reduction to the “standard” case.**

10.2.1. Write ${}^{cl}\mathcal{X}$ as $\underset{\alpha}{colim}\, X_\alpha$, where X_α are classical schemes locally of finite type. Let $\mathcal{X}_\alpha := \mathcal{X}^\wedge_{{}^{red}X_\alpha}$ be the formal completion of $\mathcal{X}$ along ${}^{red}X_\alpha$. Each $\mathcal{X}_\alpha$ is DG indscheme satisfying the assumptions of the theorem.

Since

$$\underset{\alpha}{colim}\, \mathcal{X}_\alpha \to \mathcal{X},$$

is an isomorphism (the above colimit taken in PreStk), the functors

$$\mathrm{QCoh}(\mathcal{X}) \to \underset{\alpha}{lim}\, \mathrm{QCoh}(\mathcal{X}_\alpha) \text{ and } \mathrm{IndCoh}(\mathcal{X}) \to \underset{\alpha}{lim}\, \mathrm{IndCoh}(\mathcal{X}_\alpha)$$

are both equivalences, where the first limit is taken with respect to the *-pullback functors, and the second limit is taken with respect to the !-pullback functors.

Since for $\alpha \to \beta$ the diagrams

$$\begin{array}{ccccc} \mathrm{QCoh}(\mathcal{X}) & \longrightarrow & \mathrm{QCoh}(\mathcal{X}_\beta) & \longrightarrow & \mathrm{QCoh}(\mathcal{X}_\alpha) \\ {\scriptstyle \Upsilon_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \Upsilon_{\mathcal{X}_\beta}} & & \downarrow{\scriptstyle \Upsilon_{\mathcal{X}_\alpha}} \\ \mathrm{IndCoh}(\mathcal{X}) & \longrightarrow & \mathrm{IndCoh}(\mathcal{X}_\beta) & \longrightarrow & \mathrm{IndCoh}(\mathcal{X}_\alpha) \end{array}$$

are commutative, it suffices to show that each of the functors

$$\Upsilon_{\mathcal{X}_\alpha} : \mathrm{QCoh}(\mathcal{X}_\alpha) \to \mathrm{IndCoh}(\mathcal{X}_\alpha) \tag{10.1}$$

is an equivalence.

So, from now on we will assume that $\mathcal{X}$ is formal.

10.2.2. By [GL:IndCoh, Proposition 4.2.1], the functor IndCoh satisfies Zariski descent. So, the statement about equivalence in (10.1) is local in the Zariski topology. Therefore, we can assume that $\mathcal{X}$ is affine, and thus apply Proposition 9.5.2.

10.2.3. Since the statement of the theorem survives taking retracts and colimits of DG indschemes, we can assume that $\mathcal{X}$ is *elementary* (see Sect. 8.4.7). The proof that the functor $\Upsilon_{\mathcal{X}}$ is an equivalence in this case is a rather straightforward but somewhat tedious verification, which we shall presently perform.

10.3. **The functor $\Upsilon^{\vee}_{\mathcal{X}}$.**

10.3.1. Recall the notation of Sect. 8.4.4. Let us denote by $\mathcal{Y}$ the DG indscheme $\mathbb{A}^{n,m}$. Let $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ denote the corresponding closed embedding.

Since

$$\mathcal{X} \simeq 0 \underset{\mathbb{A}^k}{\times} \mathcal{Y},$$

by [GL:QCoh, Proposition 3.2.1], we have:

$$\operatorname{QCoh}(\mathcal{X}) \simeq \operatorname{Vect} \underset{\operatorname{QCoh}(\mathbb{A}^k)}{\otimes} \operatorname{QCoh}(\mathcal{Y}). \tag{10.2}$$

By Sect. 7.2, the indscheme $\mathcal{Y}$ is quasi-perfect (i.e., the category $\operatorname{QCoh}(\mathcal{Y})$ is compactly generated and its compact objects are perfect). We claim:

Lemma 10.3.2. *The DG indscheme $\mathcal{X}$ is quasi-perfect.*

Proof. From (10.2) we obtain that a generating set of compact objects of $\operatorname{QCoh}(\mathcal{X})$ is obtained as the essential image under the functor f^* of compact objects of $\operatorname{QCoh}(\mathcal{Y})$. The assertion of the lemma follows from the fact that the pullback functor preserves perfectness. □

