IND-COHERENT SHEAVES

DENNIS GAITSGORY

To the memory of I. M. Gelfand

Abstract. We develop the theory of ind-coherent sheaves on schemes and stacks. The category of ind-coherent sheaves is closely related, but inequivalent, to the category of quasi-coherent sheaves, and the difference becomes crucial for the formulation of the categorical Geometric Langlands Correspondence.

Contents

Date: October 14, 2012.

Introduction

0.1. **Why ind-coherent sheaves?** This paper grew out of a series of digressions in an attempt to write down the formulation of the categorical Geometric Langlands Conjecture.

Let us recall that the categorical Geometric Langlands Conjecture is supposed to say that the following two categories are equivalent. One category is the (derived) category of D-modules on the stack Bun_G classifying principal G-bundles on a smooth projective curve X. The other category is the (derived) category of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$ classifying $\check{G}$-local systems on X, where $\check{G}$ is the Langlands dual of G.

However, when G is not a torus, the equivalence between

$$\text{D-mod}(\mathrm{Bun}_G) \text{ and } \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

does not hold, but it is believed it will hold once we slightly modify the categories D-mod(Bun_G) and $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$.

0.1.1. So, the question is what "modify slightly" means. However, prior to that, one should ask what kind of categories we want to consider.

Experience shows that when working with triangulated categories, [1] the following framework is convenient: we want to consider categories that are cocomplete (i.e., admit arbitrary direct sums), and that are generated by a set of compact objects. The datum of such a category is equivalent (up to passing to the Karoubian completion) to the datum of the corresponding subcategory of compact objects. So, let us stipulate that this is the framework in which we want to work. As functors between two such triangulated categories we will take those triangulated functors that commute with direct sums; we call such functors continuous.

0.1.2. *Categories of D-modules.* Let X be a scheme of finite type (over a field of characteristic zero). We assign to it the (derived) category D-mod(X), by which we mean the unbounded derived category with quasi-coherent cohomologies. This category is compactly generated; the corresponding subcategory D-mod$(X)^c$ of compact objects consists of those complexes that are cohomologically bounded and coherent (i.e., have cohomologies in finitely many degrees, and each cohomology is locally finitely generated).

The same goes through when X is no longer a scheme, but a quasi-compact algebraic stack, under a mild additional hypothesis, see [DrGa1]. Note, however, that in this case, compact objects of the category D-mod$(X)^c$ are less easy to describe, see [DrGa1, Sect. 8.1].

However, the stack $\mathrm{Bun}_G(X)$ is not quasi-compact, so the compact generation of the category D-mod(Bun_G), although true, is not obvious. The proof of the compact generation of D-mod(Bun_G) is the subject of the paper [DrGa2].

[1] In the main body of the paper, instead of working with triangulated categories we will work with DG categories and functors.

In any case, in addition to the subcategory $\text{D-mod}(\text{Bun}_G)^c$ of actual compact objects in $\text{D-mod}(\text{Bun}_G)$, there are several other small subcategories of $\text{D-mod}(\text{Bun}_G)$, all of which would have been the same, had Bun_G been a quasi-compact scheme. Any of these categories can be used to "redefine" the derived category on Bun_G.

Namely, if $\text{D-mod}(\text{Bun}_G)^{c\prime}$ is such a category, one can consider its ind-completion

$$\text{D-mod}(\text{Bun}_G)' := \text{Ind}(\text{D-mod}(\text{Bun}_G)^{c\prime}).$$

Replacing the initial $\text{D-mod}(\text{Bun}_G)$ by $\text{D-mod}(\text{Bun}_G)'$ is what we mean by "slightly modifying" our category.

0.1.3. *Categories of quasi-coherent sheaves.* Let X be again a scheme of finite type, or more generally, a DG scheme almost of finite type (see Sect. 0.6.6), over a field of characteristic zero. It is well-known after [TT] that the category $\text{QCoh}(X)$ is compactly generated, where the subcategory $\text{QCoh}(X)^c$ is the category $\text{QCoh}(X)^{\text{perf}}$ of perfect objects.

In this case also, there is another natural candidate to replace the category $\text{QCoh}(X)^{\text{perf}}$, namely, the category $\text{Coh}(X)$ which consists of bounded complexes with coherent cohomologies. The ind-completion $\text{QCoh}(X)' := \text{Ind}(\text{Coh}(X))$ is the category that we denote $\text{IndCoh}(X)$, and which is the main object of study in this paper.

There may be also other possibilities for a natural choice of a small subcategory in $\text{QCoh}(X)$. Specifically, if X is a locally complete intersection, one can attach a certain subcategory

$$\text{QCoh}(X)^{\text{perf}} \subset \text{QCoh}(X)^{\text{perf}}_{\mathcal{N}} \subset \text{Coh}(S)$$

to any Zariski-closed conical subset $\mathcal{N}$ of $\text{Spec}_X(\text{Sym}(H^1(T_X)))$, where T_X denotes the tangent complex of X, see [AG] where this theory is developed.

0.1.4. An important feature of the examples considered above is that the resulting modified categories $\text{D-mod}(\text{Bun}_G)'$ and $\text{QCoh}(X)'$ all carry a t-structure, such that their eventually coconnective subcategories (i.e., $\text{D-mod}(\text{Bun}_G)'^{+}$ and $\text{QCoh}(X)'^{+}$, respectively) are in fact equivalent to the corresponding old categories (i.e., $\text{D-mod}(\text{Bun}_G)^{+}$ and $\text{QCoh}(X)^{+}$, respectively).

I.e., the difference between the new categories and the corresponding old ones is that the former are not left-complete in their t-structures, i.e., Postnikov towers do not necessarily converge (see Sect. 1.3 where the notion of left-completeness is reviewed).

However, this difference is non-negligible from the point of view of Geometric Langlands: the conjectural equivalence between the modified categories

$$\text{D-mod}(\text{Bun}_G)' \simeq \text{QCoh}(\text{LocSys}_{\check{G}})'$$

has an unbounded cohomological amplitude, so an object which is trivial with respect to the t-structure on one side (i.e., has all of its cohomologies equal to 0), may be non-trivial with respect to the t-structure on the other side. An example is provided by the "constant sheaf" D-module

$$k_{\text{Bun}_G} \in \text{D-mod}(\text{Bun}_G)^{+} = \text{D-mod}(\text{Bun}_G)'^{+}.$$

0.2. **So, why ind-coherent sheaves?** Let us however return to the question why one should study specifically the category $\text{IndCoh}(X)$. In addition to the desire to understand the possible candidates for the spectral side of Geometric Langlands, there are three separate reasons of general nature.

0.2.1. Reason number one is that when X is not a scheme, but an ind-scheme, the category $\text{IndCoh}(X)$ is usually much more manageable than $\text{QCoh}(X)$. This is explained in [GR1, Sect. 2].

0.2.2. Reason number two has to do with the category of D-modules. For any scheme or stack X, the category $\text{D-mod}(X)$ possesses two forgetful functors:

$$\mathbf{oblv}^l : \text{D-mod}(X) \to \text{QCoh}(X) \text{ and } \mathbf{oblv}^r : \text{D-mod}(X) \to \text{IndCoh}(X),$$

that realize $\text{D-mod}(X)$ as "left" D-modules and "right" D-modules, respectively.

However, the "right" realization is much better behaved, see [GR2] for details.

For example, it has the nice feature of being compatible with the natural t-structures on both sides: an object $\mathcal{F} \in \text{D-mod}(X)$ is coconnective (i.e., cohomologically ≥ 0) if and only if $\mathbf{oblv}^r(\mathcal{F})$ is. [2]

So, if we want to study the category of D-modules in its incarnation as "right" D-modules, we need to study the category $\text{IndCoh}(X)$.

0.2.3. Reason number three is even more fundamental. Recall that the Grothendieck-Serre duality constructs a functor $f^! : \text{QCoh}(Y)^+ \to \text{QCoh}(X)^+$ for a proper morphism $f : X \to Y$, which is the right adjoint to the functor $f_* : \text{QCoh}(X) \to \text{QCoh}(Y)$.

However, if we stipulate that we want to stay within the framework of cocomplete categories and functors that commute with direct sums, a natural way to formulate this adjunction is for the categories $\text{IndCoh}(X)$ and $\text{IndCoh}(Y)$. Indeed, the right adjoint to f_*, considered as a functor $\text{QCoh}(X) \to \text{QCoh}(Y)$ will not be continuous, as f_* does not necessarily send compact objects of $\text{QCoh}(X)$, i.e., $\text{QCoh}(X)^{\text{perf}}$, to $\text{QCoh}(Y)^{\text{perf}}$. But it does send $\text{Coh}(X)$ to $\text{Coh}(Y)$.

We should remark that developing the formalism of the !-pullback and the base change formulas that it satisfies, necessitates passing from the category or ordinary schemes to that of DG schemes. Indeed, if we start with with a diagram of ordinary schemes

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g_Y} & Y, \end{array}$$

their Cartesian product $X' := X \underset{Y}{\times} Y'$ must be considered as a DG scheme, or else the base change formula

$$g_Y^! \circ f_* \simeq f'_* \circ g_X^! \qquad (0.1)$$

would fail, where $f' : X' \to Y'$ and $g_X : X' \to X$ denotes the resulting base changed maps.

[2]When X is smooth, the "left" realization is also compatible with the t-structures up to a shift by $\dim(X)$. However, when X is not smooth, the functor $\mathbf{oblv}^l$ is not well-behaved with respect to the t-structures, which is the reason for the popular misconception that "left D-modules do not make sense for singular schemes." More precisely, the derived category of left D-modules makes sense always, but the corresponding abelian category is indeed difficult to see from the "left" realization.

0.2.4. *What is done in this paper.* We shall presently proceed to review the actual contents of this paper.

We should say right away that there is no particular main theorem toward which other results are directed. Rather, we are trying to provide a systematic exposition of the theory that we will be able to refer to in subsequent papers on foundadtions of derived algebraic geometry needed for Geometric Langlands.

The present paper naturally splits into three parts:

Part I consists of Sections 1-4, which deal with properties of IndCoh on an individual DG scheme, and functorialities for an individual morphism between DG schemes. The techniques used in this part essentially amount to homological algebra, and in this sense are rather elementary.

Part II consists of Sections 5-9 where we establish and exploit the functoriality properties of the assignment

$$S \mapsto \mathrm{IndCoh}(S)$$

on the category of DG schemes as a whole. This part substantially relies on the theory of ∞-categories.

Part III consists of Sections 10-11, where we extend IndCoh to stacks and prestacks.

0.3. **Contents of the paper: the "elementary" part.**

0.3.1. In Sect. 1 we introduce the category $\mathrm{IndCoh}(S)$ where S is a Noetherian DG scheme. We note that when S is a classical scheme (i.e., structure sheaf $\mathcal{O}_X$ satisfies $H^i(\mathcal{O}_X) = 0$ for $i < 0$), we recover the category that was introduced by Krause in [Kr].

The main result of Sect. 1 is Proposition 1.2.4. It says that the eventually coconnective part of $\mathrm{IndCoh}(X)$, denoted $\mathrm{IndCoh}(X)^+$, maps isomorphically to $\mathrm{QCoh}(X)^+$. This is a key to transferring many of the functorialities of QCoh to IndCoh; in particular, the construction of direct images for IndCoh relies on this equivalence.

0.3.2. Section 2 is a digression that will not be used elsewhere in the present paper. Here we try to spell out the definition of $\mathrm{IndCoh}(S)$ for a scheme S which is not necessarily Noetherian.

The reason we do this is that we envisage the need of IndCoh when studying objects such as the loop group $G((t))$. In practice, $G((t))$ can be presented as an inverse limit of ind-schemes of finite type, i.e., in a sense we can treat $\mathrm{IndCoh}(G((t)))$ be referring back to the Noetherian case. However, the theory would be more elegant if we could give an independent definition, which is what is done here.

0.3.3. In Sect. 3 we introduce the three basic functors between the categories $\mathrm{IndCoh}(S_1)$ and $\mathrm{IndCoh}(S_2)$ for a map $f : S_1 \to S_2$ of Noetherian DG schemes.

The first is direct image, denoted

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S_1) \to \mathrm{IndCoh}(S_2),$$

which is "transported" from the usual direct image $f_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2)$.

The second is

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1),$$

which is also transported from the usual inverse image $f^* : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1)$. However, unlike the quasi-coherent setting, the functor $f^{\mathrm{IndCoh},*}$ is only defined for morphisms that are *eventually coconnective*, i.e., those for which the usual f^* sends $\mathrm{QCoh}(S_2)^+$ to $\mathrm{QCoh}(S_1)^+$.

The third functor is a specialty of IndCoh: this is the !-pullback functor $f^!$ for a proper map $f : S_1 \to S_2$, and which is defined as the right adjoint of f_*^{IndCoh}.

0.3.4. In Sect. 4 we discuss some of the basic properties of IndCoh, which essentially say that there are no unpleasant surprises vis-à-vis the familiar properties of QCoh. E.g., Zariski descent, localization with respect to an open embedding, etc., all hold like they do for QCoh.

A pleasant feature of IndCoh (which does not hold for QCoh) is the convergence property: for a DG scheme S, the category $\mathrm{IndCoh}(S)$ is the limit of the categories $\mathrm{IndCoh}(\tau^{\leq n}(S))$, where $\tau^{\leq n}(S)$ is the n-coconnective truncation of S.

0.4. **Interlude: ∞-categories.** A significant portion of the work in the present paper goes into enhancing the functoriality of IndCoh for an individual morphism f to a functor from the category of $\mathrm{DGSch}_{\mathrm{Noeth}}$ of Noetherian DG schemes to the category of DG categories.

We need to work with DG categories rather than with triangulated categories as the former allow for such operation as taking limits. The latter is necessary in order to extend IndCoh to stacks and to formulate descent.

Since we are working with DG schemes, $\mathrm{DGSch}_{\mathrm{Noeth}}$ form an $(\infty,1)$-category rather than an ordinary category. Thus, homotopy theory naturally enters the picture. For now we regard the category of DG categories, denoted DGCat, also as an $(\infty,1)$-category (rather than as a $(\infty,2)$-category).

0.4.1. Throughout the paper we formulate statements of ∞-categorical nature in a way which is independent of a particular model for the theory of ∞-categories (although the model that we have in mind is that of quasi-categories as treated in [Lu0]).

However, formulating things at the level of ∞-categories is not such an easy thing to do. Any of the models for ∞-categories is a very convenient computational/proof-generating machine. However, given a particular algebro-geometric problem, such as the assignment

$$S \mapsto \mathrm{IndCoh}(S),$$

it is rather difficult to feed it into this machine as an input.

E.g., in the model of quasi-categories, an ∞-category is a simplicial set, and a functor is a map of simplicial sets. It appears as a rather awkward endeavor to organize DG schemes and ind-coherent sheaves on them into a particular simplicial set... [3]

0.4.2. So, the question is: how, for example, do me make IndCoh into a functor

$$\mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat},$$

e.g., with respect to the push-forward operation, denoted f_*^{IndCoh}? The procedure is multi-step and can be described as follows:

Step 0: We start with the assignment

$$A \mapsto A\text{-mod} : (\text{DG-rings})^{\mathrm{op}} \to \mathrm{DGCat},$$

which is elementary enough that it can be carried out in any given model. We view it as a functor

$$\mathrm{QCoh}_{\mathrm{DGSch}^{\mathrm{aff}}} : \mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{DGCat},$$

[3]It should be remarked that when one works with a specific manageable diagram of DG categories arising from algebraic geometry (e.g., the category of D-modules on the loop group viewed as a monoidal category), it is sometimes possible to work in a specific model, and to carry out the constructions "at the chain level", as is done, e.g., in [FrenGa]. But in slightly more general situations, this approach does not have much promise.

with respect to f_*.

Step 1: Passing to left adjoints, we obtain a functor

$$\mathrm{QCoh}^*_{\mathrm{DGSch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat},$$

with respect to f^*.

Step 2: We apply the right Kan extension along the natural functor

$$\mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{PreStk}$$

(here PreStk is the $(\infty, 1)$-category category of derived ∞-prestacks, see [GL:Stacks], Sect. 1.1.1.) to obtain a functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}\,.$$

Step 3: We restrict along the natural functor DGSch $\to$ PreStk (see [GL:Stacks], Sect. 3.1) to obtain a functor

$$\mathrm{QCoh}^*_{\mathrm{DGSch}} : (\mathrm{DGSch})^{\mathrm{op}} \to \mathrm{DGCat},$$

with respect to f^*.

Step 4: Passing to right adjoints, we obtain a functor

$$\mathrm{QCoh}_{\mathrm{DGSch}} : \mathrm{DGSch} \to \mathrm{DGCat},$$

with respect to f_*.

Step 5: Restricting to $\mathrm{DGSch}_{\mathrm{Noeth}} \subset \mathrm{DGSch}$, and taking the eventually coconnective parts, we obtain a functor

$$\mathrm{QCoh}^+_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}\,.$$

Step 6: Finally, using Proposition 1.2.4, in Proposition 3.2.4 we use the latter functor do construct the desired functor

$$\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}\,.$$

0.4.3. In subsequent sections of the paper we will extend the functor $\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}}$, most ambitiously, to a functor out of a certain $(\infty, 1)$-category of correspondences between prestacks.

However, the procedure will always be of the same kind: we will never be able to do it "by hand" by saying where the objects and 1-morphisms go, and specifying associativity up to coherent homotopy. Rather, we will iterate many times the operations of restriction, and left and right Kan extension, and passage to the adjoint functor. [4]

[4] One observes that higher algebra, i.e., algebra done in ∞-categories, loses one of the key features of usual algebra, namely, of objects being rather concrete (such as a module over an algebra is a concrete set with concrete pieces of structure).

0.4.4. *A disclaimer.* In this paper we did not set it as our goal to give complete proofs of statements of ∞-categorical nature. Most of these statements have to do with *categories of correspondences*, introduced in Sect. 5.

The corresponding proofs are largely combinatorial in nature, and would increase the length of the paper by a large factor. The proofs of most of such statements will be supplied in the forthcoming book [GR3]. In this sense, the emphasis of the present paper is to make sure that homological algebra works (maps between objects in a given DG category are isomorphisms when they are expected to be such), while the emphasis of [GR3] is to show that the assignments of the form

$$\mathbf{i} \in \mathbf{I} \mapsto \mathbf{C}_i \in \infty\text{-Cat},$$

where $\mathbf{I}$ is some ∞-category, are indeed functors $\mathbf{I} \to \infty\text{-Cat}$.

That said, we do take care to single out every statement that requires a non-standard manipulation at the level of ∞-categories. I.e., we avoid assuming that higher homotopies can be automatically arranged in every given problem at hand.

0.5. **Contents: the rest of the paper.**

0.5.1. In Sect. 5 our goal is to define the set-up for the functor of !-pullback for arbitrary morphisms (i.e., not necessarily proper, but still of finite type), such that the base change formula (0.1) holds.

As was explained to us by J. Lurie, a proper formulation of the existence of !-pullback together with the base change property is encoded by enlarging the category DGSch: we leave the same objects, but 1-morphisms from S_1 to S_2 are now correspondences

$$\begin{array}{ccc} S_{1,2} & \xrightarrow{g} & S_1 \\ {\scriptstyle f}\downarrow & & \\ S_2. & & \end{array} \tag{0.2}$$

and compositions of morphisms are given by forming Cartesian products. I.e., when we want to compose a morphism $S_1 \to S_2$ as above with a morphism $S_2 \to S_3$ given by

$$\begin{array}{ccc} S_{2,3} & \xrightarrow{g'} & S_2 \\ {\scriptstyle f'}\downarrow & & \\ S_3, & & \end{array}$$

the composition is given by the diagram

$$\begin{array}{ccc} S_{1,2} \underset{S_2}{\times} S_{2,3} & \longrightarrow & S_1 \\ \downarrow & & \\ S_3. & & \end{array}$$

For a morphism (0.2), the corresponding functor $\text{IndCoh}(S_1) \to \text{IndCoh}(S_2)$ is given by

$$f_*^{\text{IndCoh}} \circ g^!.$$

Suppose now that we have managed to define IndCoh as a functor out of the category of correspondences. However, there are still multiple compatibilities that are supposed to hold that express how the isomorphisms (0.1) are compatible with the adjunction $(f_*^{\text{IndCoh}}, f^!)$ when f is a proper morphism.

Another idea of J. Lurie's is that this structure is most naturally encoded by further expanding our category of correspondences, by making it into a 2-category, where we allow 2-morphisms to be morphisms between correspondences that are not necessarily isomorphisms, but rather proper maps.

0.5.2. In Sect. 6 we indicate the main steps in the construction of IndCoh as a functor out of the category of correspondences. The procedure mimics the classical construction of the !-pullback functor, but is done in the ∞-categorical language. Full details of this construction will be supplied in [GR3].

0.5.3. In Sect. 7 we study the behavior of the !-pullback functor under eventually coconnective, Gorenstein and smooth morphisms.

0.5.4. The goal of Sect. 8 is to prove that the category IndCoh satisfies faithfully flat descent with respect to the !-pullback functor. The argument we give was explained to us by J. Lurie.

0.5.5. In Sect. 9 we discuss the self-duality property of the category $\operatorname{IndCoh}(S)$ for a DG scheme S almost of finite type over the ground field. The self-duality boils down to the classical Serre duality anti-equivalence of the category $\operatorname{Coh}(S)$, and is automatic from the formalism of IndCoh as a functor on the category of correspondences.

0.5.6. In Sect. 10 we take the theory as far as it goes when $\mathcal{Y}$ is an arbitrary prestack.

0.5.7. In Sect. 11 we specialize to the case of Artin stacks. The main feature of $\operatorname{IndCoh}(\mathcal{Y})$ for Artin stacks is that this category can be recovered from looking at just affine schemes equipped with a *smooth* map to $\mathcal{Y}$. This allows us to introduce a t-structure on $\operatorname{IndCoh}(\mathcal{Y})$, and establish a number of properties that make the case of Artin stacks close to that of schemes.

0.6. **Conventions, terminology and notation.**

0.6.1. *Ground field.* Throughout this paper we will be working with schemes and DG schemes defined over a ground field k of characteristic 0.

0.6.2. *∞-categories.* By an ∞-category we shall always mean an $(\infty, 1)$-category. By a slight abuse of language we will sometimes talk about "categories" when we actually mean ∞-categories. [5]

As was mentioned above, our usage of ∞-categories is not tied to any particular model, however, the basic reference for us is Lurie's book [Lu0].

For an ∞-category $\mathbf{C}$ and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we let

$$\operatorname{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$$

denote the corresponding ∞-groupoid of maps. We also denote

$$\operatorname{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) = \pi_0(\operatorname{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)).$$

For a functor $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$ between ∞-categories, and a functor $F : \mathbf{C}_1 \to \mathbf{D}$, where $\mathbf{D}$ is another ∞-category that contains limits (resp., colimits), we let

$$\operatorname{RKE}_{\Phi}(F) : \mathbf{C}_2 \to \mathbf{D} \text{ and } \operatorname{LKE}_{\Phi}(F) : \mathbf{C}_2 \to \mathbf{D}$$

the right (resp., left) Kan extension of F along Φ.

[5] Throughout the paper we will ignore set-theoretic issues. The act of ignoring can be replaced by assuming that the ∞-categories and functors involved are κ-accessible for some cardinal κ in the sense of [Lu0], Sect. 5.4.2.

0.6.3. *Subcategories.* Let $\mathbf{C}$ and $\mathbf{C}'$ be ∞-categories, and $\phi : \mathbf{C}' \to \mathbf{C}$ be a functor.

We shall say that ϕ is *0-fully faithful*, or just *fully faithful* if for any $\mathbf{c}'_1, \mathbf{c}'_2 \in \mathbf{C}'$, the map

$$\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}'_1, \mathbf{c}'_2) \to \mathrm{Maps}_{\mathbf{C}}(\phi(\mathbf{c}'_1), \phi(\mathbf{c}'_2)) \tag{0.3}$$

is an isomorphism (=homotopy equivalence) of ∞-groupoids. In this case we shall say that ϕ makes $\mathbf{C}'$ into a *0-full* (or just *full*) subcategory of $\mathbf{C}'$.

Below are two weaker notions:

We shall say that ϕ is *1-fully faithful*, or just *faithful*, if for any $\mathbf{c}'_1, \mathbf{c}'_2 \in \mathbf{C}'$, the map (0.3) is a fully faithful map of ∞-groupoids. Equivalently, the map (0.3) induces an injection on π_0 and a bijection on the homotopy groups π_i, $i \geq 1$ on each connected component of the space $\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}'_1, \mathbf{c}'_2)$.

I.e., 2- and higher morphisms between 1-morphisms in $\mathbf{C}'$ are the same in $\mathbf{C}'$ and $\mathbf{C}$, up to homotopy.

We shall say that ϕ is *faithful and groupoid-full* if it is faithful, and for any $\mathbf{c}'_1, \mathbf{c}'_2 \in \mathbf{C}'$, the map (0.3) is surjective on those connected components of $\mathrm{Maps}_{\mathbf{C}}(\phi(\mathbf{c}'_1), \phi(\mathbf{c}'_2))$ that correspond to isomorphisms. In this case we shall say that ϕ is an equivalence onto a *1-full* subactegory of $\mathbf{C}$.

0.6.4. *DG categories.* Our conventions on DG categories follow those of [GL:DG]. For the purposes of this paper one can replace DG categories by (the equivalent) $(\infty, 1)$-category of stable ∞-categories tensored over Vect, where the latter is the DG category of complexes of k-vector spaces.

By $\mathrm{DGCat}_{\text{non-cocomplete}}$ we shall denote the category of all DG-categories.

By DGCat we will denote the full subcategory of $\mathrm{DGCat}_{\text{non-cocomplete}}$ that consists of cocomplete DG categories (i.e., DG categories closed under direct sums, which is equivalent to being closed under all colimits).

By $\mathrm{DGCat}_{\mathrm{cont}}$ we shall denote the 1-full subcategory of DGCat where we restrict 1-morphisms to be continuous (i.e., commute with directs sums, or equivalently with all colimits).

Let $\mathbf{C}$ be a DG category and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$ two objects. We shall denote by

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$$

the corresponding object of Vect. The ∞-groupoid $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ is obtained from the truncation $\tau^{\leq 0}(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$ by the Dold-Kan functor

$$\mathrm{Vect}^{\leq 0} \to \infty\text{-Grpd}\,.$$

0.6.5. *t-structures.* Given $\mathbf{C} \in \mathrm{DGCat}_{\text{non-cocomplete}}$ equipped with a t-structure, we shall denote by $\mathbf{C}^+$ the corresponding full subcategory of $\mathbf{C}$ that consists of *eventually coconnective* objects, i.e.,

$$\mathbf{C}^+ = \underset{n}{\cup} \mathbf{C}^{\geq -n}.$$

Similarly, we denote

$$\mathbf{C}^- = \underset{n}{\cup} \mathbf{C}^{\leq n}, \quad \mathbf{C}^b = \mathbf{C}^+ \cap \mathbf{C}^-, \quad \mathbf{C}^\heartsuit = \mathbf{C}^{\leq 0} \cap \mathbf{C}^{\geq 0}.$$

0.6.6. *(Pre)stacks and DG schemes.* Our conventions regarding (pre)stacks and DG schemes follow [GL:Stacks].

We we shall abuse the terminology slightly and use the expression "classical scheme" for a DG scheme which is 0-coconnective, see [GL:Stacks], Sect. 3.2.1. For $S \in \mathrm{Sch}$, we shall use the same symbol S to denote the corresponding derived scheme. Correspondingly, for a derived scheme S we shall use the notation ${}^{cl}S$, rather than $\tau^{cl}(S)$, to denote its 0-coconnective truncation.

0.6.7. *Quasi-coherent sheaves.* Conventions regarding the category of quasi-coherent sheaves on (pre)stacks follow those of [GL:QCoh].

In particular, we shall denote by $\mathrm{QCoh}^*_{\mathrm{DGSch}^{\mathrm{aff}}}$ the corresponding functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

by $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ its right Kan extension along the tautological functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}},$$

and by $\mathrm{QCoh}^*_{\mathbf{C}}$ the restriction of the latter to various subcategories $\mathbf{C}$ of PreStk, such as $\mathbf{C} = \mathrm{DGSch}, \mathrm{Stk}, \mathrm{Stk}_{\mathrm{Artin}}$, etc.

The superscript "$*$" stands for the fact that for a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the corresponding functor $\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$ is f^*.

We remark (see [GL:QCoh, Corollary 1.3.12]) that for a classical scheme S, the category $\mathrm{QCoh}(S)$ is the same whether we undertand S as a classical or DG scheme.

0.6.8. *Noetherian DG schemes.* We shall say that an affine DG scheme $S = \mathrm{Spec}(A)$ is Noetherian if

- $H^0(A)$ is a Noetherian ring.
- Each $H^i(A)$ is a finitely generated as a module over $H^0(A)$.

We shall say that a DG scheme is locally Noetherian if it admits a Zariski cover by Noetherian affine DG schemes. It is easy to see that this is equivalent to requiring that any open affine subscheme is Noetherian. We shall say that a DG scheme is Noetherian if it is locally Noetherian and quasi-compact; we shall denote the full subcategory of DGSch spanned by Noetherian DG schemes by $\mathrm{DGSch}_{\mathrm{Noeth}}$.

0.6.9. *DG schemes almost of finite type.* Replacing the condition on $H^0(A)$ of being Noetherian by that of being of finite type over k, we obtain the categories of DG schemes *locally almost of finite type* and *almost of finite type* over k, denoted $\mathrm{DGSch}_{\mathrm{laft}}$ and $\mathrm{DGSch}_{\mathrm{aft}}$, respectively.

0.7. **Acknowledgements.** The author's debt to J. Lurie is obvious. In particular, it should be emphasized that the following two crucial points in this paper: the idea to use the category of correspondences as the source of the IndCoh functor, and the assertion that IndCoh satisfies faithfully flat descent, are both due to him.

The author would like to thank V. Drinfeld and N. Rozenblyum for their collaboration on the project of writing down the foundations of ind-coherent sheaves for the purposes of Geometric Langlands, and for sharing their ideas related to the subject of this paper.

The author would also like to thank D. Arinkin, J. Barlev and S. Raskin for many helpful discussions.

Finally, the author would like to emphasize that part of the material in this paper has been independently developed by T. Preygel.

The author was supported by NSF grant DMS-1063470.

Part I. Elementary properties.

1. Ind-coherent sheaves

Let S be a DG scheme. In this section we will be assuming that S is Noetherian, see Sect. 0.6.8, i.e., quasi-compact and locally Noetherian.

1.1. **The set-up.**

1.1.1. Consider the category $\mathrm{QCoh}(S)$. It carries a natural t-structure, whose heart $\mathrm{QCoh}(S)^\heartsuit$ is canonically isomorphic to $\mathrm{QCoh}({}^{cl}S)^\heartsuit$, where ${}^{cl}S$ is the underlying classical scheme.

Definition 1.1.2. *Let* $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$ *be the full subcategory that consists of objects of bounded cohomological amplitude and coherent cohomologies.*

The assumption that ${}^{cl}S$ is Noetherian, implies that the subcategory $\mathrm{Coh}(S)$ is stable under the operation of taking cones, so it is a DG subcategory of $\mathrm{QCoh}(S)$.

However, $\mathrm{Coh}(S)$ is, of course, *not* cocomplete.

Definition 1.1.3. *The DG category* $\mathrm{IndCoh}(S)$ *is defined as the ind-completion of* $\mathrm{Coh}(S)$.

Remark 1.1.4. When S is a classical scheme, the category $\mathrm{IndCoh}(S)$ was first introduced by Krause in [Kr]. Assertion (2) of Theorem 1.1 of *loc.cit.* can be stated as $\mathrm{IndCoh}(S)$ being equivalent to the DG category of *injective complexes* on S. Many of the results of this section are simple generalizations of Krause's results to the case of a DG scheme.

1.1.5. By construction, we have a canonical 1-morphism in $\mathrm{DGCat}_{\mathrm{cont}}$:

$$\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S),$$

obtained by ind-extending the tautological embedding $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$.

Lemma 1.1.6. *Assume that S is a regular classical scheme. Then Ψ_S is an equivalence.*

Remark 1.1.7. As we shall see in Sect. 1.6, the converse is also true.

Proof. Recall that by [BFN], Prop. 3.19, extending the arguments of [Ne], the category $\mathrm{QCoh}(S)$ is compactly generated by its subcategory $\mathrm{QCoh}(S)^c = \mathrm{QCoh}(S)^{\mathrm{perf}}$ consisting of perfect complexes.

Suppose that S is a regular classical scheme. Then the subcategories

$$\mathrm{Coh}(S) \subset \mathrm{QCoh}(S) \supset \mathrm{QCoh}(S)^{\mathrm{perf}}$$

coincide, and the assertion is manifest.

□

1.2. **The t-structure.**

1.2.1. The category $\mathrm{Coh}(S)$ carries a natural t-structure. Hence, $\mathrm{IndCoh}(S)$ acquires a canonical t-structure, characterized by the properties that

(1) It is compatible with filtered colimits (i.e., the truncation functors commute with filtered colimits), and
(2) The tautological embedding $\mathrm{Coh}(S) \to \mathrm{IndCoh}(S)$ is t-exact.

We have:

Lemma 1.2.2. *The functor Ψ_S is t-exact.*

Proof. Follows from the fact that the t-structure on $\mathrm{QCoh}(S)$ is compatible with filtered colimits. □

1.2.3. The next proposition, although simple, is crucial for the rest of the paper. It says that the eventually coconnective subcategories of $\mathrm{IndCoh}(S)$ and $\mathrm{QCoh}(S)$ are equivalent:

Proposition 1.2.4. *For every n, the induced functor*

$$\Psi_S : \mathrm{IndCoh}(S)^{\geq n} \to \mathrm{QCoh}(S)^{\geq n}$$

is an equivalence.

Proof. With no restricion of generality we can assume that $n = 0$. We first prove fully-faithfulness. We will show that the map

$$\mathrm{Maps}_{\mathrm{IndCoh}(S)}(\mathcal{F}, \mathcal{F}') \to \mathrm{Maps}_{\mathrm{QCoh}(S)}(\Psi_S(\mathcal{F}), \Psi_S(\mathcal{F}')) \tag{1.1}$$

is an isomorphism for $\mathcal{F}' \in \mathrm{IndCoh}(S)^{\geq 0}$ and *any* $\mathcal{F} \in \mathrm{IndCoh}(S)$.

By definition, it suffices to consider the case $\mathcal{F} \in \mathrm{Coh}(S)$.

Every $\mathcal{F}' \in \mathrm{IndCoh}(S)^{\geq 0}$ can be written as a filtered colimit

$$\underset{i}{colim}\, \mathcal{F}'_i,$$

where $\mathcal{F}'_i \in \mathrm{Coh}(S)^{\geq 0}$.

By the definition of $\mathrm{IndCoh}(S)$, the left-hand side of (1.1) is the colimit of

$$\mathrm{Maps}_{\mathrm{IndCoh}(S)}(\mathcal{F}, \mathcal{F}'_i),$$

where each term, again by definition, is isomorphic to

$$\mathrm{Maps}_{\mathrm{Coh}(S)}(\mathcal{F}, \mathcal{F}'_i) \simeq \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{F}'_i).$$

So, it suffices to show that for $\mathcal{F} \in \mathrm{Coh}(S)$, the functor $\mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, -)$ commutes with filtered colimits *taken* within $\mathrm{QCoh}(S)^{\geq 0}$.

The assumptions on S imply that for any $\mathcal{F} \in \mathrm{Coh}(S)$ there exists an object $\mathcal{F}_0 \in \mathrm{QCoh}(S)^c$ (i.e., $\mathcal{F}_0$ is perfect, see [GL:QCoh], Sect. 4.1) together with a map $\mathcal{F}_0 \to \mathcal{F}$ such that

$$\mathrm{Cone}(\mathcal{F}_0 \to \mathcal{F}) \in \mathrm{QCoh}(S)^{<0}.$$

The functor $\mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}_0, -)$ does commute with filtered colimits since $\mathcal{F}_0$ is compact, and the induced map

$$\mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}_0, \mathcal{F}') \to \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{F}')$$

is an isomorphism for any $\mathcal{F}' \in \mathrm{QCoh}(S)^{\geq 0}$. This implies the needed result.

It remains to show that $\Psi_S : \text{IndCoh}(S)^{\geq 0} \to \text{QCoh}(S)^{\geq 0}$ is essentially surjective. However, this follows from the fact that any object $\mathcal{F} \in \text{QCoh}(S)^{\geq 0}$ can be written as a filtered colimit $\underset{i}{colim}\, \mathcal{F}_i$ with $\mathcal{F}_i \in \text{Coh}(S)^{\geq 0}$.

□

Corollary 1.2.5. *An object of* IndCoh(S) *is connective if and only if its image in* QCoh(S) *under* Ψ_S *is.*

The proof is immediate from Proposition 1.2.4.

Corollary 1.2.6. *The subcategory* $\text{Coh}(S) \subset \text{IndCoh}(S)$ *equals* $\text{IndCoh}(S)^c$*, i.e., is the category of* all *compact objects of* IndCoh(S)*.*

Proof. A priori, every compact object of IndCoh(S) is a direct summand of an object of Coh(S). Hence, all compact objects of IndCoh(S) are eventually coconnective, i.e., belong to $\text{IndCoh}(S)^{\geq -n}$ for some n. Now, the assertion follows from Proposition 1.2.4, as a direct summand of every coherent object in QCoh(S) is itself coherent.

□

1.2.7. Let $\text{IndCoh}(S)_{nil} \subset \text{IndCoh}(S)$ be the full subcategory of infinitely connective, i.e., t-nil objects:

$$\text{IndCoh}(S)_{nil} := \underset{n\in\mathbb{N}}{\cap} \text{IndCoh}(S)^{\leq -n}.$$

Since the t-structure on QCoh(S) is separated (i.e., if an object has zero cohomologies with respect to the t-structure, then it is zero), we have:

$$\text{IndCoh}(S)_{nil} = \ker(\Psi_S).$$

At the level of homotopy categories, we obtain that Ψ_S factors as

$$\text{Ho}(\text{IndCoh}(S))/\,\text{Ho}(\text{IndCoh}(S)_{nil}) \to \text{Ho}(\text{QCoh}(S)). \tag{1.2}$$

However, in general, the functor in (1.2) is not an equivalence.

1.3. QCoh **as the left completion of** IndCoh**.**

1.3.1. Recall that a DG category $\mathbf{C}$ equipped with a t-structure is said to be *left-complete* in its t-structure if the canonical functor

$$\mathbf{C} \to \underset{n\in\mathbb{N}^{\text{op}}}{lim}\, \mathbf{C}^{\geq -n}, \tag{1.3}$$

is an equivalence, where for $n_1 \geq n_2$, the functor

$$\mathbf{C}^{\geq -n_1} \to \mathbf{C}^{\geq -n_2}$$

is $\tau^{\geq -n_2}$. The functor in (1.3) is given by the family

$$n \mapsto \tau^{\geq -n}.$$

1.3.2. Any DG category equipped with a t-structure admits a left completion. This is a DG category $\mathbf{C}'$ equipped with a t-structure in which it is left-complete, such that it receives a t-exact functor $\mathbf{C} \to \mathbf{C}'$, and which is universal with respect to these properties.

Explicitly, the left completion of $\mathbf{C}$ is given by the limit $\underset{n\in\mathbb{N}^{\text{op}}}{lim}\, \mathbf{C}^{\geq -n}$.

1.3.3. We now claim:

Proposition 1.3.4. *For $S \in \mathrm{DGSch}_{\mathrm{Noeth}}$, the functor $\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$ identifies $\mathrm{QCoh}(S)$ with the left completion of $\mathrm{IndCoh}(S)$ in its t-structure.*

Proof. First, it easy to see that the category $\mathrm{QCoh}(S)$ is left-complete in its t-structure. (Proof: the assertion reduces to the case when S is affine. In the latter case, the category $\mathrm{QCoh}(S)$ admits a conservative limit-preserving t-exact functor to a left-complete category, namely, $\Gamma(S,-) : \mathrm{QCoh}(S) \to \mathrm{Vect}$.)

Now, the assertion of the proposition follows from Proposition 1.2.4.

□

1.4. **The action of $\mathrm{QCoh}(S)$ on $\mathrm{IndCoh}(S)$.** The category $\mathrm{QCoh}(S)$ has a natural (symmetric) monoidal structure. We claim that $\mathrm{IndCoh}(S)$ is naturally a module over $\mathrm{QCoh}(S)$.

To define an action of $\mathrm{QCoh}(S)$ on $\mathrm{IndCoh}(S)$ it suffices to define the action of the (non-cocomplete) monoidal DG category $\mathrm{QCoh}(S)^{\mathrm{perf}}$ on the (non-cocomplete) DG category $\mathrm{Coh}(S)$.

For the latter, it suffices to notice that the action of $\mathrm{QCoh}(S)^{\mathrm{perf}}$ on $\mathrm{QCoh}(S)$ preserves the non-cocomplete subcategory $\mathrm{Coh}(S)$.

By construction, the action functor

$$\mathrm{QCoh}(S) \otimes \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S)$$

sends compact objects to compact ones.

We will use the notation

$$\mathcal{E} \in \mathrm{QCoh}(S), \mathcal{F} \in \mathrm{IndCoh}(S) \mapsto \mathcal{E} \otimes \mathcal{F} \in \mathrm{IndCoh}(S).$$

1.4.1. From the construction, we obtain:

Lemma 1.4.2. *The functor Ψ_S has a natural structure of 1-morphism between $\mathrm{QCoh}(S)$-module categories.*

At the level of individual objects, the assertion of Lemma 1.4.2 says that for $\mathcal{E} \in \mathrm{QCoh}(S)$ and $\mathcal{F} \in \mathrm{IndCoh}(S)$, we have a canonical isomorphism

$$\Psi_S(\mathcal{E} \otimes \mathcal{F}) \simeq \mathcal{E} \otimes \Psi_S(\mathcal{F}). \tag{1.4}$$

1.5. **Eventually coconnective case.** Assume now that S is eventually coconnective, see [GL:Stacks], Sect. 3.2.6 where the terminology is introduced.

We remind that by definition, this means that S is covered by affines $\mathrm{Spec}(A)$ with $H^{-i}(A) = 0$ for all i large enough. Equivalently, S is eventually coconnective if and only if $\mathcal{O}_S$ belongs to $\mathrm{Coh}(S)$.

For example, any classical scheme is 0-coconnective, and hence eventually coconnective when viewed as a DG scheme.

Remark 1.5.1. In Sect. 4.3 we will show that in a certain sense the study of IndCoh reduces to the eventually coconnective case.

1.5.2. Under the above circumstances we claim:

Proposition 1.5.3. *The functor Ψ_S admits a left adjoint Ξ_S. Moreover, Ξ_S is fully faithful, i.e., the functor Ψ realizes* $\mathrm{QCoh}(S)$ *is a co-localization of* $\mathrm{IndCoh}(S)$ *with respect to* $\mathrm{IndCoh}(S)_{nil}$.

Shortly, we shall see that a left adjoint to Ψ exists *if and only if* S is eventually coconnective.

Proof. To prove the proposition, we have to show that the left adjoint Ξ_S is well-defined and fully faithful on $\mathrm{QCoh}(S)^{\mathrm{perf}} = \mathrm{QCoh}(S)^c$.

However, this is clear: for S eventually coconnective, we have a natural fully faithful inclusion

$$\mathrm{QCoh}(S)^{\mathrm{perf}} \hookrightarrow \mathrm{Coh}(S).$$

□

1.5.4. By adjunction, from Lemma 1.4.2, and using [GL:DG, Corollary 6.2.4], we obtain:

Corollary 1.5.5. *The functor Ξ_S has a natural structure of* 1*-morphism between* $\mathrm{QCoh}(S)$*-module categories.*

At the level of individual objects, the assertion of Corollary 1.5.5 says that for $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{QCoh}(S)$, we have a canonical isomorphism

$$\Xi_S(\mathcal{E}_1 \otimes \mathcal{E}_2) \simeq \mathcal{E}_1 \otimes \Xi_S(\mathcal{E}_2), \tag{1.5}$$

where in the right-hand side $\otimes$ denotes the action of Sect. 1.4.

In particular, for $\mathcal{E} \in \mathrm{QCoh}(S)$, we have

$$\Xi_S(\mathcal{E}) \simeq \mathcal{E} \otimes \Xi_S(\mathcal{O}_S).$$

1.5.6. We note the following consequence of Proposition 1.5.3:

Corollary 1.5.7. *If S is eventually coconnective, the functor of triangulated categories* (1.2) *is an equivalence.*

The following observation is useful:

Lemma 1.5.8. *Let $\mathcal{F}$ be an object of* $\mathrm{QCoh}(S)$ *such that* $\Xi(S) \in \mathrm{Coh}(S) \subset \mathrm{IndCoh}(S)$. *Then* $\mathcal{F} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$.

Proof. Since Ξ_S is a fully faithful functor that commutes with filtered colimits, an object of $\mathrm{QCoh}(S)$ is compact if its image under Ξ_S is compact. □

1.6. Some converse implications.

1.6.1. We are now going to prove:

Proposition 1.6.2. *Assume that the functor* $\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$ *admits a left adjoint. Then S is eventually coconnective.*

Proof. Let $\Xi_S : \mathrm{QCoh}(S) \to \mathrm{IndCoh}(S)$ denote the left adjoint in question. Since Ψ_S commutes with colimits, the functor Ξ_S sends compact objects to compact ones. In particular, by Corollary 1.2.6, $\Xi_S(\mathcal{O}_S)$ belongs to $\mathrm{Coh}(S)$, and in particular, to $\mathrm{IndCoh}(S)^{\geq -n}$ for some n. Hence, by Proposition 1.2.4, the map $\mathcal{O}_S \to \Psi_S(\Xi_S(\mathcal{O}_S))$ induces an isomorphism on Homs to any eventually coconnective object of $\mathrm{QCoh}(S)$. Since the t-structure on $\mathrm{QCoh}(S)$ is separated, we obtain that $\mathcal{O}_S \to \Psi_S(\Xi_S(\mathcal{O}_S))$ is an isomorphism. In particular, $\mathcal{O}_S \in \mathrm{Coh}(S)$.

□

1.6.3. Let us now prove the converse to Lemma 1.1.6:

Proposition 1.6.4. *Let S be a DG scheme such that Ψ_S is an equivalence. Then S is a regular classical scheme.*

Proof. By Proposition 1.6.2 we obtain that S is eventually coconnective. Since Ψ_S is an equivalence, it induces an equivalence between the corresponding categories of compact objects. Hence, we obtain that

$$\text{Coh}(S) = \text{QCoh}(S)^{\text{perf}}. \tag{1.6}$$

as subcategories of $\text{QCoh}(S)$.

The question of being classical and regular, and the equality (1.6), are local. So, we can assume that S is affine, $S = \text{Spec}(A)$.

The inclusion $\subset$ in (1.6) implies that $H^0(A)$ admits a finite resolution by projective A-modules. However, it is easy to see that this is only possible when all $H^i(A)$ with $i \neq 0$ vanish. Hence A is classical.

In the latter case, the inclusion $\subset$ in (1.6) means that every A module is of finite projective dimension. Serre's theorem implies that A is regular.

□

2. IndCoh in the non-Noetherian case

This section will not be used in the rest of the paper. We will indicate the definition of the category $\text{IndCoh}(S)$ in the case when S is not necessarily locally Noetherian. We shall first treat the case when $S = \text{Spec}(A)$ is affine, and then indicate how to treat the case of a general scheme if some additional condition is satisfied.

2.1. **The coherent case.** Following J. Lurie, give the following definitions:

Definition 2.1.1. *A classical ring A_0 is said to be* coherent *if the category of finitely presented A_0 modules is abelian, i.e., is stable under taking kernels and cokernels.*

Definition 2.1.2. *A DG ring A is said to be coherent if:*

1. *The classical ring $H^0(A)$ is coherent.*
2. *Each $H^i(A)$ is finitely presented as a $H^0(A)$-module.*

2.1.3. We note that the entire discussion in the preceeding section goes through when the assumption that A be Noetherian is replaced by that of it being coherent.

In what follows, we shall show how to define $\text{IndCoh}(S)$ without the coherence assumption either.

2.2. **Coherent sheaves in the non-Noetherian setting.**

2.2.1. Informally, we say that an object $\text{QCoh}(S)$ is coherent if it is cohomologically bounded and can be approximated by a perfect object to any level of its Postnikov tower.

A formal definition is as follows:

Definition 2.2.2. *A cohomologically bounded object $\mathcal{F} \in \text{QCoh}(S)$ is said to be coherent if for any n there exists an object $\mathcal{F}_n \in \text{QCoh}(S)^{\text{perf}}$ together with a morphism $\mathcal{F}_n \to \mathcal{F}$ such that* $\text{Cone}(\mathcal{F}_n \to \mathcal{F}) \in \text{QCoh}(S)^{\leq -n}$.