In particular, from Lemma 10.3.2 we obtain a self-duality equivalence

$$\mathbf{D}^{\text{naive}}_{\mathcal{X}} : (\operatorname{QCoh}(\mathcal{X}))^{\vee} \simeq \operatorname{QCoh}(\mathcal{X}), \tag{10.3}$$

Using also

$$\mathbf{D}^{\text{Serre}}_{\mathcal{X}} : (\operatorname{IndCoh}(\mathcal{X}))^{\vee} \simeq \operatorname{IndCoh}(\mathcal{X}),$$

we can consider the functor

$$\Upsilon^{\vee}_{\mathcal{X}} : \operatorname{IndCoh}(\mathcal{X}) \to \operatorname{QCoh}(\mathcal{X}),$$

dual to $\Upsilon_{\mathcal{X}}$.

Showing that $\Upsilon_{\mathcal{X}}$ is an equivalence is equivalent to showing that $\Upsilon^{\vee}_{\mathcal{X}}$ is an equivalence.

10.3.3. Let ${}^{\sim}\mathcal{Y} := {}^{\sim}\mathbb{A}^{n,m}$ be the *scheme* introduced in the course of the proof of Proposition 8.4.6. Let ${}^{\sim}\mathcal{X}$ be the DG *scheme*

$$0 \underset{\mathbb{A}^k}{\times} {}^{\sim}\mathcal{Y}.$$

As we saw in Sect. 8.4.4, the DG scheme $\mathcal{X}$ is also 0-coconnective.

The formal (DG) scheme $\mathcal{X}$ is obtained as a formal completion of ${}^{\sim}\mathcal{X}$ along a Zariski-closed subset X. Let $\widehat{i} : \mathcal{X} \to {}^{\sim}\mathcal{X}$ denote the corresponding map, and let

$${}^{\sim}\mathcal{X} - X =: U_{\mathcal{X}} \overset{j}{\hookrightarrow} {}^{\sim}\mathcal{X}$$

be the complementary open embedding.

Since the DG scheme ${}^{\sim}\mathcal{X}$ is Noetherian, the category $\text{IndCoh}(\mathcal{X})$ and the functor

$$\Psi_{\mathcal{X}} : \text{IndCoh}(\mathcal{X}) \to \text{QCoh}(\mathcal{X})$$

are well-defined (see [GL:IndCoh, Sect. 1.1]).

10.3.4. We will deduce the fact that $\Upsilon^\vee_{\mathcal{X}}$ is an equivalence from the following statement:

Proposition 10.3.5. *The diagram of functors*

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{X}) & \xrightarrow{\widehat{i}^{\text{IndCoh}}_*} & \text{IndCoh}({}^{\sim}\mathcal{X}) \\ {\scriptstyle \Upsilon^\vee_{\mathcal{X}}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{{}^{\sim}\mathcal{X}}} \\ \text{QCoh}(\mathcal{X}) & \xrightarrow{\widehat{i}_?} & \text{QCoh}({}^{\sim}\mathcal{X}) \end{array}$$

commutes.

Remark 10.3.6. This proposition does not formally follow from the commutativity of (7.13), because the latter relied on the finite type assumption of the ambient DG scheme (in our case the ambient scheme is ${}^{\sim}\mathcal{X}$, and it is not of finite type).

10.3.7. Let us assume this proposition for a moment and finish the proof of the fact that $\Upsilon^\vee_{\mathcal{X}}$ is an equivalence (and thereby of Theorem 10.1.1).

Proof. We have a commutative diagram

$$\begin{array}{ccc} \text{IndCoh}({}^{\sim}\mathcal{X}) & \xrightarrow{j^{\text{IndCoh},*}} & \text{IndCoh}(U_{\mathcal{X}}) \\ {\scriptstyle \Psi_{{}^{\sim}\mathcal{X}}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{U_{\mathcal{X}}}} \\ \text{QCoh}({}^{\sim}\mathcal{X}) & \xrightarrow{j^*} & \text{QCoh}(U_{\mathcal{X}}) \end{array}$$

(see [GL:IndCoh, Proposition 3.5.4]).

By Proposition 7.1.3, the category $\text{QCoh}(\mathcal{X})$ identifies with the kernel of the functor

$$j^* : \text{QCoh}({}^{\sim}\mathcal{X}) \to \text{QCoh}(U_{\mathcal{X}}).$$

By Proposition 7.4.5, the category $\text{IndCoh}(\mathcal{X})$ identifies with the kernel of the functor

$$j^{\text{IndCoh},*} : \text{IndCoh}({}^{\sim}\mathcal{X}) \to \text{IndCoh}(U_{\mathcal{X}}).$$

(We remark that in Proposition 7.4.5 it was assumed that the ambient scheme is almost of finite type over the field, but the proof applies in the case when it is only assumed Noetherian, which is the case for ${}^{\sim}\mathcal{X}$.)

The required assertion follows from the fact that the functors $\Psi_{{}^\sim\mathcal{X}}$ and $\Psi_{U_{\mathcal{X}}}$ are equivalences, since the corresponding DG schemes are 0-coconnective and the underlying classical schemes are regular (see [GL:IndCoh, Lemma 1.1.6]).

□

10.4. **Proof of Proposition 10.3.5.** We shall compare the functors

$$\mathrm{QCoh}({}^\sim\mathcal{X})\otimes\mathrm{IndCoh}(\mathcal{X})\rightrightarrows\mathrm{Vect}$$

that arise from the two circuits of the diagram and the duality pairing

$$\langle -,-\rangle_{\mathrm{QCoh}({}^\sim\mathcal{X})}:\mathrm{QCoh}({}^\sim\mathcal{X})\otimes\mathrm{QCoh}({}^\sim\mathcal{X})\to\mathrm{Vect},$$

corresponding to the functor $\mathbf{D}^{\mathrm{naive}}_{\mathcal{X}}$ of (10.3).

For $\mathcal{F}\in\mathrm{QCoh}({}^\sim\mathcal{X})$ and $\mathcal{F}'\in\mathrm{IndCoh}(\mathcal{X})$ we have:

$$\langle\mathcal{F},\widehat{i}_?\circ\Upsilon^\vee_{\mathcal{X}}(\mathcal{F}_1)\rangle_{\mathrm{QCoh}({}^\sim\mathcal{X})}\simeq\langle\widehat{i}^*(\mathcal{F}),\Upsilon^\vee_{\mathcal{X}}(\mathcal{F}_1)\rangle_{\mathrm{QCoh}(\mathcal{X})}\simeq\Gamma^{\mathrm{IndCoh}}(\mathcal{X},\widehat{i}^*(\mathcal{F})\underset{\mathcal{O}_{\mathcal{X}}}{\otimes}\mathcal{F}_1),\tag{10.4}$$

where the first isomorphism follows from Corollary 7.2.6, and the second one from Sect. 7.5.2.

The description of the functor $\widehat{i}^{\mathrm{IndCoh}}_*$ given in Sect. 7.4.3 (which is valid for all Noetherian schemes) implies that we have a canonical isomorphism

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{X},-)\simeq\Gamma^{\mathrm{IndCoh}}({}^\sim\mathcal{X},-)\circ\widehat{i}^{\mathrm{IndCoh}}_*.$$

Hence, the expression in (10.4) can be further rewritten as

$$\Gamma^{\mathrm{IndCoh}}\left({}^\sim\mathcal{X},\widehat{i}^{\mathrm{IndCoh}}_*(\widehat{i}^*(\mathcal{F})\underset{\mathcal{O}_{\mathcal{X}}}{\otimes}\mathcal{F}_1)\right),$$

which by the projection formula is canonically isomorphic to

$$\Gamma^{\mathrm{IndCoh}}\left({}^\sim\mathcal{X},\mathcal{F}\underset{\mathcal{O}_{{}^\sim\mathcal{X}}}{\otimes}\widehat{i}^{\mathrm{IndCoh}}_*(\mathcal{F}_1)\right).$$

Now,

$$\langle\mathcal{F},\Psi_{{}^\sim\mathcal{X}}\circ\widehat{i}^{\mathrm{IndCoh}}_*(\mathcal{F}_1)\rangle_{\mathrm{QCoh}({}^\sim\mathcal{X})}\simeq\Gamma^{\mathrm{IndCoh}}({}^\sim\mathcal{X},\mathcal{F}\underset{\mathcal{O}_{{}^\sim\mathcal{X}}}{\otimes}\widehat{i}^{\mathrm{IndCoh}}_*(\mathcal{F}_1)),$$

as required.

□

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