We have:

Lemma 2.2.3. *For a cohomologically bounded object $\mathcal{F} \in \mathrm{QCoh}(S)$ the following conditions are equivalent:*

(1) *$\mathcal{F}$ is coherent.*
(2) *For any n, the functor on $\mathrm{QCoh}(S)^{\heartsuit}$ given by $\mathcal{F}' \mapsto \mathrm{Hom}(\mathcal{F}, \mathcal{F}'[n])$ commutes with filtered colimits.*

2.2.4. Let $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$ denote the full subcategory spanned by coherent objects. It follows from Lemma 2.2.3 that $\mathrm{Coh}(S)$ is stable under taking cones, so it is a (non-cocomplete) DG subcategory of $\mathrm{QCoh}(S)$.

Definition 2.2.5. *We define the DG category $\mathrm{IndCoh}(S)$ to be the ind-completion of $\mathrm{Coh}(S)$.*

By construction, we have a tautological functor $\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$.

2.2.6. We shall now define a t-structure on $\mathrm{IndCoh}(S)$. We let $\mathrm{IndCoh}(S)^{\leq 0}$ be generated under colimits by objects from $\mathrm{Coh}(S) \cap \mathrm{QCoh}(S)^{\leq 0}$. Since all these objects are compact in $\mathrm{IndCoh}(S)$, the resulting t-structure on $\mathrm{IndCoh}(S)$ is compatible with filtered colimits.

Remark 2.2.7. The main difference between the present situation and one when S is coherent is that now the subcategory $\mathrm{Coh}(S) \subset \mathrm{IndCoh}(S)$ is not necessarily preserved by the truncation functors. In fact, it is easy to show that the latter condition is equivalent to S being coherent.

By construction, the functor Ψ_S is right t-exact.

2.3. **Eventual coherence.**

2.3.1. Note that unless some extra conditions on S are imposed, it is not clear that the category $\mathrm{Coh}(S)$ contains any objects besides 0. Hence, in general, we cannot expect that an analog of Proposition 1.2.4 should hold. We are now going to introduce a condition on S which would guarantee that an appropriate analog of Proposition 1.2.4 does hold:

Definition 2.3.2. *We shall say that S is eventually coherent if there exists an integer N, such that for all $n \geq N$ the truncation $\tau^{\geq -n}(\mathcal{O}_S)$ is coherent.*

We are going to prove:

Proposition 2.3.3. *Assume that S is eventually coherent. Then:*

(a) *The functor Ψ_S is t-exact.*
(b) *For any m, the resulting functor $\Psi_S : \mathrm{IndCoh}(S)^{\geq m} \to \mathrm{QCoh}(S)^{\geq m}$ is an equivalence.*

Proof. Let $\mathcal{F} \simeq \underset{i, \mathrm{IndCoh}(S)}{colim}\, \mathcal{F}_i$ be an object of $\mathrm{IndCoh}(S)^{>0}$, where $\mathcal{F}_i \in \mathrm{Coh}(S)$. To prove that Ψ_S is left t-exact, we need to show that $\underset{i, \mathrm{QCoh}(S)}{colim}\, \mathcal{F}_i$ belongs to $\mathrm{QCoh}(S)^{>0}$. Since S is affine, the latter condition is equivalent to

$$\mathrm{Maps}(\mathcal{O}_S, \underset{i, \mathrm{QCoh}(S)}{colim}\, \mathcal{F}_i) = 0,$$

and since $\mathcal{O}_S$ is a compact object of $\mathrm{QCoh}(S)$, the left-hand side of the latter expression is

$$\underset{i}{colim}\, \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{O}_S, \mathcal{F}_i). \tag{2.1}$$

Since each $\mathcal{F}_i$ is cohomologically bounded, the map

$$\mathrm{Maps}_{\mathrm{QCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i) \to \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{O}_S, \mathcal{F}_i)$$

is an isomorphism for $n \gg 0$. Hence, the map

$$\underset{n}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i) \to \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{O}_S, \mathcal{F}_i)$$

is an isomorphism. Therefore,

$$\underset{i}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{O}_S, \mathcal{F}_i) \simeq \underset{i}{colim}\, \underset{n}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i) \simeq \simeq \underset{n}{colim}\, \underset{i}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i). \tag{2.2}$$

However, by assumption, for $n \geq N$, $\tau^{\geq -n}(\mathcal{O}_S) \in \mathrm{Coh}(S)$, and for such n

$$\mathrm{Maps}_{\mathrm{QCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i),$$

and hence

$$\underset{i}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i) \simeq \underset{i}{colim}\ \mathrm{Maps}_{\mathrm{IndCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}_i) \simeq \simeq \mathrm{Maps}_{\mathrm{IndCoh}(S)}(\tau^{\geq -n}(\mathcal{O}_S), \mathcal{F}),$$

which vanishes by the definition of the t-structure on $\mathrm{IndCoh}(S)$. Hence, the expression in (2.2) vanishes, as required. This proves point (a).

To prove point (b), it is enough to consider the case $m = 0$. We first prove fully-faithfulness. As in the proof of Proposition 1.2.4, it is sufficient to show that for $\mathcal{F} \in \mathrm{Coh}(S)$ and $\mathcal{F}' \in \mathrm{IndCoh}(S)^{\geq 0}$, the map

$$\mathrm{Maps}_{\mathrm{IndCoh}(S)}(\mathcal{F}, \mathcal{F}') \to \mathrm{Maps}_{\mathrm{QCoh}(S)}(\Psi_S(\mathcal{F}), \Psi_S(\mathcal{F}'))$$

is an isomorphism.

Let $i \mapsto \mathcal{F}'_i$ be a diagram in $\mathrm{Coh}(S)$, such that $\mathcal{F}' := \underset{i,\mathrm{IndCoh}(S)}{colim} \mathcal{F}'_i$ belongs to $\mathrm{IndCoh}(S)^{\geq 0}$. We need to show that the map

$$\underset{i}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{F}'_i) \to \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, \underset{i,\mathrm{QCoh}(S)}{colim} \mathcal{F}'_i) \tag{2.3}$$

is an isomorphism.

Let $\mathcal{F}_0 \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ be an object equipped with a map $\mathcal{F}_0 \to \mathcal{F}$ such that

$$\mathrm{Cone}(\mathcal{F}_0 \to \mathcal{F}) \in \mathrm{QCoh}(S)^{<0}.$$

Since $\mathcal{F}$ is cohomologically bounded, for $n \gg 0$, the above morphism factors through a morphism

$$\tau^{\geq -n}(\mathcal{O}_S) \otimes \mathcal{F}_0 =: \mathcal{F}_0^n \to \mathcal{F};$$

moreover, by the eventually coherent assumption on S, we can take n to be large enough so that $\mathcal{F}_0^n \in \mathrm{Coh}(S)$.

Consider the commutative diagram:

$$\begin{array}{ccc}
\underset{i}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}_0, \mathcal{F}'_i) & \longrightarrow & \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}_0, \underset{i,\mathrm{QCoh}(S)}{colim} \mathcal{F}'_i) \\
\uparrow & & \uparrow \\
\underset{i,n}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}_0^n, \mathcal{F}'_i) & \longrightarrow & \underset{n}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}_0^n, \underset{i,\mathrm{QCoh}(S)}{colim} \mathcal{F}'_i) \\
\uparrow & & \uparrow \\
\underset{i}{colim}\ \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{F}'_i) & \longrightarrow & \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{F}, \underset{i,\mathrm{QCoh}(S)}{colim} \mathcal{F}'_i).
\end{array} \tag{2.4}$$

We need to show that in the bottom horizontal arrow in this diagram is an isomorphism. We will do so by showing that all other arrows are isomorphisms.

For all $n \gg 0$ we have a commutative diagram

$$\begin{array}{ccc} \underset{i}{colim}\ \operatorname{Maps}_{\operatorname{QCoh}(S)}(\mathcal{F}_0^n, \mathcal{F}'_i) & \xrightarrow{\sim} & \operatorname{Maps}_{\operatorname{IndCoh}(S)}(\mathcal{F}_0^n, \mathcal{F}') \\ \uparrow & & \uparrow \\ \underset{i}{colim}\ \operatorname{Maps}_{\operatorname{QCoh}(S)}(\mathcal{F}, \mathcal{F}'_i) & \xrightarrow{\sim} & \operatorname{Maps}_{\operatorname{IndCoh}(S)}(\mathcal{F}, \mathcal{F}'), \end{array}$$

in which the right vertical arrow is an isomorphism since $\operatorname{Cone}(\mathcal{F}_0^n \to \mathcal{F}) \in \operatorname{IndCoh}(S)^{<0}$ and $\mathcal{F}' \in \operatorname{IndCoh}(S)^{\geq 0}$. Hence, the left vertical arrow is an isomorphism as well. This implies that the lower left vertical arrow in (2.4) is an isomorphism.

The upper left vertical arrow in (2.4) is an isomorphism by the same argument as in the proof of point (a) above. The top horizontal arrow is an isomorphism since $\mathcal{F}_0$ is compact in $\operatorname{QCoh}(S)$. Finally, both right vertical arrows are isomorphisms since $\underset{i,\operatorname{QCoh}(S)}{colim} \mathcal{F}'_i \in \operatorname{QCoh}(S)^{\geq 0}$, as was established in point (a).

Finally, let us show that the functor

$$\Psi_S : \operatorname{IndCoh}(S)^{\geq 0} \to \operatorname{QCoh}(S)^{\geq 0}$$

is essentially surjective. By fully faithfulness, it suffices to show that the essential image of $\Psi_S(\operatorname{IndCoh}(S)^{\geq 0})$ generates $\operatorname{QCoh}(S)^{\geq 0}$ under filtered colimits.

Since $\operatorname{QCoh}(S)$ is generated under filtered colimits by objects of the form $\mathcal{F}_0 \in \operatorname{QCoh}(S)^{\operatorname{perf}}$, the category $\operatorname{QCoh}(S)^{\geq 0}$ is generated under filtered colimits by objects of the form $\tau^{\geq 0}(\mathcal{F}_0)$ for $\mathcal{F}_0 \in \operatorname{QCoh}(S)^{\operatorname{perf}}$. Hence, it suffices to show that such objects are in the essential image of Ψ_S. However,

$$\tau^{\geq 0}(\mathcal{F}_0) \simeq \tau^{\geq 0}(\Psi_S(\mathcal{F}_0^n)) \simeq \Psi_S(\tau^{\geq 0}(\mathcal{F}_0^n))$$

for $n \gg 0$ and $\mathcal{F}_0^n = \tau^{\geq -n}(\mathcal{O}_S) \otimes \mathcal{F}_0 \in \operatorname{Coh}(S)$.

□

2.3.4. *The eventually coconnective case.* Assume now that S is eventually coconnective; in particular, it is automatically eventually coherent.

In this case $\operatorname{QCoh}(S)^{\operatorname{perf}}$ is contained in $\operatorname{Coh}(S)$. As in Sect. 1.5, this gives rise to a functor

$$\Xi_S : \operatorname{QCoh}(S) \to \operatorname{IndCoh}(S),$$

left adjoint to Ψ_S, obtained by ind-extension of the tautological functor

$$\operatorname{QCoh}(S)^{\operatorname{perf}} \hookrightarrow \operatorname{Coh}(S).$$

As in Proposition 1.5.3, it is immediate that the unit of adjunction

$$\operatorname{Id} \to \Psi_S \circ \Xi_S$$

is an isomorphism. I.e., Ξ_S is fully faithful, and Ψ_S realizes $\operatorname{QCoh}(S)$ as a colocalization of $\operatorname{IndCoh}(S)$ with respect to the subcategory $\operatorname{IndCoh}(S)_{nil}$.

Remark 2.3.5. Here is a quick way to see that Ψ_S is left t-exact in the eventually coconnective case: indeed, it is the right adjoint of Ξ_S, and the latter is right t-exact by construction.

2.4. **The non-affine case.** In this subsection we will indicate how to extend the definition of $\operatorname{IndCoh}(S)$ to the case when S is not necessarily affine.

2.4.1. First, we observe that if $S_1 \hookrightarrow S_2$ is an open embedding of affine DG schemes and S_2 is eventually coherent, then so is S_1. This observation gives rise to the following definition:

Definition 2.4.2. *We say that a scheme S is locally eventually coherent if for it admits an open cover by eventually coherent affine DG schemes.*

2.4.3. Assume that S is locally eventually coherent. We define the category $\text{IndCoh}(S)$ as

$$\underset{U \to S}{lim} \text{IndCoh}(U),$$

where the limit is taken over the category of DG schemes U that are disjoint unions of eventually coherent affine DG schemes, equipped with an open embedding into S.

Proposition 2.3.3 allows us to prove an analog of Proposition 4.2.1, which implies that for S affine, we recover the original definition of $\text{IndCoh}(S)$.

3. Basic Functorialities

In this section all DG schemes will be assumed Noetherian.

3.1. **Direct images.** Let $f : S_1 \to S_2$ be a morphism of DG schemes.

Proposition 3.1.1. *There exists a unique continuous functor*

$$f_*^{\text{IndCoh}} : \text{IndCoh}(S_1) \to \text{IndCoh}(S_2),$$

which is left t-exact with respect to the t-structures, and which makes the diagram

$$\begin{array}{ccc} \text{IndCoh}(S_1) & \xrightarrow{f_*^{\text{IndCoh}}} & \text{IndCoh}(S_2) \\ \Psi_{S_1} \downarrow & & \downarrow \Psi_{S_2} \\ \text{QCoh}(S_1) & \xrightarrow{f_*} & \text{QCoh}(S_2) \end{array}$$

commute.

Proof. By definition, we need to construct a functor

$$\text{Coh}(S_1) \to \text{IndCoh}(S_2)^+,$$

such that the diagram

$$\begin{array}{ccc} \text{Coh}(S_1) & \longrightarrow & \text{IndCoh}(S_2)^+ \\ \Psi_{S_1} \downarrow & & \downarrow \Psi_{S_2} \\ \text{QCoh}(S_1) & \xrightarrow{f_*} & \text{QCoh}(S_2)^+ \end{array}$$

commutes. Since the right vertical arrow is an equivalence (by Proposition 1.2.4), the functor in question is uniquely determined to be the composition

$$\text{Coh}(S_1) \hookrightarrow \text{QCoh}(S_1)^+ \xrightarrow{f_*} \text{QCoh}(S_2)^+.$$

□

3.1.2. Recall that according to Sect. 1.4, we can regard $\mathrm{IndCoh}(S_i)$ as a module category over the monoidal category $\mathrm{QCoh}(S_i)$. In particular, we can view both $\mathrm{IndCoh}(S_1)$ and $\mathrm{IndCoh}(S_2)$ as module categories over $\mathrm{QCoh}(S_2)$ via the monoidal functor

$$f^* : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1).$$

We now claim:

Proposition 3.1.3. *The functor* $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S_1) \to \mathrm{IndCoh}(S_2)$ *has a unique structure of* 1*-morphism of* $\mathrm{QCoh}(S_2)$*-module categories that makes the diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(S_2) & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2) \end{array}$$

commute.

At the level of individual objects, the assertion of Proposition 3.1.3 says that for $\mathcal{E}_2 \in \mathrm{QCoh}(S_2)$ and $\mathcal{F}_1 \in \mathrm{IndCoh}(S_1)$ we have a canonical isomorphism

$$\mathcal{E}_2 \otimes f_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \simeq f_*^{\mathrm{IndCoh}}(f^*(\mathcal{E}_2) \otimes \mathcal{F}_1), \tag{3.1}$$

where $\otimes$ is understood in the sense of the action of Sect. 1.4.

Proof. It is enough to show that the functor

$$f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(S_1)} : \mathrm{Coh}(S_1) \to \mathrm{IndCoh}(S_2)$$

has a unique structure of 1-morphism of module categories over $\mathrm{QCoh}(S_2)^{\mathrm{perf}}$, which makes the diagram

$$\begin{array}{ccc} \mathrm{Coh}(S_1) & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(S_2) & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2) \end{array}$$

commute.

However, by construction, the latter structure equals the one induced by the functors

$$f_*|_{\mathrm{Coh}(S_1)} : \mathrm{Coh}(S_1) \to \mathrm{QCoh}(S_2)^+$$

and

$$\mathrm{QCoh}(S_2)^+ \overset{\Psi_{S_2}}{\simeq} \mathrm{IndCoh}(S_2)^+ \hookrightarrow \mathrm{IndCoh}(S_2).$$

□

3.2. **Upgrading to a functor.**

3.2.1. We now claim that the assigment

$$S \rightsquigarrow \mathrm{IndCoh}(S), \quad f \rightsquigarrow f_*^{\mathrm{IndCoh}}$$

upgrades to a functor

$$\mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{3.2}$$

to be denoted $\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}}$.

3.2.2. First, we recall the functor

$$\mathrm{QCoh}^*_{\mathrm{DGSch}} : (\mathrm{DGSch})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

see Sect. 0.6.7.

Passing to right adjoints, we obtain a functor

$$\mathrm{QCoh}_{\mathrm{DGSch}} : \mathrm{DGSch} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Restricting to $\mathrm{DGSch}_{\mathrm{Noeth}} \subset \mathrm{DGSch}$ we obtain a functor

$$\mathrm{QCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

3.2.3. Now, we claim:

Proposition 3.2.4. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

equipped with a natural transformation

$$\Psi_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}} \to \mathrm{QCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}},$$

which at the level of objects and 1*-morphisms is given by the assignment*

$$X \rightsquigarrow \mathrm{IndCoh}(X), \quad f \rightsquigarrow f_*^{\mathrm{IndCoh}}.$$

The rest of this subsection is devoted to the proof of this proposition.

3.2.5. Consider the following $(\infty,1)$-categories:

$$\mathrm{DGCat}^{+_{\mathrm{cont}}} \text{ and } \mathrm{DGCat}^t_{\mathrm{cont}}:$$

The category $\mathrm{DGCat}^{+_{\mathrm{cont}}}$ consists if of non-cocomplete DG categories $\mathbf{C}$, endowed with a t-structure, such that $\mathbf{C} = \mathbf{C}^+$. We also require that $\mathbf{C}^{\geq 0}$ contain filtered colimits and that the embedding $\mathbf{C}^{\geq 0} \hookrightarrow \mathbf{C}$ commute with filtered colimits. As 1-morphisms we take those exact functors $F : \mathbf{C}_1 \to \mathbf{C}_2$ that are *left t-exact up to a finite shift*, and such that $F|_{\mathbf{C}_1^{\geq 0}}$ commutes with filtered colimits. The higher categorical structure is uniquely determined by the requirement that the forgetful functor

$$\mathrm{DGCat}^{+_{\mathrm{cont}}} \to \mathrm{DGCat}_{\text{non-cocomplete}}$$

be 1-fully faithful.

The category $\mathrm{DGCat}^t_{\mathrm{cont}}$ consists of cocomplete DG categories $\mathbf{C}$, endowed with a t-structure, such that $\mathbf{C}^{\geq 0}$ is closed under filtered colimits, and such that $\mathbf{C}$ is compactly generated by objects from $\mathbf{C}^+$. As 1-morphisms we allow those exact functors $F : \mathbf{C}_1 \to \mathbf{C}_2$ that are continuous and *left t-exact up to a finite shift*. The higher categorical structure is uniquely determined by the requirement that the forgetful functor

$$\mathrm{DGCat}^t_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

be 1-fully faithful.

We have a naturally defined functor

$$\mathrm{DGCat}^t_{\mathrm{cont}} \to \mathrm{DGCat}^{+_{\mathrm{cont}}}, \quad \mathbf{C} \mapsto \mathbf{C}^+. \tag{3.3}$$

Lemma 3.2.6. *The functor* (3.3) *is 1-fully faihful.*

3.2.7. We will use the following general assertion. Let $T : \mathbf{D}' \to \mathbf{D}$ be a 1-fully faithful functor between $(\infty, 1)$-categories. Let $\mathbf{I}$ be another $(\infty, 1)$-category, and let

$$(\mathbf{i} \in \mathbf{I}) \rightsquigarrow (F'(\mathbf{i}) \in \mathbf{D}'), \tag{3.4}$$

be an assignment, such that the assignment

$$\mathbf{i} \mapsto T \circ F'(\mathbf{i})$$

has been extended to a functor $F : \mathbf{I} \to \mathbf{D}$.

Lemma 3.2.8. *Suppose that for every $\alpha \in \mathrm{Maps}_{\mathbf{I}}(\mathbf{i}_1, \mathbf{i}_2)$, the point $F(\alpha) \in \mathrm{Maps}_{\mathbf{D}}(F(\mathbf{i}_1), F(\mathbf{i}_2))$ lies in the connected component corresponding to the image of*

$$\mathrm{Maps}_{\mathbf{D}'}(F'(\mathbf{i}_1), F'(\mathbf{i}_2)) \to \mathrm{Maps}_{\mathbf{D}}(F(\mathbf{i}_1), F(\mathbf{i}_2)).$$

Then there exists a unique extension of (3.4) *to a functor $F' : \mathbf{I} \to \mathbf{D}'$ equipped with an isomorphism $T \circ F' \simeq F$.*

Let now F'_1 and F'_2 be two assignments as in (3.4), satisfying the assumption of Lemma 3.2.8. Let us be given an assignment

$$\mathbf{i} \rightsquigarrow \psi'_{\mathbf{i}} \in \mathrm{Maps}_{\mathbf{D}'}(F'_1(\mathbf{i}), F'_2(\mathbf{i})). \tag{3.5}$$

Lemma 3.2.9. *Suppose that the assignment*

$$\mathbf{i} \rightsquigarrow T(\psi'_{\mathbf{i}}) \in \mathrm{Maps}_{\mathbf{D}}(F_1(\mathbf{i}), F_2(\mathbf{i}))$$

has been extended to a natural transformation $\psi : F_1 \to F_2$. Then there exists a unique extension of (3.5) *to a natural transformation $\psi : F'_1 \to F'_2$ equipped with an isomorphism $T \circ \psi' \simeq \psi$.*

3.2.10. We are now ready to prove Proposition 3.2.4:

Step 1. We start with the functor

$$\mathrm{QCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and consider

$$\mathbf{I} = \mathrm{DGSch}_{\mathrm{Noeth}},\ \mathbf{D} = \mathrm{DGCat}_{\mathrm{cont}},\ \mathbf{D}' := \mathrm{DGCat}^t_{\mathrm{cont}},\ F = \mathrm{QCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}},$$

and the assignment

$$(X \in \mathrm{DGSch}_{\mathrm{Noeth}}) \rightsquigarrow (\mathrm{QCoh}(X) \in \mathrm{DGCat}^t_{\mathrm{cont}}).$$

Apping Lemma 3.2.8, we obtain a functor

$$\mathrm{QCoh}^t_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}^t_{\mathrm{cont}}. \tag{3.6}$$

Step 2. Note that Proposition 3.1.1 defines a functor

$$\mathrm{IndCoh}^t_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{DGSch}_{\mathrm{Noeth}} \to \mathrm{DGCat}^t_{\mathrm{cont}},$$

and the natural transformation

$$\Psi^t_{\mathrm{DGSch}_{\mathrm{Noeth}}} : \mathrm{IndCoh}^t_{\mathrm{DGSch}_{\mathrm{Noeth}}} \to \mathrm{QCoh}^t_{\mathrm{DGSch}_{\mathrm{Noeth}}}$$

at the level of objects and 1-morphisms.

Since the functor $\mathrm{DGCat}^t_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$ is 1-fully faithful, by Lemmas 3.2.8 and 3.2.9, the existence and uniqueness of the pair $(\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}}, \Psi_{\mathrm{DGSch}_{\mathrm{Noeth}}})$ with a fixed behavior on objects and 1-morphisms, is equivalent to that of $(\mathrm{IndCoh}^t_{\mathrm{DGSch}_{\mathrm{Noeth}}}, \Psi^t_{\mathrm{DGSch}_{\mathrm{Noeth}}})$.

Step 3. By Lemma 3.2.6, combined with Lemmas 3.2.8 and 3.2.9, we obtain that the existence and uniqueness of the pair $(\mathrm{IndCoh}^t_{\mathrm{DGSch}_{\mathrm{Noeth}}}, \Psi^t_{\mathrm{DGSch}_{\mathrm{Noeth}}})$, with a fixed behavior on objects and 1-morphisms is equivalent to the existence and uniqueness of the pair

$$(\mathrm{IndCoh}^+_{\mathrm{DGSch}_{\mathrm{Noeth}}}, \Psi^+_{\mathrm{DGSch}_{\mathrm{Noeth}}}),$$

obtained by composing with the functor (3.3).

The latter, however, is given by

$$\mathrm{IndCoh}^+_{\mathrm{DGSch}_{\mathrm{Noeth}}} := \mathrm{QCoh}^+_{\mathrm{DGSch}_{\mathrm{Noeth}}},$$

obtained from (3.6) by composing with (3.3), and taking

$$\Psi^+_{\mathrm{DGSch}_{\mathrm{Noeth}}} := \mathrm{Id}\,.$$

□

3.3. **The !-pullback functor for proper maps.**

3.3.1. Let $f : S_1 \to S_2$ be a map between Noetherian schemes.

Definition 3.3.2. *We shall say that f is of almost of finite type if the corresponding map of classical schemes ${}^{cl}S_1 \to {}^{cl}S_2$ is.*

In particular, for $S_2 = \mathrm{Spec}(k)$, the above notion is equivalent to S_1 being almost of finite type over k, see Sect. 0.6.9.

We let $(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}$ denote the 1-full subcategory of $\mathrm{DGSch}_{\mathrm{Noeth}}$, where we restrict 1-morphisms to be maps almost of finite type.

Definition 3.3.3. *A map $f : S_1 \to S_2$ between DG schemes is said to be proper/finite/closed embedding if the corresponding map of classical schemes ${}^{cl}S_1 \to {}^{cl}S_2$ has this property.*

In particular, any proper map is almost of finite type, as the latter is included in the definition of properness for morphisms between classical schemes. We let

$$(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{cl.emb.}} \subset (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{finite}} \subset (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}} \subset (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}$$

denote the corresponding inclusions of 1-full subcategories.

3.3.4. We have the following basic feature of proper maps:

Lemma 3.3.5. *If $f : S_1 \to S_2$ is proper, then the functor $f_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2)$ sends $\mathrm{Coh}(S_1)$ to $\mathrm{Coh}(S_2)$.*

Proof. It is enough to show that f_* sends $\mathrm{Coh}(S_1)^\heartsuit$ to $\mathrm{Coh}(S_2)$. Let $\imath_i$ denote the canonical maps ${}^{cl}S_i \to S_i$, $i = 1, 2$. Since

$$(\imath_i)_* : \mathrm{Coh}({}^{cl}S_i)^\heartsuit \to \mathrm{Coh}(S_i)^\heartsuit$$

are equivalences, the required assertion follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}({}^{cl}S_1) & \xrightarrow{(\imath_1)_*} & \mathrm{QCoh}(S_1) \\ {\scriptstyle ({}^{cl}f)_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}({}^{cl}S_2) & \xrightarrow{(\imath_2)_*} & \mathrm{QCoh}(S_2), \end{array}$$

and the well-known fact that direct image preserves coherence for proper maps between classical Noetherian schemes.

□

Corollary 3.3.6. *If f is proper, the functor*

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S_1) \to \mathrm{IndCoh}(S_2)$$

sends compact objects to compact ones.

Proof. We need to show that for $\mathcal{F} \in \mathrm{Coh}(S_1) \subset \mathrm{IndCoh}(S_1)$, the object

$$f_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(S_2)$$

belongs to $\mathrm{Coh}(S_2) \subset \mathrm{IndCoh}(S_2)$. By Proposition 3.1.1, $f_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(S_2)^+$. Hence, by Proposition 1.2.4, it suffices to show that

$$\Psi_{S_2}(f_*^{\mathrm{IndCoh}}(\mathcal{F})) \in \mathrm{Coh}(S_2) \subset \mathrm{QCoh}(S_2).$$

We have

$$\Psi_{S_2}(f_*^{\mathrm{IndCoh}}(\mathcal{F})) \simeq f_*(\Psi_{S_1}(\mathcal{F})),$$

and the assertion follows from Lemma 3.3.5.

□

3.3.7. Consider the adjoint functor

$$f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$$

(it exists for general ∞-category reasons, see [Lu0], Corollary 5.5.2.9).

Now, the fact that f_*^{IndCoh} sends compacts to compacts implies that the functor $f^!$ is continuous. I.e., $f^!$ is a 1-morphism in $\mathrm{DGCat}_{\mathrm{cont}}$.

3.3.8. By adjunction from Proposition 3.2.4 we obtain:

Corollary 3.3.9. *The assignment $S \mapsto \mathrm{IndCoh}(S)$ upgrades to a functor*

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}}} : ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

3.3.10. By Proposition 3.1.3, the functor f_*^{IndCoh} has a natural structure of 1-morphism between $\mathrm{QCoh}(S_2)$-module categories. Hence, from [GL:DG, Corollary 6.2.4] we obtain:

Corollary 3.3.11. *The functor $f^!$ has a natural structure of 1-morphism between* $\mathrm{QCoh}(S_2)$*-module categories.*

At the level of individual objects, the assertion of Corollary 3.3.11 says that for $\mathcal{E} \in \mathrm{QCoh}(S_2)$ and $\mathcal{F} \in \mathrm{IndCoh}(S_2)$, we have a canonical isomorphism:

$$f^!(\mathcal{E} \otimes \mathcal{F}) \simeq f^*(\mathcal{E}) \otimes f^!(\mathcal{F}), \tag{3.7}$$

where $\otimes$ is understood in the sense of the action of Sect. 1.4.

3.4. **Proper base change.**

3.4.1. Let $f : S_1 \to S_2$ be a proper map between Noetherian DG schemes, and let $g_2 : S'_2 \to S_2$ be an arbitrary map (according to the conventions of this section, the DG scheme S'_2 is also assumed Noetherian).

Since f is almost of finite type, the Cartesian product $S'_1 := S'_2 \underset{S_2}{\times} S_1$ is also Noetherian, and the resulting map $f' : S'_1 \to S'_2$ is proper. Let g_1 denote the map $S'_1 \to S_1$:

$$\begin{array}{ccc} S'_1 & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S'_2 & \xrightarrow{g_2} & S_2. \end{array}$$

The isomorphism of functors

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \simeq (g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}}$$

by adjunction gives rise to a natural transformation

$$(g_1)_*^{\mathrm{IndCoh}} \circ (f')^! \to f^! \circ (g_2)_*^{\mathrm{IndCoh}} \tag{3.8}$$

between the two functors $\mathrm{IndCoh}(S'_2) \rightrightarrows \mathrm{IndCoh}(S_1)$.

Proposition 3.4.2. *The natural transformation* (3.8) *is an isomorphism.*

The rest of this subsection is devoted to the proof of this proposition.

3.4.3. For a proper morphism $f : S_1 \to S_2$, let

$$f^{\mathrm{QCoh},!} : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1)$$

denote the *not necessarily continuous* right adjoint to $f_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2)$.

Since f_* is right t-exact *up to a cohomological shift*, the functor $f^{\mathrm{QCoh},!}$ is left t-exact *up to a cohomological shift.* Hence, it maps $\mathrm{QCoh}(S_2)^+$ to $\mathrm{QCoh}(S_1)^+$.

Lemma 3.4.4. *The diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1)^+ & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1)^+ \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{QCoh},!}} \\ \mathrm{IndCoh}(S_2)^+ & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2)^+ \end{array}$$

obtained by passing to right adjoints along the horizontal arrows in

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1)^+ & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1)^+ \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(S_2)^+ & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2)^+, \end{array}$$

commutes.

Proof. Follows from the fact that the vertical arrows are equivalences, by Proposition 1.2.4. □

Remark 3.4.5. It is not in general true that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{QCoh},!}} \\ \mathrm{IndCoh}(S_2) & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2) \end{array}$$

obtained by passing to right adjoints along the horizontal arrows in

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(S_2) & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2), \end{array}$$

commutes.

For example, take $S_1 = \mathrm{pt} = \mathrm{Spec}(k)$, $S_2 = \mathrm{Spec}(k[t]/t^2)$ and $0 \neq \mathcal{F} \in \mathrm{IndCoh}(S_2)$ be in the kernel of the functor Ψ_{S_2}. Then $\Psi_X \circ f^!(\mathcal{F}) \neq 0$. Indeed, Ψ_{S_1} is an equivalence, and $f^!$ is conservative, see Corollary 4.1.8.

Proof of Proposition 3.4.2. Since all functors involved are continuous, it is enough to show that the map

$$(g_1)_*^{\mathrm{IndCoh}} \circ (f')^! \to f^! \circ (g_2)_*^{\mathrm{IndCoh}}(\mathcal{F})$$

is an isomorphism for $\mathcal{F} \in \mathrm{Coh}(S_1)$. Hence, it is enough to show that (3.8) is an isomorphism when restricted to $\mathrm{IndCoh}(S_1)^+$.

By Lemma 3.4.4 and Proposition 1.2.4, this reduces the assertion to showing that the natural transformation

$$(g_1)_* \circ (f')^{\mathrm{QCoh},!} \to f^{\mathrm{QCoh},!} \circ (g_2)_* \tag{3.9}$$

is an isomorphism for the functors

$$\begin{array}{ccc} \mathrm{QCoh}(S_1')^+ & \xrightarrow{(g_1)_*} & \mathrm{QCoh}(S_1)^+ \\ {\scriptstyle (f')^{\mathrm{QCoh},!}}\big\uparrow & & \big\uparrow{\scriptstyle f^{\mathrm{QCoh},!}} \\ \mathrm{QCoh}(S_2')^+ & \xrightarrow{(g_2)_*} & \mathrm{QCoh}(S_2)^+, \end{array}$$

where the natural transformation comes from the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S_1')^+ & \xrightarrow{(g_1)_*} & \mathrm{QCoh}(S_1)^+ \\ {\scriptstyle f'_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(S_2')^+ & \xrightarrow{(g_2)_*} & \mathrm{QCoh}(S_2)^+ \end{array}$$

by passing to right adjoint along the vertical arrows.

We consider the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S_1') & \xrightarrow{(g_1)_*} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f'_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(S_2') & \xrightarrow{(g_2)_*} & \mathrm{QCoh}(S_2), \end{array}$$

and the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S_1') & \xrightarrow{(g_1)_*} & \mathrm{QCoh}(S_1) \\ {\scriptstyle (f')^{\mathrm{QCoh},!}}\big\uparrow & & \big\uparrow{\scriptstyle f^{\mathrm{QCoh},!}} \\ \mathrm{QCoh}(S_2') & \xrightarrow{(g_2)_*} & \mathrm{QCoh}(S_2), \end{array}$$

obtained by passing to right adjoints along the vertical arrows. (Note, however, that the functors involved are no longer continuous).

We claim that the resulting natural transformation

$$(g_1)_* \circ (f')^{\mathrm{QCoh},!} \to f^{\mathrm{QCoh},!} \circ (g_2)_* \tag{3.10}$$

between functors

$$\mathrm{QCoh}(S'_2) \rightrightarrows \mathrm{QCoh}(S_1)$$

is an isomorphism.

Indeed, the map in (3.10) is obtained by passing to right adjoints in the natural transformation

$$(g_2)^* \circ f_* \to f'_* \circ (g_1)^* \tag{3.11}$$

as functors

$$\mathrm{QCoh}(S_1) \rightrightarrows \mathrm{QCoh}(S'_2)$$

in the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S'_1) & \xleftarrow{(g_1)^*} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(S'_2) & \xleftarrow{(g_2)^*} & \mathrm{QCoh}(S_2). \end{array}$$

Now, (3.11) is an isomorphism by the usual base change for QCoh. Hence, (3.10) is an isomorphism as well.

□

3.5. **The** (IndCoh, $*$)**-pullback.**

3.5.1. Let $f : S_1 \to S_2$ be a morphism between Noetherian DG schemes.

Definition 3.5.2. *We shall say that f is* eventually coconnective *if the functor*

$$f^* : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1)$$

sends $\mathrm{Coh}(S_2)$ *to* $\mathrm{QCoh}(S_1)^+$*, in which case it automatically sends it to* $\mathrm{Coh}(S_1)$.

It is easy to see that the following conditions are equivalent:

- f is eventually coconnective;
- For a closed embedding $S'_2 \to S_2$ with S'_2 eventually coconnective, the Cartesian product $S'_2 \underset{S_2}{\times} S_1$ is eventually coconnective.
- For a closed embedding $S'_2 \to S_2$ with S'_2 classical, the Cartesian product $S'_2 \underset{S_2}{\times} S_1$ is eventually coconnective.

3.5.3. By the same logic as in Sect. 3.1, we have:

Proposition 3.5.4. *Suppose that f is eventually coconnective. Then there exists a unique functor*

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1),$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f^{\mathrm{IndCoh},*}}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{IndCoh}(S_2) & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2) \end{array}$$

commute.

Furthermore, the functor $f^{\mathrm{IndCoh},*}$ has a unique structure of 1-morphism of $\mathrm{QCoh}(S_2)$-module categories, so that the above diagram commutes as a diagram of $\mathrm{QCoh}(S_2)$-modules.

We note that the last assertion in Proposition 3.5.4 at the level of individual objects says that for $\mathcal{E} \in \mathrm{QCoh}(S_2)$ and $\mathcal{F} \in \mathrm{IndCoh}(S_2)$, we have a canonical isomorphism

$$f^{\mathrm{IndCoh},*}(\mathcal{E} \otimes \mathcal{F}) \simeq f^*(\mathcal{E}) \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}), \tag{3.12}$$

where $\otimes$ is understood in the sense of the action of Sect. 1.4.

3.5.5. Let $(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-coconn}}$ denote the 1-full subcategory of $\mathrm{DGSch}_{\mathrm{Noeth}}$ where we allow only eventually coconnective maps as 1-morphisms. As in Proposition 3.2.4 we obtain:

Corollary 3.5.6. *The assignment*

$$S \mapsto \mathrm{IndCoh}(S) \text{ and } \Psi_S$$

upgrade to a functor

$$\mathrm{IndCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-coconn}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})^{\mathrm{op}}_{\text{ev-coconn}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and a natural transformation

$$\begin{aligned}\Psi^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-coconn}}} : \mathrm{IndCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-coconn}}} \to \mathrm{QCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-coconn}}} &:= \\ = \mathrm{QCoh}^*_{\mathrm{DGSch}_{\mathrm{Noeth}}}|_{((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-coconn}})^{\mathrm{op}}}&.\end{aligned}$$

3.5.7. *The (inverse,direct) image adjunction.* We are now going to show that under the assumptions of Proposition 3.5.4, the functors f^{IndCoh}_* and $f^{\mathrm{IndCoh},*}$ satisfy the usual adjunction property:

Lemma 3.5.8. *Let $f : S_1 \to S_2$ be eventually coconnective. Then the functor*

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$$

is the left adjoint of $f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(S_1) \to \mathrm{IndCoh}(S_2)$.

Proof. We need to construct a functorial isomorphism

$$\mathrm{Maps}_{\mathrm{IndCoh}(S_1)}(f^{\mathrm{IndCoh},*}(\mathcal{F}_2), \mathcal{F}_1) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(S_2)}(\mathcal{F}_2, f^{\mathrm{IndCoh}}_*(\mathcal{F}_1)), \tag{3.13}$$

for $\mathcal{F}_1 \in \mathrm{IndCoh}(S_1)$ and $\mathcal{F}_2 \in \mathrm{IndCoh}(S_2)$.

By the definition of $\mathrm{IndCoh}(S_2)$, it suffices to do this for $\mathcal{F}_2 \in \mathrm{Coh}(S_2)$. Now, since the functor $f^{\mathrm{IndCoh},*}$ sends compact objects to compact ones, for $\mathcal{F}_2 \in \mathrm{Coh}(S_2)$, each side in (3.13) commutes with filtered colimits. Hence, it suffices to construct the isomorphism (3.13) when $\mathcal{F}_2 \in \mathrm{Coh}(S_2)$ and $\mathcal{F}_1 \in \mathrm{Coh}(S_1)$.

In this case,

$$\mathrm{Maps}_{\mathrm{IndCoh}(S_1)}(f^{\mathrm{IndCoh},*}(\mathcal{F}_2),\mathcal{F}_1) \simeq \mathrm{Maps}_{\mathrm{Coh}(S_1)}(f^*(\mathcal{F}_2),\mathcal{F}_1) \simeq \mathrm{Maps}_{\mathrm{QCoh}(S_1)}(f^*(\mathcal{F}_2),\mathcal{F}_1),$$

and

$$\mathrm{Maps}_{\mathrm{IndCoh}(S_2)}(\mathcal{F}_2, f_*^{\mathrm{IndCoh}}(\mathcal{F}_1)) = \mathrm{Maps}_{\mathrm{IndCoh}(S_2)^+}(\mathcal{F}_2, f_*^{\mathrm{IndCoh}}(\mathcal{F}_1)),$$

which by Proposition 1.2.4 identifies with

$$\mathrm{Maps}_{\mathrm{QCoh}(S_2)^+}(\mathcal{F}_2, f_*(\mathcal{F}_1)) \simeq \mathrm{Maps}_{\mathrm{QCoh}(S_2)}(\mathcal{F}_2, f_*(\mathcal{F}_1)).$$

Hence, the required isomorphism follows from the isomorphism

$$\mathrm{Maps}_{\mathrm{QCoh}(S_1)}(f^*(\mathcal{F}_2),\mathcal{F}_1) \simeq \mathrm{Maps}_{\mathrm{QCoh}(S_2)}(\mathcal{F}_2, f_*(\mathcal{F}_1)),$$

which expresses the (f^*, f_*)-adjunction for QCoh.

□

In fact, a statement converse to Lemma 3.5.8 holds:

Proposition 3.5.9. *Let $f: S_1 \to S_2$ be a morphism between DG schemes, such that the functor $f_*^{\mathrm{IndCoh}}: \mathrm{IndCoh}(S_1) \to \mathrm{IndCoh}(S_2)$ admits a left adjoint. Then f is eventually coconnective.*

Proof. Suppose f_*^{IndCoh} admits a left adjoint; let us denote it by

$$'f^{\mathrm{IndCoh},*}: \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1).$$

Being a left adjoint to a functor that commutes with colimits, $'f^{\mathrm{IndCoh},*}$ sends compact objects to compacts, i.e., it is the ind-extension of a functor

$$'f^{\mathrm{IndCoh},*}: \mathrm{Coh}(S_2) \to \mathrm{Coh}(S_1).$$

To prove the proposition, it suffices to show that the natural map

$$f^* \circ \Psi_{S_2} \to \Psi_{S_1} \circ {}'f^{\mathrm{IndCoh},*} \tag{3.14}$$

is an isomorphism.

Since the t-structure on $\mathrm{QCoh}(S_1)$ is left-complete, it suffices to show that for $\mathcal{F} \in \mathrm{Coh}(S_2)$ and any n, the induced map

$$\tau^{\geq n}(f^* \circ \Psi_{S_2}(\mathcal{F})) \to \tau^{\geq n}\left(\Psi_{S_1} \circ {}'f^{\mathrm{IndCoh},*}(\mathcal{F})\right)$$

is an isomorphism, i.e., that the induced map

$$\mathrm{Hom}_{\mathrm{QCoh}(S_1)}(\Psi_{S_1} \circ {}'f^{\mathrm{IndCoh},*}(\mathcal{F}),\mathcal{F}') \to \mathrm{Hom}_{\mathrm{QCoh}(S_1)}(f^* \circ \Psi_{S_2}(\mathcal{F}),\mathcal{F}') \tag{3.15}$$

is an isomorphism for any $\mathcal{F}' \in \mathrm{QCoh}(S_1)^{\geq n}$. By Proposition 1.2.4, we can take $\mathcal{F}' = \Psi_{S_1}(\mathcal{F}_1)$ for some $\mathcal{F}_1 \in \mathrm{IndCoh}(S_1)^{\geq n}$.

The object $'f^{\mathrm{IndCoh},*}(\mathcal{F})$ belongs to $\mathrm{Coh}(S_1) \subset \mathrm{IndCoh}(S_1)^+$. Hence, by Proposition 1.2.4, the left-hand side of (3.15) identifies with

$$\mathrm{Hom}_{\mathrm{IndCoh}(S_1)}('f^{\mathrm{IndCoh},*}(\mathcal{F}),\mathcal{F}_1),$$

which in turn identifies with

$$\mathrm{Hom}_{\mathrm{IndCoh}(S_2)}(\mathcal{F}, f_*^{\mathrm{IndCoh}}(\mathcal{F}_1)) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S_2)}(\Psi_{S_2}(\mathcal{F}), \Psi_{S_2}(f_*^{\mathrm{IndCoh}}(\mathcal{F}_1))) \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{QCoh}(S_2)}(\Psi_{S_2}(\mathcal{F}), f_*(\Psi_{S_1}(\mathcal{F}_1))),$$

which identifies by adjunction with the right-hand side of (3.15).

□

3.5.10. Assume again that S_1 and S_2 are eventually coconnective. In this case, by adjunction from Proposition 3.1.1, we obtain:

Lemma 3.5.11. *The diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xleftarrow{\Xi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f^{\mathrm{IndCoh},*}}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{IndCoh}(S_2) & \xleftarrow{\Xi_{S_2}} & \mathrm{QCoh}(S_2) \end{array}$$

commutes as well.

3.5.12. In particular, we obtain that the assignment $S \mapsto \Xi_S$ extends to a natural transformation

$$\Xi^*_{({}^{<\infty}\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}} : \mathrm{QCoh}^*_{({}^{<\infty}\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}} \to \mathrm{IndCoh}^*_{({}^{<\infty}\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}},$$

where

$$\mathrm{QCoh}^*_{({}^{<\infty}\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}} \text{ and } \mathrm{IndCoh}^*_{({}^{<\infty}\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}}$$

denote he restrictions of the functors

$$\mathrm{QCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}} \text{ and } \mathrm{IndCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}},$$

respectively, to $(({}^{<\infty}\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \subset ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}}$.

3.6. Morphisms of bounded Tor dimension.

3.6.1. We shall say that a morphism $f : S_1 \to S_2$ between DG schemes is *of bounded Tor dimension* if the functor

$$f^* : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1)$$

is left t-exact *up to a finite shift*, i.e., is of bounded cohomological amplitude.

3.6.2. First, we claim:

Lemma 3.6.3. *Let $f : S_1 \to S_2$ be a morphism almost of finite type. Then following conditions are equivalent:*

(a) *f is eventually coconnective.*

(b) *f is of bounded Tor dimension.*

Proof. The implication (b) ⇒ (a) is tautological. Let us prove that (a) implies (b).

The question is local in Zariski topology, so we can assume that f can be factored as a composition

$$S_1 \xrightarrow{f'} S_2 \times \mathbb{A}^n \to S_2,$$

where f' is a closed embedding. It is easy to see that f satisfies condition (a) (resp., (b)) if and only if f' does. Hence, we can assume that f is itself a closed embedding.

To test that f is of bounded Tor dimension, it is sufficient to test it on objects from $\mathrm{Coh}(S_2)^\heartsuit$. Since such objects come as direct images from ${}^{cl}S_2$, by base change, we can assume that S_2 is classical. Since f was assumed eventually coconnective, we obtain that S_1 is itself eventually coconnective, i.e., $f_*(\mathcal{O}_{S_1}) \in \mathrm{Coh}(S_2)$.

We need to show that $f_*(\mathcal{O}_{S_1})$ is of bounded Tor dimension. This will follow from the following general assertion:

Lemma 3.6.4. *Let S is a Noetherian DG scheme, and let $\mathcal{F} \in \operatorname{Coh}(S)$ be such that for every geometric point $s : \operatorname{Spec}(K) \to S$, the fiber $s^*(\mathcal{F})$ lives in finitely many degrees, then $\mathcal{F}$ is perfect.*

□

Proof of Lemma 3.6.4. We need to show that the functor $\mathcal{F} \underset{\mathcal{O}_S}{\otimes} - : \operatorname{QCoh}(S) \to \operatorname{QCoh}(S)$ is of bounded cohomological amplitude.

In fact, we will show that if for *some* geometric point $s : \operatorname{Spec}(K) \to S$, the fiber $s^*(\mathcal{F})$ lives in degrees $[-n, 0]$, then on some Zariski neighborhood of s, the above functor is of amplitude $[-n, 0]$.

First, it is enough to test the functor $\mathcal{F} \underset{\mathcal{O}_S}{\otimes} -$ on objects from $\operatorname{QCoh}(S)^\heartsuit$. Since such objects come as direct images under ${}^{cl}S \to S$, we can replace S by ${}^{cl}S$. Now, out assertion becomes a familiar statement from commutative algebra. Let us prove it for completeness:

We shall argue by induction on n. For $n = -1$ the assertion follows from Nakayama's lemma: if $s^*(\mathcal{F}) = 0$, then $\mathcal{F}$ vanishes on a Zariski neighborhood of s.

To execute the induction step, choose a locally free $\mathcal{O}_S$-module $\mathcal{P}$ equipped with a map $\mathcal{P} \to \mathcal{F}$ which induces an isomorphism

$$H^0(s^*(\mathcal{P})) \to H^0(s^*(\mathcal{M})).$$

Set $\mathcal{F}' := \operatorname{Cone}(\mathcal{F} \to \mathcal{P})[-1]$. By construction, $s^*(\mathcal{F}')$ lives in cohomological degrees $[-n+1, 0]$, as desired.

□

Corollary 3.6.5. *If $f : S_1 \to S_2$ is eventually coconnective and almost of finite type, the functor f^* sends $\operatorname{QCoh}(S_2)^+$ to $\operatorname{QCoh}(S_1)^+$, and the functor $f^{\operatorname{IndCoh},*}$ sends $\operatorname{IndCoh}(S_2)^+$ to $\operatorname{IndCoh}(S_1)^+$.*

3.6.6. Assume now that the DG schemes S_1 and S_2 are eventually coconnective. Note that the isomorphism

$$\Psi_{S_2} \circ f_*^{\operatorname{IndCoh}} \simeq f_* \circ \Psi_{S_1}$$

induces a natural transformation

$$\Xi_{S_2} \circ f_* \to f_*^{\operatorname{IndCoh}} \circ \Xi_{S_1}. \tag{3.16}$$

Proposition 3.6.7. *Assume that f is of bounded Tor dimension and almost of finite type. Then* (3.16) *is an isomorphism.*

Proof. As we shall see in Lemma 4.1.1, the assertion is local in the Zariski topology on both S_1 and S_2. Therefore, we can assume that S_1 and S_2 are affine and that f can be factored as

$$S_1 \overset{f'}{\to} S_2 \times \mathbb{A}^n \to S_2,$$

where f' is a closed embedding.

As in the proof of Lemma 3.6.3, the assumption that f is of bounded Tor dimension implies the same for f'. Hence, it suffices to prove the proposition separately in the following two cases: (a) f is the projection $S_1 = S_2 \times \mathbb{A}^n \to S_2$; and (b) f is a closed embedding.

Note that the two sides of (3.16) become isomorphic after applying the functor Ψ_{S_2}. Hence, by Proposition 1.2.4, the assertion of the proposition is equivalent to the fact that for $\mathcal{F} \in \operatorname{QCoh}(S_1)^{\operatorname{perf}}$, the object

$$\Xi_{S_2}(f_*(\mathcal{F})) \in \operatorname{IndCoh}(S_2)$$

belongs to $\text{IndCoh}(S_2)^+$. Since S_1 is assumed affine, it suffices to consider the case of $\mathcal{F} = \mathcal{O}_{S_1}$.

Thus, in case (a) we need to show that $\Xi_{S_2}(\mathcal{O}_{S_2}[t_1, ..., t_n])$ belongs to $\text{IndCoh}(S_2)^+$. However, $\mathcal{O}_{S_2}[t_1, ..., t_n]$ is isomorphic to a direct sum of copies of $\mathcal{O}_{S_2}$, and therefore $\Xi_{S_2}(\mathcal{O}_{S_2}[t_1, ..., t_n])$ is isomorphic to a direct sum of copies of $\Xi_{S_2}(\mathcal{O}_{S_2})$. Hence, the assertion follows from the fact that the t-structure on $\text{IndCoh}(S_2)$ is compatible with filtered colimits.

In case (b) the object $f_*(\mathcal{O}_{S_1})$ belongs to $\text{QCoh}(S_2)^{\text{perf}}$, and the assertion follows by definition.

□

3.6.8. *A base change formula.* Let $f : S_1 \to S_2$ be again a map of bounded Tor dimension and almost of finite type. Let $g_2 : S'_2 \to S_2$ be an arbitrary map between Noetherian DG schemes.

The almost finite type assumption implies that the Cartesian product $S'_1 := S'_2 \underset{S_2}{\times} S_1$ is also Noetherian. Moreover, the resulting morphism $f' : S'_1 \to S'_2$

$$\begin{array}{ccc} S'_1 & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S'_2 & \xrightarrow{g_2} & S_2 \end{array}$$

is also of bounded Tor dimension.

Lemma 3.6.9. *Under the above circumstances, the map*

$$f^{\text{IndCoh},*} \circ (g_2)_*^{\text{IndCoh}} \to (g_1)_*^{\text{IndCoh}} \circ (f')^{\text{IndCoh},*}$$

induced by the $(f^{\text{IndCoh},*}, f_*^{\text{IndCoh}})$ *adjunction, is an isomorphism.*

Proof. Follows from Corollary 3.6.5 and the usual base change formula for QCoh, by evaluating both functors on $\text{Coh}(S_1) \subset \text{QCoh}(S_1)^+$.

□

3.6.10. *A projection formula.* Let $f : S_1 \to S_2$ be again a map of bounded Tor dimension. For $\mathcal{E}_1 \in \text{QCoh}(S_1)$ and $\mathcal{F}_2 \in \text{IndCoh}(S_2)$ consider the canonical map

$$f_*(\mathcal{E}_1) \otimes \mathcal{F}_2 \to f_*^{\text{IndCoh}}(\mathcal{E}_1 \otimes f^{\text{IndCoh},*}(\mathcal{F}_2)) \tag{3.17}$$

that comes by adjunction from

$$f^{\text{IndCoh},*}(f_*(\mathcal{E}_1) \otimes \mathcal{F}_2) \simeq f^*(f_*(\mathcal{E}_1)) \otimes f^{\text{IndCoh},*}(\mathcal{F}_2) \to \mathcal{E}_1 \otimes f^{\text{IndCoh},*}(\mathcal{F}_2).$$

Here $\otimes$ denotes the action of $\text{QCoh}(-)$ on $\text{IndCoh}(-)$ given by Sect. 1.4.

Proposition 3.6.11. *The map* (3.17) *is an isomorphism.*

Remark 3.6.12. Note that there is another version of the projection formula: namely, (3.1). It says that for $\mathcal{E}_2 \in \text{QCoh}(S_2)$ and $\mathcal{F}_1 \in \text{IndCoh}(S_1)$ we have

$$f_*^{\text{IndCoh}}(f^*(\mathcal{E}_2) \otimes \mathcal{F}_1) \simeq \mathcal{E}_2 \otimes f_*^{\text{IndCoh}}(\mathcal{F}_1).$$

The formula holds nearly tautologically and expresses the fact that the functor f_*^{IndCoh} is $\text{QCoh}(S_2)$-linear, see Proposition 3.1.3.

Proof. It is enough to show that the isomorphism holds for $\mathcal{E}_1 \in \mathrm{QCoh}(S_1)^{\mathrm{perf}}$ and $\mathcal{F}_2 \in \mathrm{Coh}(S_2) \subset \mathrm{IndCoh}(S_2)$.

We also note that the map (3.17) becomes an isomorphism after applying the functor Ψ_{S_2}, by the usual projection formula for QCoh. For $\mathcal{E}_1 \in \mathrm{QCoh}(S_1)^{\mathrm{perf}}$ and $\mathcal{F}_2 \in \mathrm{Coh}(S_2)$ we have

$$f_*^{\mathrm{IndCoh}}(\mathcal{E}_1 \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_2)) \in \mathrm{IndCoh}(S_2)^+.$$

Hence, by Proposition 1.2.4, it suffices to show that in this case

$$f_*(\mathcal{E}_1) \otimes \mathcal{F}_2 \in \mathrm{IndCoh}(S_2)^+.$$

We note that the object $f_*(\mathcal{E}_1) \in \mathrm{QCoh}(S_1)^b$ is of bounded Tor dimension. The required fact follows from the next general observation:

Lemma 3.6.13. *For $S \in \mathrm{DGSch}_{\mathrm{Noeth}}$ and $\mathcal{E} \in \mathrm{QCoh}(S)^b$, whose Tor dimension is bounded on the left by an integer n, the functor*

$$\mathcal{E} \otimes - : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S)$$

has a cohomological amplitude bounded on the left by n.

□

3.6.14. *Proof of Lemma 3.6.13.* We need to show that the functor $\mathcal{E} \otimes -$ sends $\mathrm{IndCoh}(S)^{\geq 0}$ to $\mathrm{IndCoh}(S)^{\geq -n}$. It is sufficient to show that this functor sends $\mathrm{Coh}(S)^{\geq 0}$ to $\mathrm{IndCoh}(S)^{\geq -n}$. By cohomological devissage, the latter is equivalent to sending $\mathrm{Coh}(S)^{\heartsuit}$ to $\mathrm{IndCoh}(S)^{\geq -n}$.

Let i denote the closed embedding ${}^{cl}S =: S' \to S$. The functor i_*^{IndCoh} induces an equivalence $\mathrm{Coh}(S')^{\heartsuit} \to \mathrm{Coh}(S)^{\heartsuit}$. So, it is enough to show that for $\mathcal{F}' \in \mathrm{Coh}(S')^{\heartsuit}$, we have

$$\mathcal{E} \otimes i_*^{\mathrm{IndCoh}}(\mathcal{F}') \in \mathrm{IndCoh}(S)^{\geq -n}.$$

We have:

$$\mathcal{E} \otimes i_*^{\mathrm{IndCoh}}(\mathcal{F}') \simeq i_*^{\mathrm{IndCoh}}(i^*(\mathcal{E}) \otimes \mathcal{F}').$$

Note that the functor i_*^{IndCoh} is t-exact (since i_* is), and $i^*(\mathcal{E})$ has Tor dimension bounded by the same integer n.

This reduces the assertion of the lemma to the case when S is classical. Further, by Corollary 4.2.3 (which will be proved independently later), the statement is Zariski local, so we can assume that S is affine.

In the latter case, the assumption on $\mathcal{E}$ implies that it can be represented by a complex of flat $\mathcal{O}_S$-modules that lives in the cohomological degrees $\geq -n$. This reduces the assertion further to the case when $\mathcal{E}$ is a flat $\mathcal{O}_S$-module in degree 0. In this case we claim that the functor

$$\mathcal{E} \otimes - : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S)$$

is t-exact.

The latter follows from Lazard's lemma: such $\mathcal{E}$ is a filtered colimit of locally free $\mathcal{O}_S$-modules $\mathcal{E}_0$, while for each such $\mathcal{E}_0$, the functor $\mathcal{E}_0 \otimes - : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S)$ is by definition the ind-extension of the functor

$$\mathcal{E}_0 \otimes - : \mathrm{Coh}(S) \to \mathrm{Coh}(S),$$

and the latter is t-exact.

□

4. Properties of IndCoh inherited from QCoh

In this section we retain the assumption that all DG schemes considered are Noetherian.

4.1. **Localization.** Let S be a DG scheme, and let $j:\overset{\circ}{S}\hookrightarrow S$ be an open embedding. By Lemma 3.5.8, we have a pair of mutually adjoint functors

$$j^{\mathrm{IndCoh},*}:\mathrm{IndCoh}(S)\rightleftarrows \mathrm{IndCoh}(\overset{\circ}{S}):j_*^{\mathrm{IndCoh}}.$$

Lemma 4.1.1. *The functor j_*^{IndCoh} is fully faithful.*

Proof. To prove that j_*^{IndCoh} is fully faithful, we have to show that the counit of the adjunction

$$j^{\mathrm{IndCoh},*}\circ j_*^{\mathrm{IndCoh}}\to \mathrm{Id}_{\mathrm{IndCoh}(\overset{\circ}{S})} \tag{4.1}$$

is an isomorphism.

Since both sides are continuous functors, it is sufficient to do so when evaluated on objects of $\mathrm{Coh}(\overset{\circ}{S})$, in which case both sides of (4.1) belong to $\mathrm{IndCoh}(\overset{\circ}{S})^+$. Therefore, by Proposition 1.2.4, it is sufficient to show that the map

$$\Psi_{\overset{\circ}{S}}\circ j^{\mathrm{IndCoh},*}\circ j_*^{\mathrm{IndCoh}}\to \Psi_{\overset{\circ}{S}}$$

is an isomorphism. However, by Propositions 3.1.1 and 3.5.4, we have a commutative diagram

$$\begin{array}{ccc}
\Psi_{\overset{\circ}{S}}\circ j^{\mathrm{IndCoh},*}\circ j_*^{\mathrm{IndCoh}} & \longrightarrow & \Psi_{\overset{\circ}{S}} \\
\sim\downarrow & & \downarrow \mathrm{id} \\
j^*\circ j_*\circ \Psi_{\overset{\circ}{S}} & \longrightarrow & \Psi_{\overset{\circ}{S}}
\end{array}$$

and the assertion follows from the fact that $j^*\circ j_*\to \mathrm{Id}_{\mathrm{QCoh}(\overset{\circ}{S})}$ is an isomorphism.

□

4.1.2. Let now $i:S'\hookrightarrow S$ be a closed embedding whose image is complementary to $\overset{\circ}{S}$. Let $\mathrm{IndCoh}(S)_{S'}$ denote the full subcategory of $\mathrm{IndCoh}(S)$ equal to

$$\ker\left(j^{\mathrm{IndCoh},*}:\mathrm{IndCoh}(S)\to \mathrm{IndCoh}(\overset{\circ}{S})\right);$$

we denote the tautological functor $\mathrm{IndCoh}(S)_{S'}\hookrightarrow \mathrm{IndCoh}(S)$ by $\widehat{i}_*^{\mathrm{IndCoh}}$.

Remark 4.1.3. It is shown in [GR2, Proposition 7.4.5] that the category $\mathrm{IndCoh}(S)_{S'}$ is intrinsic to the ind-scheme equal to the formal completion of S at S'.

4.1.4. From Lemma 4.1.1 we obtain:

Corollary 4.1.5. *The functor $\widehat{i}_*^{\mathrm{IndCoh}}$ admits a continuous right adjoint (denoted $\widehat{i}^!$) making $\mathrm{IndCoh}(S)_{S'}$ into a colocalization of $\mathrm{IndCoh}(S)$. The kernel of $\widehat{i}^!$ is the essential image of the functor j_*^{IndCoh}. the kernel of $\widehat{i}^!$ is the essential image of the functor j_*^{IndCoh}.*

We can depict the assertion of Corollary 4.1.5 as a "short exact sequence of DG categories":

$$\mathrm{IndCoh}(S)_{S'}\underset{\widehat{i}^!}{\overset{\widehat{i}_*^{\mathrm{IndCoh}}}{\rightleftarrows}}\mathrm{IndCoh}(S)\underset{j_*^{\mathrm{IndCoh}}}{\overset{j^{\mathrm{IndCoh},*}}{\rightleftarrows}}\mathrm{IndCoh}(\overset{\circ}{S}).$$

4.1.6. It is clear that the essential image of the functor

$$i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S') \to \mathrm{IndCoh}(S)$$

lies in $\mathrm{IndCoh}(S)_{S'}$, and that its right adjoint

$$i^! : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S')$$

factors through the colocalization $\widehat{i}^!$. We will denote the resulting pair of adjoint functors as follows:

$$'i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S') \rightleftarrows \mathrm{IndCoh}(S)_{S'} : {}'i^!.$$

Proposition 4.1.7. (a) *The functor* $'i^!$ *is conservative. The essential image of* $'i_*^{\mathrm{IndCoh}}$ *generates the category* $\mathrm{IndCoh}(S)_{S'}$.
(b) *The category* $\mathrm{IndCoh}(S)_{S'}$ *identifies with the ind-completion of*

$$\mathrm{Coh}(S)_{S'} := \ker\left(j^* : \mathrm{Coh}(S) \to \mathrm{Coh}(\overset{\circ}{S})\right).$$

Proof. First, we observe that the two statements of point (a) are equivalent. We will prove the second statement.

Observe also that the functor $j^{\mathrm{IndCoh},*}$ is t-exact, so the subcategory

$$\mathrm{IndCoh}(S)_{S'} \subset \mathrm{IndCoh}(S)$$

is stable under taking truncations. In particular, it inherits a t-structure.

The category $\mathrm{IndCoh}(S)$ is generated by $\mathrm{IndCoh}(S)^+$. The functor $\widehat{i}^!$ is explicitly given by

$$\mathcal{F} \mapsto \mathrm{Cone}\left(\mathcal{F} \to j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*}(\mathcal{F})\right)[-1],$$

from which it is clear that $\mathrm{IndCoh}(S)_{S'}$ is also generated by

$$\mathrm{IndCoh}(S)^+_{S'} = \mathrm{IndCoh}(S)_{S'} \cap \mathrm{IndCoh}(S)^+.$$

Note $\mathrm{QCoh}(S)$ is right-complete in its t-structure (every object is isomorphic to the colimit to its $\tau^{\leq n}$-truncations). Hence, by Proposition 1.2.4, the same is true for $\mathrm{IndCoh}(S)^+$, and threrefore also for $\mathrm{IndCoh}(S)^+_{S'}$. From here we obtain that $\mathrm{IndCoh}(S)_{S'}$ is generated by

$$\mathrm{IndCoh}(S)^b_{S'} = \mathrm{IndCoh}(S)_{S'} \cap \mathrm{IndCoh}(S)^b,$$

and hence, by devissage, by

$$\mathrm{IndCoh}(S)^\heartsuit_{S'} = \mathrm{IndCoh}(S)_{S'} \cap \mathrm{IndCoh}(S)^\heartsuit.$$

Therefore, it is sufficient to show that every object of $\mathrm{IndCoh}(S)^\heartsuit_{S'}$ admits an (increasing) filtration with subquotients belonging to the essential image of the functor

$$i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S')^\heartsuit \to \mathrm{IndCoh}(S)^\heartsuit.$$

However, the latter is obvious: by Proposition 1.2.4, the functor Ψ gives rise to a commutative diagram

$$\begin{array}{ccccc}
\mathrm{IndCoh}(S')^\heartsuit & \xrightarrow{i_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(S)^\heartsuit & \xrightarrow{j^{\mathrm{IndCoh},*}} & \mathrm{IndCoh}(\overset{\circ}{S})^\heartsuit \\
\Psi_{S'}\downarrow & & \downarrow\Psi_S & & \downarrow\Psi_{\overset{\circ}{S}} \\
\mathrm{QCoh}(S')^\heartsuit & \xrightarrow{i_*} & \mathrm{QCoh}(S)^\heartsuit & \xrightarrow{j^*} & \mathrm{QCoh}(\overset{\circ}{S})^\heartsuit
\end{array}$$

where the vertical arrows are equivalences, and the corresponding assertion for

$$\ker\left(j^*:\mathrm{QCoh}(S)^{\heartsuit}\to\mathrm{QCoh}(\overset{\circ}{S})^{\heartsuit}\right)$$

is manifest.

To prove point (b) we note that ind-extending the tautological embedding

$$\mathrm{Coh}(S)_{S'}\hookrightarrow\mathrm{Coh}(S),$$

we obtain a fully faithful functor

$$\mathrm{Ind}(\mathrm{Coh}(S)_{S'})\to\mathrm{IndCoh}(S)_{S'}.$$

Therefore, it remains to show that it is essentially surjective, but this is implied by point (a). □

Corollary 4.1.8. *Let $f:S_1\to S_2$ be a morphism that induces an isomorphism of the underlying reduced classical schemes: $({}^{cl}S_1)_{red}\simeq({}^{cl}S_2)_{red}$. Then the essential image of f_*^{IndCoh} generates the target category, and, equivalently, the functor $f^!$ is conservative.*

4.1.9. Let us note that an argument similar to that in the proof of Proposition 4.1.7 shows the following:

Let η be a generic point of $({}^{cl}S)_{red}$, and let S_η be the corresponding localized DG scheme. We have a natural restriction functor

$$\mathrm{IndCoh}(S)\to\mathrm{IndCoh}(S_\eta).$$

Lemma 4.1.10. *The subcategory* $\ker(\mathrm{IndCoh}(S)\to\mathrm{IndCoh}(S_\eta))$ *is generated by the union of the essential images of* $\mathrm{IndCoh}(S')$ *for closed embeddings $S'\to S$ such that $({}^{cl}S')_{red}\cap\eta=\emptyset$.*

4.2. **Zariski descent.** Let $f:S'\to S$ be a Zariski cover of a scheme S (i.e., S' is a finite disjoint union of open subsets in S that cover it). Let $S'^{\bullet}/S$ be its Čech nerve.

Restriction along open embedding makes $\mathrm{IndCoh}(S)\to\mathrm{IndCoh}(S'^{\bullet}/S)$ into an augmented cosimplicial object in $\mathrm{DGCat}_{\mathrm{cont}}$.

Proposition 4.2.1. *Under the above circumstances, the natural map*

$$\mathrm{IndCoh}(S)\to|\,\mathrm{Tot}(\mathrm{IndCoh}(S'^{\bullet}/S))|$$

is an equivalence.

Proof. The usual argument reduces the assertion of the proposition to the following. Let $S=U_1\cup U_2$; $U_{12}=U_1\cap U_2$. Let

$$U_1\overset{j_1}{\hookrightarrow}S,\ U_2\overset{j_2}{\hookrightarrow}S,\ U_{12}\overset{j_{12}}{\hookrightarrow}S,\ U_{12}\overset{j_{12,1}}{\hookrightarrow}U_1,\ U_{12}\overset{j_{12,2}}{\hookrightarrow}U_2$$

denote the corresponding open embeddings.

We need to show that the functor

$$\mathrm{IndCoh}(S)\to\mathrm{IndCoh}(U_1)\underset{\mathrm{IndCoh}(U_{12})}{\times}\mathrm{IndCoh}(U_1)$$

that sends $\mathcal{F}\in\mathrm{IndCoh}(S)$ to the datum of

$$\{j_1^{\mathrm{IndCoh},*}(\mathcal{F}),j_2^{\mathrm{IndCoh},*}(\mathcal{F}),j_{12,1}^{\mathrm{IndCoh},*}(j_1^{\mathrm{IndCoh},*}(\mathcal{F}))\simeq j_{12}^{\mathrm{IndCoh},*}(\mathcal{F})\simeq j_{12,2}^{\mathrm{IndCoh},*}(j_2^{\mathrm{IndCoh},*}(\mathcal{F}))\}$$

is an equivalence.

We construct a functor

$$\mathrm{IndCoh}(U_1) \underset{\mathrm{IndCoh}(U_{12})}{\times} \mathrm{IndCoh}(U_1) \to \mathrm{IndCoh}(S)$$

by sending

$$\{\mathcal{F}_1 \in \mathrm{IndCoh}(U_1), \mathcal{F}_2 \in \mathrm{IndCoh}(U_2), \mathcal{F}_{12} \in \mathrm{IndCoh}(U_{12}), j_{12,1}^{\mathrm{IndCoh},*}(\mathcal{F}_1) \simeq \mathcal{F}_{12} \simeq j_{12,2}^{\mathrm{IndCoh},*}(\mathcal{F}_2)\}$$

to

$$\mathrm{Cone}\Big(\big((j_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \oplus (j_2)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)\big) \to (j_{12})_*^{\mathrm{IndCoh}}(\mathcal{F}_{12})\Big)[-1],$$

where the maps $(j_i)_*^{\mathrm{IndCoh}}(\mathcal{F}_i) \to (j_{12})_*^{\mathrm{IndCoh}}(\mathcal{F}_{12})$ are

$$(j_i)_*^{\mathrm{IndCoh}}(\mathcal{F}_i) \to (j_i)_*^{\mathrm{IndCoh}} \circ (j_{12,i})_*^{\mathrm{IndCoh}} \circ (j_{12,i})^{\mathrm{IndCoh},*}(\mathcal{F}_i) =$$
$$= (j_{12})_*^{\mathrm{IndCoh}} \circ (j_{12,i})^{\mathrm{IndCoh},*}(\mathcal{F}_i) \simeq (j_{12})_*^{\mathrm{IndCoh}}(\mathcal{F}_{12}).$$

It is strightforward to see from Lemmas 4.1.1 and 3.6.9 that the composition

$$\mathrm{IndCoh}(U_1) \underset{\mathrm{IndCoh}(U_{12})}{\times} \mathrm{IndCoh}(U_1) \to \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(U_1) \underset{\mathrm{IndCoh}(U_{12})}{\times} \mathrm{IndCoh}(U_1)$$

is canonically isomorphic to the identity functor.

To prove that the composition

$$\mathrm{IndCoh}(S) \to \mathrm{IndCoh}(U_1) \underset{\mathrm{IndCoh}(U_{12})}{\times} \mathrm{IndCoh}(U_1) \to \mathrm{IndCoh}(S)$$

is also isomorphic to the identity functor, it is sufficient to show that for $\mathcal{F} \in \mathrm{IndCoh}(S)$, the canonical map from it to

$$\mathrm{Cone}\Big(\big((j_1)_*^{\mathrm{IndCoh}} \circ (j_1)^{\mathrm{IndCoh},*}(\mathcal{F}) \oplus (j_2)_*^{\mathrm{IndCoh}} \circ (j_2)^{\mathrm{IndCoh},*}(\mathcal{F})\big) \to \to (j_{12})_*^{\mathrm{IndCoh}} \circ j_{12}^{\mathrm{IndCoh},*}(\mathcal{F})\Big)[-1] \tag{4.2}$$

is an isomorphism.

Since all functors in question are continuous, it is sufficient to do so for $\mathcal{F} \in \mathrm{Coh}(S)$. In this case, both sides of (4.2) belong to $\mathrm{IndCoh}(S)^+$. So, it is enough to prove that the map in question becomes an isomorphism after applying the functor Ψ_S. However, in this case we are dealing with the map

$$\Psi_S(\mathcal{F}) \to \mathrm{Cone}\Big(\big((j_1)_* \circ (j_1)^*(\Psi_S(\mathcal{F})) \oplus (j_2)_* \circ (j_2)^*(\Psi_S(\mathcal{F}))\big) \to (j_{12})_* \circ j_{12}^*(\Psi_S(\mathcal{F}))\Big)[-1],$$

which is an isomorphism as it expresses Zariski descent for QCoh.

□

4.2.2. As a consequence of the above proposition we obtain:

Corollary 4.2.3. *The t-structure on* $\mathrm{IndCoh}(S)$ *is Zariski-local. I.e., an object is connective/coconnective if and only if it is such when restricted to a Zariski open cover.*

Proof. It is clear that the functor $j^{\mathrm{IndCoh},*}$ for an open embedding is t-exact. So, if $\mathcal{F} \in \mathrm{IndCoh}(S)$ is connective/coconnective, then so is $f^{\mathrm{IndCoh},*}(\mathcal{F})$.

Vice versa, suppose first that $f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(S')$ is connective, and we wish to show that $\mathcal{F}$ itself is connective. By Corollary 1.2.5, it is sufficient to show that $\Psi_S(\mathcal{F})$ is connective as an object of $\mathrm{QCoh}(S)$. But the latter follows from the fact that $f^*(\Psi_S(\mathcal{F})) \simeq \Psi_{S'}(f^{\mathrm{IndCoh},*}(\mathcal{F}))$ is connective.

Now, let $\mathcal{F} \in \mathrm{IndCoh}(S)$ be such that $f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(S')$ is coconnective. We need to show that $\mathcal{F}$ itself is coconnective, i.e., that for every $\mathcal{F}' \in \mathrm{IndCoh}(S')^{<0}$, we have $\mathrm{Maps}_{\mathrm{IndCoh}(S)}(\mathcal{F}', \mathcal{F}) = 0$. However, this follows from Proposition 4.2.1 as all the terms in $\mathrm{Maps}_{\mathrm{IndCoh}(S'^\bullet/S)}(\mathcal{F}'|_{S'^\bullet/S}, \mathcal{F}|_{S'^\bullet/S}[i])$ are zero for $i \leq 0$. □

4.3. **The convergence property of** IndCoh.

4.3.1. Let us recall the notion of n-coconnective DG scheme (see [GL:Stacks], Sect. 3.2). For a DG scheme S and an integer n, let $\tau^{\leq n}(S)$ denote its n-coconnective truncation (see [GL:Stacks], Sects. 1.1.2, 1.1.3).

Let us denote by i_n the corresponding map

$$\tau^{\leq n}(S) \hookrightarrow S,$$

and for $n_1 \leq n_2$, by i_{n_1,n_2} the map

$$\tau^{\leq n_1}(S) \hookrightarrow \tau^{\leq n_2}(S).$$

Remark 4.3.2. It follows from [GL:Stacks], Lemma 3.1.5 and Sect. 1.2.6, the map

$$\underset{n}{colim}\, \tau^{\leq n}(S) \to S$$

is an isomorphism, where the colimit is taken in the category DGSch.

4.3.3. We have an $\mathbb{N}$-diagram of categories

$$n \mapsto \mathrm{IndCoh}(\tau^{\leq n}(S)),$$

with the functors

$$\mathrm{IndCoh}(\tau^{\leq n_1}(S)) \to \mathrm{IndCoh}(\tau^{\leq n_2}(S))$$

given by $(i_{n_1,n_2})_*^{\mathrm{IndCoh}}$.

These functors admit right adjoints, given by $(i_{n_1,n_2})^!$, which gives rise to the corresponding $\mathbb{N}^{\mathrm{op}}$-diagram of categories. According to [GL:DG], Lemma. 1.3.3, we have:

$$\underset{\mathbb{N},(i_{n_1,n_2})_*^{\mathrm{IndCoh}}}{colim} \mathrm{IndCoh}(\tau^{\leq n}(S)) \simeq \underset{\mathbb{N},(i_{n_1,n_2})^!}{lim} \mathrm{IndCoh}(\tau^{\leq n}(S)). \tag{4.3}$$

Moreover, the functors $(i_n)_*^{\mathrm{IndCoh}}$ define a functor

$$\underset{\mathbb{N},(i_{n_1,n_2})_*^{\mathrm{IndCoh}}}{colim} \mathrm{IndCoh}(\tau^{\leq n}(S)) \to \mathrm{IndCoh}(S), \tag{4.4}$$

and the functors $(i_n)^!$ define a functor

$$\underset{\mathbb{N},(i_{n_1,n_2})^!}{lim} \mathrm{IndCoh}(\tau^{\leq n}(S)) \leftarrow \mathrm{IndCoh}(S), \tag{4.5}$$

which is the right adjoint of the functor (4.4) under the identification (4.3).

Proposition 4.3.4. *The functors* (4.4) *and* (4.5) *are mutually inverse equivalences.*

Remark 4.3.5. Proposition 4.3.4 can be viewed as an expression of the fact that the study of the category IndCoh reduces to the case of eventually coconnective DG schemes.

Proof. The essential image of the functor (4.4) generates the target category by Corollary 4.1.8. Since it also maps compact objects to compact ones, it is sufficient to show that it is fully faithful when restricted to compact objects.

In other words, we have to show that for $\mathcal{F}', \mathcal{F}'' \in \mathrm{Coh}(\tau^{\leq m}(S))$ for some m, the map

$$\underset{n \geq m}{colim}\ \mathrm{Hom}\left((i_{n,m})_*(\mathcal{F}'), (i_{n,m})_*(\mathcal{F}'')\right) \to \mathrm{Hom}\left((i_n)_*(\mathcal{F}'), (i_n)_*(\mathcal{F}'')\right) \tag{4.6}$$

is an isomorphism.

We claim that the colimit (4.6) stabilizes at some finite n, and the stable value maps isomorphically to the right-hand side of (4.6). This follows from Proposition 4.2.1 and the next general observation:

Let A be a connective DG ring, and let M' and M'' be two A-modules belonging to $(A\text{-mod})^b$. In particular, we can view M' and M'' as $\tau^{\geq -n}(A)$-modules for all n large enough. Suppose that $M' \in (A\text{-mod})^{\leq m'}$ and that $M'' \in (A\text{-mod})^{\geq -m''}$.

Lemma 4.3.6. *Under the above circumstances, the map*

$$\mathrm{Hom}_{\tau^{\geq -n}(A)\text{-mod}}(M', M'') \to \mathrm{Hom}_{A\text{-mod}}(M', M'')$$

is an isomorphism whenever $n > m' + m''$.

□

4.4. **Tensoring up.**

4.4.1. Let $f : S_1 \to S_2$ be a map of bounded Tor dimension. By Proposition 3.5.4, $f^{\mathrm{IndCoh},*}$ induces a functor

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1). \tag{4.7}$$

Proposition 4.4.2. *The functor in* (4.7) *is fully faithful.*

Proof. We note that the left-hand side in (4.7) is compactly generated by objects of the form

$$\mathcal{E}_1 \otimes \mathcal{F}_2 \in \mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2),$$

where $\mathcal{E}_1 \in \mathrm{QCoh}(S_1)^{\mathrm{perf}}$ and $\mathcal{F}_2 \in \mathrm{Coh}(S_2)$. Moreover, the functor $(\mathrm{Id}_{\mathrm{QCoh}(S_1)} \otimes f^{\mathrm{IndCoh},*})$ sends these objects to compact objects in $\mathrm{IndCoh}(S_1)$.

Hence, it is enough to show that for $\mathcal{E}'_1, \mathcal{E}''_1$ and $\mathcal{F}'_2, \mathcal{F}''_2$ as above, the map

$$\begin{split} \mathrm{Maps}_{\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2)}(\mathcal{E}'_1 \otimes \mathcal{F}'_2, \mathcal{E}''_1 \otimes \mathcal{F}''_2) \to \\ \to \mathrm{Maps}_{\mathrm{IndCoh}(S_1)}(\mathcal{E}'_1 \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}'_2), \mathcal{E}''_1 \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}''_2)) \end{split} \tag{4.8}$$

is an isomorphism, where in the right-hand side $\otimes$ denotes the action of QCoh on IndCoh.

We can rewrite the map in (4.8) as

$$\begin{split} \mathrm{Maps}_{\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2)}(\mathcal{O}_{S_1} \otimes \mathcal{F}'_2, \mathcal{E}_1 \otimes \mathcal{F}''_2) \to \\ \to \mathrm{Maps}_{\mathrm{IndCoh}(S_1)}(\mathcal{O}_{S_1} \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}'_2), \mathcal{E}_1 \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}''_2)), \end{split} \tag{4.9}$$

where $\mathcal{E}_1 \simeq \mathcal{E}''_1 \otimes (\mathcal{E}'_1)^\vee$.

We note that the functor right adjoint to

$$\mathrm{IndCoh}(S_2) \simeq \mathrm{QCoh}(S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \overset{f^* \otimes \mathrm{Id}_{\mathrm{IndCoh}(S_2)}}{\longrightarrow} \mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2)$$

is given by

$$\mathrm{IndCoh}(S_2) \simeq \mathrm{QCoh}(S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \overset{f_* \otimes \mathrm{Id}_{\mathrm{IndCoh}(S_2)}}{\longleftarrow} \mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2).$$

Hence, we can rewrite the map in (4.9) as the map

$$\mathrm{Maps}_{\mathrm{IndCoh}(S_2)}(\mathcal{F}_2', f_*(\mathcal{E}_1) \otimes \mathcal{F}_2'') \to \mathrm{Maps}_{\mathrm{IndCoh}(S_2)}(\mathcal{F}_2', f_*^{\mathrm{IndCoh}}(\mathcal{E}_1 \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_2'')))$$

coming from (3.17).

Hence, the required isomorphism follows from Proposition 3.6.11. □

As a formal corollary we obtain:

Corollary 4.4.3. *In the situation of Proposition 4.4.2, the natural map of endo-functors of* $\mathrm{IndCoh}(S_2)$

$$f_*(\mathcal{O}_{S_1}) \otimes - \to f_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh},*}$$

is an isomorphism.

4.4.4. The next corollary expresses the category IndCoh on an open subscheme:

Corollary 4.4.5. *Ler $j : \overset{\circ}{S} \hookrightarrow S$ be an open embedding. Then the functor*

$$\mathrm{QCoh}(\overset{\circ}{S}) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(\overset{\circ}{S})$$

of (4.7) *is an equivalence.*

Proof. By Proposition 4.4.2, the functor in question is fully faithful. Thus, it remains to show that its essential image generates $\mathrm{IndCoh}(\overset{\circ}{S})$, for which it would suffice to show that the essential image of the composition

$$\mathrm{IndCoh}(S) \to \mathrm{QCoh}(\overset{\circ}{S}) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(\overset{\circ}{S})$$

generates $\mathrm{IndCoh}(\overset{\circ}{S})$. However, the latter follows from Lemma 4.1.1. □

4.4.6. Let now

$$\begin{array}{ccc} \overset{\circ}{S}_1 & \overset{j_1}{\longrightarrow} & S_1 \\ {\scriptstyle \overset{\circ}{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \overset{\circ}{S}_2 & \overset{j_2}{\longrightarrow} & S_2 \end{array}$$

be a Cartesian diagram of DG schemes, where the maps j_1 and j_2 are open embeddings and the map f is proper.

The base change isomorphism

$$\overset{\circ}{f}{}_*^{\mathrm{IndCoh}} \circ j_2^{\mathrm{IndCoh},*} \simeq j_1^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}}$$

of Lemma 3.6.9 gives rise to a natural transformation

$$j_1^{\mathrm{IndCoh},*} \circ f^! \to \overset{\circ}{f}{}^! \circ j_2^{\mathrm{IndCoh},*} \tag{4.10}$$

of functors $\mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(\overset{\circ}{S}_1)$.

We claim:

Corollary 4.4.7. *The natural transformation* (4.10) *is an isomorphism.*

Remark 4.4.8. As we shall see in Proposition 7.1.6, the assertion of Corollary 4.4.7 holds more generally when the maps $j_i : \overset{\circ}{S}_i \to S_i$ just have a bounded Tor dimension.

Remark 4.4.9. There exists another natural transformation

$$j_1^{\mathrm{IndCoh},*} \circ f^! \to \overset{\circ}{f}{}^! \circ j_2^{\mathrm{IndCoh},*}, \tag{4.11}$$

namely one coming by adjunction from the base change isomorphism

$$f^! \circ (j_2)_*^{\mathrm{IndCoh}} \simeq (j_1)_*^{\mathrm{IndCoh}} \circ \overset{\circ}{f}{}^!$$

of Proposition 3.4.2. A diagram chase that the natural transformations (4.10) and (4.11) are canonically isomorphic.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{IndCoh}(S_1) & \xrightarrow{\mathrm{Id}_{\mathrm{IndCoh}(S_1)} \otimes j_2^*} & \mathrm{IndCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{QCoh}(\overset{\circ}{S}_2) & \longrightarrow & \mathrm{IndCoh}(\overset{\circ}{S}_1) \\
{\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^! \otimes \mathrm{Id}_{\mathrm{QCoh}(\overset{\circ}{S}_2)}} & & \uparrow{\scriptstyle \overset{\circ}{f}{}^!} \\
\mathrm{IndCoh}(S_2) & \xrightarrow{\mathrm{Id}_{\mathrm{IndCoh}(S_2)} \otimes j_2^*} & \mathrm{IndCoh}(S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{QCoh}(\overset{\circ}{S}_2) & \longrightarrow & \mathrm{IndCoh}(\overset{\circ}{S}_2),
\end{array}$$

where both right horizontal arrows are those of (4.7), and the middle vertical arrow is well-defined in view of Corollary 3.3.11.

The left square commutes tautologically. Hence, it remains to show that the right square commutes as well. As the horizontal arrows are equivalences (by Corollary 4.4.5), it would be sufficient to show that the corresponding square commutes, when we replace the vertical arrows by their respective left adjoints.

The commutativity of

$$\begin{array}{ccc}
\mathrm{IndCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{QCoh}(\overset{\circ}{S}_2) & \longrightarrow & \mathrm{IndCoh}(\overset{\circ}{S}_1) \\
{\scriptstyle f_*^{\mathrm{IndCoh}} \otimes \mathrm{Id}_{\mathrm{QCoh}(\overset{\circ}{S}_2)}}\downarrow & & \downarrow{\scriptstyle \overset{\circ}{f}{}_*^{\mathrm{IndCoh}}} \\
\mathrm{IndCoh}(S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{QCoh}(\overset{\circ}{S}_2) & \longrightarrow & \mathrm{IndCoh}(\overset{\circ}{S}_2)
\end{array}$$

as $\mathrm{QCoh}(\overset{\circ}{S}_2)$-module categories is equivalent to the commutativity of

$$\begin{array}{ccc}
\mathrm{IndCoh}(S_1) & \xrightarrow{j_1^{\mathrm{IndCoh},*}} & \mathrm{IndCoh}(\overset{\circ}{S}_1) \\
{\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle \overset{\circ}{f}{}_*^{\mathrm{IndCoh}}} \\
\mathrm{IndCoh}(S_2) & \xrightarrow{j_2^{\mathrm{IndCoh},*}} & \mathrm{IndCoh}(\overset{\circ}{S}_2)
\end{array}$$

as $\mathrm{QCoh}(S_2)$-module categories. However, the latter is the original isomorphism of functors that gives rise to (4.10). □

4.5. **Smooth maps.**

4.5.1. Let us recall that a morphism $f: S_1 \to S_2$ of DG schemes is called *flat* (resp., *smooth*) if:

- The fiber product DG scheme ${}^{cl}S_2 \underset{S_2}{\times} S_1$ is classical;
- The resulting map of classical schemes ${}^{cl}S_2 \underset{S_2}{\times} S_1 \to {}^{cl}S_2$ is flat (resp., smooth).

It is easy to see that a flat morphism is of bounded Tor dimension, and in particular, eventually coconnective.

4.5.2. We shall now prove a generalization of Corollary 4.4.5 when instead of an open embedding we have a smooth map between DG schemes.

Proposition 4.5.3. *Assume that in the situation of Proposition 4.4.2, the map f is smooth. Then the functor*

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$$

of (4.7) *is an equivalence.*

Proof. By Proposition 4.4.2, the functor in question is fully faithful, so we only need to show that it its essential image generates $\mathrm{IndCoh}(S_1)$.

We have a canonical isomorphism

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(({}^{cl}S_2)_{red}) \simeq \mathrm{QCoh}(({}^{cl}S_1)_{red}) \underset{\mathrm{QCoh}(({}^{cl}S_2)_{red})}{\otimes} \mathrm{IndCoh}(({}^{cl}S_2)_{red}),$$

and by Corollary 4.1.8 this reduces the assertion to the case when S_2 is classical and reduced.

By Noetherian induction, we can assume that the assertion holds for all proper closed subschemes of S_2. By Lemma 4.1.10, this reduces the assertion to the case when $S_2 = \mathrm{Spec}(K)$, where K is a field.

However, in the latter case, the assertion follows from Lemma 1.1.6.

□

4.6. **Behavior with respect to products.** We shall now establish one more property of IndCoh. In this we will be assuming that all our schemes are almost of finite type over the ground field k, i.e., $\mathrm{DGSch}_{\mathrm{aft}} \subset \mathrm{DGSch}_{\mathrm{Noeth}}$.

4.6.1. Thus, let S_1 and S_2 be two DG schemes almost of finite type over k. Consider their product $S_1 \times S_2$, which also has the same property.

External tensor product defines a functor

$$\mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1 \times S_2),$$

such that if $\mathcal{F}_i \in \mathrm{Coh}(S_i)$, we have $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \mathrm{Coh}(S_1 \times S_2)$.

Consider the resulting functor

$$\mathrm{IndCoh}(S_1) \otimes \mathrm{IndCoh}(S_2) \overset{\Psi_{S_1} \otimes \Psi_{S_2}}{\longrightarrow} \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1 \times S_2).$$

By the above, it sends compact objects in $\mathrm{IndCoh}(S_1) \otimes \mathrm{IndCoh}(S_2)$ to $\mathrm{Coh}(S_1 \times S_2)$. Hence, we obtain a functor

$$\mathrm{IndCoh}(S_1) \otimes \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1 \times S_2), \tag{4.12}$$

which makes the diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(S_1)\otimes \mathrm{IndCoh}(S_2) & \longrightarrow & \mathrm{IndCoh}(S_1\times S_2) \\
{\scriptstyle \Psi_{S_1}\otimes\Psi_{S_2}}\downarrow & & \downarrow{\scriptstyle \Psi_{S_1\times S_2}} \\
\mathrm{QCoh}(S_1)\otimes \mathrm{QCoh}(S_2) & \longrightarrow & \mathrm{QCoh}(S_1\times S_2)
\end{array} \tag{4.13}
$$

commute.

Proposition 4.6.2. *The functor* (4.12) *is an equivalence.*

Proof. Both categories are compactly generated, and the functor in question sends compact objects to compacts, by construction.

Hence, to prove that it is fully faithful, it is sufficient to show that for

$$\mathcal{F}_1',\mathcal{F}_1''\in \mathrm{Coh}(S_1),\ \mathcal{F}_2',\mathcal{F}_2''\in \mathrm{Coh}(S_2),$$

the map

$$\mathcal{M}aps_{\mathrm{Coh}(S_1)}(\mathcal{F}_1',\mathcal{F}_1'')\otimes \mathcal{M}aps_{\mathrm{Coh}(S_2)}(\mathcal{F}_2',\mathcal{F}_2'')\to \mathcal{M}aps_{\mathrm{Coh}(S_1\times S_2)}(\mathcal{F}_1'\boxtimes\mathcal{F}_2',\mathcal{F}_1''\boxtimes\mathcal{F}_2'')$$

is an isomorphism (see Sect. 0.6.4 for the notation $\mathcal{M}aps(-,-)$).

I.e., we have to show that for $\mathcal{F}_i'$, $\mathcal{F}_i''$ as above, the map

$$\mathcal{M}aps_{\mathrm{QCoh}(S_1)}(\mathcal{F}_1',\mathcal{F}_1'')\otimes \mathcal{M}aps_{\mathrm{QCoh}(S_2)}(\mathcal{F}_2',\mathcal{F}_2'')\to \mathcal{M}aps_{\mathrm{QCoh}(S_1\times S_2)}(\mathcal{F}_1'\boxtimes\mathcal{F}_2',\mathcal{F}_1''\boxtimes\mathcal{F}_2'') \tag{4.14}$$

is an isomorphism.

Recall that if $\mathbf{C}_1,\mathbf{C}_2$ are DG categories and $\mathbf{c}_i',\mathbf{c}_i''\in\mathbf{C}_i$, $i=1,2$ are such that $\mathbf{c}_1'$ and $\mathbf{c}_2'$ are compact, then the natural map

$$\mathcal{M}aps_{\mathbf{C}_1}(\mathbf{c}_1',\mathbf{c}_1'')\otimes \mathcal{M}aps_{\mathbf{C}_2}(\mathbf{c}_2',\mathbf{c}_2'')\to \mathcal{M}aps_{\mathbf{C}_1\otimes\mathbf{C}_2}(\mathbf{c}_1'\otimes\mathbf{c}_2',\mathbf{c}_1''\otimes\mathbf{c}_2'') \tag{4.15}$$

is an isomorphism.

The isomorphism(4.14) is not immediate since the objects $\mathcal{F}_i'\in \mathrm{QCoh}(S_i)$ are not compact. To circumvent this, we proceed as follows.

It is enough to show that

$$\tau^{\leq -n}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_1)}(\mathcal{F}_1',\mathcal{F}_1'')\otimes \mathcal{M}aps_{\mathrm{QCoh}(S_2)}(\mathcal{F}_2',\mathcal{F}_2'')\right)\to$$
$$\to \tau^{\leq -n}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_1\times S_2)}(\mathcal{F}_1'\boxtimes\mathcal{F}_2',\mathcal{F}_1''\boxtimes\mathcal{F}_2'')\right)$$

is an isomorphism for any n.

Choose $\alpha_1:\widetilde{\mathcal{F}}_1'\to\mathcal{F}_1'$ (resp., $\alpha_2:\widetilde{\mathcal{F}}_2'\to\mathcal{F}_2'$) with $\widetilde{\mathcal{F}}_1'$ (resp., $\widetilde{\mathcal{F}}_2'$) in $\mathrm{QCoh}(S_1)^{\mathrm{perf}}$ (resp., $\mathrm{QCoh}(S_2)^{\mathrm{perf}}$), such that

$$\mathrm{Cone}(\alpha_1)\in \mathrm{QCoh}(S_1)^{\leq -N} \text{ and } \mathrm{Cone}(\alpha_2)\in \mathrm{QCoh}(S_2)^{\leq -N}$$

for $N\gg 0$.

We choosing N large enough, we can ensure that

$$\tau^{\leq -m}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_i)}(\mathcal{F}_i',\mathcal{F}_i'')\right)\to \tau^{\leq -m}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_i)}(\widetilde{\mathcal{F}}_i',\mathcal{F}_i'')\right)$$

is an isomorphism for m large enough, which in turn implies that

$$\tau^{\leq -n}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_1)}(\mathcal{F}'_1,\mathcal{F}''_1)\otimes \mathcal{M}aps_{\mathrm{QCoh}(S_2)}(\mathcal{F}'_2,\mathcal{F}''_2)\right)\to \\ \to \tau^{\leq -n}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_1)}(\widetilde{\mathcal{F}}'_1,\mathcal{F}''_1)\otimes \mathcal{M}aps_{\mathrm{QCoh}(S_2)}(\widetilde{\mathcal{F}}'_2,\mathcal{F}''_2)\right)$$

and

$$\tau^{\leq -n}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_1\times S_2)}(\mathcal{F}'_1\boxtimes\mathcal{F}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)\right)\to \tau^{\leq -n}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_1\times S_2)}(\widetilde{\mathcal{F}}'_1\boxtimes\widetilde{\mathcal{F}}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)\right)$$

are isomorphisms.

Hence, it is enough to show that

$$\mathcal{M}aps_{\mathrm{QCoh}(S_1)}(\widetilde{\mathcal{F}}'_1,\mathcal{F}''_1)\otimes \mathcal{M}aps_{\mathrm{QCoh}(S_2)}(\widetilde{\mathcal{F}}'_2,\mathcal{F}''_2)\to \mathcal{M}aps_{\mathrm{QCoh}(S_1\times S_2)}(\widetilde{\mathcal{F}}'_1\boxtimes\widetilde{\mathcal{F}}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)$$

is an isomorphism. But this follows from the isomorphism (4.15) and the fact that the fuctor

$$\mathrm{QCoh}(S_1)\boxtimes \mathrm{QCoh}(S_2)\to \mathrm{QCoh}(S_1\times S_2)$$

is an equivalence, by [GL:QCoh], Prop. 1.4.4.

Thus, it remains to show that the essential image of (4.12) generates the target category. It is sufficient to show that the essential image of

$$\mathrm{IndCoh}((^{cl}S_1)_{red})\otimes \mathrm{IndCoh}((^{cl}S_2)_{red})\to \mathrm{IndCoh}((^{cl}S_1)_{red}\times(^{cl}S_1)_{red})\to \\ \to \mathrm{IndCoh}((^{cl}(S_1\times S_2)_{red})\to \mathrm{IndCoh}(S_1\times S_2)$$

generates $\mathrm{IndCoh}(S_1\times S_2)$. Hence, by Corollary 4.1.8, we can assume that S_1 and S_2 are both classical and reduced. Moreover, by Noetherian induction, we can assume that the statement holds for all proper closed subschemes of S_1 and S_2. So, it is sufficient to show that the functor

$$\mathrm{IndCoh}(\overset{\circ}{S}_1)\otimes \mathrm{IndCoh}(\overset{\circ}{S}_2)\to \mathrm{IndCoh}(\overset{\circ}{S}_1\times \overset{\circ}{S}_2)$$

is an equivalence for some non-empty open subschemes $\overset{\circ}{S}_i\subset S_i$.

Now, the $\mathrm{char}(k)=0$ assumption implies that S_1 and S_2 are generically smooth over k. I.e., we obtain that it is sufficient to show that the functor (4.12) is an equivalence when S_i are smooth. In this case $S_1\times S_2$ is also smooth, and in particular regular. Thus, all vertical maps in the diagram (4.13) are equivalences. However, the bottom arrow in (4.13) is an equivalence as well, which implies the required assertion.

□

Part II. Correspondences, !-pullback and duality.

5. The !-pullback functor for morphisms almost of finite type

All DG schemes in this section will be Noetherian. The goal of this section is to extend the functor $f^!$ to maps between DG schemes that are not necessarily proper.

5.1. **The paradigm of correspondences.**

5.1.1. Let $\mathbf{C}$ be a $(\infty,1)$-category. Let *vert* and *horiz* be two classes of 1-morphisms in $\mathbf{C}$, such that:

(1) Both classes contain all isomorphisms.
(2) If a given 1-morphism belongs to a given class, then so do all isomorphic 1-morphisms.
(3) Both classes are stable under compositions.
(4) Given a pair of morphisms $f:\mathbf{c}_1\to\mathbf{c}_2$ with $f\in vert$ and $g_2:\mathbf{c}_2'\to\mathbf{c}_2$ with $g_2\in horiz$, the Cartesian square square

$$\begin{array}{ccc} \mathbf{c}_1' & \xrightarrow{g_1} & \mathbf{c}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{c}_2' & \xrightarrow{g_2} & \mathbf{c}_2 \end{array} \tag{5.1}$$

exists, and $f'\in vert$ and $g_1\in horiz$.

We let $\mathbf{C}_{vert}\subset\mathbf{C}$ and $\mathbf{C}_{horiz}\subset\mathbf{C}$ denote the corresponding 1-full subcategories of $\mathbf{C}$.

5.1.2. We let $\mathbf{C}_{\mathrm{corr}:vert;horiz}$ denote a new $(\infty,1)$-category whose objects are the same as objects of $\mathbf{C}$. For $\mathbf{c}_1,\mathbf{c}_2\in\mathbf{C}_{\mathrm{corr}}$, the groupoid of 1-morphisms $\mathrm{Hom}_{\mathbf{C}_{\mathrm{corr}:vert;horiz}}(\mathbf{c}_1,\mathbf{c}_2)$ is the ∞-groupoid of diagrams

$$\begin{array}{ccc} \mathbf{c}_{1,2} & \xrightarrow{g} & \mathbf{c}_1 \\ \downarrow{\scriptstyle f} & & \\ \mathbf{c}_2 & & \end{array} \tag{5.2}$$

where $f\in\mathbf{C}_{vert}$ and $g\in\mathbf{C}_{horiz}$ and where compositions are given by taking Cartesian squares.

The higher categorical structure on $\mathbf{C}_{\mathrm{corr}:vert;horiz}$ is described in terms of the corresponding *complete Segal space*, i.e., an object of $\infty\text{-Grpd}^{\mathbf{\Delta}^{\mathrm{op}}}$, see Sect. 7.6.1 for a brief review.

Namely, the ∞-groupoid of n-fold compositions is that of diagrams

$$\begin{array}{ccccccccc} \mathbf{c}_{0,n} & \longrightarrow & \mathbf{c}_{0,n-1} & \longrightarrow & ... & \longrightarrow & \mathbf{c}_{0,1} & \longrightarrow & \mathbf{c}_0 \\ \downarrow & & \downarrow & & & & \downarrow & & \\ \mathbf{c}_{1,n} & \longrightarrow & \mathbf{c}_{2,n-1} & \longrightarrow & ... & \longrightarrow & \mathbf{c}_1 & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ ... & \longrightarrow & ... & \longrightarrow & ... & & & & \\ \downarrow & & \downarrow & & & & & & \\ \mathbf{c}_{n-1,n} & \longrightarrow & \mathbf{c}_{n-1} & & & & & & \\ \downarrow & & & & & & & & \\ \mathbf{c}_n, & & & & & & & & \end{array} \tag{5.3}$$

where we require that all vertical (resp., horizontal) maps belong to *vert* (resp., horiz), and each square be Cartesian.

Note that $\mathbf{C}_{\mathrm{corr}:vert;horiz}$ contains $\mathbf{C}_{vert}$ and $(\mathbf{C}_{horiz})^{\mathrm{op}}$ as 1-full subcategories, obtained by restricting 1-morphisms to those diagrams (5.2), for which g (resp., f) is an isomorphism.

5.1.3. Our input will be a functor

$$P_{\text{corr}:vert;horiz} : \mathbf{C}_{\text{corr}:vert;horiz} \to \text{DGCat}_{\text{cont}}.$$

We shall denote its value on objects simply by $P(\mathbf{c})$.

Consider the restriction of $P_{\text{corr}:vert;horiz}$ to $\mathbf{C}_{vert}$, which we denote by P_{vert}. For

$$(f : \mathbf{c}_1 \to \mathbf{c}_2) \in \mathbf{C}_{vert}$$

we let $P_{vert}(f)$ denote the corresponding 1-morphism $P(\mathbf{c}_1) \to P(\mathbf{c}_2)$ in $\text{DGCat}_{\text{cont}}$.

Consider also the restriction of $P_{\text{corr}:vert;horiz}$ to $(\mathbf{C}_{horiz})^{\text{op}}$, which we denote by $P^!_{horiz}$. For

$$(g : \mathbf{c}_1 \to \mathbf{c}_2) \in \mathbf{C}_{horiz}$$

we let $P^!_{horiz}(g)$ denote the corresponding 1-morphism $P(\mathbf{c}_2) \to P(\mathbf{c}_1)$ in $\text{DGCat}_{\text{cont}}$.

Remark 5.1.4. Note that a datum of a functor $P_{\text{corr}:vert;horiz}$ can be thought of as assignment to every Cartesian square as in (5.1), with $f, f' \in \mathbf{C}_{vert}$ and $g_1, g_2 \in \mathbf{C}_{horiz}$ an isomorphism of functors $P(\mathbf{c}_1) \rightrightarrows P(\mathbf{c}'_2)$:

$$P_{vert}(f') \circ P^!_{horiz}(g_1) \simeq P^!_{horiz}(g_2) \circ P_{vert}(f). \tag{5.4}$$

Formulating it as a functor $P_{\text{corr}:vert;horiz}$ out of the category of correspondences is a way to formulate the compatibilities that the isomorphisms (5.4) need to satisfy. The author learned this idea from J. Lurie.

5.2. **A provisional formulation.**

5.2.1. In the above framework we take $\mathbf{C} := \text{DGSch}_{\text{Noeth}}$, and *vert* to be all 1-morphisms. We take *horiz* to be morphisms almost of finite type.

Let $(\text{DGSch}_{\text{Noeth}})_{\text{corr:all;aft}}$ denote the corresponding category of correspondences.

Theorem 5.2.2. *There exists a canonically defined functor*

$$\text{IndCoh}_{(\text{DGSch}_{\text{Noeth}})_{\text{corr:all;aft}}} : (\text{DGSch}_{\text{Noeth}})_{\text{corr:all;aft}} \to \text{DGCat}_{\text{cont}}$$

with the following properties:

(a) *The restriction* $\text{IndCoh}_{(\text{DGSch}_{\text{Noeth}})_{\text{corr:all;aft}}}|_{((\text{DGSch}_{\text{Noeth}})_{\text{proper}})^{\text{op}}}$ *is canonically isomorphic to the functor* $\text{IndCoh}^!_{(\text{DGSch}_{\text{Noeth}})^{\text{op}}}$ *of Corollary 3.3.9.*

(b) *The restriction* $\text{IndCoh}_{(\text{DGSch}_{\text{Noeth}})_{\text{corr:all;aft}}}|_{((\text{DGSch}_{\text{Noeth}})_{\text{open}})^{\text{op}}}$ *is canonically isomorphic to the functor*

$$\text{IndCoh}^*_{(\text{DGSch}_{\text{Noeth}})_{\text{open}}} := \text{IndCoh}^*_{(\text{DGSch}_{\text{Noeth}})_{\text{ev-coconn}}}|_{((\text{DGSch}_{\text{Noeth}})_{\text{open}})^{\text{op}}}$$

given by Corollary 3.5.6.

(c) *The restriction* $\text{IndCoh}_{(\text{DGSch}_{\text{Noeth}})_{\text{corr:all;aft}}}|_{\text{DGSch}_{\text{Noeth}}}$ *is canonically isomorphic to the functor* $\text{IndCoh}_{\text{DGSch}_{\text{Noeth}}}$ *of Proposition 3.2.4.*

5.2.3. As one of the main corollaries of Theorem 5.2.2, we obatin:

Corollary 5.2.4. *There exists a well-defined functor*

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}} : ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

such that

(a) *The restriction* $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}|_{((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}})^{\mathrm{op}}}$ *is canonically isomorphic to the functor* $\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}}}$ *of Corollary 3.3.9.*

(b) *The restriction* $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}|_{((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{open}})^{\mathrm{op}}}$ *is canonically isomorphic to the restriction of the functor* $\mathrm{IndCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{open}}}$ *given by Corollary 3.5.6.*

5.2.5. A crucial property of the functor $\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}$ is the following base change property, see Remark 5.1.4. Let

$$\begin{array}{ccc} S_1' & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S_2' & \xrightarrow{g_2} & S_2 \end{array} \tag{5.5}$$

be a Cartesian diagram in $\mathrm{DGSch}_{\mathrm{Noeth}}$, where the horizontal arrows are almost of finite type. Then there exists a canonical isomorphism of functors

$$g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^!, \tag{5.6}$$

which we shall refer to as *base change*.

Note that, unlike the case when the morphisms g_i are proper or open embeddings, there is a priori no map in (5.6) in either direction.

One can say that the real content of Theorem 5.2.2 is the well-definedness of the functor of the !-pullback such that the isomorphism (5.6) holds.

5.3. **Compatibility with adjunction for open embeddings.**

5.3.1. Let $j : \overset{\circ}{S} \hookrightarrow S$ be an open embedding. Note that the square

$$\begin{array}{ccc} \overset{\circ}{S} & \xrightarrow{\mathrm{id}} & \overset{\circ}{S} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle j} \\ \overset{\circ}{S} & \xrightarrow{j} & S \end{array}$$

is Cartesian.

Hence, the base change isomorphism (5.6) defines an isomorphism of functors

$$j^! \circ j_*^{\mathrm{IndCoh}} \simeq \mathrm{Id}_{\mathrm{IndCoh}(\overset{\circ}{S})},$$

and in particular, a map in one direction

$$j^! \circ j_*^{\mathrm{IndCoh}} \to \mathrm{Id}_{\mathrm{IndCoh}(\overset{\circ}{S})}. \tag{5.7}$$

It will follow from the construction of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$ (specifically, from Theorem 6.1.2) that the map (5.7) is canonically isomorphic to the counit of the adjunction for $(j^!, j_*^{\mathrm{IndCoh}})$, where we are using the identification

$$j^! \simeq j^{\mathrm{IndCoh},*}$$

of Corollary 5.2.4(b).

We can interpret this is saying that the datum of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$, encodes the datum for the $(j^!, j_*^{\mathrm{IndCoh}})$-adjunction for an open embedding j.

5.3.2. From here we are going to deduce:

Proposition 5.3.3. *Consider a Cartesian diagram* (5.5) *and the corresponding isomorphism of functors*

$$g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^! \tag{5.8}$$

of (5.6).

(a) *Suppose that the morphism f (and hence f') is an open embedding. Then the natural transformation $\to$ in* (5.8) *comes via the adjunctions of $(f^!, f_*^{\mathrm{IndCoh}})$ and $((f')^!, (f')_*^{\mathrm{IndCoh}})$ from the isomorphism*

$$(f')^! \circ g_2^! \simeq g_1^! \circ f^!.$$

(b) *Suppose that the morphism g_2 (and hence g_1) is an open embedding. Then the natural transformation $\to$ in* (5.8) *comes via the adjunctions of $(g_1^!, (g_1)_*^{\mathrm{IndCoh}})$ and $(g_2^!, (g_2)_*^{\mathrm{IndCoh}})$ from the isomorphism*

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \simeq (g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}}.$$

Proof. We will prove point (a); point (b) is proved similarly. Applying the $(f^!, f_*^{\mathrm{IndCoh}})$-adjunction, the assertion is equivalent to the fact that the two natural transformations

$$(f')^! \circ g_2^! \circ f_*^{\mathrm{IndCoh}} \rightrightarrows g_1^! \tag{5.9}$$

are canonically isomorphic. The first natural transformation is

$$(f')^! \circ g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq g_1^! \circ f^! \circ f_*^{\mathrm{IndCoh}} \overset{\mathrm{id}\circ\mathrm{counit}}{\longrightarrow} g_1^!,$$

and the second natural transformation is

$$(f')^! \circ g_2^! \circ f_*^{\mathrm{IndCoh}} \overset{(5.6)}{\simeq} (f')^! \circ (f')_*^{\mathrm{IndCoh}} \circ g_1^! \overset{\mathrm{counit}\circ\mathrm{id}}{\longrightarrow} g_1^!.$$

Taking into account that the counit maps

$$f^! \circ f_*^{\mathrm{IndCoh}} \to \mathrm{Id} \text{ and } (f')^! \circ (f')_*^{\mathrm{IndCoh}} \to \mathrm{Id}$$

are given by base change for the squares

$$\begin{array}{ccc} S_1 & \xrightarrow{\mathrm{id}} & S_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ S_1 & \xrightarrow{f} & S_2 \end{array}$$

and

$$\begin{array}{ccc} S_1' & \xrightarrow{\mathrm{id}} & S_1' \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f'} \\ S_1' & \xrightarrow{f'} & S_2', \end{array}$$

respectively, we obtain that the two natural transformations in question come as base change from the Cartesian square

$$\begin{array}{ccc} S_1' & \xrightarrow{g_1} & S_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ S_1' & \xrightarrow{g_2\circ f'=f\circ g_1} & S_2, \end{array}$$

for the first one factoring it as

$$\begin{array}{ccccc} S_1' & \xrightarrow{g_1} & S_1 & \xrightarrow{\mathrm{id}} & S_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ S_1' & \xrightarrow{g_1} & S_1 & \xrightarrow{f} & S_2 \end{array}$$

and for the second one factoring it as

$$\begin{array}{ccccc} S_1' & \xrightarrow{\mathrm{id}} & S_1' & \xrightarrow{g_1} & S_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S_1' & \xrightarrow{f'} & S_2' & \xrightarrow{g_2} & S_2. \end{array}$$

□

5.4. **Compatibility with adjunction for proper maps.**

5.4.1. Let $f : S_1 \to S_2$ be a proper map. By contrast with the case of open embeddings, the datum of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$ does not contain the datum for either the unit

$$\mathrm{Id}_{\mathrm{IndCoh}(S_1)} \to f^! \circ f_*^{\mathrm{IndCoh}}$$

or the counit

$$f^{\mathrm{IndCoh}} \circ f^! \to \mathrm{Id}_{\mathrm{IndCoh}(S_2)}$$

for the $(f_*^{\mathrm{IndCoh}}, f^!)$-adjunction. [6] The situation can be remedied by considering the 2-category of correspondences, see Sect. 5.4.3 below.

The following property of the base change isomorphisms (5.6) follows from the construction of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$:

Proposition 5.4.2. *Consider a Cartesian diagram* (5.5) *and the corresponding isomorphism of functors*

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \simeq g_2^! \circ f_*^{\mathrm{IndCoh}} \tag{5.10}$$

of (5.6).

(a) *Suppose that the morphism f (and hence also f') is proper, and g_2 (and hence g_1) almost of finite type. Then the map $\to$ in the isomorphism* (5.10) *comes via the adjunctions of* $(f_*^{\mathrm{IndCoh}}, f^!)$ *and* $((f')_*^{\mathrm{IndCoh}}, (f')^!)$ *from the isomorphism*

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!.$$

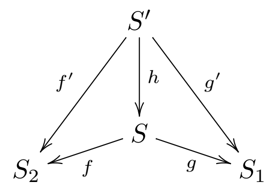
(b) *Suppose that the morphism g_2 (and hence also g_1) is proper. Then the map $\to$ in the isomorphism* (5.10) *comes via the adjunctions of* $((g_1)_*^{\mathrm{IndCoh}}, g_1^!)$ *and* $((g_2)_*^{\mathrm{IndCoh}}, g_2^!)$ *from the isomorphism*

$$(g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}} \simeq f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}}.$$

[6] We remark that it does encode this adjunction in the world of D-modules, see [FrGa], Sect. 1.4.5.

5.4.3. *A 2-categorical formulation.* As was explained to us by J. Lurie, the most natural formulation of the existence of the !-pullback functors that would encode the adjunction for proper maps used the language of $(\infty, 2)$-categories.

Namely, according to Lurie, one needs to consider an $(\infty, 2)$-category $(\mathrm{DGSch}_{\mathrm{Noeth}})^{\text{2-Cat}}_{\mathrm{corr:all;aft}}$, whose objects and 1-morphisms are the same as in $(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}$, however, we allow non-invertible 2-morphisms which are commutative diagrams



(5.11)

where the map h is required to be a proper.

The higher categorical structure on $\mathrm{IndCoh}^{\text{2-Cat}}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$ is described via the corresponding higher Segal space. Namely, the corresponding $(\infty, 1)$-category of n-fold compositions is has as objects diagrams as in (5.3), where 1-morphisms between such diagrams are maps that induce proper maps on veritices.

In this framework, one will consider $\mathrm{IndCoh}^{\text{2-Cat}}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$ as a functor

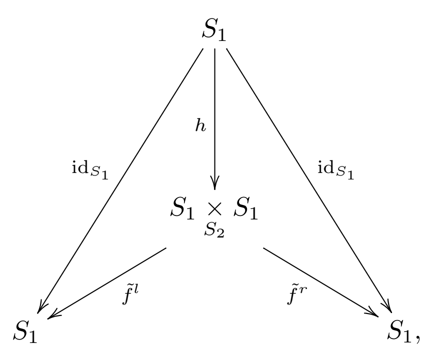
$$(\mathrm{DGSch}_{\mathrm{Noeth}})^{\text{2-Cat}}_{\mathrm{corr:all;aft}} \to \mathrm{DGCat}^{\text{2-Cat}}_{\mathrm{cont}},$$

where DGCat $\text{2-Cat}_{\mathrm{cont}}$ is the 2-category of cocomplete DG categories and continuous functors, where we now allow non-invertible natural transformations as 2-morphisms.

The construction of this functor will be carried out in [GR3].

5.4.4. Let explain how the datum of the functor $\mathrm{IndCoh}^{\text{2-Cat}}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$ encodes the adjunction for $(f^{\mathrm{IndCoh}}_*, f^!)$ for a proper map $f : S_1 \to S_2$.

Consider the diagram



(5.12)

where $h = \Delta_{S_1/S_2}$.

Then the resulting 2-morphism

$$\mathrm{Id}_{\mathrm{IndCoh}(S)} \simeq (\mathrm{id}_{S_1})^{\mathrm{IndCoh}}_* \circ \mathrm{id}^!_{S_1} \to (\tilde{f}^l)^{\mathrm{IndCoh}}_* \circ (\tilde{f}^r)^! \simeq f^! \circ f^{\mathrm{IndCoh}}_*$$

is the unit of the $(f^{\mathrm{IndCoh}}_*, f^!)$-adjunction.

5.4.5. As an illustration, assuming the above description of the unit of the adjunction for $(f_*^{\mathrm{IndCoh}}, f^!)$, let us deduce the assertion of Proposition 5.4.2(b).

As in the proof of Proposition 5.3.3, it suffices to show that the two natural transformations

$$g_1^! \rightrightarrows (f')^! \circ g_2^! \circ f_*^{\mathrm{IndCoh}} \tag{5.13}$$

are canonically isomorphic, the first one being

$$g_1^! \overset{\mathrm{id} \circ \mathrm{unit}}{\longrightarrow} g_1^! \circ f^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')^! \circ g_2^! \circ f_*^{\mathrm{IndCoh}},$$

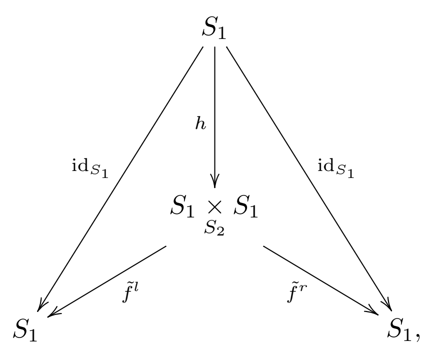
and the second one

$$g_1^! \overset{\mathrm{unit} \circ \mathrm{id}}{\longrightarrow} (f')^! \circ (f')_*^{\mathrm{IndCoh}} \circ g_1^! \overset{(5.6)}{\longrightarrow} (f')^! \circ g_2^! \circ f_*^{\mathrm{IndCoh}}.$$

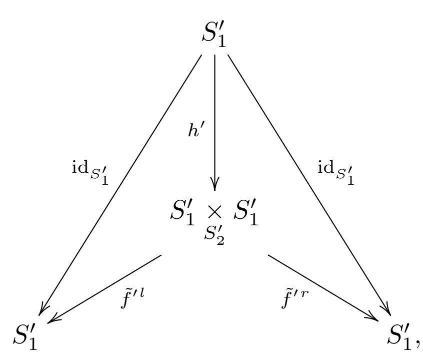
Recall that the above unit maps

$$\mathrm{Id} \to f^! \circ f_*^{\mathrm{IndCoh}} \text{ and } \mathrm{Id} \to (f')^! \circ (f')_*^{\mathrm{IndCoh}}$$

are given by the 2-morphisms coming from the diagrams

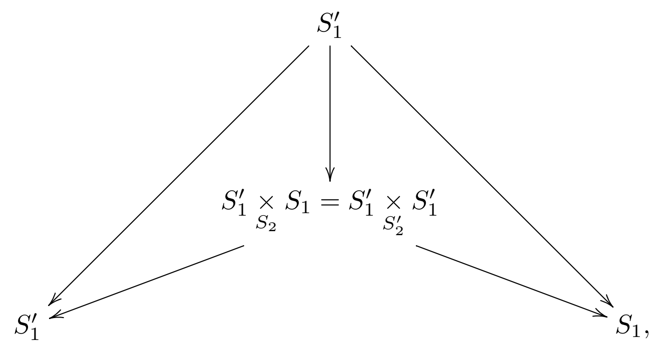
(5.14)


and

(5.15)


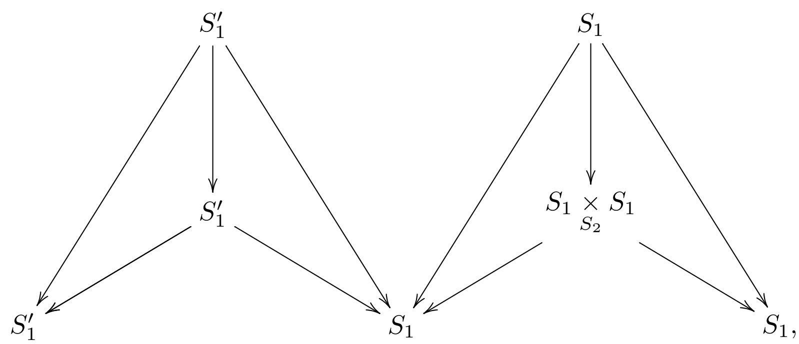
respectively.

Hence, both maps in (5.13) are given by the diagram



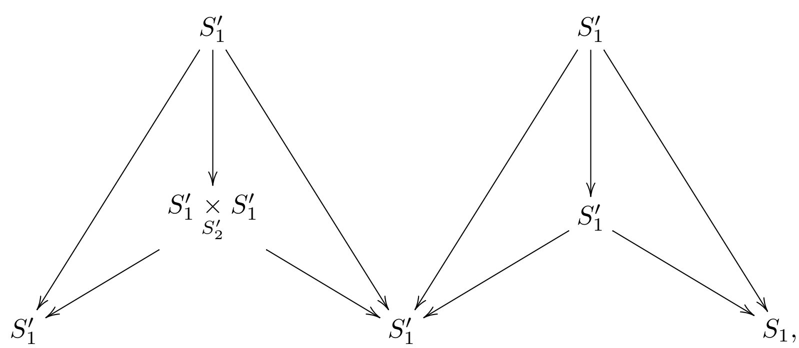
(5.16)

one time presenting it as a composition



(5.17)

and another time as a composition



(5.18)

5.5. **Compatibility with the action of** QCoh.

5.5.1. Recall (see Sect. 1.4) that for an individual DG scheme S, the category $\mathrm{IndCoh}(S)$ is naturally a module category for $\mathrm{QCoh}(S)$.

Recall also that for a map $f : S_1 \to S_2$, which is proper (resp., open), the functor

$$f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$$

has a natural structure of 1-morphism of $\mathrm{QCoh}(S_2)$-module categories.

Indeed, for f proper, this structure is given by Corollary 3.3.11. For f an open embedding, we have $f^! = f^{\mathrm{IndCoh},*}$ and this structure is given by Proposition 3.5.4.

5.5.2. Consider now the ∞-category $\mathrm{DGCat}^{\mathrm{SymMon+Mod}}_{\mathrm{cont}}$, whose objects are pairs

$$(\mathbf{O}, \mathbf{C}),$$

where $\mathbf{O} \in \mathrm{DGCat}^{\mathrm{SymMon}}$ and $\mathbf{C}$ is an $\mathbf{O}$-module category.

Morphisms in $\mathrm{DGCat}^{\mathrm{SymMon+Mod}}_{\mathrm{cont}}$ between $(\mathbf{O}_1, \mathbf{C}_1)$ and $(\mathbf{O}_2, \mathbf{C}_2)$ are pairs $(F_{\mathbf{O}}, F_{\mathbf{C}})$, where $F_{\mathbf{O}} : \mathbf{O}_1 \to \mathbf{O}_2$ is a morphism in $\mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}$, and

$$F_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_2$$

is a morhism of $\mathbf{O}_1$-module categories.

The higher categorical structure on $\mathrm{DGCat}^{\mathrm{SymMon+Mod}}_{\mathrm{cont}}$ is defined naturally.

5.5.3. The assignments

$$S \rightsquigarrow (\mathrm{QCoh}(S), \mathrm{IndCoh}(S)),\ (f : S_1 \to S_2) \rightsquigarrow (f^*, f^!)$$

naturally upgrade to functors

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}}} : ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon+Mod}}_{\mathrm{cont}},$$

and

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{open}}} : ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{open}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon+Mod}}_{\mathrm{cont}},$$

respectively.

5.5.4. The following ehnancement of Corollary 5.2.4 will be proved in [GR3]:

Theorem 5.5.5. *There exists a canonical upgrading of the functor*

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{Noeth}}} : ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to a functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}} : ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon+Mod}}_{\mathrm{cont}},$$

whose restrictions to

$$((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}})^{\mathrm{op}} \text{ and } ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{open}})^{\mathrm{op}}$$

identify with

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{proper}}} \text{ and } (\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{open}}},$$

respectively.

A plausibility check for this theorem follows the outline of the proof of Theorem 5.2.2 given in the next section.

5.5.6. Let us rewrite the statement of Theorem 5.5.5 in concrete terms for an individual morphism $f : S_1 \to S_2$:

It says that that for $\mathcal{E} \in \mathrm{QCoh}(S_2)$ and $\mathcal{F} \in \mathrm{IndCoh}(S_2)$, there exists a canonical isomorphism

$$f^!(\mathcal{E} \otimes \mathcal{F}) \simeq f^*(\mathcal{E}) \otimes f^!(\mathcal{F}). \tag{5.19}$$

5.6. **The multiplicative structure.** In this subsection we will specialize to the full subcategory

$$\mathrm{DGSch}_{\mathrm{aft}} \subset \mathrm{DGSch}_{\mathrm{Noeth}}$$

of DG schemes almost of finite type over k.

5.6.1. We consider the full subcategory

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}} \subset (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}},$$

obtained by taking as objects DG schemes that are almost of finite type.

Let us denote by $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ the restriction of the functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to the subcategory $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}$.

We let

$$\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{DGSch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

denote the restriction of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ to the 1-full subcategory

$$\mathrm{DGSch}_{\mathrm{aft}} \subset (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}.$$

Similarly, we let

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}} : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

denote the restriction of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ to the 1-full subcategory

$$(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \subset (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}.$$

5.6.2. Let us observe that the categories $\mathrm{DGSch}_{\mathrm{aft}}$ and $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}$ possess natural symmetric monoidal structures given by Cartesian product over $\mathrm{pt} := \mathrm{Spec}(k)$.

The category $\mathrm{DGCat}_{\mathrm{cont}}$ also possesses a natural symmetric monoidal structure given by tensor product. We state the following result without a proof, as it is obtained by retracing the argument of Theorem 5.2.2 (the full proof will be given in [GR3]):

Theorem 5.6.3. *The functor*

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}} : (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

has a natural right-lax symmetric monoidal structure.

Combined with Proposition 4.6.2, we obtain:

Corollary 5.6.4. *The right-lax symmetric monoidal structure on* $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ *is strict, i.e., is symmetric monoidal.*

5.6.5. Restricting the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ to

$$(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

from Theorem 5.6.3 and Corollary 5.6.4, we obtain:

Corollary 5.6.6. *The functor*

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}} : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

has a natural symmetric monoidal structure.

Note that at the level of objects, the statement of Corollary 5.6.6 coincides with that of Proposition 4.6.2.

At the level of 1-morphisms, it says that for two pairs of objects of $\mathrm{DGSch}_{\mathrm{aft}}$:

$$(f_1 : S_1 \to S_1') \text{ and } (f_2 : S_2 \to S_2'),$$

the diagram

$$\begin{array}{ccc} \operatorname{IndCoh}(S_1)\otimes\operatorname{IndCoh}(S_2) & \xrightarrow{\boxtimes} & \operatorname{IndCoh}(S_1\times S_2) \\ {\scriptstyle f_1^!\otimes f_2^!}\uparrow & & \uparrow{\scriptstyle (f_1\times f_2)^!} \\ \operatorname{IndCoh}(S_1')\otimes\operatorname{IndCoh}(S_2') & \xrightarrow{\boxtimes} & \operatorname{IndCoh}(S_1'\times S_2') \end{array}$$

canonically commutes.

5.6.7. Note now that for any $S\in\operatorname{DGSch}_{\mathrm{aft}}$, the diagonal morphism on S defines on it a structure of commutative coalgebra in $\operatorname{DGSch}_{\mathrm{aft}}$. Hence, from Corollary 5.6.6 we obtain:

Corollary 5.6.8. *For $S\in\operatorname{DGSch}_{\mathrm{aft}}$, the category $\operatorname{IndCoh}(S)$ has a natural symmetric monoidal structure. Furthermore, the assignement*

$$S\rightsquigarrow(\operatorname{IndCoh}(S),\overset{!}{\otimes})$$

naturally extends to a functor

$$(\operatorname{DGSch}_{\mathrm{aft}})^{\mathrm{op}}\to\operatorname{DGCat}_{\mathrm{cont}}^{\mathrm{SymMon}}.$$

Concretely, the monoidal operation on $\operatorname{IndCoh}(S)$ is the functor

$$\operatorname{IndCoh}(S)\otimes\operatorname{IndCoh}(S)\overset{\boxtimes}{\longrightarrow}\operatorname{IndCoh}(S\times S)\overset{\Delta_S^!}{\longrightarrow}\operatorname{IndCoh}(S).$$

We shall use the notation

$$\mathcal{F}_1,\mathcal{F}_2\in\operatorname{IndCoh}(S)\ \mapsto\ \mathcal{F}_1\overset{!}{\otimes}\mathcal{F}_2\in\operatorname{IndCoh}(S).$$

The unit in this symmetric monoidal category is ω_S, defined as

$$\omega_S:=(p_S)^!(k)\in\operatorname{IndCoh}(S).$$

We shall refer to ω_S as the "dualizing sheaf of S."

5.7. **Compatibility between the multiplicative structure and the action of** QCoh**.** We are now going to discuss a common refinement of Corollary 5.6.6 and Theorem 5.5.5:

5.7.1. Note that the category $\operatorname{DGCat}_{\mathrm{cont}}^{\mathrm{SymMon+Mod}}$ also has a natural symmetric monoidal structure, given by

$$(\mathbf{O}_1,\mathbf{C}_1)\otimes(\mathbf{O}_2,\mathbf{C}_2):=(\mathbf{O}_1\otimes\mathbf{O}_2,\mathbf{C}_1\otimes\mathbf{C}_2).$$

We have:

Theorem 5.7.2. *The symmetric monoidal structures on the functors*

$$\operatorname{IndCoh}^!_{\operatorname{DGSch}_{\mathrm{aft}}}:(\operatorname{DGSch}_{\mathrm{aft}})^{\mathrm{op}}\to\operatorname{DGCat}_{\mathrm{cont}}$$

and

$$\operatorname{QCoh}^*_{\operatorname{DGSch}_{\mathrm{aft}}}:(\operatorname{DGSch}_{\mathrm{aft}})^{\mathrm{op}}\to\operatorname{DGCat}_{\mathrm{cont}}^{\mathrm{SymMon}}$$

naturally combine to a symmetric monoidal structure on the functor

$$(\operatorname{QCoh}^*,\operatorname{IndCoh}^!)_{\operatorname{DGSch}_{\mathrm{aft}}}:(\operatorname{DGSch}_{\mathrm{aft}})^{\mathrm{op}}\to\operatorname{DGCat}_{\mathrm{cont}}^{\mathrm{SymMon+Mod}}.$$

5.7.3. As a corollary, we obtain:

Corollary 5.7.4. *For* $S \in \mathrm{DGSch}_{\mathrm{aft}}$*, the symmetric monoidal structure on* $\mathrm{IndCoh}(S)$ *has a natural* $\mathrm{QCoh}(S)$*-linear structure. Furthermore, the assignment*

$$S \rightsquigarrow (\mathrm{QCoh}(S), \otimes) \to (\mathrm{IndCoh}(S), \overset{!}{\otimes})$$

naturally extends to a functor

$$(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{Funct}([1], \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}),$$

where $[1]$ *is the category* $0 \to 1$.

In the above corollary, the symmetric monoidal functor

$$\mathrm{QCoh}(S) \to \mathrm{IndCoh}(S), \tag{5.20}$$

is given by the action on the unit, i.e.,

$$\mathcal{E} \mapsto \mathcal{E} \otimes \omega_S,$$

where the action is understood in the sense of Sect. 1.4. We shall denote the functor by the symbol Υ_S.

In concrete terms, the structure of $\mathrm{QCoh}(S)$-linearity on the symmetric monoidal category $\mathrm{IndCoh}(S)$ means that for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{IndCoh}(S)$ and $\mathcal{E} \in \mathrm{QCoh}(S)$, we have canonical isomorphisms

$$\mathcal{E} \otimes (\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2) \simeq (\mathcal{E} \otimes \mathcal{F}_1) \overset{!}{\otimes} \mathcal{F}_2 \simeq \mathcal{F}_1 \overset{!}{\otimes} (\mathcal{E} \otimes \mathcal{F}_2). \tag{5.21}$$

5.7.5. Thus, we can consider a natural transformation

$$\Upsilon_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{QCoh}^*_{\mathrm{DGSch}_{\mathrm{aft}}} \to \mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}, \tag{5.22}$$

where $\mathrm{QCoh}^*_{\mathrm{DGSch}_{\mathrm{aft}}}$ and $\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$ are both considered as functors

$$(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \rightrightarrows \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}.$$

The functoriality statement of Corollary 5.7.4 says that at the level of 1-morphisms, for $f : S_1 \to S_2$, we have a commutative diagram of symmetric monoidal categories.

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xleftarrow{\Upsilon_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f^!}\big\uparrow & & \big\uparrow{\scriptstyle f^*} \\ \mathrm{IndCoh}(S_2) & \xleftarrow{\Upsilon_{S_2}} & \mathrm{QCoh}(S_2). \end{array} \tag{5.23}$$

5.7.6. Note that if we regard $\mathrm{QCoh}^*_{\mathrm{DGSch}_{\mathrm{aft}}}$ and $\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$ just as functors

$$(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \rightrightarrows \mathrm{DGCat}_{\mathrm{cont}},$$

then the structure of natural transfomration on $\Upsilon_{\mathrm{DGSch}_{\mathrm{aft}}}$ is given by Theorem 5.5.5, i.e., we do not need to consider the finer structure given by Theorem 5.7.2.

6. Proof of Theorem 5.2.2

6.1. **Framework for the construction of the functor.**

6.1.1. We first discuss the most basic example of a construction of a functor out of a category of correspondences.

Let $(\mathbf{C}, vert, horiz)$ be as in Sect. 5.1.1, and suppose that $horiz \subset vert$.

Let $P_{vert} : \mathbf{C}_{vert} \to \mathrm{DGCat}_{\mathrm{cont}}$ be a functor. Assume that for every $(g : \mathbf{c}_1 \to \mathbf{c}_2) \in \mathbf{C}_{horiz}$, the functor $P_{vert}(g) : P(\mathbf{c}_1) \to P(\mathbf{c}_2)$ admits a continuous right (resp., left) adjoint; we denote it $P^!_{horiz}(g)$.

The passage to adjoints defines a functor

$$P^!_{horiz} : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Consider a Cartesian square (5.1). By adjunction, we obtain a map

$$P_{vert}(f') \circ P^!_{horiz}(g_1) \to P^!_{horiz}(g_2) \circ P_{vert}(f) \tag{6.1}$$

(in the case of right adjoints), and

$$P^!_{horiz}(g_2) \circ P_{vert}(f) \to P_{vert}(f') \circ P^!_{horiz}(g_1) \tag{6.2}$$

(in the case of left adjoints).

We shall say that P_{vert} satisfies the *left base change* (resp., *right base change*) condition with respect to *horiz* if the corresponds adjoints $P^!_{horiz}(g)$, $g \in horiz$ exist, and map (6.1) (resp., (6.2)) is an isomorphism for any Cartesian diagram as above.

Theorem 6.1.2. *Suppose that P_{vert} satisfies the left (resp., right) base change condition with respect to horiz. Then there exists a canonically defined functor*

$$P_{\mathrm{corr}:vert;horiz} : \mathbf{C}_{\mathrm{corr}:vert;horiz} \to \mathrm{DGCat}_{\mathrm{cont}},$$

equipped with isomorphisms

$$P_{vert} \simeq P_{\mathrm{corr}:vert;horiz}|_{\mathbf{C}_{vert}} \text{ and } P^!_{horiz} \simeq P_{\mathrm{corr}:vert;horiz}|_{(\mathbf{C}_{horiz})^{\mathrm{op}}}.$$

The proof of this theorem when $\mathbf{C}$ is an ordinary category is easy. The higher-categorical version will appear in [GR3].

6.1.3. Let $\mathbf{C}, vert, horiz$ be as in Sect. 5.1.1. Let *adm* be a subclass of *vert*, which satisfies the following conditions:

(A) *adm* satisfies conditions (1)-(3) of Sect. 5.1.1.

(B) The pairs of classes $(vert, adm)$, $(adm, horiz)$ satisfy condition (4) of Sect. 5.1.1. (Note that since $adm \subset vert$, the pair (adm, adm) also satisfies condition (4).)

(C) If in a diagram (5.1), we take $g_2 = f$ and it belongs to both *horiz* and *adm*, then the arrows f' and g_1 are isomorphisms.

(D) For *horiz* and *adm*, if $h = h_1 \circ h_2$ and h and h_1 belong to the given class, then so does h_2.

6.1.4. Consider the category $\mathbf{C}_{\mathrm{corr}:vert;horiz}$, and let $P_{\mathrm{corr}:vert;horiz}$ be a functor

$$P_{\mathrm{corr}:vert;horiz}:\mathbf{C}_{\mathrm{corr}:vert;horiz}\to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

We will impose the following conditions:

(I) The functor $P_{vert}:\mathbf{C}_{vert}\to \mathrm{DGCat}_{\mathrm{cont}}$ satisfies the left base change condition with respect to adm.

(II) For a Cartesian square as in (5.1), with $f,f'\in adm$ and $g_1,g_2\in horiz$ the morphism between the resulting two functors $P(\mathbf{c}_2)\rightrightarrows P(\mathbf{c}_1')$

$$P^!_{horiz}(g_1)\circ P^!_{adm}(f)\to P^!_{adm}(f')\circ P^!_{horiz}(g_2)$$

that comes by the $(P_{vert}(-),P^!_{adm}(-))$-adjunction from the isomorphism (5.4), is an isomorphism.

6.1.5. For a morphism $f:\mathbf{c}_1\to\mathbf{c}_2$ with $f\in adm\cap horiz$ consider the diagram

$$\begin{array}{ccc}\mathbf{c}_1 & \xrightarrow{\mathrm{id}} & \mathbf{c}_1\\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f}\\ \mathbf{c}_1 & \xrightarrow{f} & \mathbf{c}_2,\end{array}$$

which is Cartesian due to Condition (C).

Note that from Condition (II) we obtain a canonical isomorphism

$$P^!_{horiz}(f)\simeq P^!_{adm}(f).$$

6.1.6. We let $horiz_{\mathrm{new}}$ be yet another class of 1-morphisms in $\mathbf{C}$. We impose the following two conditions on $horiz_{\mathrm{new}}$:

(i) $horiz_{\mathrm{new}}$ satisfies conditions (1)-(3) of Sect. 5.1.1.

(ii) $horiz_{\mathrm{new}}$ contains both $horiz$ and adm.

(iii) Every morphism $h\in horiz_{\mathrm{new}}$ can be factored as $h=f\circ g$ with $g\in horiz$ and $f\in adm$.

Remark 6.1.7. It follows formally that $horiz_{\mathrm{new}}$ is precisely the class of 1-morphisms that can be factored as in condition (iii). Thus, the actual condition is that this class of 1-morphisms is stable under compositions.

6.1.8. Finally, we impose the following crucial condition on the classes $horiz$ and adm. Let us call it condition $(\star)$:

For given 1-morphism $(h:\mathbf{c}_1\to\mathbf{c}_2)\in horiz_{\mathrm{new}}$ consider the category $\mathsf{Factor}(h)$, whose objects are factorizations of h into a composition

$$\mathbf{c}_1\xrightarrow{g}\mathbf{c}_{3/2}\xrightarrow{f}\mathbf{c}_2$$

with $g\in horiz$ and $f\in adm$. Morphisms in this category are commutative diagrams

$$\begin{array}{ccccc} & & \mathbf{c}'_{3/2} & & \\ & \overset{g'}{\nearrow} & & \overset{f'}{\searrow} & \\ \mathbf{c}_1 & & \downarrow{\scriptstyle e} & & \mathbf{c}_2. \\ & \underset{g''}{\searrow} & & \underset{f''}{\nearrow} & \\ & & \mathbf{c}''_{3/2} & & \end{array} \tag{6.3}$$

(Note that by condition (D), the arrow $e:\mathbf{c}'_{3/2}\to\mathbf{c}''_{3/2}$ is also in adm.)

Condition $(\star)$ reads as follows: the above category $\mathsf{Factor}(h)$ is contractible.

6.1.9. Consider the category $\mathbf{C}_{\mathrm{corr}:vert;horiz_{\mathrm{new}}}$. Note that $\mathbf{C}_{\mathrm{corr}:vert;horiz_{\mathrm{new}}}$ contains as 1-full subcategories both $\mathbf{C}_{\mathrm{corr}:vert;adm}$ and $\mathbf{C}_{\mathrm{corr}:vert;horiz}$.

We have the following assertion, generalizing Theorem 6.1.2

Theorem 6.1.10. *There exists a canonically define functor*

$$P_{\mathrm{corr},vert,horiz_{\mathrm{new}}} : \mathbf{C}_{\mathrm{corr}:vert;horiz_{\mathrm{new}}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

equipped with identifications

$$P_{\mathrm{corr},vert,horiz_{\mathrm{new}}}|_{\mathbf{C}_{\mathrm{corr}:vert;horiz}} \simeq P_{\mathrm{corr},vert,horiz}$$

and

$$P_{\mathrm{corr},vert,horiz_{\mathrm{new}}}|_{\mathbf{C}_{\mathrm{corr}:vert;adm}} \simeq P_{\mathrm{corr},vert,adm},$$

which is compatible with the further restriction to

$$\mathbf{C}_{\mathrm{corr}:vert;adm} \leftarrow \mathbf{C}_{vert} \to \mathbf{C}_{\mathrm{corr}:vert;horiz}.$$

6.1.11. *Sketch of proof of Theorem 6.1.10.* In this section we will indicate the proof of Theorem 6.1.10, modulo homotopy-theoretic issues (i.e., a proof that works when $\mathbf{C}$ is an ordinary category). The full proof will appear in [GR3].

First, we are going to construct the functor $P_{horiz_{\mathrm{new}}} : (\mathbf{C}_{horiz_{\mathrm{new}}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$. Let $h : \mathbf{c}_1 \to \mathbf{c}_2$ be a 1-morphism in $horiz_{\mathrm{new}}$, and let us factor it as a composition $f \circ g$ as in Condition (iii). First, we need to show that the functor

$$P^!_{horiz}(g) \circ P^!_{adm}(f) : P(\mathbf{c}_2) \to P(\mathbf{c}_1)$$

is canonically independent of the factorization. Since by condition $(\star)$ the category of factorizations is contractible, it suffices to show that for any 1-morphism between factorizations give as in diagram (6.3), the resulting two functors

$$P^!_{horiz}(g') \circ P^!_{adm}(f') \text{ and } P^!_{horiz}(g'') \circ P^!_{adm}(f'')$$

are canonically isomorphic.

This easily reduces to the case of a diagram

$$\begin{array}{ccc} & & \mathbf{c}_1 \\ & \overset{g_1}{\nearrow} & \\ \mathbf{c} & & \Big\downarrow f \\ & \underset{g_2}{\searrow} & \\ & & \mathbf{c}_2 \end{array}$$

with $g', g'' \in horiz$ and $f \in adm$, and we need to establish an isomorphism

$$P^!_{horiz}(g_2) \simeq P^!_{horiz}(g_1) \circ P^!_{adm}(f)$$

as functors $P(\mathbf{c}_2) \rightrightarrows P(\mathbf{c})$.

We will show more generally that given a commutative (but not necessarily Cartesian) square

$$\begin{array}{ccc} \mathbf{c}'_1 & \xrightarrow{g_1} & \mathbf{c}_1 \\ {\scriptstyle \tilde{f}}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ \mathbf{c}'_2 & \xrightarrow{g_2} & \mathbf{c}_2 \end{array}$$

with $f, \tilde{f} \in adm$ and $g_1, g_2 \in horiz$, we have a canonical isomorphism

$$P^!_{horiz}(g_1) \circ P^!_{adm}(f) \simeq P^!_{adm}(\tilde{f}) \circ P^!_{horiz}(g_2) \tag{6.4}$$

as functors $P(\mathbf{c}_2) \rightrightarrows P(\mathbf{c}'_1)$.

Consider the diagram

$$\begin{array}{ccccc} \mathbf{c}'_1 & \xrightarrow{h} & \mathbf{c}'_2 \underset{\mathbf{c}_2}{\times} \mathbf{c}_1 & \xrightarrow{g'_2} & \mathbf{c}_1 \\ & & {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ & & \mathbf{c}'_2 & \xrightarrow{g_2} & \mathbf{c}_2 \end{array}$$

where $g'_2 \circ h \simeq g_1$ and $f' \circ h \simeq \tilde{f}$.

By Condition (D) we obtain that

$$h \in adm \cap horiz.$$

Therefore, we have:

$$\begin{aligned} P^!_{horiz}(g_1) \circ P^!_{adm}(f) \simeq P^!_{horiz}(h) \circ P^!_{horiz}(g'_2) \circ P^!_{adm}(f) &\overset{\text{Condition II}}{\simeq} \\ \simeq P^!_{horiz}(h) \circ P^!_{adm}(f') \circ P^!_{horiz}(g_2) &\overset{\text{Sect. 6.1.5}}{\simeq} \\ \simeq P^!_{adm}(h) \circ P^!_{adm}(f') \circ P^!_{horiz}(g_2) &\simeq P^!_{adm}(\tilde{f}) \circ P^!_{horiz}(g_2). \end{aligned}$$

Isomorphism (6.4) allows one to define the functor $P_{horiz_{\mathrm{new}}}$ on compositions of morphisms.

The base change data, needed to extend the functors $P_{horiz_{\mathrm{new}}}$ and P_{vert} to a functor

$$P_{\mathrm{corr},vert,horiz_{\mathrm{new}}} : \mathbf{C}_{\mathrm{corr}:vert;horiz_{\mathrm{new}}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

follows from the corresponding data of $P_{\mathrm{corr}:vert;horiz}$ and $P_{\mathrm{corr}:vert;adm}$ by construction.

6.2. Construction of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$.

6.2.1. *Step 1.* We start with $\mathbf{C} := \mathrm{DGSch}_{\mathrm{Noeth}}$, and we take *vert* to be the class of all morphisms, and *horiz* to be the class of open embeddings.

Consider the functor $P_{vert} := \mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{Noeth}}}$ of Proposition 3.2.4. It satisfies the right base change condition with respect to the class of open embeddings by Proposition 3.2.4.

Applying Theorem 6.1.2, we obtain a functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;open}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;open}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

6.2.2. *Step 2.* We take $\mathbf{C} := \mathrm{DGSch}_{\mathrm{Noeth}}$, *vert* to be the class of all morphisms, *horiz* to be the class of open embeddings, and *adm* to be the class of proper morphisms. It is easy to see that conditions (A)-(D) of Sect. 6.1.3 are satisfied.

We consider the functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;open}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;open}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

constructed in Step 1.

We claim that it satisfies Conditions (I) and (II) of Sect. 6.1.4. Indeed, Condition (I) is given by Proposition 3.4.2, and Condition (II) is given by Corollary 4.4.7.

We take $horiz_{\mathrm{new}}$ to be the class of separated morphisms almost of finite type. We claim that it satisfies Conditions (i), (ii) and (iii) of Sect. 6.1.6 and condition $(\star)$ of Sect. 6.1.8.

Conditions (i) and (ii) are evident. Condition ($\star$), which contains Condition (iii) as a particular case, will be verified in Sect. 6.3.

Applying Theorem 6.1.10, we obtain a functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where

$$\text{aft-sep} \subset \text{aft}$$

denotes the class of separated morphisms almost of finite type.

It is clear from the construction that the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}}}$ satisfies the Conditions (a), (b) and (c) of Theorem 5.2.2.

6.2.3. *Interlude.* In order to extend the functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

to a functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

we will use the following construction.

Let

$$(\mathbf{C}^1, vert^1, horiz^1) \text{ and } (\mathbf{C}^2, vert^2, horiz^2)$$

be a pair of categories and classes of morphisms as in Sect. 5.1.1.

Let $\Phi : \mathbf{C}^1 \to \mathbf{C}^2$ be a functor. Assume that Φ sends morphisms from $vert^1$ (resp., $horiz^1$) to morphisms from $vert^2$ (resp., $horiz^2$). In particular, Φ induces functors

$$\Phi_{vert} : \mathbf{C}^1_{vert} \to \mathbf{C}^2_{vert} \text{ and } \Phi_{horiz} : \mathbf{C}^1_{horiz} \to \mathbf{C}^2_{horiz},$$

and a functor

$$\Phi_{\mathrm{corr}:vert;horiz} : \mathbf{C}^1_{\mathrm{corr}:vert;horiz} \to \mathbf{C}^2_{\mathrm{corr}:vert;horiz}.$$

Let now

$$P_{\mathrm{corr}:vert;horiz} : \mathbf{C}^1_{\mathrm{corr}:vert;horiz} \to \mathbf{D}$$

be a functor, where $\mathbf{D}$ is an $(\infty, 1)$-category that contains limits. Consider the functors

$$\mathrm{RKE}_{\Phi_{\mathrm{corr}:vert;horiz}}(P_{\mathrm{corr}:vert;horiz}) : \mathbf{C}^2_{\mathrm{corr}:vert;horiz} \to \mathbf{D}$$

and

$$\mathrm{RKE}_{(\Phi_{horiz})^{\mathrm{op}}}(P^!_{horiz}) : (\mathbf{C}^2_{horiz})^{\mathrm{op}} \to \mathbf{D}.$$

We claim:

Proposition 6.2.4. *Assume that for any $\mathbf{c}^1 \in \mathbf{C}^1$, the functor Φ induces an equivalence*

$$(\mathbf{C}^1_{vert})_{/\mathbf{c}^1} \to (\mathbf{C}^2_{vert})_{/\Phi(\mathbf{c}^1)}.$$

Then the natural map

$$\mathrm{RKE}_{\Phi_{\mathrm{corr}:vert;horiz}}(P_{\mathrm{corr}:vert;horiz})|_{(\mathbf{C}^2_{horiz})^{\mathrm{op}}} \to \mathrm{RKE}_{(\Phi_{horiz})^{\mathrm{op}}}(P^!_{horiz})$$

is an equivalence.

Proof. It is enough to show that the map in question induces an isomorphism at the level of objects.

The value of $\mathrm{RKE}_{\Phi_{\mathrm{corr}:vert;horiz}}(P_{\mathrm{corr}:vert;horiz})$ on $\mathbf{c}^2 \in \mathbf{C}^2$ is

$$lim\, P(\mathbf{c}^1),$$

where the limit is taken over the category of diagrams

$$\begin{array}{ccc} \mathbf{c}'^2 & \xrightarrow{g} & \mathbf{c}_2 \\ {\scriptstyle f^2}\big\downarrow & & \\ \Phi(\mathbf{c}^1), & & \end{array}$$

with $f^2 \in vert^2$ and $g \in horiz^2$.

The condition of the proposition implies that this category is equivalent to that of diagrams

$$\begin{array}{ccc} \Phi(\mathbf{c}'^1) & \xrightarrow{g} & \mathbf{c}_2 \\ {\scriptstyle \Phi(f^1)}\big\downarrow & & \\ \Phi(\mathbf{c}^1), & & \end{array}$$

with $f^1 \in vert^1$ and $g \in horiz^2$.

However, it is clear that cofinal in the (opposite of the above) category is the full subcategory consisting of diagrams with f^1 being an isomorphism. The latter category is the same as

$$(\mathbf{C}^1 \underset{\mathbf{C}^2}{\times} (\mathbf{C}^2_{horiz})_{/\mathbf{c}^2})^{\mathrm{op}}.$$

I.e., the value of $\mathrm{RKE}_{\Phi_{\mathrm{corr}:vert;horiz}}(P_{\mathrm{corr}:vert;horiz})$ on $\mathbf{c}^2$ maps isomorphically to

$$\underset{\mathbf{c}^1 \in (\mathbf{C}^1 \underset{\mathbf{C}^2}{\times} (\mathbf{C}^2_{horiz})_{/\mathbf{c}^2})^{\mathrm{op}}}{lim} P^!(\mathbf{c}^1),$$

while the latter limit computes the value of $\mathrm{RKE}_{(\Phi_{horiz})^{\mathrm{op}}}(P^!_{horiz})$ on $\mathbf{c}^2$.

□

6.2.5. *Step 3.* In the set-up of Sect. 6.2.3 we take $\mathbf{C}^1 = \mathbf{C}^2 = \mathrm{DGSch}_{\mathrm{Noeth}}$, $vert^1 = vert^2 = \mathrm{all}$ and

$$horiz^1 = \text{aft-sep} \text{ and } horiz^2 = \text{aft}\,.$$

We define the functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

as the right Kan extension of

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}}} : (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

along the tautological functor

$$(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}} \to (\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}.$$

We claim that the resulting functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}$ satisfies the conditions of Theorem 5.2.2. By Step 2, it remains to show that for $S \in \mathrm{DGSch}_{\mathrm{Noeth}}$, the natural map

$$\mathrm{IndCoh}(S) \to \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}(S)$$

is an isomorphism in $\mathrm{DGCat}_{\mathrm{cont}}$.

Denote

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}} := \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{corr:all;aft-sep}}}|_{((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{corr:all;aft-sep}})^{\mathrm{op}}}$$

and

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}} := \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{corr:all;aft}}}|_{((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{corr:all;aft}})^{\mathrm{op}}}.$$

Note that by Proposition 6.2.4, we have

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}} \simeq \mathrm{RKE}_{((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}})^{\mathrm{op}} \to ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}}).$$

Consider the full subcategory

$$\mathrm{DGSch}_{\mathrm{Noeth,sep}} \subset \mathrm{DGSch}_{\mathrm{Noeth}}$$

that consists of separated DG schemes. Denote

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}}} := \mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}}|_{((\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}})^{\mathrm{op}}}.$$

We claim:

Lemma 6.2.6. *The map*

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}} \to \mathrm{RKE}_{((\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}})^{\mathrm{op}} \to ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}}}) \tag{6.5}$$

is an isomorphism.

The proof of this lemma will be given in Sect. 6.4.4.

From the above lemma, we obtain:

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}} \simeq \mathrm{RKE}_{((\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}})^{\mathrm{op}} \to ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}}}).$$

Hence, it remains to show that for $S \in \mathrm{DGSch}_{\mathrm{Noeth}}$, the map

$$\mathrm{IndCoh}(S) \to \underset{S'}{lim}\, \mathrm{IndCoh}(S')$$

is an isomorphism in $\mathrm{DGCat}_{\mathrm{cont}}$, where the limit is taken over the category (opposite to)

$$(\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}} \underset{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}{\times} ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})_{/S}.$$

However, the above category is the same as

$$(\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}} \underset{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}{\times} ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}})_{/S} \tag{6.6}$$

(indeed, a map from a separated DG scheme to any DG scheme is separated). Note, however, that the limit of $\mathrm{IndCoh}(S')$ over the category (opposite to one) in (6.6) is the value of

$$\mathrm{RKE}_{((\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}})^{\mathrm{op}} \to ((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}}})$$

on S. In particular, the map to it from $\mathrm{IndCoh}(S)$ is an isomorphism because (6.5) is an isomorphism.

□

6.3. **Factorization of separated morphisms.** Let $f : S_1 \to S_2$ be a separated map between Noetherian DG schemes. In this subsection we will prove that the category $\mathsf{Factor}(f)$ of factorizations of f as

$$S_1 \xrightarrow{j} S_{3/2} \xrightarrow{g} S_2, \tag{6.7}$$

where j is an open embedding and g proper, is contractible (in particular, non-empty).

6.3.1. *Step 1.* First we show that $\mathsf{Factor}(f)$ is non-empty. By Nagata's theorem, we can factor the morphism

$$({}^{cl}S_1)_{red} \to ({}^{cl}S_2)_{red}$$

as

$$({}^{cl}S_1)_{red} \to S'_{3/2} \to ({}^{cl}S_2)_{red},$$

where $S'_{3/2}$ is a reduced classical scheme, with the morphism $({}^{cl}S_1)_{red} \to S'_{3/2}$ being an open embedding and $S'_{3/2} \to ({}^{cl}S_2)_{red}$ proper.

We define an object of $\mathsf{Factor}(f)$ by setting

$$S_{3/2} := S_1 \underset{({}^{cl}S_1)_{red}}{\sqcup} S'_{3/2}.$$

(we refer the reader to [GR2, Sect. 3.3], where the existence and properties of push-out for DG schemes are reviewed).

6.3.2. *Digression.* For a map of Noetherian DG schemes $h : T_1 \to T_2$ we let

$$(\mathrm{DGSch}_{\mathrm{Noeth}})_{T_1/,\,\mathrm{closed\ in}\ T_2} \subset (\mathrm{DGSch}_{\mathrm{Noeth}})_{T_1//T_2}$$

be the full subcategory spanned by those objects

$$T_1 \to T_{3/2} \to T_2,$$

where the map $T_{3/2} \to T_2$ is a closed embedding (see Definition 3.3.3).

The following is established in [GR2, Sect. 3.1]:

Proposition 6.3.3.

(a) *The category* $(\mathrm{DGSch}_{\mathrm{Noeth}})_{T_1/,\,\mathrm{closed\ in}\ T_2}$ *contains finite colimits (in particular, an initial object).*

(b) *The formation of finite colimits in* $(\mathrm{DGSch}_{\mathrm{Noeth}})_{T_1/,\,\mathrm{closed\ in}\ T_2}$ *commutes with Zariski localization with respect to* T_2*.*

The initial object in $(\mathrm{DGSch}_{\mathrm{Noeth}})_{T_1/,\,\mathrm{closed\ in}\ T_2}$ will be denoted $\overline{f(T_1)}$ and referred to as *the closure of* T_1 *in* T_2.

It is easy to see that if h is a closed embedding that the canonical map

$$T_1 \to \overline{f(T_1)}$$

is an isomorphism.

We will need the following transitivity property of the operation of taking the closure. Let

$$T_1 \xrightarrow{h_{1,2}} T_2 \xrightarrow{h_{2,3}} T_3$$

be a pair of morphisms between Noetherian DG schemes. Set $h_{1,3} = h_{2,3} \circ h_{1,2}$ and $T'_2 := \overline{h_{1,2}(T_1)}$.

By the universal property of closure, we have a canonically defined map

$$\overline{h_{1,3}(T_1)} \to \overline{h_{1,2}(T'_2)} \tag{6.8}$$

in $(\mathrm{DGSch}_{\mathrm{Noeth}})_{T_1/,\,\mathrm{closed\ in}\ T_3}$. We have:

Lemma 6.3.4. *The map* (6.8) *is an isomorphism.*

6.3.5. *Step 2.* Let $\mathsf{Factor}_{\mathrm{dense}}(f) \subset \mathsf{Factor}(f)$ be the full subcategory consisting of those objects

$$S_1 \overset{j}{\to} S_{3/2} \overset{g}{\to} S_2,$$

for which the map

$$\overline{j(S_1)} \to S_{3/2}$$

is an isomorphism.

We claim that the tautological embedding

$$\mathsf{Factor}_{\mathrm{dense}}(f) \hookrightarrow \mathsf{Factor}(f)$$

admits a right adjoint that sends a given object (6.7) to

$$S_1 \to \overline{j(S_1)} \to S_2.$$

Indeed, the fact that the map $S_1 \to \overline{j(S_1)}$ is an open embedding follows from Proposition 6.3.3(b). The fact that the above operation indeed produces a right adoint follows from Lemma 6.3.4.

Hence, it suffices to show that the category $\mathsf{Factor}_{\mathrm{dense}}(f)$ is contractible.

6.3.6. *Step 3.* We will show that the category $\mathsf{Factor}_{\mathrm{dense}}(f)$ contains products.

Given two objects

$$(S_1 \to S'_{3/2} \to S_2) \text{ and } (S_1 \to S''_{3/2} \to S_2)$$

of $\mathsf{Factor}_{\mathrm{dense}}(f)$ consider

$$T := S'_{3/2} \underset{S_2}{\times} S''_{3/2},$$

and let h denote the resulting map $S_1 \to T$.

Set $S_{3/2} := \overline{h(S_1)}$. We claim that the map $S_1 \to S_{3/2}$ is an open embedding. Indeed, consider the open subscheme of $\overset{\circ}{T} \subset T$ equal to $S_1 \underset{S_2}{\times} S_1$. By Proposition 6.3.3(b),

$$\overset{\circ}{S}_{3/2} := S_{3/2} \cap \overset{\circ}{T}$$

is the closure of the map

$$\Delta_{S_1/S_2} : S_1 \to S_1 \underset{S_2}{\times} S_1.$$

However, $S_1 \to \overline{\Delta_{S_1/S_2}(S_1)}$ is an isomorphism since Δ_{S_1/S_2} is a closed embedding.

6.3.7. *Step 4.* Finally, we claim that the resulting object

$$S_1 \to S_{3/2} \to S_2$$

is the product of $S_1 \to S'_{3/2} \to S_2$ and $S_1 \to S''_{3/2} \to S_2$ in $\mathsf{Factor}_{\mathrm{dense}}(f)$.

Indeed, let

$$S_1 \to \widetilde{S}_{3/2} \to S_2$$

be another object of $\mathsf{Factor}_{\mathrm{dense}}(f)$, endowed with maps to

$$S_1 \to S'_{3/2} \to S_2 \text{ and } S_1 \to S''_{3/2} \to S_2.$$

Let i denote the resulting morphism

$$\widetilde{S}_{3/2} \to S'_{3/2} \underset{S_2}{\times} S''_{3/2} = T.$$

We have a canonical map in $\mathsf{Factor}(f)$

$$(S_1 \to \widetilde{S}_{3/2} \to S_2) \to (S_1 \to \overline{i(\widetilde{S}_{3/2})} \to S_2).$$

However, from Lemma 6.3.4 we obtain that the natural map

$$S_{3/2} \to \overline{i(\widetilde{S}_{3/2})}$$

is an isomorphism. This gives rise to the desired map

$$(S_1 \to \widetilde{S}_{3/2} \to S_2) \to (S_1 \to S_{3/2} \to S_2).$$

□

6.4. **The notion of density.** In this subsection we shall discuss the notion of density of a full subcategory in an ∞-category.

6.4.1. Let $\mathbf{C}$ be an ∞-category with fiber products, and equipped with a Grothendieck topology. Let

$$i : \mathbf{C}' \hookrightarrow \mathbf{C}$$

be a full subcategory.

We define a Grothendieck topology on $\mathbf{C}'$ be declaring that a 1-morphism is a covering if its image in $\mathbf{C}$ is.

We shall say that $\mathbf{C}'$ is dense in $\mathbf{C}$ if:

- Every object in $\mathbf{C}$ admits a covering $\underset{\alpha}{\sqcup}\, \mathbf{c}'_\alpha \to \mathbf{c}$, $\mathbf{c}'_\alpha \in \mathbf{C}'$.
- If $\underset{\alpha}{\sqcup}\, \mathbf{c}'_{1,\alpha} \to \mathbf{c}$, and $\underset{\alpha}{\sqcup}\, \mathbf{c}'_{2,\beta} \to \mathbf{c}$, are coverings and $\mathbf{c}'_{1,\alpha}, \mathbf{c}'_{2,\beta} \in \mathbf{C}'$, then each $\mathbf{c}'_{1,\alpha} \underset{\mathbf{c}}{\times} \mathbf{c}'_{2,\beta}$ belongs to $\mathbf{C}'$.

6.4.2. Let $\mathbf{D}$ be an ∞-category that contains limits. We let

$$\mathrm{Funct}(\mathbf{C}^{op}, \mathbf{D})_{\mathrm{descent}} \subset \mathrm{Funct}(\mathbf{C}^{op}, \mathbf{D})$$

be the full subcategory of functors that satisfy descent with respect to the given Grothendieck topology. I.e., these are $\mathbf{D}$-valued sheaves as a subcategory of $\mathbf{D}$-valued presheaves.

We will prove:

Proposition 6.4.3. *Suppose $\mathbf{C}' \subset \mathbf{C}$ is dense. Then for any $\mathbf{D}$ as above, the adjoint functors*

$$\mathrm{Res}_i : \mathrm{Funct}(\mathbf{C}^{\mathrm{op}}, \mathbf{D}) \rightleftarrows \mathrm{Funct}(\mathbf{C}'^{\mathrm{op}}, \mathbf{D}) : \mathrm{RKE}_i$$

define mutually inverse equivalences

$$\mathrm{Funct}(\mathbf{C}^{\mathrm{op}}, \mathbf{D})_{\mathrm{descent}} \rightleftarrows \mathrm{Funct}(\mathbf{C}'^{\mathrm{op}}, \mathbf{D})_{\mathrm{descent}}.$$

The proposition is apparently well-known. For the sake of completeness, we will give a proof in Sect. 6.4.5.

6.4.4. Let us show how Proposition 6.4.3 implies Lemma 6.2.6. Indeed, we take

$$\mathbf{C} := (\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}, \ \mathbf{C}' := (\mathrm{DGSch}_{\mathrm{Noeth,sep}})_{\mathrm{aft}},$$

and we consider the functor

$$((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

equal to $\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}}$.

We equip $\mathbf{C} := (\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}$ with the Zariski topology, i.e., coverings are surjective maps $\underset{\alpha}{\sqcup} S'_\alpha \to S$, where each S'_α is an open DG subscheme of S.

The fact that the functor $\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{aft-sep}}}$ satisfies Zariski descent follows from Proposition 4.2.1 and the fact that the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft\text{-}sep}}}$ satisfies condition (b) of Theorem 5.2.2.

□

6.4.5. *Proof of Proposition 6.4.3.* First, it easy to see that the functors Res_i and RKE_i indeed send the subcategories in question to one another. It is equally easy to see that the functor Res_i is conservative.

Hence, it remains to show that for $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ that satisfies descent, the natural map

$$F \to \mathrm{RKE}_i(F|_{\mathbf{C}'^{\mathrm{op}}})$$

is an isomorphism.

For $\mathbf{c} \in \mathbf{C}$ we have

$$\mathrm{RKE}_i(F|_{\mathbf{C}'^{\mathrm{op}}}) \simeq \underset{\mathbf{c}' \in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}').$$

Choose a covering $\mathbf{c}'_A := \underset{\alpha}{\sqcup} \mathbf{c}'_\alpha \to \mathbf{c}$, with $\mathbf{c}'_\alpha \in \mathbf{C}'$. Let $\mathbf{c}'^{\bullet}_A/\mathbf{c}$ be its Čech nerve. Restriction defines a map

$$\underset{\mathbf{c}' \in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}') \to \mathrm{Tot}(F(\mathbf{c}'^{\bullet}_A/\mathbf{c})),$$

and the composition

$$F(\mathbf{c}) \to \underset{\mathbf{c}' \in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}') \to \mathrm{Tot}(F(\mathbf{c}'^{\bullet}_A/\mathbf{c}))$$

is the natural map

$$F(\mathbf{c}) \to \mathrm{Tot}(F(\mathbf{c}'^{\bullet}_A/\mathbf{c})),$$

and hence is an isomorphism, since F satisfies descent.

Consider now the object

$$\mathrm{Tot}\left(\underset{\mathbf{c}' \in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}' \underset{\mathbf{c}}{\times} (\mathbf{c}'^{\bullet}_A/\mathbf{c}))\right).$$

We shall complete the above maps to a commutative diagram

$$
\begin{array}{ccc}
F(\mathbf{c}) & \longrightarrow & \underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}'). \\
\downarrow & \swarrow & \downarrow \\
\mathrm{Tot}(F(\mathbf{c}'^{\bullet}_A/\mathbf{c})) & \longrightarrow & \mathrm{Tot}\left(\underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{\bullet}_A/\mathbf{c}))\right).
\end{array}
\tag{6.9}
$$

in which the right vertical arrow is an isomorphism. This will prove that the top horizontal arrow is also an isomorphism, as required.

The map

$$\mathrm{Tot}(F(\mathbf{c}'^{\bullet}_A/\mathbf{c})) \to \mathrm{Tot}\left(\underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{\bullet}_A/\mathbf{c}))\right)$$

is the simplex-wise map

$$F(\mathbf{c}'^{\bullet}_A/\mathbf{c}) \to \underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{\bullet}_A/\mathbf{c})),$$

given by restriction.

We shall now consider three maps

$$
\begin{aligned}
\underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}') \to \mathrm{Tot}\left(\underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{\bullet}_A/\mathbf{c}))\right) \simeq \\
\simeq \underset{([n]\times\mathbf{c}')\in\mathbf{\Delta}\times(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{n}_A/\mathbf{c})) \simeq \underset{\mathbf{c}'\in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}}{lim} \mathrm{Tot}(F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{\bullet}_A/\mathbf{c}))).
\end{aligned}
\tag{6.10}
$$

The first map in (6.10) corresponds to the map of index categories that sends

$$([n]\times\mathbf{c}')\in\mathbf{\Delta}\times(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}} \mapsto \mathbf{c}'^{n}_A/\mathbf{c}\in(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}$$

and the map

$$F(\mathbf{c}'^{n}_A/\mathbf{c}) \to F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{n}_A/\mathbf{c}))$$

is given by the projection

$$\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{n}_A/\mathbf{c}) \to \mathbf{c}'^{n}_A/\mathbf{c}.$$

It is clear that for this map the lower traingle in (6.9) commutes.

The second map in (6.10) corresponds to the map of index categories that sends

$$([n]\times\mathbf{c}')\in\mathbf{\Delta}\times(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}} \mapsto \mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{n}_A/\mathbf{c})\in(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}},$$

and the identity map

$$F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{n}_A/\mathbf{c})) \to F(\mathbf{c}'\underset{\mathbf{c}}{\times}(\mathbf{c}'^{n}_A/\mathbf{c})).$$

We note, however, that the first and the second maps are canonically homotopic.

The third map in (6.10) corresponds to the map of index categories that sends

$$([n]\times\mathbf{c}')\in\mathbf{\Delta}\times(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}} \mapsto \mathbf{c}'\in(\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}},$$

and the map

$$F(\mathbf{c}') \to F(\mathbf{c}' \underset{\mathbf{c}}{\times} (\mathbf{c}'^{\,n}_A/\mathbf{c}))$$

is given by the projection

$$\mathbf{c}' \underset{\mathbf{c}}{\times} (\mathbf{c}'^{\,n}_A/\mathbf{c}) \to \mathbf{c}'.$$

Again, we note that the third and the second map are canonically homotopic.

Finally, it remains to see that the third map is an isomorphism. This follows from the fact that for each $\mathbf{c}' \in (\mathbf{C}'_{/\mathbf{c}})^{\mathrm{op}}$, the map

$$F(\mathbf{c}') \to \mathrm{Tot}(F(\mathbf{c}' \underset{\mathbf{c}}{\times} (\mathbf{c}'^{\,\bullet}_A/\mathbf{c})))$$

is an isomorphism, since F satisfies descent.

□

6.4.6. In what follows we will use the following variant of Proposition 6.4.3. Let $\mathbf{C}$ be a category with fiber products and equipped with a Grothendick topology.

Let $\mathbf{C}_0$ be a 1-full subcategory of $\mathbf{C}$. We will assume that the following is satisfied:

Whenever $\underset{\alpha}{\sqcup} \mathbf{c}_\alpha \to \mathbf{c}$ in $\mathbf{C}$ is a covering, then each arrow $\mathbf{c}_\alpha \to \mathbf{c}$ belongs to $\mathbf{C}_0$.

Note that in this case we can talk about descent for presheaves on $\mathbf{C}_0$: the fiber products used in the formulation of the are taken inside the ambient category $\mathbf{C}$.

6.4.7. Let $\mathbf{C}' \subset \mathbf{C}$ be a full subactegory satisfying the assumptions of Sect. 6.4.1. Let $\mathbf{C}'_0$ be the corresponding 1-full subcategory of $\mathbf{C}'$.

We shall assume that the following additional condition on $\mathbf{C}_0 \to \mathbf{C}$ is satisfied:

- If $\underset{\alpha}{\sqcup} \mathbf{c}_\alpha \to \mathbf{c}$ is a covering and $\widetilde{\mathbf{c}} \to \mathbf{c}$ is a 1-morphism in $\mathbf{C}_0$, then each of the maps $\mathbf{c}_\alpha \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}} \to \mathbf{c}_\alpha$ belongs to $\mathbf{C}_0$.

In this case, the proof of Proposition 6.4.3 applies and gives the following:

Proposition 6.4.8. *The adjoint functors*

$$\mathrm{Res}_i : \mathrm{Funct}((\mathbf{C}_0)^{\mathrm{op}}, \mathbf{D}) \rightleftarrows \mathrm{Funct}((\mathbf{C}'_0)^{\mathrm{op}}, \mathbf{D}) : \mathrm{RKE}_i$$

define mutually inverse equivalences

$$\mathrm{Funct}((\mathbf{C}_0)^{\mathrm{op}}, \mathbf{D})_{\mathrm{descent}} \rightleftarrows \mathrm{Funct}((\mathbf{C}'_0)^{\mathrm{op}}, \mathbf{D})_{\mathrm{descent}}.$$

7. Eventually coconnective, Gorenstein and smooth morphisms

In this section we continue to assume that all DG schemes are Noetherian.

The results in this section were obtained in collaboration with D. Arinkin and some of them appear also in [AG, Appendix E].

7.1. **The !-pullback functor for eventually coconnective morphisms.**

7.1.1. First, we notice:

Lemma 7.1.2. *Let $S_1 \to S_2$ be a morphism almost of finite type. Then the following conditions are equivalent:*

- *The morphism f is eventually coconnective.*
- *The functor $f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$ sends $\mathrm{Coh}(S_2)$ to $\mathrm{Coh}(S_1)$.*

In this case, the functor $f^!$ has a bounded cohomological amplitude.

Proof. As in the proof of Proposition 3.6.7, the assertion reduces to the case when f is a closed embedding. Denote $\mathcal{E} := f_*(\mathcal{O}_{S_1}) \in \mathrm{Coh}(S_2)$. Then the same argument as in Lemma 3.6.3 shows that both conditions in the lemma are equivalent to $\mathcal{E}$ being perfect.

□

7.1.3. Let now

$$\begin{array}{ccc} S_1' & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S_2' & \xrightarrow{g_2} & S_2 \end{array}$$

be a Cartesian diagram where horizontal arrows are almost of finite type, and vertical arrows are of bounded Tor dimension (in particular, eventually coconnective).

Starting from the base change isomorphism

$$g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^!$$

by the $(f^{\mathrm{IndCoh},*}, f_*^{\mathrm{IndCoh}})$- and $((f')^{\mathrm{IndCoh},*}, (f')_*^{\mathrm{IndCoh}})$-adjunctions, we obtain a natural transformation

$$(f')^{\mathrm{IndCoh},*} \circ g_2^! \to g_1^! \circ f^{\mathrm{IndCoh},*}. \tag{7.1}$$

Remark 7.1.4. If the map g_2 (and hence g_1) is proper, then diagram chase shows that the map in (7.1) is canonically isomorphic to one obtained from the isomorphism

$$(g_1)_*^{\mathrm{IndCoh}} \circ (f')^{\mathrm{IndCoh},*} \simeq f^{\mathrm{IndCoh},*} \circ (g_2)_*^{\mathrm{IndCoh}}$$

of Lemma 3.6.9.

7.1.5. We claim:

Proposition 7.1.6. *The map* (7.1) *is an isomorphism.*

Proof. First, it is easy to see that the assertion is Zariski-local in S_2 and S_2'. By the construction of the !-pullback functor, we can consider separately the cases of g_2 (and hence g_1) proper and an open embedding, respectively.

The case of an open embedding is immediate. So, in what follows we shall assume that the maps g_1 and g_2 are proper.

Note that the statement is Zariski-local also in S_1. Hence, we can assume that the morphism f (and hence f') is affine.

Since all the functors involved are continuous, it is enough to show that the map (7.1) is an isomorphism when evaluated on $\mathrm{Coh}(S_2)$. We will show more generally that it is an isomorphism, when evaluated on objects of $\mathrm{IndCoh}(S_2)^+$.

The assumption of f implies that the functor $f^{\mathrm{IndCoh},*}$ (resp., $(f')^{\mathrm{IndCoh},*}$) sends the category $\mathrm{IndCoh}(S_2)^+$ (resp., $\mathrm{IndCoh}(S_2')^+$) to $\mathrm{IndCoh}(S_1)^+$ (resp., $\mathrm{IndCoh}(S_1')^+$).

We claim:

Lemma 7.1.7. *For any morphism $g : T_1 \to T_2$ amost of finite type, the functor $g^!$ is left t-exact, up to a finite shift.*

The proof of the lemma is given below.

Assuminh the lemma, we obtain that both functors in (7.1) send the category $\mathrm{IndCoh}(S_2)^+$ to $\mathrm{IndCoh}(S_1')^+$. Since f (and hence f') is affine, the functor f'_* is conservative. It follows from Proposition 1.2.4 that the functor $(f')^{\mathrm{IndCoh}}_*$ is conservative when restricted to $\mathrm{IndCoh}(S_1')^+$.

Hence, it remains to show that the map

$$(f')^{\mathrm{IndCoh}}_* \circ (f')^{\mathrm{IndCoh},*} \circ g_2^!(\mathcal{F}) \to (f')^{\mathrm{IndCoh}}_* \circ g_1^! \circ f^{\mathrm{IndCoh},*}(\mathcal{F}) \tag{7.2}$$

is an isomorphism for any $\mathcal{F} \in \mathrm{IndCoh}(S_2)$.

By Corollary 4.4.3, we have canonical isomorphisms

$$(f')^{\mathrm{IndCoh}}_* \circ (f')^{\mathrm{IndCoh},*} \simeq f'_*(\mathcal{O}_{S_1'}) \otimes - \text{ and } f^{\mathrm{IndCoh}}_* \circ f^{\mathrm{IndCoh},*} \simeq f'_*(\mathcal{O}_{S_1}) \otimes -.$$

Note also that

$$f'_*(\mathcal{O}_{S_1'}) \simeq g_2^*(f_*(\mathcal{O}_{S_1})).$$

Hence, we can rewrite the left-hand side in (7.2) as

$$g_2^*(f_*(\mathcal{O}_{S_1})) \otimes g_2^!(\mathcal{F}) \overset{\text{Equation (3.7)}}{\simeq} g_2^!(f_*(\mathcal{O}_{S_1}) \otimes \mathcal{F}).$$

We rewrite the right-hand side in (7.2) as

$$g_2^! \circ f^{\mathrm{IndCoh}}_* \circ f^{\mathrm{IndCoh},*}(\mathcal{F}) \simeq g_2^!(f_*(\mathcal{O}_{S_1}) \otimes \mathcal{F}).$$

It follows by unwinding the constructions that the map in (7.2) identifies with the identity map on $g_2^!(f_*(\mathcal{O}_{S_1}) \otimes \mathcal{F})$.

□

Proof of Lemma 7.1.7. By the construction of $g^!$, it suffices to consider separately the cases of g being an open embedding and a proper map. The case of an open embedding is evident; in this case the functor $g^! = g^{\mathrm{IndCoh},*}$ is t-exact.

For a proper map, the functor $g^!$ is by definition the right adjoint of g^{IndCoh}_*. The required assertion follows from the fact that the functor g^{IndCoh}_* is right t-exact, up to a finite shift, which in turn follows from the corresponding property of g_*.

□

7.2. **The !-pullback functor on** QCoh. In this subsection we let $f : S_1 \to S_2$ be an eventually coconnective morphism almost of finite type.

7.2.1. We claim:

Proposition 7.2.2. *There exists a uniquely defined continuous functor*

$$f^{\mathrm{QCoh},!} : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1),$$

which is of bounded cohomological amplitude, and makes the following diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(S_1) & \xrightarrow{\Psi_{S_1}} & \mathrm{QCoh}(S_1) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{QCoh},!}} \\ \mathrm{IndCoh}(S_2) & \xrightarrow{\Psi_{S_2}} & \mathrm{QCoh}(S_2) \end{array}$$

commute. Furthermore, the functor $f^{\mathrm{QCoh},!}$ has a unique structure of 1-morphism of $\mathrm{QCoh}(S_2)$-module categories, for which the above diagram commutes as a diagram of $\mathrm{QCoh}(S_2)$-module categories.

Proof. This follows using Proposition 1.3.4 from the following general observation.

Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a continuous functor between cocomplete DG categories. Suppose that $\mathbf{C}_1$ and $\mathbf{C}_2$ are endowed with t-structures compatible with filtered colimits. Suppose that F is right t-exact up to a finite shift.

Let $\mathbf{C}'_1$ and $\mathbf{C}'_2$ be the left completions of $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, in their t-structures.

Lemma 7.2.3. *Under the above circumstances there exists a uniquely defined continuous functor $F' : \mathbf{C}'_1 \to \mathbf{C}'_2$ which makes the diagram*

$$\begin{array}{ccc} \mathbf{C}_1 & \longrightarrow & \mathbf{C}'_1 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathbf{C}_2 & \longrightarrow & \mathbf{C}'_2 \end{array}$$

commute.

□

7.2.4. Note that the $\mathrm{QCoh}(S_2)$-linearity of $f^{\mathrm{QCoh},!}$ tautologically implies:

Corollary 7.2.5. *There is a canonical isomorphism*

$$f^*(\mathcal{E}) \otimes f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \simeq f^{\mathrm{QCoh},!}(\mathcal{E}), \quad \mathcal{E} \in \mathrm{QCoh}(S_2).$$

We now claim:

Proposition 7.2.6. *There exists a uniquely defined natural transformation*

$$f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}) \to f^!(\mathcal{F}), \quad \mathcal{F} \in \mathrm{IndCoh}(S_2) \tag{7.3}$$

that makes the diagram

$$\begin{array}{ccc} \Psi_{S_1}(f^!(\mathcal{F})) & \xrightarrow{\sim} & f^{\mathrm{QCoh},!}(\Psi_{S_2}(\mathcal{F})) \\ {\scriptstyle \Psi_{S_1}((7.3))}\uparrow & & \sim\uparrow{\scriptstyle \textit{Corollary 7.2.5}} \\ \Psi_{S_1}(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\mathcal{F})) & & f^*(\Psi_{S_2}(\mathcal{F})) \otimes f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \\ \sim\uparrow & & \uparrow\sim \\ f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes \Psi_{S_1}(f^{\mathrm{IndCoh},*}(\mathcal{F})) & \xrightarrow[\textit{Proposition 3.5.4}]{\sim} & f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^*(\Psi_{S_2}(\mathcal{F})) \end{array}$$

commute.

Proof. It suffuces to construct (and prove the uniquness of) the isomorphism in question on the compact generators of $\mathrm{IndCoh}(S_2)$, i.e., for $\mathcal{F} \in \mathrm{Coh}(S_2)$. For such $\mathcal{F}$, we have $f^!(\mathcal{F}) \in \mathrm{IndCoh}(S_1)^+$. Hence, it suffices to construct (and prove the uniquness of) a map

$$\tau^{\geq -n}\left(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\mathcal{F})\right) \to f^!(\mathcal{F})$$

for $n \gg 0$ that makes the corresponding diagram commute.

By Proposition 1.2.4, the latter map is equivalent to a map

$$\begin{aligned} &\tau^{\geq -n}\left(\Psi_{S_1}\left(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right) \simeq \\ &\qquad \simeq \Psi_{S_1}\left(\tau^{\geq -n}\left(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right) \to \Psi_{S_1}\left(f^!(\mathcal{F})\right). \end{aligned} \tag{7.4}$$

We have:

$$\Psi_{S_1}\left(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})\otimes f^{\mathrm{IndCoh},*}(\mathcal{F})\right)\simeq f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})\otimes\left(\Psi_{S_1}\circ f^{\mathrm{IndCoh},*}(\mathcal{F})\right)\simeq$$
$$\simeq f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})\otimes\left(f^*\circ\Psi_{S_2}(\mathcal{F})\right),$$

which, by Corollary 7.2.5, identifies with

$$f^{\mathrm{QCoh},!}(\Psi_{S_2}(\mathcal{F}))\simeq\Psi_{S_1}(f^!(\mathcal{F})).$$

In particular,

$$\tau^{\geq -n}\left(\Psi_{S_1}\left(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})\otimes f^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right)\simeq\Psi_{S_1}(f^!(\mathcal{F}))$$

for all $n\gg 0$, and the required map in (7.4) is the identity.

□

Remark 7.2.7. As we shall see in Propositions 7.3.8 and 7.4.5, the map in (7.3) is an isomorphism *if and only if* the map f is Gorenstein.

7.2.8. Here are some properties of the functor of the functor $f^{\mathrm{QCoh},!}$ introduced in Proposition 7.2.2.

Proposition 7.2.9.
(a) *If f is proper, the functor $f^{\mathrm{QCoh},!}$ is the right adjoint of f_*.*
(b) *Let*

$$\begin{array}{ccc} S'_1 & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S'_2 & \xrightarrow{g_2} & S_2 \end{array}$$

be a Cartesian diagram. Then there exists a canonical isomorphism of functors

$$(g_1)_*\circ(f')^{\mathrm{QCoh},!}\simeq f^{\mathrm{QCoh},!}\circ(g_2)_* \tag{7.5}$$

Moreover, if f is proper, the map $\to$ in (7.5) *comes by the $(f_*,f^{\mathrm{QCoh},!})$- and $(f'_*,(f')^{\mathrm{QCoh},!})$-adjunctions from the isomorphism*

$$f_*\circ(g_1)_*\simeq(g_2)_*\circ f'_*.$$

(c) *The map*

$$g_1^*\circ f^{\mathrm{QCoh},!}\to(f')^{\mathrm{QCoh},!}\circ g_2^* \tag{7.6}$$

obtained from (7.5) *by the $(g_1^*,(g_1)_*)$- and $(g_2^*,(g_2)_*)$-adjunctions, is an isomorphism.*

Remark 7.2.10. A diagram chase shows that if f (and hence f') is proper, the map in (7.6) is canonically isomorphic to one obtained by the $(f_*,f^{\mathrm{QCoh},!})$- and $(f'_*,(f')^{\mathrm{QCoh},!})$-adjunctions from the usual base change isomorphism for QCoh:

$$f'_*\circ g_1^*\simeq g_2^*\circ f_*.$$

Proof. We first prove point (a). Since $\mathrm{QCoh}(S_1)$ and $\mathrm{QCoh}(S_2)$ are left-complete in their respective t-structures and since $f^{\mathrm{QCoh},!}$ is right t-exact up to a finite shift, it suffices construct a canonical isomorphism

$$\mathrm{Maps}_{\mathrm{QCoh}(S_2)}(f_*(\mathcal{E}_1),\mathcal{E}_2)\to\mathrm{Maps}_{\mathrm{QCoh}(S_2)}(\mathcal{E}_1,f^{\mathrm{QCoh},!}(\mathcal{E}_2)) \tag{7.7}$$

for $\mathcal{E}_2\in\mathrm{QCoh}(S_2)^+$.

Since $f^{\mathrm{QCoh},!}$ is also left t-exact up to a finite shift, and since f_* is left t-exact, we can also take $\mathcal{E}_1 \in \mathrm{QCoh}(S_1)^+$. In this case, Proposition 1.2.4 reduces the isomorphism (7.7) for one for IndCoh.

Point (b) follows from Lemma 7.2.3 and the base change isomorphism for IndCoh.

Let us prove point (c). As in the proof of Proposition 7.1.6, we can assume that f (and hence f') is proper, and g_2 (and hence g_1) is affine.

In this case, the functor $(g_1)_*$ is conservative. Hence, it is sufficient to show that the natural transformation

$$(g_1)_* \circ g_1^* \circ f^{\mathrm{QCoh},!} \to (g_1)_* \circ (f')^{\mathrm{QCoh},!} \circ g_2^* \tag{7.8}$$

is an isomorphism.

The value of the left-hand side of (7.8) on $\mathcal{E} \in \mathrm{QCoh}(S_2)$ is canonically isomorphic to

$$(g_1)_*(\mathcal{O}_{S_1'}) \otimes f^{\mathrm{QCoh},!}(\mathcal{E}) \simeq f^*((g_2)_*(\mathcal{O}_{S_2'})) \otimes f^{\mathrm{QCoh},!}(\mathcal{E}) \simeq f^{\mathrm{QCoh},!}((g_2)_*(\mathcal{O}_{S_2'}) \otimes \mathcal{E}).$$

The value of the right-hand side of (7.8) on $\mathcal{E} \in \mathrm{QCoh}(S_2)$ is, by point (b), canonically isomorphic to

$$f^{\mathrm{QCoh},!} \circ (g_2)_* \circ g_2^*(\mathcal{E}) \simeq f^{\mathrm{QCoh},!}((g_2)_*(\mathcal{O}_{S_2'}) \otimes \mathcal{E}).$$

By unwinding the constructions, we obtain that the map in (7.8) is the identity map on $f^{\mathrm{QCoh},!}((g_2)_*(\mathcal{O}_{S_2'}) \otimes \mathcal{E})$.

□

7.3. **Gorenstein morphisms.**

7.3.1. Let $f : S_1 \to S_2$ be a morphism between Noetherian DG schemes.

Definition 7.3.2. *We shall say that f is Gorenstein if it is eventually coconnective, locally almost of finite type, and the object $f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \in \mathrm{QCoh}(S_1)$ is a graded line bundle.*

In what follows, for a Gorenstein morphism morphism $f : S_1 \to S_2$, we shall denote by $\mathcal{K}_{S_1/S_2}$ the graded line bundle appearing in the above definition, and refer to it as the "relative dualizing line bundle."

7.3.3. We shall say that $S \in \mathrm{DGSch}_{\mathrm{Noeth}}$ is Gorenstein if the map

$$p_S : S \to \mathrm{pt} := \mathrm{Spec}(k)$$

is Gorenstein. I.e., if S is almost of finite type, eventually coconnective and the object

$$\omega_S := (p_S)^!(k) \in \mathrm{Coh}(S),$$

thought of as an object of $\mathrm{QCoh}(S)$, is a graded line bundle.

7.3.4. It follows from Proposition 7.2.9 that the class of Gorenstein morphisms is stable under base change and that the formation of $\mathcal{K}_{S_1/S_2}$ is compatible with base change, i.e., for a Cartesian square

$$\begin{array}{ccc} S_1' & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S_2' & \xrightarrow{g_2} & S_2 \end{array}$$

we have a canonical isomorphism

$$g_1^*(\mathcal{K}_{S_1/S_2}) \simeq \mathcal{K}_{S_1'/S_2'}. \tag{7.9}$$

In addition, we claim:

Corollary 7.3.5. *Let $f : S_1 \to S_2$ be an eventually coconnective morphism almost of finite type. Then f is Gorenstein if only if its geometric fibers are Gorenstein.*

Proof. Assume that the geometric fibers of f are Gorenstein. We need to show that $f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})$ is a graded line bundle. As in Lemma 3.6.3, it suffices to show that the *-restrictions of $f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})$ to the geometric fibers of f are graded line bundles. The latter follows from Proposition 7.2.9(c).

□

Corollary 7.3.6. *Let $f : S_1 \to S_2$ be an eventually coconnective morphism almost of finite type. If the base change of f by $i : {}^{cl}S_2 \to S_2$ is Gorenstein, then f is Gorenstein.*

7.3.7. We are going to prove:

Proposition 7.3.8. *Let $f : S_1 \to S_2$ be Gorenstein. Then the map*

$$\mathcal{K}_{S_1/S_2} \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}) \to f^!(\mathcal{F}), \quad \mathcal{F} \in \mathrm{IndCoh}(S_2)$$

of (7.3) *is an isomorphism.*

Proof. Going back to the proof of Proposition 7.2.6, it suffices to show that for $\mathcal{F} \in \mathrm{Coh}(S)$, the object

$$\mathcal{K}_{S_1/S_2} \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(S_1)$$

belongs to $\mathrm{IndCoh}(S_1)^+$. We have: $f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(S_1)^+$, and the required assertion follows from the fact that $\mathcal{K}_{S_1/S_2} \otimes -$ shifts degrees by a finite amount.

□

7.4. **Characterizations of Gorenstein morphisms.** The material of this subsection is included for the sake of completeness and will not be needed elsewhere in the paper.

7.4.1. We have:

Proposition 7.4.2. *Let $f : S_1 \to S_2$ be an eventually coconnective morphism almost of finite type. Suppose that the object $f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})$ belongs to $\mathrm{QCoh}(S_1)^{\mathrm{perf}}$. Then f is Gorenstein.*

As a particular case, we have the following characterization of Gorenstein DG schemes, suggested by Drinfeld:

Corollary 7.4.3. *Let $S \in \mathrm{DGSch}_{\mathrm{aft}}$ be eventually coconnective. Suppose that the object*

$$\omega_S \in \mathrm{Coh}(S)$$

belongs to $\mathrm{QCoh}(S)^{\mathrm{perf}} \subset \mathrm{Coh}(S)$. Then S is Gorenstein.

Proof. It is easy to see that an object of $\mathrm{QCoh}(S_1)^{\mathrm{perf}}$ is a line bundle if and only if such is the retsriction to all of its geometric fibers over S_2. Using Proposition 7.2.9(c), this reduces the statement of the proposition to that of Corollary 7.4.3. The latter will be proved in Sect. 9.6.15.

□

7.4.4. We are now going to prove a converse to Proposition 7.3.8.

Proposition 7.4.5. *Let $f : S_1 \to S_2$ be an eventually coconnective morphism almost of finite type, such that the natural transformation* (7.3) *is an isomorphism. Then f is Gorenstein.*

Proof. By Proposition 7.1.6, the assumption of the proposition is stable under base change. Using Corollary 7.3.6, we reduce the assertion to the case when S_2 is classical, in particular, eventually coconnective as a DG scheme.

By assumption,

$$f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\Xi_{S_2}(\mathcal{O}_{S_2})) \simeq f^!(\Xi_{S_2}(\mathcal{O}_{S_2})) \in \mathrm{Coh}(S_1).$$

However,

$$f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes f^{\mathrm{IndCoh},*}(\Xi_{S_2}(\mathcal{O}_{S_2})) \simeq f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes \Xi_{S_1}(\mathcal{O}_{S_1}) \simeq \Xi_{S_1}(f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})).$$

From Lemma 1.5.8, we obtain that $f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \in \mathrm{QCoh}(S_1)^{\mathrm{perf}}$. Hence, the assertion follows from Proposition 7.4.2.

□

7.4.6. We end this subsection with the following observation. Let $f : S_1 \to S_2$ be an eventually coconnective morphism almost of finite type, where S_2 (and hence S_1) is itself eventually coconnective. Then from the isomorphism

$$f^{\mathrm{QCoh},!} \circ \Psi_{S_2} \simeq \Psi_{S_1} \circ f^!$$

we obtain a natural transformation:

$$\Xi_{S_1} \circ f^{\mathrm{QCoh},!} \to f^! \circ \Xi_{S_2}. \tag{7.10}$$

We claim:

Proposition 7.4.7. *The map f is Gorenstein if and only if the natural transformation* (7.10) *is an isomorphism.*

Proof. For $\mathcal{E} \in \mathrm{QCoh}(S_2)$ we have

$$\Xi_{S_1} \circ f^{\mathrm{QCoh},!}(\mathcal{E}) \simeq f^*(\mathcal{E}) \otimes (\Xi_{S_1} \circ f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2})) \simeq f^*(\mathcal{E}) \otimes f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \otimes \Xi_{S_1}(\mathcal{O}_{S_1}).$$

If f is Gorenstein, then by Proposition 7.3.8, we have:

$$f^! \circ \Xi_{S_2}(\mathcal{E}) \simeq \mathcal{K}_{S_1/S_2} \otimes (f^{\mathrm{IndCoh},*} \circ \Xi_{S_2}(\mathcal{E})) \simeq \mathcal{K}_{S_1/S_2} \otimes f^*(\mathcal{E}) \otimes (f^{\mathrm{IndCoh},*} \circ \Xi_{S_2}(\mathcal{O}_{S_2})) \simeq$$
$$\simeq \mathcal{K}_{S_1/S_2} \otimes f^*(\mathcal{E}) \otimes (\Xi_{S_1} \circ f^*(\mathcal{O}_{S_2})) \simeq \mathcal{K}_{S_1/S_2} \otimes f^*(\mathcal{E}) \otimes \Xi_{S_1}(\mathcal{O}_{S_1}),$$

and the isomorphism is manifest.

Vice versa, assume that (7.10) holds. We obtain that

$$\Xi_{S_1} \circ f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \in \mathrm{Coh}(S_1).$$

Applying Lemma 1.5.8, we deduce that $f^{\mathrm{QCoh},!}(\mathcal{O}_{S_2}) \in \mathrm{QCoh}(S_1)^{\mathrm{perf}}$. Applying Proposition 7.4.2, we deduce that f is Gorenstein.

□

7.5. **The functor of !-pullback for smooth maps.**

7.5.1. Note that from Corollary 7.3.6 we obtain:

Corollary 7.5.2. *A smooth map is Gorenstein.*

7.5.3. As a corollary of Proposition 7.3.8, we obtain:

Corollary 7.5.4. *Let $f : S_1 \to S_2$ be smooth. There exists a unique isomorphism of functors*

$$\mathcal{K}_{S_1/S_2} \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}) \simeq f^!(\mathcal{F}), \quad \mathcal{F} \in \mathrm{IndCoh}(S_2) \tag{7.11}$$

that makes the diagram

$$\begin{array}{ccc}
\Psi_{S_1}(f^!(\mathcal{F})) & \xrightarrow{\sim} & f^{\mathrm{QCoh},!}(\Psi_{S_2}(\mathcal{F})) \\
{\scriptstyle (7.11)}\big\uparrow & & \sim\big\uparrow{\scriptstyle \mathit{Corollary\ 7.2.5}} \\
\Psi_{S_1}(\mathcal{K}_{S_1/S_2} \otimes f^{\mathrm{IndCoh},*}(\mathcal{F})) & & f^*(\Psi_{S_2}(\mathcal{F})) \otimes \mathcal{K}_{S_1/S_2} \\
\sim\big\uparrow & & \big\uparrow\sim \\
\mathcal{K}_{S_1/S_2} \otimes \Psi_{S_1}(f^{\mathrm{IndCoh},*}(\mathcal{F})) & \xrightarrow[\mathit{Proposition\ 3.5.4}]{\sim} & \mathcal{K}_{S_1/S_2} \otimes f^*(\Psi_{S_2}(\mathcal{F}))
\end{array}$$

commute.

Corollary 7.5.5. *For a smooth map f, the functors $f^!$ and $f^{\mathrm{IndCoh},*}$ differ by an* $\mathrm{QCoh}(S_1)$*-linear automorphism of* $\mathrm{IndCoh}(S_1)$.

7.5.6. Combining this with Proposition 4.5.3, we obtain:

Corollary 7.5.7. *For a smooth map $f : S_1 \to S_2$, the functor $f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$ defines an equivalence*

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1).$$

7.5.8. We are now going to use Corollary 7.5.7 to deduce:

Proposition 7.5.9. *Let $f : S_1 \to S_2$ be eventually coconnective and almost of finite type. Then the functor*

$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1),$$

induced by $f^!$, is fully faithful.

Proof of Proposition 7.5.9. By Corollary 4.4.5, the assertion of the proposition is Zariski-local with respect to both S_1 and S_2.

As in the proof of Lemma 3.6.3, we can assume that f can be factored as a composition $f' \circ f''$, where f' is smooth, and f'' is an eventually coconnective closed embedding. By transitivity, it suffices to prove the assertion for each of these two cases separately.

For smooth maps, the required assertion is given by Corollary 7.5.7. For an eventually coconnective closed embedding, the argument repeats that given in Proposition 4.4.2, where we use the $(f_*^{\mathrm{IndCoh}}, f^!)$-adjunction instead of the $(f^{\mathrm{IndCoh},*}, f_*^{\mathrm{IndCoh}})$-adjunction.

□

7.6. **A higher-categorical compatibility of ! and $*$-pullbacks.** The material of this subsection will be needed for developing IndCoh on Artin stacks, and may be skipped on the first pass.

We will introduce a setting in which we can view IndCoh as a functor out of the category of Noetherian DG schemes, where there are two kinds of pullback functors: one with ! that can be applied to morphisms locally almost of finite type, and another with $*$ that can be applied to morphisms of bounded Tor dimension.

Specifically, we will explain an ∞-categorical framework that encodes the compatibiltiy of the two pullbacks, which at the level of 1-morphisms is given by Proposition 7.1.6.

7.6.1. For an ∞-category $\mathbf{C}$ we let

$$\mathrm{Seg}^\bullet(\mathbf{C}) := \mathrm{Funct}([\bullet], \mathbf{C}) \in \infty\text{-Grpd}^{\mathbf{\Delta}^{\mathrm{op}}}$$

denote the Segal construction applied to $\mathbf{C}$.

I.e., $\mathrm{Seg}^\bullet(\mathbf{C})$ is a simplicial ∞-groupoid, whose n-simplices is the ∞-groupoid of functors

$$[n] \to \mathbf{C},$$

where $[n]$ is the category corresponding to the ordered set

$$0 \to 1 \to ... \to n.$$

The assignment

$$\mathbf{C} \rightsquigarrow \mathrm{Seg}^\bullet(\mathbf{C})$$

is a functor

$$\mathrm{Seg}^\bullet(-) : \infty\text{-Cat} \to \infty\text{-Grpd}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

The following is well-known:

Theorem 7.6.2. *The functor* $\mathrm{Seg}^\bullet(-)$ *is fully faithful.*

The objects of $\infty\text{-Grpd}^{\mathbf{\Delta}^{\mathrm{op}}}$ that lie in the essential image of the functor $\mathrm{Seg}^\bullet(-)$ are called "complete Segal spaces." Thus, Theorem 7.6.2 implies that the category of complete Segal spaces is equivalent to ∞-Cat.

7.6.3. We let

$$\mathrm{Seg}^{\bullet,\bullet}(\mathbf{C}) := \mathrm{Seg}^\bullet(\mathrm{Seg}^\bullet(\mathbf{C})) \in \infty\text{-Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}}$$

be the iteration of the above construction. I.e.,

$$\mathrm{Seg}^{m,n}(\mathbf{C}) := \mathrm{Funct}([m] \times [n], \mathbf{C}) \in \infty\text{-Grpd}.$$

Let π_h, π_v denote the two maps $\mathbf{\Delta} \times \mathbf{\Delta} \rightrightarrows \mathbf{\Delta}$. For a simplicial object $\mathbf{e}^\bullet$ of some ∞-category $\mathbf{E}$ we let

$$\pi_h(\mathbf{e}^\bullet) \text{ and } \pi_v(\mathbf{e}^\bullet)$$

denote the corresponding bi-simplicial objects of $\mathbf{E}$.

We have canonically defined maps

$$\pi_h(\mathrm{Seg}^\bullet(\mathbf{C})) \to \mathrm{Seg}^{\bullet,\bullet}(\mathbf{C}) \leftarrow \pi_h(\mathrm{Seg}^\bullet(\mathbf{C}))$$

corresponding to taking the functors

$$[m] \times [n] \to \mathbf{C}$$

that constant along the second (for h) and first (for v) coordinate, respectively.

7.6.4. Let $\mathbf{C}$ be an ∞-category, with two distinguished classes of 1-morphisms *vert* and *horiz*, satisfying the assumptions of Sect. 5.1.1.

We let

$$\mathrm{Cart}^{\bullet,\bullet}_{vert;horiz}(\mathbf{C})$$

denote the bi-simplicial groupoid, whose (m,n)-simplices is the full subgroupoid of

$$\mathrm{Funct}([m]\times[n],\mathbf{C}),$$

that consists of commutative diagrams

$$\begin{array}{ccccccccc}
\mathbf{c}_{0,0} & \longrightarrow & \mathbf{c}_{1,0} & \longrightarrow & ... & \longrightarrow & \mathbf{c}_{m-1,0} & \longrightarrow & \mathbf{c}_{m,0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{c}_{0,1} & \longrightarrow & \mathbf{c}_{1,1} & \longrightarrow & ... & \longrightarrow & \mathbf{c}_{m-1,1} & \longrightarrow & \mathbf{c}_{m,1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
... & \longrightarrow & ... & \longrightarrow & ... & \longrightarrow & ... & \longrightarrow & ... \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{c}_{0,n-1} & \longrightarrow & \mathbf{c}_{1,n-1} & \longrightarrow & ... & \longrightarrow & \mathbf{c}_{m-1,n-1} & \longrightarrow & \mathbf{c}_{m,n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{c}_{0,n} & \longrightarrow & \mathbf{c}_{1,n} & \longrightarrow & ... & \longrightarrow & \mathbf{c}_{m-1,n} & \longrightarrow & \mathbf{c}_{m,n}
\end{array} \tag{7.12}$$

in which every square is Cartesian, and in which all vertical arrows belong to *vert* and all horizontal arrows belong to *horiz*.

Note that we have a canonically defined maps in $\infty\text{-Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}}$

$$\pi_h(\mathrm{Seg}^\bullet(\mathbf{C}_{horiz}))\to \mathrm{Cart}^{\bullet,\bullet}_{vert;horiz}(\mathbf{C})\leftarrow \pi_v(\mathrm{Seg}^\bullet(\mathbf{C}_{vert})).$$

At the level of (m,n)-simplices, they correspond to diagrams (7.12), in which (for h) the vertical maps are identity maps, and (for v) the horizontal maps are identity maps.

7.6.5. Consider the functors

$$\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}:((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}}\to \mathrm{DGCat}_{\mathrm{cont}}$$

(see Proposition 3.2.4), and

$$\mathrm{IndCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}}}:((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}})^{\mathrm{op}}\to \mathrm{DGCat}_{\mathrm{cont}}$$

(see Corollary 3.5.6), where

$$(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}}\subset (\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{ev-conconn}}$$

is the 1-full subcategory, where we retsrict 1-morphisms to those of bounded Tor dimension.

Consider the corresponding maps in $\infty\text{-Grpd}^{\mathbf{\Delta}^{\mathrm{op}}}$

$$\mathrm{Seg}^\bullet(\mathrm{IndCoh}^!_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}}):\mathrm{Seg}^\bullet(((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})^{\mathrm{op}})\to \mathrm{Seg}^\bullet((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}),$$

and

$$\mathrm{Seg}^\bullet(\mathrm{IndCoh}^*_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}}}):\mathrm{Seg}^\bullet(((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}})^{\mathrm{op}})\to \mathrm{Seg}^\bullet((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}),$$

respectively.

7.6.6. We consider the category $\mathrm{DGSch}_{\mathrm{Noeth}}$ equipped with the following classes of 1-morphisms:

$$\mathit{vert} = \text{bdd-Tor} \text{ and } \mathit{horiz} = \text{aft}.$$

We will prove:

Proposition 7.6.7. *There exists a canonically defined map in* $\infty\text{-Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}}$

$$\mathrm{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor;aft}}} : \mathrm{Cart}^{\bullet,\bullet}_{\text{bdd-Tor;aft}}(\mathrm{DGSch}_{\mathrm{Noeth}}) \to \mathrm{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}})$$

that makes the following diagrams commute

$$\begin{array}{ccc} \mathrm{Cart}^{\bullet,\bullet}_{\text{bdd-Tor;aft}}(\mathrm{DGSch}_{\mathrm{Noeth}}) & \xrightarrow{\mathrm{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor;aft}}}} & \mathrm{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}) \\ \uparrow & & \uparrow \\ \pi_h(\mathrm{Seg}^{\bullet}((\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}})) & \xrightarrow[\mathrm{Seg}^{\bullet}(\mathrm{IndCoh}^{!}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{aft}}})]{} & \pi_h(\mathrm{Seg}^{\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}})) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Cart}^{\bullet,\bullet}_{\text{bdd-Tor;aft}}(\mathrm{DGSch}_{\mathrm{Noeth}}) & \xrightarrow{\mathrm{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor;aft}}}} & \mathrm{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}) \\ \uparrow & & \uparrow \\ \pi_v(\mathrm{Seg}^{\bullet}((\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}})) & \xrightarrow[\mathrm{Seg}^{\bullet}(\mathrm{IndCoh}^{*}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\text{bdd-Tor}}})]{} & \pi_v(\mathrm{Seg}^{\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}})). \end{array}$$

Proof. Given a Cartesian diagram

$$\begin{array}{ccccc} S_{0,0} & \xrightarrow{f_?} & \ldots & \xrightarrow{f_?} & S_{m,0} \\ g_? \downarrow & & \downarrow & & \downarrow g_? \\ \ldots & \longrightarrow & \ldots & \longrightarrow & \ldots \\ g_? \downarrow & & \downarrow & & \downarrow g_? \\ S_{0,n} & \xrightarrow[f_?]{} & \ldots & \xrightarrow[f_?]{} & S_{m,n} \end{array} \tag{7.13}$$

in $(\mathrm{DGSch}_{\mathrm{Noeth}})$ with horizontal arrows being locally almost of finite type, and vertical arrows of bounded Tor dimension, we need to construct a commutative diagram in $(\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}$:

$$\begin{array}{ccccc} \mathrm{IndCoh}(S_{0,0}) & \xleftarrow{(f_?)^!} & \ldots & \xleftarrow{(f_?)^!} & \mathrm{IndCoh}(S_{m,0}) \\ (g_?)^{\mathrm{IndCoh},*} \uparrow & & \uparrow & & \uparrow (g_?)^{\mathrm{IndCoh},*} \\ \ldots & \longrightarrow & \ldots & \longrightarrow & \ldots \\ (g_?)^{\mathrm{IndCoh},*} \uparrow & & \uparrow & & \uparrow (g_?)^{\mathrm{IndCoh},*} \\ \mathrm{IndCoh}(S_{0,n}) & \xleftarrow[(f_?)^!]{} & \ldots & \xleftarrow[(f_?)^!]{} & \mathrm{IndCoh}(S_{m,n}). \end{array} \tag{7.14}$$

Moreover, the assignment

$$(7.13) \rightsquigarrow (7.14)$$

must be compatible with maps

$$[m] \times [n] \to [m'] \times [n']$$

in $\mathbf{\Delta} \times \mathbf{\Delta}$.

We start with the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{IndCoh}(S_{0,0}) & \xleftarrow{(f_?)^!} & \cdots & \xleftarrow{(f_?)^!} & \mathrm{IndCoh}(S_{m,0}) \\
\downarrow{\scriptstyle (g_?)_*^{\mathrm{IndCoh}}} & & \downarrow & & \downarrow{\scriptstyle (g_?)_*^{\mathrm{IndCoh}}} \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\downarrow{\scriptstyle (g_?)_*^{\mathrm{IndCoh}}} & & \downarrow & & \downarrow{\scriptstyle (g_?)_*^{\mathrm{IndCoh}}} \\
\mathrm{IndCoh}(S_{0,n}) & \xleftarrow[(f_?)^!]{} & \cdots & \xleftarrow[(f_?)^!]{} & \mathrm{IndCoh}(S_{m,n})
\end{array}
\tag{7.15}
$$

whose datum is is given by the functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:bdd\text{-}Tor;aft}}} := \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:all;aft}}}|_{(\mathrm{DGSch}_{\mathrm{Noeth}})_{\mathrm{corr:bdd\text{-}Tor;aft}}}.$$

The diagram (7.14) is obtained from (7.15) by passage to the left adjoints along the vertical arrows. The fact that the arising natural transformations are isomorphisms is given by Proposition 7.2.6.

□

8. Descent

In this section we continue to assume that all DG schemes are Noetherian.

8.1. **A result on conservativeness.** In this subsection we will establish a technical result that will be useful in the sequel.

8.1.1. Let $f : S_1 \to S_2$ be a map almost of finite type.

Proposition 8.1.2. *Assume that f is surjective at the level of geometric points. Then the functor $f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1)$ is conservative.*

Proof. By Corollary 4.1.8 we can assume that both S_1 are S_2 are classical and reduced. By Noetherian induction, we can assume that the statement is true for all proper closed subschemes of S_2.

Hence, we can replace S_1 and S_2 be open subschemes and thus assume that f is smooth. In this case, by Corollary 7.5.5, $f^!$ differs from $f^{\mathrm{IndCoh},*}$ by an automorphism of $\mathrm{IndCoh}(S_1)$. So, it is enough to show that $f^{\mathrm{IndCoh},*}$ is fully faithful.

By Lemma 4.1.10, we can assume that S_2 is the spectrum of a field. In this case S_1 is a smooth scheme over this field, and the assertion becomes manifest.

□

Corollary 8.1.3. *Let $f : S_1 \to S_2$ be a proper map, surjective at the level of geometric points. Then the essential image of the functor $f_* : \mathrm{Coh}(S_1) \to \mathrm{Coh}(S_2)$ generates* $\mathrm{IndCoh}(S_2)$.

8.2. **h-descent.**

8.2.1. Let $f: S_1 \to S_2$ be a map locally almost of finite type. We can form the cosimplicial category $\operatorname{IndCoh}^!(S_1^\bullet/S_2)$ *using the !-pullback functors*, and !-pullback defines an augmentation

$$\operatorname{IndCoh}(S_2) \to \operatorname{Tot}\left(\operatorname{IndCoh}^!(S_1^\bullet/S_2)\right). \tag{8.1}$$

We shall say that IndCoh satisfies !-descent for f if (8.1) is an equivalence.

The main theorem of this section is:

Theorem 8.2.2. IndCoh *satisfies !-descent for maps that are covers for the h-topology.*

Taking into account Proposition 4.2.1 and [GoLi, Theorem 4.1], to prove Theorem 8.2.2, it suffices to prove the following:

Proposition 8.2.3. IndCoh *satisfies !-descent for maps that are proper and surjective at the level of geometric points.*

Remark 8.2.4. In Sect. 8.3 we will give a direct proof of the fact that IndCoh satisfies !-descent for fppf maps.

Proof of Proposition 8.2.3. Consider the forgetful functor

$$\mathrm{ev}_0 : \operatorname{Tot}\left(\operatorname{IndCoh}^!(S_1^\bullet/S_2)\right) \to \operatorname{IndCoh}(S_1)$$

given by evaluation on the 0-th term. Its composition with the functor (8.1) is the functor $f^!$.

We will show that ev_0 admits a left adjoint, to be denoted ev_0^L, such that the natural map

$$\mathrm{ev}_0 \circ \mathrm{ev}_0^L \to f^! \circ f_*^{\operatorname{IndCoh}} \tag{8.2}$$

is an isomorphism. We will also show that both pairs

$$(\mathrm{ev}_0^L, \mathrm{ev}_0) \text{ and } (f_*^{\operatorname{IndCoh}}, f^!)$$

satisfy the conditions of the Barr-Beck-Lurie theorem (see [GL:DG], Sect. 3.1). This would imply the assertion of the proposition, as the isomorphism (8.2) would imply that the resulting two monads acting on $\operatorname{IndCoh}(S_1)$ are isomorphic.

Consider the augmented simplicial scheme $S_1 \underset{S_2}{\times} (S_1^\bullet/S_2)$, which is equipped with a map of simplicial schemes

$$f^\bullet : S_1 \underset{S_2}{\times} (S_1^\bullet/S_2) \to S_1^\bullet/S_2.$$

The operation of !-pullback defines a functor between the corresponding cosimplicial categories

$$(f^\bullet)^! : \operatorname{IndCoh}(S_1^\bullet/S_2) \to \operatorname{IndCoh}(S_1 \underset{S_2}{\times} (S_1^\bullet/S_2)).$$

By Proposition 3.4.2, the term-wise adjoint $(f^\bullet)_*^{\operatorname{IndCoh}}$ is also a functor of cosimplicial categories.

In particular, we obtain a pair of adjoint functors

$$\operatorname{Tot}((f^\bullet)_*^{\operatorname{IndCoh}}) : \operatorname{Tot}\left(\operatorname{IndCoh}^!(S_1 \underset{S_2}{\times} (S_1^\bullet/S_2))\right) \rightleftarrows \operatorname{Tot}\left(\operatorname{IndCoh}^!(S_1^\bullet/S_2)\right) : \operatorname{Tot}((f^\bullet)^!).$$

However, the augmented simplicial scheme $S_1 \underset{S_2}{\times} (S_1^\bullet/S_2)$ is split by S_1, and therefore, the functors of !-pullback from the augmentation and evaluation on the splitting define mutual inverse equivalences

$$\operatorname{Tot}\left(\operatorname{IndCoh}^!(S_1 \underset{S_2}{\times} (S_1^\bullet/S_2))\right) \simeq \operatorname{IndCoh}(S_1).$$

The resulting functor

$$\mathrm{Tot}\left(\mathrm{IndCoh}^!(S_1^\bullet/S_2)\right) \overset{\mathrm{Tot}((f^\bullet)^!)}{\longrightarrow} \mathrm{Tot}\left(\mathrm{IndCoh}^!(S_1 \underset{S_2}{\times} (S_1^\bullet/S_2))\right) \simeq \mathrm{IndCoh}(S_1)$$

is easily seen to be canonically isomorphic to the functor ev_0. Thus, the functor

$$\mathrm{IndCoh}(S_1) \simeq \mathrm{Tot}\left(\mathrm{IndCoh}^!(S_1 \underset{S_2}{\times} (S_1^\bullet/S_2))\right) \overset{\mathrm{Tot}((f^\bullet)_*^{\mathrm{IndCoh}})}{\longrightarrow} \mathrm{Tot}\left(\mathrm{IndCoh}^!(S_1^\bullet/S_2)\right)$$

provides the desired functor ev_0^L.

The composition $\mathrm{ev}_0 \circ \mathrm{ev}_0^L$ identifies with the functor

$$\mathrm{pr}_1^! \circ \mathrm{pr}_2{}_*^{\mathrm{IndCoh}},$$

where

$$\begin{array}{ccc} & S_1 \underset{S_2}{\times} S_1 & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ S_1 & & S_1, \end{array}$$

and the fact that the map (8.2) is an isomorphism follows from Proposition 3.4.2.

Finally, let us verify the conditions of the Barr-Beck-Lurie theorem. The functors $f^!$ and ev_0, being continuous, commute with all colimits. The functor ev_0 is conservative by definition, and the functor $f^!$ is conservative by Proposition 8.1.2.

□

8.3. **Faithfully flat descent.**

8.3.1. Let is recall that a map $f : S_1 \to S_2$ is said to be fppf if:

- It is almost of finite type;
- It is flat;
- It is surjective on geometric points.

The following is a corollary of Theorem 8.2.2:

Theorem 8.3.2. (Lurie) IndCoh *satisfies !-descent with respect to fppf morphisms.*

Below we will give an alternative proof that does not rely on [GoLi]. This will result from Proposition 8.2.3, and the combination of the following two assertions:

Proposition 8.3.3. IndCoh *satisfies !-descent with respect to Nisnevich covers.*

Proposition 8.3.4. *Any presheaf on* $\mathrm{DGSch}_{\mathrm{Noeth}}$ *that satisfies descent with respect to proper surjective maps and Nisnevich covers satisfies fppf descent.*

8.3.5. To prove Proposition 8.3.3, we will prove a more general statement, which is itself a particular case of Theorem 8.3.2:

Proposition 8.3.6. IndCoh *satisfies !-descent with respect to smooth surjective maps.*

Proof. By Corollary 7.5.7, the co-simplicial category $\mathrm{IndCoh}^!(S_1^\bullet/S_2)$ identifies with

$$\mathrm{QCoh}^*(S_1^\bullet/S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2),$$

where $\mathrm{QCoh}^*(S_1^\bullet/S_2)$ is the usual co-simplicial category attached to the simplicial DG schemes $S_1^\bullet/S_2$ and the functor

$$\mathrm{QCoh}^*_{\mathrm{DGSch}} : (\mathrm{DGSch})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Since $\mathrm{QCoh}(S_2)$ is rigid and $\mathrm{IndCoh}(S_2)$ is dualizable, by [GL:DG], Corollaries 6.4.2 and 4.3.2, the operation

$$\mathrm{IndCoh}(S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} - : \mathrm{QCoh}(S_2)\text{-}\mathbf{mod} \to \mathrm{DGCat}_{\mathrm{cont}}$$

commutes with limits.

Hence, from we obtain that the natural map

$$\mathrm{Tot}\left(\mathrm{QCoh}^*(S_1^\bullet/S_2)\right) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \to \mathrm{Tot}\left(\mathrm{QCoh}^*(S_1^\bullet/S_2) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2)\right)$$

is an equivalence. Thus, we obtain an equivalence

$$\mathrm{Tot}\left(\mathrm{QCoh}^*(S_1^\bullet/S_2)\right) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{IndCoh}(S_2) \simeq \mathrm{Tot}\left(\mathrm{IndCoh}^!(S_1^\bullet/S_2)\right). \tag{8.3}$$

By faithfully flat descent for QCoh, we obtain that the natural map

$$\mathrm{QCoh}(S_2) \to \mathrm{Tot}\left(\mathrm{QCoh}^*(S_1^\bullet/S_2)\right)$$

is an equivalence. So, the assertion of the proposition follows from (8.3).

□

8.3.7. Note that the same proof (using Proposition 4.5.3 instead of Corollary 7.5.7) implies the following statement:

Let $f : S_1 \to S_2$ be an eventually coconnective map. We can form the cosimplicial category $\mathrm{IndCoh}^*(S_1^\bullet/S_2)$ *using the *-pullback functors*, and *-pullback defines an augmentation

$$\mathrm{IndCoh}(S_2) \to \mathrm{Tot}\left(\mathrm{IndCoh}^*(S_1^\bullet/S_2)\right). \tag{8.4}$$

We shall say that IndCoh satisfies *-descent for f if (8.4) is an equivalence. We have:

Proposition 8.3.8. IndCoh *satisfies *-descent with respect to smooth surjective maps.*

8.4. **Proof of Proposition 8.3.4.** The assertion of the proposition, as well as the proof given below, are apparently well-known. The author has learned it from J. Lurie.

8.4.1. We have the following two general lemmas, valid for any $(\infty,1)$-category $\mathbf{C}$ and a presheaf of $(\infty,1)$-categories P on it:

Lemma 8.4.2. *If a morphism f admits a section, then any presheaf satisfies descent with respect to f.*

Lemma 8.4.3. *Let $S'' \overset{g''}{\to} S' \overset{g'}{\to} S$ be maps, such that P satisfies descent with respect to g''. Then P satisfies descent with respect to g' if and only if satisfies descent with respect to $g' \circ g''$.*

Consider now a Cartesian square

$$\begin{array}{ccc} S_1' & \xrightarrow{g_1} & S_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S_2' & \xrightarrow{g_2} & S_2. \end{array}$$

Corollary 8.4.4. *If a presheaf P satisfies descent with respect to g_1, g_2 and f', then it also satisfies descent with respect to f.*

8.4.5. First, applying Corollary 8.4.4 and Proposition 8.2.3, we can replace S_2 by ${}^{cl}S_2$, and S_1 by $S_1 \underset{S_2}{\times} {}^{cl}S_2 \simeq {}^{cl}S_1$. Thus, we can assume that the DG schemes involved are classical.

8.4.6. We are going to show that for any faithfully flat map f of finite type, there exists a diagram

$$S_2'' \overset{g''}{\to} S_2' \overset{g'}{\to} S_2,$$

such that

- The composition $g' \circ g'' : S_2'' \to S_2$ lifts to a map $S_2'' \to S_1$.
- IndCoh satisfies !-descent with respect to any map obtained as base change of either g' or g''.

In view of Corollary 8.4.4, this will prove Theorem 8.3.2.

Thus, it remains to show:

Proposition 8.4.7. *For any faithfully flat map of finite type between classical Noetherian schemes $f : S_1 \to S_2$, there exist maps $S_2'' \overset{g''}{\to} S_2' \overset{g'}{\to} S_2$ such that $g' \circ g''$ lifts to a map $S_2'' \to S_1$, and such that (a) g' is a Nisnevich cover, and (b) g'' is finite, flat and surjective.*

8.5. **Proof of Proposition 8.4.7.**

8.5.1. *Step 1.* First, we claim that we can assume that S_2 is the spectrum of a local Henselian Noetherian ring.

Indeed, let $x_2 \in S_2$ be a point. Let A_2 denote the Henselisation of S_2 at x_2. Let

$$\operatorname{Spec}(B_2) \to \operatorname{Spec}(A_2)$$

be a finite, flat and surjective map such that the composition

$$\operatorname{Spec}(B_2) \to \operatorname{Spec}(A_2) \to S_2$$

lifts to a map $\operatorname{Spec}(B_2) \to S_1$.

Write $A_2 = \underset{i \in I}{colim}\, A_2^i$, where I is a filtered category, and where each $\operatorname{Spec}(A_2^i)$ is endowed with an étale map to S_2, and contains a point x_2^i that maps to x_2 inducing anisomorphism on residue fields. With no restriction of generality, we can assume that each $\operatorname{Spec}(A_2^i)$ is connected.

Since $\operatorname{Spec}(B_2) \to \operatorname{Spec}(A_2)$ is finite and flat, there exists an index $i_0 \in I$, and a finite and flat (and automatically surjective) map

$$\operatorname{Spec}(B_2^{i_0}) \to \operatorname{Spec}(A_2^{i_0})$$

equipped with an isomorphism

$$\operatorname{Spec}(B_2) \simeq \operatorname{Spec}(B_2^{i_0}) \underset{\operatorname{Spec}(A_2^{i_0})}{\times} \operatorname{Spec}(A_2).$$

For $i \in I_{i_0}$ set

$$\operatorname{Spec}(B_2^i) \simeq \operatorname{Spec}(B_2^{i_0}) \underset{\operatorname{Spec}(A_2^{i_0})}{\times} \operatorname{Spec}(A_2^i).$$

Replacing I be $I_{/i_0}$, we obtain an isomorphism

$$B_2 \simeq \underset{i \in I}{colim}\, B_2^i,$$

where I is filtered.

Since S_1 is of finite type over S_2, the morphism

$$\operatorname{Spec}(B_2) \to S_1$$

of schemes over S_2 factors as

$$\mathrm{Spec}(B_2) \to \mathrm{Spec}(B_2^{i'}) \to S_1$$

for some index $i' \in I$.

We let S_2' be the (finite) disjoint union of the above schemes $\mathrm{Spec}(A_2^{i'})$ that covers S_2, and we let S_2'' to be the corresponding union of the schemes $\mathrm{Spec}(B_2^{i'})$.

8.5.2. *Step 2.* Next, we are going to show that we can assume that the map $f : S_1 \to S_2$ is quasi-finite.

With no restriction of generality, we can assume that S_1 is affine. Let x_2 be the closed point of S_2, and let S_1' be the fiber of S_1 over it. By assumption, S_1' is a scheme of finite type over k', the residue field of x_2. By Noether normalization, there exists a map $h' : S_1' \to \mathbb{A}^n_{k'}$, which is finite and flat *at some* closed point $x_1 \in S_1'$. Replacing S_1 by an open subset containing x_1, we can assume that h' comes from a map $h : X \to \mathbb{A}^n \times S_2$. By construction, the map h is quasi-finite and flat at x_1. We can replace S_1 by an even smaller open subset containing x_1, and thus assume that h is quasi-finite and flat on all of S_1.

The sought-for quasi-finite fppf map is

$$(0 \times S_2) \underset{\mathbb{A}^n \times S_2}{\times} S_1 \to S_2.$$

8.5.3. *Step 3.* Thus, we have reduced the situation to the case when S_2 is the spectrum of a local Henselian Noetherian ring, and S_1 is affine and its map to S_2 is quasi-finite. We claim that in this case S_1 contains a connected component over which f is finite, see Lemma 8.5.4 below. This proves Proposition 8.4.7.

□

Lemma 8.5.4. *Let $\phi : Z \to Y = \mathrm{Spec}(R)$ be a quasi-finite and separated map, where R is a local Henselian Noetherian ring. Suppose, moreover, that the closed point of Y is in the image of ϕ. Then Z contains a connected component Z', such that $\phi|_{Z'}$ is finite.*

Proof. By Zariski's Main Theorem, we can factor ϕ as

$$\begin{array}{ccc} Z & \xrightarrow{\ j\ } & \overline{Z} \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\overline{\phi}} & \\ & Y, & \end{array}$$

where j is an open embedding and $\overline{\phi}$ is finite.

Since $\overline{Z}$ is finite over Y, and Y is Henselian, $\overline{Z}$ is a union of connected components, each of which is local. Let z be a closed point of Z that maps to the closed point y of Y. Let $\overline{Z}'$ be the connected component of $\overline{Z}$ that contains z. Set $Z' := \overline{Z}' \cap Z$. It remains to show that the open embedding $Z' \hookrightarrow \overline{Z}'$ is an equality.

However, since $\overline{Z}'$ is finite over Y, we obtain that z is closed in $\overline{Z}'$. We obtain that Z' contains the *unique* closed point of $\overline{Z}'$, which implies the assertion.

□

9. Serre duality

In this section we restrict our attention to the category $\mathrm{DGSch}_{\mathrm{aft}}$ of DG schemes almost of finite type over k.

9.1. **Self-duality of** IndCoh.

9.1.1. Consider the category $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}$ and the functor

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}} : (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Note that by construction, there exists a canonical involutive equivalence

$$\varpi : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}}_{\mathrm{corr:all;all}} \simeq (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}},$$

obtained by interchanging the roles of vertical and horizontal arrows.

9.1.2. Let $\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}$ denote the full subcategory of $\mathrm{DGCat}_{\mathrm{cont}}$ formed by dualizable categories. Recall also (see., e.g., [GL:DG], Sect. 2.3) that the category $\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}$ carries a canonical involutive equivalence

$$\mathrm{dualization} : (\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}})^{\mathrm{op}} \simeq \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\,.$$

given by taking the dual category: $\mathbf{C} \mapsto \mathbf{C}^\vee$.

By construction, the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ takes values in the subcategory

$$\mathrm{DGCat}^{\mathrm{comp.gener.}}_{\mathrm{cont}} \subset \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}} \subset \mathrm{DGCat}_{\mathrm{cont}}\,.$$

9.1.3. We will encode the Serre duality structure of the functor IndCoh by the following theorem:

Theorem 9.1.4. *The following diagram of functors*

$$\begin{array}{ccc} (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}}_{\mathrm{corr:all;all}} & \xrightarrow{\mathrm{IndCoh}^{\mathrm{op}}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}} & (\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}})^{\mathrm{op}} \\ \varpi \downarrow & & \downarrow \mathrm{dualization} \\ (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}} & \xrightarrow{\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}} & \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\,. \end{array} \tag{9.1}$$

is canonically commutative. Moreover, this structure is canonically compatible with the involutivity structure on the equivalences ϖ *and* dualization*.*

The rest of this subsection is devoted to the proof of this theorem.

9.1.5. Let us consider the following general paradigm. Let $\mathbf{C}$ be a symmetric monoidal category, which is rigid. Then the assignment $\mathbf{c} \mapsto \mathbf{c}^\vee$ defines a canonical involutive self-equivalence

$$\mathbf{C}^{\mathrm{op}} \overset{\mathrm{dualization}}{\longrightarrow} \mathbf{C}.$$

If $\mathbf{C}_1$ is another rigid symmetric monoidal category, and $F : \mathbf{C} \to \mathbf{C}_1$ is a symmetric monoidal functor, then the diagram

$$\begin{array}{ccc} \mathbf{C}^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} & (\mathbf{C}_1)^{\mathrm{op}} \\ \mathrm{dualization} \downarrow & & \downarrow \mathrm{dualization} \\ \mathbf{C} & \xrightarrow{F} & \mathbf{C}_1 \end{array}$$

naturally commutes in a way compatible with the involutive structure on the vertical arrows.

9.1.6. Let $\mathbf{C}^0$ be a category with fiber products and a final object, and let $\mathbf{C}^0_{\text{corr:all;all}}$ be the resulting category of correspondences. We make $\mathbf{C}^0_{\text{corr:all;all}}$ into a symmetric monoidal category by means of the product operation.

The category $\mathbf{C}^0_{\text{corr:all;all}}$ is automatically rigid: every $\mathbf{c} \in \mathbf{C}^0_{\text{corr:all;all}}$ is self-dual, where the counit of the duality datum is given by the correspodence

$$\begin{array}{ccc} \mathbf{c} & \xrightarrow{\Delta_{\mathbf{c}}} & \mathbf{c} \times \mathbf{c} \\ \downarrow & & \\ \text{pt}, & & \end{array}$$

(here pt is the final object in $\mathbf{C}^0$) and the unit is given by the correspondence

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & \text{pt} \\ {\scriptstyle \Delta_{\mathbf{c}}}\downarrow & & \\ \mathbf{c} \times \mathbf{c}. & & \end{array}$$

The following results from the definitions (a full proof will be given in [GR3]):

Lemma 9.1.7. *The involutive self-equivalence*

$$\mathbf{C}^0_{\text{corr:all;all}} \overset{\text{dualization}}{\longrightarrow} (\mathbf{C}^0_{\text{corr:all;all}})^{\text{op}},$$

is canonically isomorphic to ϖ.

9.1.8. To prove Theorem 9.1.4 we apply the discussion of Sections 9.1.5 and 9.1.6 to

$$\mathbf{C}^0 = \text{DGSch}_{\text{aft}},\ \mathbf{C} := (\text{DGSch}_{\text{aft}})_{\text{corr:all;all}},\ \mathbf{C}_1 := \text{DGCat}^{\text{dualizable}}_{\text{cont}},$$

and $F := \text{IndCoh}_{(\text{DGSch}_{\text{aft}})_{\text{corr:all;all}}}$, where the symmetric monoidal structure on F is given by Theorem 5.6.3 and Corollary 5.6.4.

□

9.2. **Properties of self-duality.** In this subsection shall explain what Theorem 9.1.4 says in concrete terms.

9.2.1. The assertion of Theorem 9.1.4 at the level of objects means that for every $S \in \text{DGSch}_{\text{aft}}$ we have a canonical equivalence

$$\mathbf{D}^{\text{Serre}}_S : (\text{IndCoh}(S))^\vee \simeq \text{IndCoh}(S), \tag{9.2}$$

which in the sequel we will refer to as Serre duality on a given scheme (see Sect. 9.5 for the relation to a more familiar formulation of Serre duality).

Moreover, the equivalence obtained by iterating (9.2)

$$((\text{IndCoh}(S))^\vee)^\vee \simeq (\text{IndCoh}(S))^\vee \simeq \text{IndCoh}(S)$$

is canonically isomorphic to the tautological equivalence $((\text{IndCoh}(S))^\vee)^\vee \simeq \text{IndCoh}(S)$.

9.2.2. The construction of the commutative diagram (9.1) at the level of objects amounts to the following description of the duality data

$$\epsilon_S : \text{IndCoh}(S) \otimes \text{IndCoh}(S) \to \text{Vect} \text{ and } \mu_S : \text{Vect} \to \text{IndCoh}(S) \otimes \text{IndCoh}(S).$$

The functor ϵ_S is the composition

$$\text{IndCoh}(S) \otimes \text{IndCoh}(S) \xrightarrow{\boxtimes} \text{IndCoh}(S \times S) \xrightarrow{\Delta_S^!} \text{IndCoh}(S) \overset{(p_S)_*^{\text{IndCoh}}}{\to} \text{Vect},$$

where p_S is the projection $S \to \text{pt}$.

The functor μ_S is the composition

$$\text{Vect} \xrightarrow{p_S^!} \text{IndCoh}(S) \overset{\Delta_*^{\text{IndCoh}}}{\longrightarrow} \text{IndCoh}(S \times S) \simeq \text{IndCoh}(S) \otimes \text{IndCoh}(S).$$

The fact that the functors (ϵ_S, μ_S) specified above indeed define a duality data for IndCoh(S) is an easy diagram chase. So, the content of Theorem 9.1.4 is the higher categorical functoriality of this construction.

9.2.3. At the level of 1-morphisms, the construction in Theorem 9.1.4 implies that for a morphism $f : S_1 \to S_2$ we have the following commutative diagrams of functors

$$\begin{array}{ccc} (\text{IndCoh}(S_1))^\vee & \xrightarrow{\mathbf{D}_{S_1}^{\text{Serre}}} & \text{IndCoh}(S_1) \\ {\scriptstyle (f_*^{\text{IndCoh}})^\vee}\uparrow & & \uparrow{\scriptstyle f^!} \\ (\text{IndCoh}(S_2))^\vee & \xrightarrow{\mathbf{D}_{S_2}^{\text{Serre}}} & \text{IndCoh}(S_2) \end{array} \tag{9.3}$$

and

$$\begin{array}{ccc} (\text{IndCoh}(S_1))^\vee & \xrightarrow{\mathbf{D}_{S_1}^{\text{Serre}}} & \text{IndCoh}(S_1) \\ {\scriptstyle (f^!)^\vee}\downarrow & & \downarrow{\scriptstyle f_*^{\text{IndCoh}}} \\ (\text{IndCoh}(S_2))^\vee & \xrightarrow{\mathbf{D}_{S_2}^{\text{Serre}}} & \text{IndCoh}(S_2) \end{array} \tag{9.4}$$

Moreover, the isomorphism obtained by iteration

$$((f_*^{\text{IndCoh}})^\vee)^\vee \simeq (f^!)^\vee \simeq f_*^{\text{IndCoh}}$$

is canonically isomorphic to the tautological isomorphism $((f_*^{\text{IndCoh}})^\vee)^\vee \simeq f_*^{\text{IndCoh}}$.

In particular, the dual of the functor $(p_S)_*^{\text{IndCoh}} : \text{IndCoh}(S) \to \text{Vect}$ is $(p_S)^!$, and vice versa.

Again, we note that the explicit description of the duality data given in Sect. 9.2.2 makes the commutativity of the above diagrams evident.

9.3. **Compatibility with duality on** QCoh**.**

9.3.1. Recall that the category $\mathrm{QCoh}(S)$, being a rigid monoidal category, is also self-dual (see [GL:DG], Sect. 6.1.). We denote the corresponding functor by

$$\mathbf{D}_S^{\mathrm{naive}} : (\mathrm{QCoh}(S))^\vee \to \mathrm{QCoh}(S). \tag{9.5}$$

The corresponding pairing

$$\mathrm{QCoh}(S) \otimes \mathrm{QCoh}(S) \to \mathrm{Vect}$$

is by definition given by

$$\mathrm{QCoh}(S) \otimes \mathrm{QCoh}(S) \overset{\boxtimes}{\longrightarrow} \mathrm{QCoh}(S \times S) \overset{\Delta_S^*}{\longrightarrow} \mathrm{QCoh}(S) \overset{\Gamma(S,-)}{\longrightarrow} k.$$

9.3.2. Recall the functor

$$\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S).$$

Passing to dual functors, and using the identifications $\mathbf{D}_S^{\mathrm{naive}}$ and $\mathbf{D}_S^{\mathrm{Serre}}$, we obtain a functor

$$\Psi_S^\vee : \mathrm{QCoh}(S) \to \mathrm{IndCoh}(S). \tag{9.6}$$

We claim:

Proposition 9.3.3. *The functor $\Psi_S^\vee$ identifies canonically with the functor Υ_S of* (5.20)*, i.e.,*

$$\mathcal{E} \mapsto \mathcal{E} \otimes \omega_S,$$

where the latter is understood in the sense of the action of $\mathrm{QCoh}(S)$ *on* $\mathrm{IndCoh}(S)$*.*

Proof. Let

$$\langle -,- \rangle_{\mathrm{IndCoh}(S)} \text{ and } \langle -,- \rangle_{\mathrm{QCoh}(S)}$$

denote the functors

$$\mathrm{IndCoh}(S) \times \mathrm{IndCoh}(S) \to \mathrm{Vect} \;\text{ and }\; \mathrm{QCoh}(S) \times \mathrm{QCoh}(S) \to \mathrm{Vect}$$

given by the counits of the duality isomorphisms $\mathbf{D}_S^{\mathrm{Serre}}$ and $\mathbf{D}_S^{\mathrm{naive}}$, respectively.

We need to show that for $\mathcal{F} \in \mathrm{IndCoh}(S)$ and $\mathcal{E} \in \mathrm{QCoh}(S)$ we have a canonical isomorphism

$$\langle \mathcal{F}, \mathcal{E} \otimes \omega_S \rangle_{\mathrm{IndCoh}(S)} \simeq \langle \Psi_S(\mathcal{F}), \mathcal{E} \rangle_{\mathrm{QCoh}(S)}. \tag{9.7}$$

By the construction of $\mathbf{D}_S^{\mathrm{Serre}}$, we have:

$$\langle \mathcal{F}, \mathcal{E} \otimes \omega_S \rangle_{\mathrm{IndCoh}(S)} \simeq \Gamma\left(S, \Psi_S(\mathcal{F} \overset{!}{\otimes} (\mathcal{E} \otimes \omega_S))\right).$$

Using (5.21), we have:

$$\mathcal{F} \overset{!}{\otimes} (\mathcal{E} \otimes \omega_S) \simeq \mathcal{E} \otimes (\mathcal{F} \overset{!}{\otimes} \omega_S) \simeq \mathcal{E} \otimes \mathcal{F}.$$

Hence, the left-hand side in (9.7) identifies with

$$\Gamma\left(S, \Psi_S(\mathcal{E} \otimes \mathcal{F})\right) \simeq \Gamma(S, \mathcal{E} \otimes \Psi_S(\mathcal{F})),$$

while the latter identifies with the left-hand side in (9.7).

□

9.4. **A higher-categorical compatibility of the dualities.** The material in this subsection will not be used elsewhere in the paper, and is included for the sake of completeness.

9.4.1. According to Theorem 9.1.4, the functor obtained from $\text{IndCoh}_{\text{DGSch}_{\text{aft}}}$ by passing to dual categories, identifies canonically with $\text{IndCoh}^!_{\text{DGSch}_{\text{aft}}}$.

Similarly, the functor obtained from $\text{QCoh}_{\text{DGSch}_{\text{aft}}}$ by passing to dual categories, identifies canonically with $\text{QCoh}^*_{\text{DGSch}_{\text{aft}}}$.

Hence, the natural transformation

$$\Psi_{\text{DGSch}_{\text{aft}}} : \text{IndCoh}_{\text{DGSch}_{\text{aft}}} \to \text{QCoh}_{\text{DGSch}_{\text{aft}}}$$

gives rise to the natural transformation

$$\Psi^\vee_{\text{DGSch}_{\text{aft}}} : \text{QCoh}^*_{\text{DGSch}_{\text{aft}}} \to \text{IndCoh}^!_{\text{DGSch}_{\text{aft}}}.$$

Proposition 9.3.3 says that at the level of objects, we have a canonical isomorphism:

$$\Psi^\vee_S \simeq \Upsilon_S, \quad S \in \text{DGSch}_{\text{aft}}. \tag{9.8}$$

Moreover, the following assertion follows from the construction of the isomorphism in Proposition 9.3.3:

Lemma 9.4.2. *For a morphism $f : S_1 \to S_2$, the isomorphisms*

$$\Psi^\vee_{S_1} \simeq \Upsilon_{S_1} \text{ and } \Psi^\vee_{S_2} \simeq \Upsilon_{S_2}$$

are compatible with the data of commutativity for the diagrams

$$\begin{array}{ccc} \text{IndCoh}(S_1) & \xleftarrow{\Psi^\vee_{S_1}} & \text{QCoh}(S_1) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^*} \\ \text{IndCoh}(S_2) & \xleftarrow{\Psi^\vee_{S_2}} & \text{QCoh}(S_2) \end{array}$$

and

$$\begin{array}{ccc} \text{IndCoh}(S_1) & \xleftarrow{\Upsilon_{S_1}} & \text{QCoh}(S_1) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^*} \\ \text{IndCoh}(S_2) & \xleftarrow{\Upsilon_{S_2}} & \text{QCoh}(S_2) \end{array}$$

The following theorem, strengthening the above lemma will be proved in [GR3]:

Theorem 9.4.3. *There exists a canonical isomorphism*

$$\Psi^\vee_{\text{DGSch}_{\text{aft}}} \simeq \Upsilon_{\text{DGSch}_{\text{aft}}}$$

as natural transformations $\text{QCoh}^*_{\text{DGSch}_{\text{aft}}} \rightrightarrows \text{IndCoh}^!_{\text{DGSch}_{\text{aft}}}$ between the functors

$$(\text{DGSch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}.$$

9.4.4. Theorem 9.4.3 is a consequence of a more general statement, described below. Recall the category $\text{DGCat}^{\text{SymMon+Mod}}$ introduced in Sect. 5.5.2.

We introduce another category $\text{DGCat}^{\text{SymMon}^{\text{op}}+\text{Mod}}$, whose objects are the same as for $\text{DGCat}^{\text{SymMon+Mod}}$, but 1-morphisms

$$(\mathbf{O}_1, \mathbf{C}_1) \to (\mathbf{O}_2, \mathbf{C}_2)$$

are now pairs $(F_{\mathbf{O}}, F_{\mathbf{C}})$, where $F_{\mathbf{O}}$ is a symmetric monoidal functor $F_{\mathbf{O}} : \mathbf{O}_2 \to \mathbf{C}_1$ (not the direction of the arrow!), and $F_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_2$, which is a morphism of $\mathbf{O}_2$-module categories.

We let

$$(\text{DGCat}^{\text{SymMon+Mod}})^{\text{dualizable}} \subset \text{DGCat}^{\text{SymMon+Mod}}$$

$$(\mathrm{DGCat}^{\mathrm{SymMon}^{\mathrm{op}}+\mathrm{Mod}})^{\mathrm{dualizable}} \subset \mathrm{DGCat}^{\mathrm{SymMon}^{\mathrm{op}}+\mathrm{Mod}}$$

denote full subcategories, spanned by those $(\mathbf{O}, \mathbf{C})$, for which $\mathbf{C}$ is dualizable as an $\mathbf{O}$-module category, see [GL:DG, Sect. 4]. (Note that if $\mathbf{O}$ is rigid, this condition is equivalent to $\mathbf{C}$ being dualizable as a plain DG category, see [GL:DG, Corollary, 6.4.2].)

Passing to duals

$$(\mathbf{O}, \mathbf{C}) \mapsto (\mathbf{O}, \mathbf{C}^\vee)$$

defines an equivalence

$$\mathrm{dualization} : ((\mathrm{DGCat}^{\mathrm{SymMon}+\mathrm{Mod}})^{\mathrm{dualizable}})^{\mathrm{op}} \to (\mathrm{DGCat}^{\mathrm{SymMon}^{\mathrm{op}}+\mathrm{Mod}})^{\mathrm{dualizable}}.$$

9.4.5. Recall (see Theorem 5.5.5) that we have a functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{DGSch}_{\mathrm{aft}}} : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}+\mathrm{Mod}}_{\mathrm{cont}}.$$

Note that the constructions in Sect. 3.2 and Proposition 3.1.3 combine to a functor

$$(\mathrm{QCoh}^*, \mathrm{IndCoh})_{\mathrm{DGSch}_{\mathrm{aft}}} : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}^{\mathrm{op}}+\mathrm{Mod}}_{\mathrm{cont}}.$$

We have the following assertion which will be proved in [GR3]:

Theorem 9.4.6. *There is a commutative diagram of functors*

$$\begin{array}{ccc} \mathrm{DGSch}_{\mathrm{aft}} & \xrightarrow{((\mathrm{QCoh}^*, \mathrm{IndCoh}^!)_{\mathrm{DGSch}_{\mathrm{aft}}})^{\mathrm{op}}} & ((\mathrm{DGCat}^{\mathrm{SymMon}+\mathrm{Mod}})^{\mathrm{dualizable}})^{\mathrm{op}} \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{dualization}} \\ \mathrm{DGSch}_{\mathrm{aft}} & \xrightarrow{(\mathrm{QCoh}^*, \mathrm{IndCoh})_{\mathrm{DGSch}_{\mathrm{aft}}}} & (\mathrm{DGCat}^{\mathrm{SymMon}^{\mathrm{op}}+\mathrm{Mod}})^{\mathrm{dualizable}}. \end{array}$$

Moreover, the above isomorphism is compatible with the symmetric monoidal structure on both functors.

9.4.7. In concrete terms, Theorem 9.4.6 says that for $S \in \mathrm{DGSch}_{\mathrm{aft}}$, the duality isomorphism

$$\mathbf{D}^{\mathrm{Serre}}_S : (\mathrm{IndCoh}(S))^\vee \simeq \mathrm{IndCoh}(S)$$

is compatible with the action of $\mathrm{QCoh}(S)$ on both sides.

Furthermore, for a map $f : S_1 \to S_2$, the isomorphism of functors

$$(f^!)^\vee \simeq f^{\mathrm{IndCoh}}_*$$

is compatible with the $\mathrm{QCoh}(S)$-linear structures on both sides.

9.5. **Relation to the classical Serre duality.** In this subsection we will explain the relation between the self-duality functor

$$\mathbf{D}^{\mathrm{Serre}}_S : \mathrm{IndCoh}(S)^\vee \simeq \mathrm{IndCoh}(S)$$

and the classical Serre duality functor

$$\mathbb{D}^{\mathrm{Serre}}_S : \mathrm{Coh}(S)^{\mathrm{op}} \to \mathrm{Coh}(S).$$

9.5.1. Recall that if $\mathbf{C}_1, \mathbf{C}_2$ are compactly generated categories, then the datum of an equivalence $\mathbf{C}_1^{\mathrm{op}} \simeq \mathbf{C}_2$ is equivalent to the datum of an equivalence $(\mathbf{C}_1^c)^{\mathrm{op}} \simeq \mathbf{C}_2^c$.

For example, the self-duality functor

$$\mathbf{D}_S^{\mathrm{naive}} : (\mathrm{QCoh}(S))^\vee \to \mathrm{QCoh}(S)$$

of (9.5) is induced by the "naive" duality on the category $\mathrm{QCoh}(S)^{\mathrm{perf}} \simeq \mathrm{QCoh}(S)^c$:

$$\mathbb{D}_S^{\mathrm{naive}} : (\mathrm{QCoh}(S)^{\mathrm{perf}})^{\mathrm{op}} \to \mathrm{QCoh}(S)^{\mathrm{perf}}, \quad \mathcal{E} \mapsto \mathcal{E}^\vee,$$

the latter being the passage to the dual object in $\mathrm{QCoh}(S)^{\mathrm{perf}}$ as a symmetric monoidal category.

In particular, we obtain that the equivalence $\mathbf{D}_S^{\mathrm{Serre}}$ of (9.2) induces a certain involutive equivalence

$$'\mathbb{D}_S^{\mathrm{Serre}} : \mathrm{Coh}(S)^{\mathrm{op}} \to \mathrm{Coh}(S). \tag{9.9}$$

9.5.2. Recall the action of the monoidal category $\mathrm{QCoh}(S)$ on $\mathrm{IndCoh}(S)$, see Sect. 1.4. For $\mathcal{F} \in \mathrm{IndCoh}(S)$ we can consider the relative internal Hom functor

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, -) : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$$

as defined in [GL:DG], Sect. 5.1. Explicitly, for $\mathcal{E} \in \mathrm{QCoh}(S)$ and $\mathcal{F}' \in \mathrm{IndCoh}(S)$ we have

$$\mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{E}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{F}')) := \mathrm{Maps}_{\mathrm{IndCoh}(S)}(\mathcal{E} \otimes \mathcal{F}, \mathcal{F}'),$$

where $- \otimes -$ denotes the action of $\mathrm{QCoh}(S)$ on $\mathrm{IndCoh}(S)$ of Sect. 1.4.

Lemma 9.5.3. *If $\mathcal{F} \in \mathrm{IndCoh}(S)^c = \mathrm{Coh}(S)$, the functor $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, -)$ is continuous.*

Proof. It is enough to show that for a set of compact generators $\mathcal{E} \in \mathrm{QCoh}(S)$, the functor

$$\mathcal{F}' \mapsto \mathrm{Maps}_{\mathrm{QCoh}(S)}(\mathcal{E}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{F}')), \quad \mathrm{IndCoh}(S) \to \infty\text{-Grpd}$$

commutes with filtered colimits.

We take $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$. In this case $\mathcal{E} \otimes \mathcal{F} \in \mathrm{IndCoh}(S)^c$, and the assertion follows.

□

9.5.4. We now claim:

Lemma 9.5.5. *For $\mathcal{F} \in \mathrm{Coh}(S)$, the functor $\mathcal{F} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, \omega_S)$ sends to $\mathrm{Coh}(S)$ to $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$.*

Proof. Let i denote the canonical map ${}^{cl}S \to S$. It is enough to prove the lemma for $\mathcal{F}$ of the form $i_*^{\mathrm{IndCoh}}(\mathcal{F}')$ for $\mathcal{F}' \in \mathrm{Coh}({}^{cl}S)$.

It is easy to see from (3.7) that for a proper map $f : S_1 \to S_2$ $\mathcal{F}_i \in \mathrm{IndCoh}(S_i)$, we have

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S_2)}(f_*^{\mathrm{IndCoh}}(\mathcal{F}_1), \mathcal{F}_2) \simeq f_*\left(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S_1)}(\mathcal{F}_1, f^!(\mathcal{F}_2))\right).$$

Hence,

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, \omega_S) \simeq i_*\left(\underline{\mathrm{Hom}}_{\mathrm{QCoh}({}^{cl}S)}(\mathcal{F}', \omega_{{}^{cl}S})\right).$$

This reduces the assertion to the case of classical schemes, where it is well-known (proved by the same manipulation as above by locally embedding into $\mathbb{A}^n$).

□

9.5.6. From the above lemma we obtain a well-defined functor $\mathrm{Coh}(S)^{\mathrm{op}} \to \mathrm{Coh}(S)$ that we denote $\mathbb{D}_S^{\mathrm{Serre}}$.

Proposition 9.5.7. *The functors* $\mathbb{D}_S^{\mathrm{Serre}}$ *and* $'\mathbb{D}_S^{\mathrm{Serre}}$ *of* (9.9) *are canonically isomorphic.*

Proof. Let $\mathcal{F}_1, \mathcal{F}_2$ be two objects of $\mathrm{Coh}(S)$. By definition,

$$\mathrm{Hom}_{\mathrm{IndCoh}(S)}(\mathcal{F}_1, '\mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F}_2)) \simeq \mathrm{Hom}_{\mathrm{IndCoh}(S)\otimes \mathrm{IndCoh}(S)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \mu_{\mathrm{IndCoh}(S)}(k)) \simeq \simeq \mathrm{Hom}_{\mathrm{IndCoh}(S\times S)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \Delta_{S*}^{\mathrm{IndCoh}}(\omega_S)). \tag{9.10}$$

Note that both $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ and $\Delta_{S*}^{\mathrm{IndCoh}}(\omega_S)$ belong to $\mathrm{IndCoh}(S)^+$, so by Proposition 1.2.4, we can rewrite the right-hand side of (9.10) as

$$\mathrm{Hom}_{\mathrm{QCoh}(S\times S)}(\Psi_S(\mathcal{F}_1) \boxtimes \Psi_S(\mathcal{F}_2), \Delta_{S*}(\Psi_S(\omega_S))) \simeq \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\Psi_S(\mathcal{F}_1) \underset{\mathcal{O}_S}{\otimes} \Psi_S(\mathcal{F}_2), \Psi_S(\omega_S)). \tag{9.11}$$

By definition, $\mathrm{Hom}_{\mathrm{IndCoh}(S)}(\mathcal{F}_1, \mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F}_2))$ is isomorphic to

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(\Psi_S(\mathcal{F}_1), \Psi_S(\mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F}_2))) := \mathrm{Hom}_{\mathrm{QCoh}(S)}(\Psi_S(\mathcal{F}_1), \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}_2, \omega_S)) := \mathrm{Hom}_{\mathrm{IndCoh}(S)}(\Psi_S(\mathcal{F}_1) \underset{\mathcal{O}_S}{\otimes} \mathcal{F}_2, \omega_S), \tag{9.12}$$

where $\Psi_S(\mathcal{F}_1) \underset{\mathcal{O}_S}{\otimes} \mathcal{F}_2$ is understood in the sense of the action of $\mathrm{QCoh}(S)$ on $\mathrm{IndCoh}(S)$.

However, since $\omega_S \in \mathrm{IndCoh}(S)^+$, by Proposition 1.2.4, the right-hand side of (9.12) can be rewritten as

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(\Psi_S(\Psi_S(\mathcal{F}_1) \underset{\mathcal{O}_S}{\otimes} \mathcal{F}_2), \Psi(\omega_S)),$$

which by Lemma 1.4.2 is isomorphic to the right-hand side of (9.11).

□

9.5.8. Combining Proposition 9.5.7 with [GL:DG, Lemma 2.3.3], we obtain:

Corollary 9.5.9. *Let* $S_1 \to S_2$ *be a morphism in* $\mathrm{DGSch}_{\mathrm{aft}}$.

(a) *Suppose that* f *is eventually coconnective. Then we have canonical isomorphisms of functors* $\mathrm{Coh}(S_2)^{\mathrm{op}} \to \mathrm{Coh}(S_1)$*:*

$$\mathbb{D}_{S_1}^{\mathrm{Serre}} \circ (f^{\mathrm{IndCoh},*})^{\mathrm{op}} \simeq f^! \circ \mathbb{D}_{S_2}^{\mathrm{Serre}};$$

and

$$\mathbb{D}_{S_1}^{\mathrm{Serre}} \circ (f^!)^{\mathrm{op}} \simeq f^{\mathrm{IndCoh},*} \circ \mathbb{D}_{S_2}^{\mathrm{Serre}}.$$

(b) *Suppose that* f *is proper. Then we have a canonical isomorphism of functors* $\mathrm{Coh}(S_1)^{\mathrm{op}} \to \mathrm{Coh}(S_2)$

$$\mathbb{D}_{S_2}^{\mathrm{Serre}} \circ (f_*^{\mathrm{IndCoh}})^{\mathrm{op}} \simeq f_*^{\mathrm{IndCoh}} \circ \mathbb{D}_{S_1}^{\mathrm{Serre}}.$$

9.6. **Serre duality in the eventually coconnective case.** In this subsection we will assume that $S \in \mathrm{DGSch}_{\mathrm{aft}}$ is eventually coconnective.

9.6.1. From Proposition 1.5.3 and [GL:DG], Sect. 2.3.2, we obtain:

Corollary 9.6.2. *The functor* $\Psi_S^\vee$ *sends compact objects to compact ones, and is fully faithful. The functor* $\Psi_S^\vee$ *realizes* $\mathrm{QCoh}(S)$ *as a colocalization of* $\mathrm{IndCoh}(S)$*.*

Taking into account Proposition 9.3.3, we obtain:

Corollary 9.6.3. *The functor* $\Upsilon_S \simeq - \underset{\mathcal{O}_S}{\otimes} \omega_S$ *sends compact objects to compact ones and is fully faithful.*

In particular:

Corollary 9.6.4. *For* S *eventually coconnective, the object* $\omega_S \in \mathrm{IndCoh}(S)$ *is compact.*

Remark 9.6.5. We have an isomorphism

$$'\mathbb{D}_S^{\mathrm{Serre}} \circ (\Xi_S)^{\mathrm{op}} \simeq \Psi_S^\vee \circ \mathbb{D}_S^{\mathrm{naive}} \tag{9.13}$$

as functors $(\mathrm{QCoh}(S)^{\mathrm{perf}})^{\mathrm{op}} \rightrightarrows \mathrm{Coh}(S)$, see [GL:DG, Lemma 2.3.3].

Applying Proposition 9.5.7 and Proposition 9.3.3 we obtain an isomorphism

$$\mathbb{D}_S^{\mathrm{Serre}} \circ (\Xi_S)^{\mathrm{op}} \simeq \Upsilon_S \circ \mathbb{D}_S^{\mathrm{naive}}. \tag{9.14}$$

It is easy to see that the isomorphism in (9.14) is the tautoloigical isomorphism

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{E}, \omega_S) \simeq \mathcal{E}^\vee \otimes \omega_S, \quad \mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}.$$

9.6.6. By [GL:DG, Sect. 2.3.2 and Lemma 2.3.3], the functor $\Psi_S^\vee$ admits a continuous right adjoint, which identifies with the functor $\Xi_S^\vee$, dual of Ξ_S.

Since $\omega_S \in \mathrm{IndCoh}(S)$ is compact, by Sect. 9.5.2, we have a continuous functor

$$\mathcal{F} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, \mathcal{F}) : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S). \tag{9.15}$$

From the definition and the isomorphism $\Psi_S^\vee \simeq \Upsilon_S$ of Proposition 9.3.3, we obtain:

Lemma 9.6.7. *The functor* $\mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$*, right adjoint of* $\Psi_S^\vee$ *is given by*

$$\mathcal{F} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, \mathcal{F}).$$

Remark 9.6.8. The functor right adjoint to $\Psi_S^\vee \simeq \Upsilon_S$ is defined for any S (i.e., not necessarily eventually coconnective), but in general it will fail to be continuous.

Hence, we obtain:

Corollary 9.6.9. *The functor*

$$\Xi_S^\vee : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S),$$

dual to $\Xi_S : \mathrm{QCoh}(S) \to \mathrm{IndCoh}(S)$*, is given by* $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, -)$*,*

We also note:

Lemma 9.6.10. *Let* $f : S_1 \to S_2$ *be an eventually coconnective map in* $\mathrm{DGSch}_{\mathrm{aft}}$*, where* S_1 *and* S_2 *are themselves eventually coconnective. Then there exists a canonical isomorphism:*

$$\Xi_{S_1}^\vee \circ f^! \simeq f^* \circ \Xi_{S_2}^\vee : \mathrm{IndCoh}(S_2) \to \mathrm{QCoh}(S_1).$$

Proof. Obtained by passing to dual functors in the isomorphism

$$\Xi_{S_2} \circ f_* \simeq f_*^{\mathrm{IndCoh}} \circ \Xi_{S_1}$$

of Proposition 3.6.7. □

9.6.11. We note that the logic of the previous discussion can be inverted, and we obtain:

Proposition 9.6.12. *The following conditions on S are equivalent:*
(a) The functor Ψ_S admits a left adjoint.
(b) The functor $\Psi_S^\vee \simeq \Upsilon_S$ sends compact objects to compact ones.
(c) The object $\omega_S \in \mathrm{IndCoh}(S)$ is compact.
(d) S is eventually coconnective.

Proof. The equivalence (b) $\Leftrightarrow$ (c) is tautological. The equivalence (a) $\Leftrightarrow$ (d) has been established in Proposition 1.6.2. The equivalence (a) $\Leftrightarrow$ (b) follows from [GL:DG], Sect. 2.3.2. □

9.6.13. Let $S \in \mathrm{DGSch}_{\mathrm{aft}}$ be affine. In this case, it is easy to see that for every $\mathcal{F} \in \mathrm{Coh}(S)$ and $k \in \mathbb{N}$ there exists an object $\mathcal{F}' \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ and a map $\mathcal{F}' \to \mathcal{F}$, such that

$$\mathrm{Cone}(\Xi_S(\mathcal{F}') \to \mathcal{F})[-1] \in \mathrm{Coh}(S)^{\leq -k}.$$

We claim that the functor $\Psi_S^\vee$ plays a dual role:

Lemma 9.6.14. *For $\mathcal{F} \in \mathrm{Coh}(S)$ and $k \in \mathbb{N}$ there exists an object $\mathcal{F}' \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ and a map $\mathcal{F} \to \Upsilon_S(\mathcal{F}')$ so that*

$$\mathrm{Cone}(\mathcal{F} \to \Upsilon_S(\mathcal{F}')) \in \mathrm{Coh}(S)^{\geq k}.$$

Proof. Let m be an integer such that $\mathbb{D}_S^{\mathrm{Serre}}$ sends

$$\mathrm{Coh}(S)^{\leq 0} \to \mathrm{Coh}(S)^{\geq -m}$$

(in fact, m can be taken to be the dimension of ${}^{cl}S$.)

For $\mathcal{F}$ as in the lemma, consider $\mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F}) \in \mathrm{Coh}(S)$, and let $\mathcal{F}'' \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ and $\Xi_S(\mathcal{F}'') \to \mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F})$ be such that

$$\mathrm{Cone}(\Xi_S(\mathcal{F}'') \to \mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F}))[-1] \in \mathrm{Coh}(S)^{\leq -(k+m)}.$$

Set $\mathcal{F}' := \mathbb{D}_S^{\mathrm{naive}}(\mathcal{F}'')$. Applying Serre duality to $\Xi_S(\mathcal{F}'') \to \mathbb{D}_S^{\mathrm{Serre}}(\mathcal{F})$ we obtain a map

$$\mathcal{F} \to \mathbb{D}_S^{\mathrm{Serre}}(\Xi_S(\mathcal{F}'')) \simeq \Upsilon_S(\mathcal{F}')$$

with the desired properties. □

9.6.15. *Proof of Corollary 7.4.3.* It is enough to show that $\Psi_S(\omega_S) \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ is invertible.

For $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{IndCoh}(S)$ and $\mathcal{E} \in \mathrm{QCoh}(S)$ consider the canonical map

$$\mathcal{E} \otimes \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}_1, \mathcal{F}_2) \to \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}_1, \mathcal{E} \otimes \mathcal{F}_2).$$

It is easy to see that this map is an isomorphism if $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$. In particular, we obtain an isomorphism

$$\Psi_S(\omega_S) \otimes \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, \Xi_S(\mathcal{O}_S)) \simeq \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, \Psi_S(\omega_S) \otimes \Xi_S(\mathcal{O}_S)).$$

By Corollary 1.5.5, we have

$$\Psi_S(\omega_S) \otimes \Xi_S(\mathcal{O}_S) \simeq \Xi_S \circ \Psi_S(\omega_S),$$

and since $\Psi_S(\omega_S) \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ we have $\Xi_S \circ \Psi_S(\omega_S) \simeq \omega_S$. Thus, we obtain:

$$\Psi_S(\omega_S) \otimes \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, \Xi_S(\mathcal{O}_S)) \simeq \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\omega_S, \omega_S) \simeq \Xi_S^\vee \circ \Psi_S^\vee(\mathcal{O}_S) \simeq \mathcal{O}_S.$$

I.e., $\Psi_S(\omega_S) \in \mathrm{QCoh}(S)$ is invertible, as required. □

Part III. Extending to stacks

10. IndCoh on prestacks

As was mentioned in the introduction, our conventions regarding prestacks and stacks follow [GL:Stacks]. In this section we shall mostly be interested in the full subcategory

$$\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$$

consisting of prestacks, *locally almost of finite type*, see [GL:Stacks, Sect. 1.3.9] for the definition.

10.1. Definition of IndCoh.

10.1.1. By definition, the category $\mathrm{PreStk}_{\mathrm{laft}}$ is the category of functors

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Grpd}\,.$$

We have the fully faithful embeddings

$${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}} \hookrightarrow {}^{<\infty}\mathrm{DGSch}_{\mathrm{aft}} \hookrightarrow \mathrm{DGSch}_{\mathrm{aft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}},$$

where the composed arrow is the Yoneda embedding.

10.1.2. We let $\mathrm{IndCoh}^!_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$ be the functor

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

obtained by retsricting the functor $\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$ under

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}}.$$

We define the functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

as the right Kan extension of $\mathrm{IndCoh}^!_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$ under the Yoneda embedding

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

The following is immediate from the definition:

Lemma 10.1.3. *The functor* $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ *introduced above takes colimits in* $\mathrm{PreStk}_{\mathrm{laft}}$ *to limits in* $\mathrm{DGCat}_{\mathrm{cont}}$*.*

10.1.4. For $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ we let $\mathrm{IndCoh}(\mathcal{Y})$ denote the value of $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ on $\mathcal{Y}$.

Tautologically, we have:

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \underset{(S,y)\in(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{IndCoh}(S), \tag{10.1}$$

where the functors for $f : S_1 \to S_2$, $y_1 = y_2 \circ f$ are

$$f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1).$$

For $(S, y) \in ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}}$ we let $y^!$ denote the corresponding evaluation functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(S).$$

For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in $\mathrm{PreStk}_{\mathrm{laft}}$ we let $f^!$ denote the corresponding functor

$$\mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1).$$

In particular, we let $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$ denote the dualizing complex of $\mathcal{Y}$, defined as $p_{\mathcal{Y}}^!(k)$, where $p_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{pt}$.

10.1.5. *The n-coconnective case.* Suppose that $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ is n-coconnective, i.e., when viewed as a functor

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow \infty\text{-Grpd},$$

it belongs to the essential image of the fully faithful embedding

$$\mathrm{Funct}(({}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \infty\text{-Grpd}) \hookrightarrow \mathrm{Funct}(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}, \infty\text{-Grpd}),$$

given by left Kan extension along

$$({}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}.$$

This is equivalent to the condition that the fully faithful embedding

$$({}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}} \hookrightarrow ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}}$$

be cofinal.

In particular, we obtain that the restriction functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \underset{(S,y)\in(({}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{IndCoh}(S) \tag{10.2}$$

is an equivalence.

Hence, to calculate the value of $\mathrm{IndCoh}(\mathcal{Y})$ on an n-coconnective prestack, it suffices to consider only n-coconnective affine DG schemes. In particular, if $\mathcal{Y}$ is 0-coconnective, i.e., is classical, it suffices to consider only classical affine schemes.

10.2. **Convergence.**

In this subsection we will discuss several applications of Proposition 4.3.4 to the study of properties of the category $\mathrm{IndCoh}(\mathcal{Y})$ for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$.

First, we note that the assertion of Proposition 4.3.4 implies::

Corollary 10.2.1. *The functor*

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}} : (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is the right Kan extension from the full subcategory

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}.$$

Proof. We wish to show that for $S \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, the functor

$$\mathrm{IndCoh}(S) \to \underset{S'\in{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}, S'\to S}{\mathit{lim}} \mathrm{IndCoh}(S') \tag{10.3}$$

is an equivalence. The right-hand side of (10.3) can be rewritten as

$$\underset{n}{\mathit{lim}} \underset{S'\in{}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}, S'\to S}{\mathit{lim}} \in \mathrm{IndCoh}(S').$$

However, since the functor ${}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ admits a right adjoint, given by $S \mapsto \tau^{\leq n}(S)$, we have:

$$\underset{S'\in{}^{<n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}, S'\to S}{\mathit{lim}} \mathrm{IndCoh}(S') \simeq \mathrm{IndCoh}(\tau^{\leq n}(S)).$$

Thus, we need to show that the functor

$$\mathrm{IndCoh}(S) \to \underset{n}{\mathit{lim}}\, \mathrm{IndCoh}(\tau^{\leq n}(S))$$

is an equivalence, but the latter is the content of Proposition 4.3.4. $\square$

Tautologically, Corollary 10.2.1 implies that in the description of $\operatorname{IndCoh}(\mathcal{Y})$ given by (10.1), instead of eventually coconnective affine schemes of finite type, we can use all affine schemes almost of finite type:

Corollary 10.2.2. *The restriction functor*

$$\underset{(S,y)\in((\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}})_{/\mathcal{Y}})^{\operatorname{op}}}{lim} \operatorname{IndCoh}(S) \to \underset{(S,y)\in(({}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}})_{/\mathcal{Y}})^{\operatorname{op}}}{lim} \operatorname{IndCoh}(S) =: \operatorname{IndCoh}(\mathcal{Y})$$

is an equivalence.

10.2.3. Equivalently, we can formulate the above corollary as saying that the functor

$$\operatorname{IndCoh}^!_{\operatorname{PreStk}_{\operatorname{laft}}} : (\operatorname{PreStk}_{\operatorname{aft}})^{\operatorname{op}} \to \operatorname{DGCat}_{\operatorname{cont}}$$

is the right Kan extension of

$$\operatorname{IndCoh}^!_{\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}}} : (\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}})^{\operatorname{op}} \to \operatorname{DGCat}_{\operatorname{cont}}$$

along the tautological embedding

$$(\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}})^{\operatorname{op}} \hookrightarrow (\operatorname{PreStk}_{\operatorname{aft}})^{\operatorname{op}}.$$

10.2.4. Finally, let us note that the value of the functor $\operatorname{IndCoh}^!_{\operatorname{PreStk}_{\operatorname{laft}}}$ on a given $\mathcal{Y} \in \operatorname{PreStk}_{\operatorname{laft}}$ can be recovered from the n-coconnective truncations $\tau^{\leq n}(\mathcal{Y})$ of $\mathcal{Y}$, where we recall that

$$\tau^{\leq n}(\mathcal{Y}) := \operatorname{LKE}_{({}^{<n}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{ft}})^{\operatorname{op}} \hookrightarrow ({}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}})^{\operatorname{op}}}(\mathcal{Y}|_{{}^{<n}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{ft}})^{\operatorname{op}}}).$$

Namely, we claim:

Lemma 10.2.5. *For $\mathcal{Y} \in \operatorname{PreStk}_{\operatorname{laft}}$, the natural map*

$$\operatorname{IndCoh}(\mathcal{Y}) \to \underset{n}{lim} \operatorname{IndCoh}(\tau^{\leq n}(\mathcal{Y}))$$

is an isomorphism.

Proof. The assertion follows from the fact that the natural map

$$\underset{n}{colim} \operatorname{LKE}_{{}^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{ft}} \hookrightarrow {}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}}}(\tau^{\leq n}(\mathcal{Y})) \to \mathcal{Y}$$

is an isomorphism in $\operatorname{PreStk}_{\operatorname{laft}}$, combined with Lemma 10.1.3. □

10.3. **Relation to** QCoh **and the multiplicative structure.**

10.3.1. Following Sect. 5.7.5, we view the assignment

$$S \in \operatorname{DGSch}_{\operatorname{aft}} \rightsquigarrow \Upsilon_S : \operatorname{QCoh}(S) \to \operatorname{IndCoh}(S)$$

as a natural transformation

$$\Upsilon_{\operatorname{DGSch}_{\operatorname{aft}}} : \operatorname{QCoh}^*_{\operatorname{DGSch}_{\operatorname{aft}}} \to \operatorname{IndCoh}^!_{\operatorname{DGSch}_{\operatorname{aft}}},$$

when we view $\operatorname{QCoh}^*_{\operatorname{DGSch}_{\operatorname{aft}}}$ and $\operatorname{IndCoh}^!_{\operatorname{DGSch}_{\operatorname{aft}}}$ as functors

$$(\operatorname{DGSch}_{\operatorname{aft}})^{\operatorname{op}} \to \operatorname{DGCat}^{\operatorname{SymMon}}_{\operatorname{cont}}.$$

Restricting to ${}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}} \subset \operatorname{DGSch}_{\operatorname{aft}}$, we obtain the corresponding functors and the natural transformation

$$\Upsilon_{{}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}}} : \operatorname{QCoh}^*_{{}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}}} \to \operatorname{IndCoh}^!_{{}^{<\infty}\operatorname{DGSch}^{\operatorname{aff}}_{\operatorname{aft}}}.$$

10.3.2. Since the forgetful functor $\mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$ commutes with limits, we obtain that the functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

naturally upgrades to a functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}.$$

In particular, for every $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{aft}}$, the category $\mathrm{IndCoh}(\mathcal{Y})$ acquires a natural symmetric monoidal structure; we denote the corresponding monoidal operation by $\overset{!}{\otimes}$. The unit in this category is $\omega_{\mathcal{Y}}$.

For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the functor $f^!$ has a natural symmetric monoidal structure.

10.3.3. Applying the functor

$$\mathrm{RKE}_{({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}}$$

to $\Upsilon_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$, we obtain a natural transformation

$$\mathrm{RKE}_{({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}}(\mathrm{QCoh}^*_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}) \to \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}.$$

Recall that the functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is defined as

$$\mathrm{RKE}_{(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}}(\mathrm{QCoh}^*_{\mathrm{DGSch}^{\mathrm{aff}}}).$$

Hence, we have a natural transformation

$$\begin{aligned} (10.4) \quad \mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}}|_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}} =: \mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{laft}}} \to \\ \to \mathrm{RKE}_{({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}}(\mathrm{QCoh}^*_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}). \end{aligned}$$

Composing, we obatin a functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$$

that we shall denote by $\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}}$.

For an individual $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ we shall denote by

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$$

the resulting functor.

Furthermore, the functors and the natural transformation in (10.4) naturally upgrade to take values in $\mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}$, and so does the natural transformation $\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}}$.

Lemma 10.3.4. *Assume that $\mathcal{Y}$ belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{laft}}$. Then the functor functor $\Upsilon_{\mathcal{Y}}$ is fully faithful.*

Proof. Follows from the isomorphism (10.2) (and the corresponding assertion for QCoh), and Corollary 9.6.3. □

10.3.5. *Behavior with respect to products.* Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be prestacks. Pulling back along the two projections

$$\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_i$$

and applying the monoidal operation, we obtain a functor

$$\text{IndCoh}(\mathcal{Y}_1) \otimes \text{IndCoh}(\mathcal{Y}_2) \to \text{IndCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2). \tag{10.5}$$

Repeating the argument of [GL:QCoh], Prop. 1.4.4, from Proposition 4.6.2 we deduce:

Corollary 10.3.6. *Assume that* $\text{IndCoh}(\mathcal{Y}_1)$ *is dualizable as a category. Then for any* $\mathcal{Y}_2$*, the functor* (10.5) *is an equivalence.*

10.4. **Descent properties.**

10.4.1. As in [GL:QCoh], Sect. 1.3.1, we can consider the notion of descent for presheaves on the category ${}^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}}$ with values in an arbitrary $(\infty, 1)$-category $\mathbf{C}$.

10.4.2. We observe that Theorem 8.3.2 implies that $\text{IndCoh}^!_{{}^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}}}$, regarded as a presheaf on ${}^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}}$ with values in $\text{DGCat}_{\text{cont}}$, is a sheaf in the fppf topology.

By [Lu0], Sect. 6.2.1, we obtain that whenever $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a map in $\text{PreStk}_{\text{laft}}$ such that $L_{\text{laft}}(\mathcal{Y}_1) \to L_{\text{laft}}(\mathcal{Y}_2)$ is an isomorphism, the map

$$\text{IndCoh}(\mathcal{Y}_2) \to \text{IndCoh}(\mathcal{Y}_1)$$

is an isomorphism.

Here L_{laft} denotes the localization functor on

$$\text{PreStk}_{\text{laft}} = \text{Funct}({}^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}}, \infty\text{-Grpd})$$

in the fppf topology.

From here we obtain:

Corollary 10.4.3. *For* $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$*, the natural map* $\mathcal{Y} \to L_{\text{laft}}(\mathcal{Y})$ *induces an isomorphism*

$$\text{IndCoh}(L_{\text{laft}}(\mathcal{Y})) \to \text{IndCoh}(\mathcal{Y}).$$

10.4.4. For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ consider the cosimplicial category $\text{IndCoh}^!(\mathcal{Y}_1^\bullet/\mathcal{Y}_2)$, formed using the !-pullback functors.

Assume now that f is a surjection for the fppf topology on ${}^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}}$ (see [GL:Stacks], Sect. 2.4.8). From [Lu0, Corollary 6.2.3.5], we obtain:

Corollary 10.4.5. *Under the above circumstances, we obtain that the functor*

$$\text{IndCoh}(\mathcal{Y}_2) \to \text{Tot}\left(\text{IndCoh}^!(\mathcal{Y}_1^\bullet/\mathcal{Y}_2)\right)$$

given by !-pullback, is an equivalence.

10.5. **Two definitions of** IndCoh **for DG schemes.**

10.5.1. Let X be an object of $\text{DGSch}_{\text{aft}}$.

Note that we have two *a priori* different definitions of $\text{IndCoh}(X)$: one given in Sect. 1.1 (which we temporarily denote $\text{IndCoh}(X)'$), and another as in Sect. 10.1.2 (which we temporarily denote $\text{IndCoh}(X)''$), when we regard X as an object of $\text{PreStk}_{\text{laft}}$.

10.5.2. Note that !-pullback defines a natural functor:

$$\text{IndCoh}(X)' \to \text{IndCoh}(X)''. \tag{10.6}$$

Proposition 10.5.3. *The functor* (10.6) *is an equivalence.*

Proof. First, assume that X is affine (but not necessarily eventually coconnective). Then the assertion follows from Corollary 10.2.1.

Next, assume that X is separated. Let $f : X' \to X$ be an affine Zariski cover, and let $X'^{\bullet}/X$ be its Čech nerve (whose terms are affine, since X was assumed separated). Then the validity of the assertion in the affine case, combined with Corollary 10.4.5 and Proposition 4.2.1, imply that (10.6) is an equivalence.

Let now X be arbitrary. We choose a Zariski cover $f : X' \to X$, where X' is separated, and repeat the same argument, using the fact that the terms of $X'^{\bullet}/X$ are now separated. □

10.5.4. The above proposition implies that in the definition of IndCoh($\mathcal{Y}$) for $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ we can use all DG schemes almost of finite type, instead of the affine ones:

Corollary 10.5.5. *For* $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$*, the restriction functor*

$$\underset{(S,y)\in((\text{DGSch}_{\text{aft}})_{/\mathcal{Y}})^{\text{op}}}{lim} \text{IndCoh}(S) \to \underset{(S,y)\in((^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}})_{/\mathcal{Y}})^{\text{op}}}{lim} \text{IndCoh}(S) := \text{IndCoh}(\mathcal{Y})$$

is an equivalence.

Proof. The left-hand side in the corollary calculates the value on $\mathcal{Y}$ of the right Kan extension of $\text{IndCoh}^!_{\text{DGSch}_{\text{aft}}}$ along

$$(\text{DGSch}_{\text{aft}})^{\text{op}} \hookrightarrow (\text{PreStk}_{\text{laft}})^{\text{op}}.$$

Hence, it is enough to show that the map

$$\text{IndCoh}^!_{\text{DGSch}_{\text{aft}}} \to \text{RKE}_{(^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}})^{\text{op}} \hookrightarrow (\text{DGSch}_{\text{aft}})^{\text{op}}}(\text{IndCoh}^!_{^{<\infty}\text{DGSch}^{\text{aff}}_{\text{aft}}})$$

is an isomorphism.

However, the latter is equivalent to the statement of Proposition 10.5.3. □

10.6. **Functoriality for direct image under schematic morphisms.** Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map in $\text{PreStk}_{\text{laft}}$. We will not be able to define the functor

$$f^{\text{IndCoh}}_* : \text{IndCoh}(\mathcal{Y}_1) \to \text{IndCoh}(\mathcal{Y}_2)$$

in general.

However, we will be able to do this in the case when f is schematic and quasi-compact morphism between arbitrary prestacks, which is goal of this subsection.

10.6.1. We shall say that a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is schematic if for any $(S_2, y_2) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$, the fiber product

$$S_1 := S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

is a DG scheme.

We shall say that f is schematic and quasi-separated and quasi-compact if for all $(S_2, y_2) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$, the above DG scheme S_1 is quasi-separated and quasi-compact.

We shall say that f is schematic and proper if for all $(S_2, y_2) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$, the resulting map $S_1 \to S_2$ of DG schemes is proper.

It is easy to see that if f is schematic, then for any $(S_2, y_2) \in \mathrm{DGSch}_{\mathcal{Y}_2}$ (i.e., S_2 is not necessarily affine), the fiber product $S_1 := S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is a DG scheme.

The next assertion results easily from the definitions:

Lemma 10.6.2. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map in* PreStk*.*

(a) *Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are convergent (see* [GL:Stacks, Sect. 1.2] *where the notion is introduced). Then the condition for f to be schematic (resp., schematic and quasi-separated and quasi-compact, schematic and proper) is enough to test on $(S_2, y_2) \in {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_2}$.*

(b) *Suppose that $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}_{\mathrm{laft}}$. Then the condition for f to be schematic (resp., schematic and quasi-separated and quasi-compact, schematic and proper) is enough to test on $(S_2, y_2) \in ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2}$.*

10.6.3. We are going to make IndCoh into a functor on the category $\mathrm{PreStk}_{\mathrm{laft}}$ with 1-morphisms being correspondences, where we allow to take direct images along morphisms that are schematic and quasi-compact.

Namely, in the framework of Sect. 5.1.1, we take $\mathbf{C} := \mathrm{PreStk}_{\mathrm{laft}}$, *horiz* to be the class of all 1-morphisms, and *vert* to be the class of 1-morphisms that are schematic and quasi-compact (the quasi-separetedness condition comes for free because of the finite type assumption).

Let $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}$ denote the resulting category of correspondences.

We shall now extend the assignment $\mathcal{Y} \mapsto \mathrm{IndCoh}(\mathcal{Y})$ to a functor

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}} : (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

such that for $g : \mathcal{Y}_1 \to \mathcal{Y}_2$ the corresponding functor

$$\mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1)$$

is $g^!$ defined in Sect. 10.1.4.

10.6.4. Consider the tautological functor

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}} \to (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}, \tag{10.7}$$

and define the functor

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}} : (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}} \to \mathrm{DGCat}_{\mathrm{cont}} \tag{10.8}$$

as the right Kan extension of $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ along the functor (10.7).

Proposition 10.6.5. *The diagram of functors*

$$\begin{array}{ccc} (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}} & \xrightarrow{\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}}} & \mathrm{DGCat}_{\mathrm{cont}} \\ \uparrow & & \uparrow \mathrm{Id} \\ (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} & \xrightarrow{\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}} & \mathrm{DGCat}_{\mathrm{cont}} \end{array}$$

is canonically commutative

Proof. This follows from Proposition 6.2.4.

□

10.7. **Adjunction for proper maps.** Let $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}proper;all}}$ be the 1-full subcategory of $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}$, where we restrict vertical morphisms to be proper.

In this subsection we shall study the restriction of the functor $\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}}$ to this subcategory.

10.7.1. Let us return to the setting of Sect. 6.2.3. Assume that $vert^1 \subset horiz^1$ and $vert^2 \subset horiz^2$.

Let us start with a functor

$$P^!_{horiz} : (\mathbf{C}^1_{horiz})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Assume that $P^!_{horiz}$ satisfies the right (resp., left) base change condition with respect to $vert^1$, see Sect. 6.1.1. I.e., for every 1-morphism $f : \widetilde{\mathbf{c}}^1 \to \mathbf{c}^1$ in $\mathbf{C}^1$, with $f \in vert^1$, the functor

$$P^!_{horiz} : P(\mathbf{c}^1) \to P(\widetilde{\mathbf{c}}^1)$$

admits a left (right) adjoint, denoted $P_{vert}(f)$, and that for a Cartesian square

$$\begin{array}{ccc} \widetilde{\mathbf{c}}'^1 & \xrightarrow{\widetilde{g}} & \widetilde{\mathbf{c}}^1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{c}'^1 & \xrightarrow{g} & \mathbf{c}^1 \end{array}$$

the resulting natural transformation

$$P_{vert}(f') \circ P^!_{horiz}(\widetilde{g}) \to P^!_{horiz}(g) \circ P_{vert}(f)$$

(in the case of left adjoints) and

$$P^!_{horiz}(g) \circ P_{vert}(f) \to P_{vert}(f') \circ P^!_{horiz}(\widetilde{g})$$

(in the case of right adjoints) is an isomorphism.

We claim:

Proposition 10.7.2. *Under the assumptions of Proposition 6.2.4, the functor*

$$Q^!_{horiz} := \mathrm{RKE}_{(\Phi_{horiz})^{\mathrm{op}}}(P^!_{horiz}) : (\mathbf{C}^2_{horiz})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

satisfies the right (resp., left) base change condition with respect to $vert^2 \subset horiz^2$.

Proof. Let $f : \widetilde{\mathbf{c}}^2 \to \mathbf{c}^2$ be a 1-morphism in $vert^2$. The condition of the proposition implies that the assignment

$$\mathbf{c}^1 \in \mathbf{C}^1 \underset{\mathbf{C}^2}{\times} (\mathbf{C}^2_{horiz})_{/\mathbf{c}^2} \rightsquigarrow \Phi_{horiz}(\mathbf{c}^1) \underset{\mathbf{c}^2}{\times} \widetilde{\mathbf{c}}_2 \simeq \Phi_{horiz}(\widetilde{\mathbf{c}}^1)$$

defines a functor

$$\mathbf{C}^1 \underset{\mathbf{C}^2}{\times} (\mathbf{C}^2_{horiz})_{/\mathbf{c}^2} \to \mathbf{C}^1 \underset{\mathbf{C}^2}{\times} (\mathbf{C}^2_{horiz})_{/\widetilde{\mathbf{c}}^2},$$

which, moreover, is cofinal.

Hence, we can calculate the value of $Q^!_{horiz}$ on $\widetilde{\mathbf{c}}^2$ as

$$\underset{\mathbf{c}^1\in(\mathbf{C}^1\underset{\mathbf{C}^2}{\times}(\mathbf{C}^2_{horiz})_{/\mathbf{c}^2})^{\mathrm{op}}}{lim} P^!_{horiz}(\widetilde{\mathbf{c}}^1).$$

The existsence of the left (resp., right) adjoint $Q_{vert}(f)$ of $Q^!_{horiz}(f)$ follows now from the following general paradigm:

Let I be an index category and

$$i\mapsto \mathbf{C}_i \text{ and } i\mapsto \widetilde{\mathbf{C}}_i$$

be two functors $I\to \mathrm{DGCat}_{\mathrm{cont}}$, and let

$$i\mapsto F_i\in \mathrm{Funct}_{\mathrm{cont}}(\widetilde{\mathbf{C}}_i,\mathbf{C}_i)$$

be a natural transformations between them.

Denote

$$\mathbf{C}:=\underset{i\in I}{lim}\,\mathbf{C}_i \text{ and } \widetilde{\mathbf{C}}:=\underset{i\in I}{lim}\,\widetilde{\mathbf{C}}_i,$$

and let F be the corresponding functor $\widetilde{\mathbf{C}}\to\mathbf{C}$.

Assume that for everi i, the functor F_i admits a left (resp., continuous right) adjoint G_i, and that for every arrow $i\to i'$ in I, the square

$$\begin{array}{ccc} \widetilde{\mathbf{C}}_i & \longrightarrow & \widetilde{\mathbf{C}}_{i'} \\ {\scriptstyle G_i}\uparrow & & \uparrow{\scriptstyle G_{i'}} \\ \mathbf{C}_i & \longrightarrow & \mathbf{C}_{i'}, \end{array}$$

obtained by adjunction from the commutative square

$$\begin{array}{ccc} \widetilde{\mathbf{C}}_i & \longrightarrow & \widetilde{\mathbf{C}}_{i'} \\ {\scriptstyle F_i}\downarrow & & \downarrow{\scriptstyle F_{i'}} \\ \mathbf{C}_i & \longrightarrow & \mathbf{C}_{i'}, \end{array}$$

which a priori commutes up to a natural transformation, actually commutes.

Lemma 10.7.3. *Under the above circumstances, the functor $F:\widetilde{\mathbf{C}}\to\mathbf{C}$ admits a left (resp., continuous right), denoted G, and for every $i\in I$ the square,*

$$\begin{array}{ccc} \widetilde{\mathbf{C}} & \longrightarrow & \widetilde{\mathbf{C}}_i \\ {\scriptstyle G}\uparrow & & \uparrow{\scriptstyle G_i} \\ \mathbf{C} & \longrightarrow & \mathbf{C}_i, \end{array}$$

obtained by adjunction from the commutative square

$$\begin{array}{ccc} \widetilde{\mathbf{C}} & \longrightarrow & \widetilde{\mathbf{C}}_i \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_{i'}} \\ \mathbf{C} & \longrightarrow & \mathbf{C}_i, \end{array}$$

which a priori commutes up to a natural transformation, actually commutes.

The fact that $Q^!_{horiz}$ satisfies the right (resp., left) base change condition with respect to $vert^2 \subset horiz^2$ is a formal consequence of the second statament in Lemma 10.7.3.

□

We shall need also the following statement which will be proved in [GR3] along with Theorem 6.1.2:

Lemma 10.7.4. *Let*

$$P_{\mathrm{corr}:vert;horiz} : \mathbf{C}^1_{\mathrm{corr}:vert;horiz} \to \mathrm{DGCat}_{\mathrm{cont}}$$

be the functor obtained from $P^!_{horiz}$ by Theorem 6.1.2 applied to $vert^1 \subset horiz^1$. Let

$$Q_{\mathrm{corr}:vert;horiz} : \mathbf{C}^1_{\mathrm{corr}:vert;horiz} \to \mathrm{DGCat}_{\mathrm{cont}}$$

be the right Kan extension of $P_{\mathrm{corr}:vert;horiz}$ along the functor

$$\Phi_{\mathrm{corr}:vert;horiz} : \mathbf{C}^1_{\mathrm{corr}:vert;horiz} \to \mathbf{C}^2_{\mathrm{corr}:vert;horiz}.$$

Then in terms of the isomorphism

$$Q_{\mathrm{corr}:vert;horiz}|_{(\mathbf{C}^2_{horiz})^{\mathrm{op}}} \simeq Q^!_{horiz}$$

of Proposition 6.2.4, the functor $Q_{\mathrm{corr}:vert;horiz}$ identifies with one obtained from $Q^!_{horiz}$ by Theorem 6.1.2 applied to $vert^2 \subset horiz^2$.

10.7.5. We apply Proposition 10.7.2 to the functor

$$\Phi : \mathrm{DGSch}_{\mathrm{aft}} \to \mathrm{PreStk}_{\mathrm{laft}}$$

with $horiz^1 = \mathrm{all}$, $horiz^2 = \mathrm{all}$ and $vert^1 = \mathrm{proper}$, $vert^1 = \text{sch-proper}$, and

$$P^!_{horiz} := \mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}.$$

By Corollary 10.5.5, the resulting functor

$$(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

identifies with $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}$.

Thus, we obtain that the functor $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}$ satisfies the right base change condition with respect the class of schematic and proper maps. Applying Theorem 6.1.2, we obtain a functor

$$(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}proper;all}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

that we shall denote by $\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}proper;all}}}$.

We now claim:

Proposition 10.7.6. *There exists a canonical isomorphism*

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}qc;all}}}|_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}proper;all}}} \simeq \mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}proper;all}}},$$

compatible with the further restriction under

$$(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{corr:sch\text{-}proper;all}}.$$

This follows by applying Lemma 10.7.4 using the following statement implicit in the proof of Theorem 5.2.2, and which contains Proposition 5.4.2(a) as a particular case:

Proposition 10.7.7. *The restriction of the functor* $\text{IndCoh}_{(\text{DGSch}_{\text{aft}})_{\text{corr:all;all}}}$ *under*

$$(\text{DGSch}_{\text{aft}})_{\text{corr:proper;all}} \to (\text{DGSch}_{\text{aft}})_{\text{corr:all;all}}$$

identifies canonically with the functor obtained from

$$\text{IndCoh}^!_{\text{DGSch}_{\text{aft}}} : (\text{DGSch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}$$

by Theorem 6.1.2 applied to proper $\subset$ all.

11. IndCoh on Artin stacks

11.1. **Recap: Artin stacks.** In this subsection we will recall some facts concerning Artin stacks. We refer the reader to [GL:Stacks], Sect. 4.9 for a more detailed discussion.

11.1.1. We let $\text{Stk}_{\text{Artin}}$ denote the full subcategory of PreStk consisting of Artin stacks. For $k \in \mathbb{N}$, we let $\text{Stk}_{k\text{-Artin}}$ denote the full subcategory of k-Artin stacks.

We set by definition:

$$\text{Stk}_{(-1)\text{-Artin}} = \text{DGSch},$$

(i.e., all DG schemes).

$$\text{Stk}_{(-2)\text{-Artin}} = \text{DGSch}_{\text{sep}},$$

(i.e., separated DG schemes)

$$\text{Stk}_{(-3)\text{-Artin}} = \text{DGSch}^{\text{aff}}.$$

11.1.2. We shall say that a morphism in PreStk is "k-representable" if its base change by any affine DG scheme yields an object of $\text{Stk}_{k\text{-Artin}}$. E.g., "$(-1)$-representable" is the same as "schematic."

We shall say that a morphism in PreStk is "eventually representable" if is k-representable for some k.

We shall say that an eventually representable morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ is smooth/flat/of bounded Tor dimension/ eventually coconnective if for every $S_2 \in \text{DGSch}^{\text{aff}}$ equipped with a map to $\mathcal{Y}_2$, and $S_1 \in \text{DGSch}^{\text{aff}}$, equipped with a smooth map to $S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$, the resulting map $S_1 \to S_2$ is smooth/flat/of bounded Tor dimension/eventually coconnective.

If $\mathcal{Y}_2$ is itself an Artin stack, it is enough to test the above condition for those maps $S_2 \to \mathcal{Y}_2$ that are smooth (or flat), and in fact for just one smooth (or flat) covering of $\mathcal{Y}_2$.

We note that the diagonal morphism of a k-Artin stack is $(k-1)$-representable.

11.1.3. We let $\text{Stk}_{\text{laft,Artin}}$ denote the full subcategory of $\text{PreStk}_{\text{laft}}$ equal to

$$\text{PreStk}_{\text{laft}} \cap \text{Stk}_{\text{Artin}},$$

and similarly,

$$\text{Stk}_{\text{laft},k\text{-Artin}} = \text{PreStk}_{\text{laft}} \cap \text{Stk}_{k\text{-Artin}}.$$

We also note that in order to check that in order to check that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in $\text{PreStk}_{\text{laft}}$ is k-representable (resp., k-representable and smooth/flat/of bounded Tor dimension/eventually coconnective) it is enough to do so for $S_2 \to \mathcal{Y}_2$ with $S_2 \in {}^{<\infty}\text{DGSch}_{\text{aft}}$.

11.1.4. The following is a basic fact concerning the subcategory

$$\mathrm{Stk}_{\mathrm{laft,Artin}} \subset \mathrm{Stk}_{\mathrm{Artin}},$$

see [GL:Stacks, Proposition 4.9.4]:

Lemma 11.1.5. *For a $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{laft,Artin}}$ and a smooth map $S \to \mathcal{Y}$, where $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the DG scheme S is almost of finite type.*

In particular:

Corollary 11.1.6. *Every object of $\mathrm{Stk}_{\mathrm{laft,Artin}}$ admits a smooth surjective map from $S \in \mathrm{DGSch}_{\mathrm{laft}}$.*

11.1.7. Let

$$\mathrm{IndCoh}^!_{\mathrm{Stk}_{\mathrm{laft,Artin}}} : (\mathrm{Stk}_{\mathrm{laft,Artin}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

(resp., $\mathrm{IndCoh}^!_{\mathrm{Stk}_{\mathrm{laft},k\text{-}\mathrm{Artin}}}$) denote the restriction of the functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to the corresponding subcategory.

11.2. **Recovering from smooth/flat/eventually coconnective maps.** In this subsection we will show that if $\mathcal{Y}$ is an Artin stack, the category $\mathrm{IndCoh}(\mathcal{Y})$ can be recovered from just looking at affine DG schemes equipped with a *smooth* map to $\mathcal{Y}$.

11.2.1. Let $\mathfrak{c}$ be a class of morphisms between Artin stacks belonging to the following set

$$\text{smooth} \subset \text{flat} \subset \text{bdd-Tor} \subset \text{ev-coconn} \subset \text{all}\,.$$

Consider the corresponding fully faithful embedding

$$(\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c}} \hookrightarrow (\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}.$$

We claim:

Proposition 11.2.2. *Let P be a presheaf on $(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}$ with values in an arbitrary ∞-category. Assume that P satisfies descent with respect to smooth surjective maps. Then the map*

$$P \to \mathrm{RKE}_{((\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c}})^{\mathrm{op}} \hookrightarrow ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})^{\mathrm{op}}}(P)$$

is an isomorphism.

Proof. It is enough to prove the claim after the restriction to

$$(\mathrm{Stk}_{k\text{-}\mathrm{Artin}})_{\mathfrak{c}} \subset (\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}$$

for every k.

We will argue by induction on k. For $k = -3$, the assertion is tautological. The induction step follows from Proposition 6.4.8 and Corollary 11.1.6, applied to:

$$\mathbf{C} := \mathrm{Stk}_{k\text{-}\mathrm{Artin}},\ \mathbf{C}' := \mathrm{Stk}_{(k-1)\text{-}\mathrm{Artin}},\ \mathbf{C}_0 := (\mathrm{Stk}_{k\text{-}\mathrm{Artin}})_{\mathfrak{c}},$$

and the smooth topology.

□

The above proposition can be reformulated as follows:

Corollary 11.2.3. *For P as in Proposition 11.2.2 and $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{Artin}}$, the natural map*

$$P(\mathcal{Y}) \to \underset{S \in ((\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} P(S)$$

is an isomorphism.

We empasize that

$$(\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})_{/\mathcal{Y}}$$

is the 1-full subcategory of $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$ spanned by those $S \to \mathcal{Y}$ whose map to $\mathcal{Y}$ belongs to $\mathfrak{c}$, and where we restrict 1-morphisms to those maps $S_1 \to S_2$ that themselves belong to $\mathfrak{c}$.

As a corollary, we obtain:

Corollary 11.2.4. *Under the assumptions of Corollary 11.2.3, the maps*

$$P(\mathcal{Y}) \to \underset{S \in ((\mathrm{DGSch})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} P(S) \to$$

$$\to \underset{S \in (((\mathrm{DGSch})_{\mathrm{qsep\text{-}qc}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} P(S)$$

are also isomorphisms.

11.2.5. Let us fix $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{Artin}}$, and let

$$(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c} \text{ over } \mathcal{Y}} \subset (\mathrm{Stk}_{\mathrm{Artin}})_{/\mathcal{Y}}$$

be the full subcategory spanned by those $f : \mathcal{Y}' \to \mathcal{Y}$, where f belongs to $\mathfrak{c}$.

For another class $\mathfrak{c}'$ from the collection

$$\mathrm{smooth} \subset \mathrm{flat} \subset \mathrm{bdd\text{-}Tor} \subset \mathrm{ev\text{-}coconn} \subset \mathrm{all},$$

consider the 1-full subcategory

$$((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'}$$

and its full subcategory $((\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'}$.

To decipher this, $((\mathrm{DGSch})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'}$ is the 1-full subcategory of $(\mathrm{DGSch}_{\mathrm{Artin}})_{/\mathcal{Y}}$, spanned by those $f : S \to \mathcal{Y}$, where f belongs to $\mathfrak{c}$, and where we restrict 1-morphisms to those maps $S_1 \to S_2$ that belong to $\mathfrak{c}'$.

11.2.6. As in Proposition 11.2.2, we have:

Proposition 11.2.7. *Let P be a presheaf on $((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'}$ with values in an arbitrary ∞-category. Assume that P satisfies descent with respect to smooth surjective maps. Then the map*

$$P \to \mathrm{RKE}_{(((\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}} \hookrightarrow (((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}}}(P)$$

is an isomorphism.

Corollary 11.2.8. *Under the assumptions of Corollary 11.2.3, the maps*

$$P(\mathcal{Y}) \to \underset{S \in (((\mathrm{DGSch})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}}}{lim} P(S) \to$$

$$\to P(\mathcal{Y}) \to \underset{S \in ((((\mathrm{DGSch})_{\mathrm{qsep\text{-}qc}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}}}{lim} P(S) \to$$

$$\to \underset{S \in (((\mathrm{DGSch}^{\mathrm{aff}})_{\mathfrak{c} \text{ over } \mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}}}{lim} P(S)$$

are isomorphisms.

11.2.9. Applying the above discussion to $\mathrm{IndCoh}^!_{\mathrm{Stk}_{\mathrm{laft,Artin}}}$, we obtain:

Proposition 11.2.10. *Let* $\mathcal{Y}$ *be an object of* $\mathrm{Stk}_{\mathrm{laft,Artin}}$. *Let* $\mathfrak{c}$ *be one of the classes*

$$\mathrm{smooth} \subset \mathrm{flat} \subset \mathrm{ev\text{-}coconn} \subset \mathrm{all}\,.$$

Then the restriction maps

$$\mathrm{IndCoh}(\mathcal{Y}) \to \underset{S\in(((\mathrm{DGSch})_{\mathrm{aft}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S) \to$$
$$\to \underset{S\in(((\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{aft}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S)$$

are isomorphisms.

Similarly, we have:

Proposition 11.2.11. *Let* $\mathcal{Y}$ *be an object of* $\mathrm{Stk}_{\mathrm{laft,Artin}}$. *Let* $\mathfrak{c}$ *and* $\mathfrak{c}'$ *be any two of the classes*

$$\mathrm{smooth} \subset \mathrm{flat} \subset \mathrm{ev\text{-}coconn} \subset \mathrm{all}\,.$$

Then the restriction maps

$$\mathrm{IndCoh}(\mathcal{Y}) \to \underset{S\in((((\mathrm{DGSch})_{\mathrm{aft}})_{\mathfrak{c}\text{ over }\mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S) \to$$
$$\to \underset{S\in((((\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{aft}})_{\mathfrak{c}\text{ over }\mathcal{Y}})_{\mathfrak{c}'})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S)$$

are isomorphisms.

11.2.12. The upshot of the above two propositions is that the category $\mathrm{IndCoh}(\mathcal{Y})$ is recovered from the knowledge of $\mathrm{IndCoh}(S)$ where S belongs to $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ (resp., $\mathrm{DGSch}_{\mathrm{aft}}$, $\mathrm{DGSch}_{\mathrm{laft}}$), endowed with a smooth/flat/eventually coconnective/arbitrary map to $\mathcal{Y}$.

Furthermore, we can either take all maps between the schemes S, or restrict them to be smooth/flat/eventually coconnective.

11.3. **The $*$-version.** We are going to introduce another functor

$$((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

denoted $\mathrm{IndCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}}$.

11.3.1. We set

$$\mathrm{IndCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}} :=$$
$$= \mathrm{RKE}_{(((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}))_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \hookrightarrow ((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}}}(\mathrm{IndCoh}^*_{(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}}),$$

where

$$\mathrm{IndCoh}^*_{(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}} : ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is the functor obtained from the functor $\mathrm{IndCoh}^*_{(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}}}$ of Corollary 3.5.6 by restriction along

$$((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \hookrightarrow ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{Noeth}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}}.$$

11.3.2. In other words,

$$\mathrm{IndCoh}^*(\mathcal{Y}) = \underset{S\to\mathcal{Y}}{lim}\, \mathrm{IndCoh}^*(S),$$

where the limit is taken over the category opposite to

$$(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\text{ev-coconn}} \underset{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}}}{\times} ((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}})_{/\mathcal{Y}},$$

which is a 1-full subcategory of $(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}}$, spanned by those $S\to\mathcal{Y}$, which are eventually coconnective, and where 1-morphisms $f: S_1\to S_2$ are restricted to also be eventually coconnective.

In the above formula, for $S\in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, the category $\mathrm{IndCoh}^*(S)$ is the usual $\mathrm{IndCoh}(S)$, and for a 1-morphism $f: S_1\to S_2$, the functor $\mathrm{IndCoh}(S_2)\to\mathrm{IndCoh}(S_1)$ is $f^{\mathrm{IndCoh},*}$.

11.3.3. We claim:

Lemma 11.3.4. *The functor* $\mathrm{IndCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}}}$ *satisfies descent with respect to smooth surjective morphisms.*

Proof. Follows from the fact that the category $\mathrm{Stk}_{\mathrm{laft,Artin}}$ embeds fully fiathfully into

$$\mathrm{Funct}((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}, \infty\text{-Grpd}),$$

combined with [Lu0, Corollary 6.2.3.5] and Proposition 8.3.8.

□

From Corollaries 11.2.3 and 11.2.4, we obtain:

Corollary 11.3.5. *Let* $\mathfrak{c}$ *be one of the classes*

$$\text{smooth} \subset \text{flat} \subset \text{ev-coconn}\,.$$

Then for $\mathcal{Y}\in\mathrm{Stk}_{\mathrm{laft,Artin}}$ *the restriction functors*

$$\mathrm{IndCoh}^*(\mathcal{Y}) \to \underset{S\in(((\mathrm{DGSch})_{\mathrm{aft}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}^*(S) \to$$
$$\to \underset{S\in(((\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{aft}})_{\mathfrak{c}} \underset{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathfrak{c}}}{\times} ((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathfrak{c}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}^*(S)$$

are isomorphisms.

11.4. **Comparing the two versions of** IndCoh**: the case of algebraic stacks.** In this subsection we will show that for $\mathcal{Y}\in\mathrm{Stk}_{\mathrm{laft,Artin}}$, which is an algebraic stack, the categories $\mathrm{IndCoh}^*(\mathcal{Y})$ and $\mathrm{IndCoh}^!(\mathcal{Y}) := \mathrm{IndCoh}(\mathcal{Y})$ are canonically equivalent.

11.4.1. Our conventions regarding algebraic stacks follow those of [DrGa1, Sect. 1.1.3]. Namely, an algebraic stack is an object of $\mathrm{Stk}_{1\text{-Artin}}$, for which the diagonal morphism

$$\mathcal{Y}\to\mathcal{Y}\times\mathcal{Y}$$

is schematic, quasi-separated and quasi-compact.

11.4.2. We are going to prove:

Proposition 11.4.3. *For an algebraic stack $\mathcal{Y}$ locally almost of finite type, there exists a canonical equivalence*

$$\mathrm{IndCoh}^!(\mathcal{Y}) \simeq \mathrm{IndCoh}^*(\mathcal{Y}),$$

such that for a Cartesian square

$$\begin{array}{ccc} S & \xrightarrow{g''} & S'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & \mathcal{Y}, \end{array}$$

with $S', S'' \in \mathrm{DGSch}_{\mathrm{aft}}$, the morphism g being arbitrary and f being eventually coconnective, the functors

$$(f')^{\mathrm{IndCoh},*} \circ g^! : \mathrm{IndCoh}^!(\mathcal{Y}) \to \mathrm{IndCoh}(S) \text{ and } (g'')^! \circ f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}^*(\mathcal{Y}) \to \mathrm{IndCoh}(S)$$

are canonically identified.

The rest of this subsection is devoted to the proof of this proposition.

11.4.4. By restricting to DG schemes almost of finite type, Proposition 7.6.7 defines a map in $\infty\text{-Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}}$:

$$\mathrm{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\text{ev-coconn;all}}} : \mathrm{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\mathrm{DGSch}_{\mathrm{aft}}) \to \mathrm{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}).$$

Using Theorem 7.6.2, the map $\mathrm{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\text{ev-coconn;all}}}$ gives rise to the following construction:

Let $\mathbf{I}'$ and $\mathbf{I}''$ be a pair of ∞-categories, and let F be a functor

$$\mathbf{I}' \times \mathbf{I}'' \to \mathrm{DGSch}_{\mathrm{aft}},$$

with the property that for every $\mathbf{i}' \in \mathbf{I}'$ and a 1-morphism $(\mathbf{i}''_0 \to \mathbf{i}''_1) \in \mathbf{I}''$, the map

$$F(\mathbf{i}' \times \mathbf{i}''_0) \to F(\mathbf{i}' \times \mathbf{i}''_1)$$

is eventually coconnective, and for any pair of 1-morphisms

$$(\mathbf{i}'_0 \to \mathbf{i}'_1) \in \mathbf{I}' \text{ and } (\mathbf{i}''_0 \to \mathbf{i}''_1) \in \mathbf{I}'',$$

the square

$$\begin{array}{ccc} F(\mathbf{i}'_0 \times \mathbf{i}''_0) & \longrightarrow & F(\mathbf{i}'_1 \times \mathbf{i}''_0) \\ \downarrow & & \downarrow \\ F(\mathbf{i}'_0 \times \mathbf{i}''_1) & \longrightarrow & F(\mathbf{i}'_1 \times \mathbf{i}''_1) \end{array}$$

is Cartesian.

Then the map $\mathrm{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\text{ev-coconn;all}}}$ canonically attaches to a functor F as above, a datum of a functor

$$\mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F : (\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Moreover, for every fixed object $\mathbf{i}' \in \mathbf{I}'$ (resp., $\mathbf{i}'' \in \mathbf{I}''$) the resulting functors

$$\mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F|_{\mathbf{i}' \times \mathbf{I}''} : (\mathbf{I}'')^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$\mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F|\mathbf{I}' \times \mathbf{i}'' : (\mathbf{I}')^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

identify with

$$\mathrm{IndCoh}^*_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F|_{\mathbf{i}' \times \mathbf{I}''} \text{ and } \mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F|\mathbf{I}' \times \mathbf{i}'',$$

respectively.

11.4.5. We let

$$\mathbf{I}' := (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}} \text{ and } \mathbf{I}'' := (\mathrm{DGSch}_{\mathrm{aft}})_{\text{ev-coconn}} \underset{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}}}{\times} ((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}})_{/\mathcal{Y}}.$$

We let F be the functor that sends

$$(g : S' \to \mathcal{Y}), (f : S'' \to \mathcal{Y}) \mapsto S' \underset{\mathcal{Y}}{\times} S''.$$

Applying the construction from Sect. 11.4.4, we obtain a functor

$$\mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F.$$

We claim that there are canonical equivalences

$$\mathrm{IndCoh}^!(\mathcal{Y}) \simeq \underset{(\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}}}{lim} \mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F \simeq \mathrm{IndCoh}^*(\mathcal{Y}),$$

which would imply the assertion of Proposition 11.4.3.

11.4.6. Let us construct the isomorphism

$$\mathrm{IndCoh}^!(\mathcal{Y}) \simeq \underset{(\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}}}{lim} \mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F.$$

We calculate

$$\underset{(\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}}}{lim} \mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F$$

as the limit over $(\mathbf{I}')^{\mathrm{op}}$ of the functor that sends $(g : S' \to \mathcal{Y})$ to

$$\underset{(f:S'' \to \mathcal{Y}) \in (\mathbf{I}'')^{\mathrm{op}}}{lim} \mathrm{IndCoh}^*(S' \underset{\mathcal{Y}}{\times} S'').$$

However, we claim:

Lemma 11.4.7. *The natural map*

$$\mathrm{IndCoh}(S') \to \underset{(f:S'' \to \mathcal{Y}) \in (\mathbf{I}'')^{\mathrm{op}}}{lim} \mathrm{IndCoh}^*(S' \underset{\mathcal{Y}}{\times} S'')$$

is an isomorphism.

Hence, we obtain that the above functor on $(\mathbf{I}')^{\mathrm{op}}$ identifies canonically with

$$(\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}})|_{(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}}}.$$

The limit of the latter is $\mathrm{IndCoh}^!(\mathcal{Y})$, by definition.

11.4.8. The isomorphism

$$\underset{(\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}}}{lim} \mathrm{IndCoh}^{*!}_{\mathrm{DGSch}_{\mathrm{aft}}} \circ F \simeq \mathrm{IndCoh}^*(\mathcal{Y})$$

is constructed similarly.

11.4.9. *Proof of Lemma 11.4.7.* The functor

$$(f:S''\to \mathcal{Y})\mapsto \operatorname{IndCoh}^*(S'\underset{\mathcal{Y}}{\times} S'')$$

is a presheaf on $\mathbf{I}''$ that satisfies descent with respect to smooth surjective maps.

Hence, the limit

$$\underset{(f:S''\to \mathcal{Y})\in (\mathbf{I}'')^{\mathrm{op}}}{lim} \operatorname{IndCoh}^*(S'\underset{\mathcal{Y}}{\times} S'')$$

can be calculated as

$$\operatorname{Tot}(\operatorname{IndCoh}^*(S'\underset{\mathcal{Y}}{\times} (S''^{\bullet}_0/\mathcal{Y}))),$$

where $S''_0\to \mathcal{Y}$ is some smooth cover, and $S''^{\bullet}_0/\mathcal{Y}$ is its Čech nerve.

However, $S'\underset{\mathcal{Y}}{\times} (S''^{\bullet}_0/\mathcal{Y})$ is the Čech nerve of the map

$$S'\underset{\mathcal{Y}}{\times} S''_0\to S',$$

and the required isomorphism follows from smooth descent for $\operatorname{IndCoh}^*_{\mathrm{DGSch}_{\mathrm{aft}}}$.

11.5. **Comparing the two versions of** IndCoh**: general Artin stacks.** In this subsection we will generalize the construction of Sect. 11.4 to arbitrary Artin stacks.

11.5.1. We consider the category $\mathrm{Stk}_{\mathrm{laft,Artin}}$ endowed with the classes of 1-morphisms

$$(\text{ev-coconn};\text{all}),$$

and we consider the corresponding object

$$\operatorname{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft,Artin}})\in \infty\text{-}\mathrm{Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}}.$$

We claim:

Proposition 11.5.2. *There exists a uniquely defined map in* $\infty\text{-}\mathrm{Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}}$

$$\operatorname{IndCoh}^{*!}_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn;all}}}:\operatorname{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft,Artin}})\to \operatorname{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}})$$

that makes the following diagrams commute

$$\begin{array}{ccc}
\operatorname{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft,Artin}}) & \xrightarrow{\operatorname{IndCoh}^{*!}_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn;all}}}} & \operatorname{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}) \\
\uparrow & & \uparrow \\
\pi_h(\operatorname{Seg}^{\bullet}(\mathrm{Stk}_{\mathrm{laft,Artin}}) & \xrightarrow[\operatorname{Seg}^{\bullet}(\operatorname{IndCoh}^!_{\mathrm{Stk}_{\mathrm{laft,Artin}}})]{} & \pi_h(\operatorname{Seg}^{\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}))
\end{array}$$

and

$$\begin{array}{ccc}
\operatorname{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft,Artin}}) & \xrightarrow{\operatorname{IndCoh}^{*!}_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn;all}}}} & \operatorname{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}) \\
\uparrow & & \uparrow \\
\pi_v(\operatorname{Seg}^{\bullet}((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}})) & \xrightarrow[\operatorname{Seg}^{\bullet}(\operatorname{IndCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\text{ev-coconn}}})]{} & \pi_v(\operatorname{Seg}^{\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}})),
\end{array}$$

and which extends the map

$$\operatorname{IndCoh}^{*!}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\text{ev-coconn;all}}}:\operatorname{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\mathrm{DGSch}_{\mathrm{aft}})\to \operatorname{Seg}^{\bullet,\bullet}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}).$$

Corollary 11.5.3. *For $\mathcal{Y} \in \operatorname{Stk}_{\text{laft,Artin}}$ there exists a canonical equivalence*

$$\operatorname{IndCoh}^!(\mathcal{Y}) \simeq \operatorname{IndCoh}^*(\mathcal{Y}),$$

such that for a Cartesian square of Artin stacks

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g''} & \mathcal{Z}'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z}' & \xrightarrow{g} & \mathcal{Y}, \end{array} \tag{11.1}$$

where the morphism g is arbitrary and f is eventually coconnective, the functors

$$(f')^{\operatorname{IndCoh},*} \circ g^! : \operatorname{IndCoh}^!(\mathcal{Y}) \to \operatorname{IndCoh}(\mathcal{Z}) \text{ and } (g'')^! \circ f^{\operatorname{IndCoh},*} : \operatorname{IndCoh}^*(\mathcal{Y}) \to \operatorname{IndCoh}(\mathcal{Z})$$

are canonically identified.

The rest of this subsection is devoted to a sketch of the proof of Proposition 11.5.2; details will be supplied elsewhere.

11.5.4. We proceed by induction on k. Thus, we assume that the existence and uniqueness of the map

$$\operatorname{IndCoh}^{*!}_{(\operatorname{Stk}_{\text{laft},(k-1)\text{-Artin}})_{\text{ev-coconn;all}}} : \operatorname{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\operatorname{Stk}_{\text{laft},(k-1)\text{-Artin}}) \to \operatorname{Seg}^{\bullet,\bullet}((\operatorname{DGCat}_{\text{cont}})^{\text{op}})$$

with the required properties.

The base of the induction is $k = -3$, in which case the assertion follows by restriction from Proposition 7.6.7 inder $\operatorname{DGSch}^{\text{aff}}_{\text{aft}} \hookrightarrow \operatorname{DGSch}_{\text{aft}}$.

We shall now perform the induction step.

Remark 11.5.5. Note that by repeating the construction in the proof of Proposition 11.4.3, we obtain that for an individual $\mathcal{Y} \in \operatorname{Stk}_{\text{laft},k\text{-Artin}}$ there exists a canonical equivalence

$$\operatorname{IndCoh}^!(\mathcal{Y}) \simeq \operatorname{IndCoh}^*(\mathcal{Y}),$$

such that for every diagram (11.1) with $\mathcal{Z}', \mathcal{Z}'' \in \operatorname{Stk}_{\text{laft},(k-1)\text{-Artin}}$, the diagram

$$\begin{array}{ccccccc} \operatorname{IndCoh}^*(\mathcal{Z}) & \xleftarrow{\sim} & \operatorname{IndCoh}^!(\mathcal{Z}) & \xleftarrow{(g'')^!} & \operatorname{IndCoh}^!(\mathcal{Z}'') & \xleftarrow{\sim} & \operatorname{IndCoh}^*(\mathcal{Z}'') \\ {\scriptstyle (f')^{\operatorname{IndCoh},*}}\uparrow & & & & & & \uparrow{\scriptstyle f^{\operatorname{IndCoh},*}} \\ \operatorname{IndCoh}^*(\mathcal{Z}') & \xleftarrow{\sim} & \operatorname{IndCoh}^!(\mathcal{Z}') & \xleftarrow{g^!} & \operatorname{IndCoh}^!(\mathcal{Y}) & \xleftarrow{\sim} & \operatorname{IndCoh}^*(\mathcal{Y}) \end{array}$$

is commutative.

The above amounts to calculating the value of the map $\operatorname{IndCoh}^{*!}_{(\operatorname{Stk}_{\text{laft},k\text{-Artin}})_{\text{ev-coconn;all}}}$ on $[0] \times [0] \in \mathbf{\Delta} \times \mathbf{\Delta}$. To make this construction functorial in $\mathcal{Y}$ amounts to constructing the map $\operatorname{IndCoh}^{*!}_{(\operatorname{Stk}_{\text{laft},k\text{-Artin}})_{\text{ev-coconn;all}}}$. The construction of the latter follows the same idea: approximating Cartesian diagrams of k-Artin stacks by Cartesian diagrams of $(k-1)$-Artin stacks.

11.5.6. As in Sect. 11.4.4, the datum of $\operatorname{IndCoh}^{*!}_{(\operatorname{Stk}_{\text{laft},(k-1)\text{-Artin}})_{\text{ev-coconn;all}}}$ gives rise to the following construction:

For a pair of ∞-categories $\mathbf{I}'$ and $\mathbf{I}''$ and a functor

$$F : \mathbf{I}' \times \mathbf{I}'' \to \operatorname{Stk}_{\text{laft},(k-1)\text{-Artin}}$$

with the corresponding properties, we obtain a functor

$$\operatorname{IndCoh}^{*!}_{(\operatorname{Stk}_{\text{laft},(k-1)\text{-Artin}})_{\text{ev-coconn;all}}} \circ F : (\mathbf{I}' \times \mathbf{I}'')^{\text{op}} \to \operatorname{DGCat}_{\text{cont}}.$$

11.5.7. To carry out the induction step we need to construct, for every

$$[m] \times [n] \in \mathbf{\Delta} \times \mathbf{\Delta},$$

a map of ∞-groupoids

$$\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}}) \to \mathrm{Seg}^{m,n}((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}),$$

in a way functorial in $[m] \times [n]$.

We fix an object $\mathcal{Y}^{m,n} \in \mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}})$ and will construct the corresponding functor

$$\mathrm{IndCoh}^{*!}(\mathcal{Y}^{m,n}) : ([m] \times [n])^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

11.5.8. We introduce an ∞-category

$$_{\mathrm{all}}\,\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}}),$$

which has the same objects as $\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}})$, but where we now allow arbitrary maps between such diagrams.

Similarly, we introduce an ∞-category

$$_{\text{ev-coconn}}\,\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}}),$$

which has the same objects as $\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}})$, but where we allow arbitrary maps between diagrams that are eventually coconnective in each coordinate.

We let $\mathbf{J}'$ be the full subcategory in

$$\left(_{\mathrm{all}}\,\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}})\right)_{/\mathcal{Y}^{m,n}},$$

whose objects are diagrams with entries from $\mathrm{Stk}_{\mathrm{laft},(k-1)\text{-Artin}}$.

Similarly, we let $\mathbf{J}''$ be the full subcategory in

$$\left(_{\text{ev-coconn}}\,\mathrm{Cart}^{m,n}_{\text{ev-coconn;all}}(\mathrm{Stk}_{\mathrm{laft},k\text{-Artin}})\right)_{/\mathcal{Y}^{m,n}}$$

whose objects are diagrams with entries from $\mathrm{Stk}_{\mathrm{laft},(k-1)\text{-Artin}}$.

We set

$$\mathbf{I}' := \mathbf{J}' \times [m] \text{ and } \mathbf{I}'' := \mathbf{J}'' \times [n].$$

We define the functor

$$\mathbf{I}' \times \mathbf{I}'' \to \mathrm{Stk}_{\mathrm{laft},(k-1)\text{-Artin}}$$

by sending

$$(g : \mathcal{Z}'^{m,n} \to \mathcal{Y}^{m,n}) \in \mathbf{J}',\ (f : \mathcal{Z}''^{m,n} \to \mathcal{Y}^{m,n}) \in \mathbf{J}'',\ a \in [0,m],\ b \in [0,n]$$

to

$$\mathcal{Z}'^{m,n}_{a,b} \underset{\mathcal{Y}^{m,n}_{a,b}}{\times} \mathcal{Z}''^{m,n}_{a,b},$$

where the subscript "a,b" stands for the corresponding entry of $\mathcal{Z}'^{m,n}$, $\mathcal{Z}''^{m,n}$ and $\mathcal{Y}^{m,n}$, respectively.

Consider the corresponding functor

$$\mathrm{IndCoh}^{*!}_{(\mathrm{Stk}_{\mathrm{laft},(k-1)\text{-Artin}})_{\text{ev-coconn;all}}} \circ F : (\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Finally, we define the sought-for functor $\text{IndCoh}^{*!}(\mathcal{Y}^{m,n})$ as the right Kan extension of $\text{IndCoh}^{*!}_{(\text{Stk}_{\text{laft},(k-1)\text{-Artin}})_{\text{ev-coconn;all}}} \circ F$ along the projection

$$(\mathbf{I}' \times \mathbf{I}'')^{\text{op}} \to ([m] \times [n])^{\text{op}}.$$

11.5.9. To extend the assignment

$$\mathcal{Y}^{m,n} \rightsquigarrow \text{IndCoh}^{*!}(\mathcal{Y}^{m,n})$$

to a functor

$$\text{IndCoh}^{*!}_{(\text{Stk}_{\text{laft},k\text{-Artin}})_{\text{ev-coconn;all}}} : \text{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\text{Stk}_{\text{laft},(k-1)\text{-Artin}}) \to \text{Seg}^{\bullet,\bullet}((\text{DGCat}_{\text{cont}})^{\text{op}}),$$

we need to define the following data.

Let

$$\phi : ([m'] \times [n']) \to ([m] \times [n])$$

be a map in $\mathbf{\Delta} \times \mathbf{\Delta}$.

For $\mathcal{Y}^{m,n} \in \text{Cart}^{m,n}_{\text{ev-coconn;all}}(\text{Stk}_{\text{laft},k\text{-Artin}})$ as above, we denote by $\mathcal{Y}^{m',n'}$ the corresponding object of $\text{Cart}^{m',n'}_{\text{ev-coconn;all}}(\text{Stk}_{\text{laft},k\text{-Artin}})$ obtained by restriction.

We need to construct a canonical isomorphism between

$$\text{IndCoh}^{*!}(\mathcal{Y}^{m,n}) \circ (\phi)^{\text{op}} \text{ and } \text{IndCoh}^{*!}(\mathcal{Y}^{m',n'})$$

as functors $([m'] \times [n'])^{\text{op}} \rightrightarrows \text{DGCat}_{\text{cont}}$.

Furthermore, we need the above isomorphism to be homotopy coherent with respect to compositions of morphisms in $\mathbf{\Delta} \times \mathbf{\Delta}$.

The map

$$\text{IndCoh}^{*!}(\mathcal{Y}^{m,n}) \circ (\phi)^{\text{op}} \to \text{IndCoh}^{*!}(\mathcal{Y}^{m',n'})$$

follows from the universal property of right Kan extensions. The fact that it is an isomorphism is an easy cofinality argument.

11.6. **The *-pullback for eventually representable morphisms.** In this section we will further extend the assignment

$$\mathcal{Y} \to \text{IndCoh}(\mathcal{Y}), \quad \mathcal{Y} \in \text{PreStk}_{\text{laft}}$$

where we can !-pullback along any morphisms, and $*$-pullback along eventually representable eventually coconnective morphisms of prestacks.

11.6.1. Let

$$\text{ev-repr-coconn} \subset \text{all}$$

denote the class of eventually representable eventually coconnective morphisms in $\text{PreStk}_{\text{laft}}$.

Consider the corresponding object

$$\text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}}) \in \infty\text{-Grpd}^{(\mathbf{\Delta} \times \mathbf{\Delta})^{\text{op}}}.$$

We claim:

Proposition 11.6.2. *There exists a uniquely defined map in* ∞-$\text{Grpd}^{(\mathbf{\Delta}\times\mathbf{\Delta})^{\text{op}}}$

$$\text{IndCoh}^{*!}_{\text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}})} : \text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}}) \to \text{Seg}^{\bullet,\bullet}((\text{DGCat}_{\text{cont}})^{\text{op}})$$

that makes the following diagram commute

$$\begin{array}{ccc} \text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}}) & \xrightarrow{\text{IndCoh}^{*!}_{\text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}})}} & \text{Seg}^{\bullet,\bullet}((\text{DGCat}_{\text{cont}})^{\text{op}}) \\ \uparrow & & \uparrow \\ \pi_h(\text{Seg}^{\bullet}(\text{PreStk}_{\text{laft}}) & \xrightarrow[\text{Seg}^{\bullet}(\text{IndCoh}^!_{\text{PreStk}_{\text{laft}}})]{} & \pi_h(\text{Seg}^{\bullet}((\text{DGCat}_{\text{cont}})^{\text{op}})) \end{array}$$

and which extends the map

$$\text{IndCoh}^{*!}_{(\text{Stk}_{\text{laft,Artin}})_{\text{ev-coconn;all}}} : \text{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\text{Stk}_{\text{laft,Artin}}) \to \text{Seg}^{\bullet,\bullet}((\text{DGCat}_{\text{cont}})^{\text{op}}).$$

The rest of this subsection is devoted to a sketch of the proof of this proposition.

11.6.3. The idea of the proof is similar to that of Proposition 11.5.2.

First, from the map

$$\text{IndCoh}^{*!}_{(\text{Stk}_{\text{laft,Artin}})_{\text{ev-coconn;all}}} : \text{Cart}^{\bullet,\bullet}_{\text{ev-coconn;all}}(\text{Stk}_{\text{laft,Artin}}) \to \text{Seg}^{\bullet,\bullet}((\text{DGCat}_{\text{cont}})^{\text{op}})$$

we obtain that for a pair of ∞-categories $\mathbf{I}'$ and $\mathbf{I}''$ and a functor

$$F : \mathbf{I}' \times \mathbf{I}'' \to \text{Stk}_{\text{laft,Artin}}$$

with the corresponding properties, we can produce a functor

$$\text{IndCoh}^{*!}_{(\text{Stk}_{\text{laft,Artin}})_{\text{ev-coconn;all}}} \circ F : (\mathbf{I}' \times \mathbf{I}'')^{\text{op}} \to \text{DGCat}_{\text{cont}}.$$

11.6.4. To define the map $\text{IndCoh}^{*!}_{\text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}})}$, we need to construct, for every

$$[m] \times [n] \in \mathbf{\Delta} \times \mathbf{\Delta},$$

a map of ∞-groupoids

$$\text{Cart}^{m,n}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}}) \to \text{Seg}^{m,n}((\text{DGCat}_{\text{cont}})^{\text{op}}),$$

in a way functorial in $[m] \times [n]$.

We fix an object $\mathcal{Y}^{m,n} \in \text{Cart}^{\bullet,\bullet}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}})$ and will construct the corresponding functor

$$\text{IndCoh}^{*!}(\mathcal{Y}^{m,n}) : ([m] \times [n])^{\text{op}} \to \text{DGCat}_{\text{cont}}.$$

11.6.5. We introduce an ∞-category

$${}_{\text{all}}\,\text{Cart}^{m,n}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}}),$$

which has the same objects as $\text{Cart}^{m,n}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}})$, but where we now allow arbitrary maps between such diagrams.

We let $\mathbf{J}$ denote the full subcategory in

$$\left({}_{\text{all}}\,\text{Cart}^{m,n}_{\text{ev-repr-coconn;all}}(\text{PreStk}_{\text{laft}})\right)_{/\mathcal{Y}^{m,n}}$$

whose objects are diagrams with entries from $\text{Stk}_{\text{laft,Artin}}$.

We set

$$\mathbf{I}' := \mathbf{J} \times [m] \text{ and } \mathbf{I}'' := [n].$$

We define the functor

$$F : (\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}} \to \mathrm{Stk}_{\mathrm{laft,Artin}}$$

by sending

$$(g : \mathcal{Z}^{m,n} \to \mathcal{Y}^{m,n}) \in \mathbf{J},\ a \in [0,m],\ b \in [0,n]$$

to $\mathcal{Z}^{m,n}_{a,b}$.

Consider the resulting functor

$$\mathrm{IndCoh}^{*!}_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn;all}}} \circ F : (\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

We define the sought-for functor $\mathrm{IndCoh}^{*!}(\mathcal{Y}^{m,n})$ as the right Kan extension of the functor $\mathrm{IndCoh}^{*!}_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn;all}}} \circ F$ along the projection

$$(\mathbf{I}' \times \mathbf{I}'')^{\mathrm{op}} \to ([m] \times [n])^{\mathrm{op}}.$$

The functoriality of this construction in

$$([m] \times [n]) \in (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}}$$

is established in the same way as in Proposition 11.5.2.

11.7. **Properties of** IndCoh **of Artin stacks.** According to Proposition 11.5.2, if $\mathcal{Y}$ is an Artin stack locally almost of finite type, there exists a well-defined category $\mathrm{IndCoh}(\mathcal{Y})$, and pullback functors

$$f^! : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(S),$$

for $(f : S \to \mathcal{Y}) \in (\mathrm{DGSch}_{\mathrm{aft}})_{\mathcal{Y}}$, and also

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(S),$$

if f is eventually coconnective.

This will allow to define certain additional pieces of structure on $\mathrm{IndCoh}(\mathcal{Y})$, which are less obvious to see when we view $\mathrm{IndCoh}(\mathcal{Y})$ just as $\mathrm{IndCoh}^!(\mathcal{Y})$.

11.7.1. *The functor* Ψ. We claim that the functor $\mathrm{IndCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}}$ comes equipped with a natural transformation

$$\Psi^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}} : \mathrm{IndCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}} \to \mathrm{QCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}},$$

where $\mathrm{QCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}}$ denotes the restriction of $\mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{laft}}}$ to

$$((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \to (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

Indeed, this natural transformation is obtained as the right Kan extension of

$$\Psi^*_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}} : \mathrm{IndCoh}^*_{(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}} \to \mathrm{QCoh}^*_{(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}},$$

given by Corollary 3.5.6.

Above we have used the fact that the natural map

$$\mathrm{QCoh}^*_{(\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}} \to$$
$$\to \mathrm{RKE}_{(((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}))_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}} \hookrightarrow ((\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}})^{\mathrm{op}}}(\mathrm{QCoh}^*_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}})$$

is an isomorphism, which holds due to Proposition 11.2.2.

11.7.2. In plain terms, the existence of the above natural transformation means that for $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{laft,Artin}}$, there exists a canonically defined functor

$$\Psi_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}),$$

such that for a smooth map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ between Artin stacks we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y}_1) & \xrightarrow{\Psi_{\mathcal{Y}_1}} & \mathrm{QCoh}(\mathcal{Y}_1) \\ {\scriptstyle f^{\mathrm{IndCoh},*}}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{IndCoh}(\mathcal{Y}_2) & \xrightarrow{\Psi_{\mathcal{Y}_2}} & \mathrm{QCoh}(\mathcal{Y}_2). \end{array}$$

11.7.3. By a similar token, we claim that if $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{laft,Artin}}$ is eventually coconnective, i.e., n-coconnective as a stack for some n (see [GL:Stacks], Sect. 4.6.3 for the terminology), the functor $\Psi_{\mathcal{Y}}$ admits a left adjoint, denoted $\Xi_{\mathcal{Y}}$, such that for $S \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ endowed with an eventually coconnective map $f : S \to \mathcal{Y}$, the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & \xleftarrow{\Xi_S} & \mathrm{QCoh}(S) \\ {\scriptstyle f^{\mathrm{IndCoh},*}}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{IndCoh}(\mathcal{Y}) & \xleftarrow{\Xi_{\mathcal{Y}}} & \mathrm{QCoh}(\mathcal{Y}) \end{array}$$

commutes. Moreover, the functor $\Xi_{\mathcal{Y}}$ is fully faithful.

Indeed, we construct the corresponding natural transformation

$$\Xi^*_{({}^{\leq n}\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}} : \mathrm{QCoh}^*_{({}^{\leq n}\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}} \to \mathrm{IndCoh}^*_{({}^{\leq n}\mathrm{Stk}_{\mathrm{laft,Artin}})_{\mathrm{ev\text{-}coconn}}},$$

as the right Kan extension of the corresponding natural transformation

$$\Xi^*_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}} : \mathrm{QCoh}^*_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}} \to \mathrm{IndCoh}^*_{({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{ev\text{-}coconn}}},$$

the latter being given by Sect. 3.5.12.

The adjunction for $(\Xi_{\mathcal{Y}}, \Psi_{\mathcal{Y}})$ follows from Lemma 10.7.3.

In particular, we obtain that for $\mathcal{Y}$ eventually coconnective, the functor $\Psi_{\mathcal{Y}}$ realizes $\mathrm{QCoh}(\mathcal{Y})$ as a co-localization of $\mathrm{IndCoh}(\mathcal{Y})$.

11.7.4. *t-structure.* We now claim:

Proposition 11.7.5. *Let $\mathcal{Y}$ be an object of $\mathrm{Stk}_{\mathrm{laft,Artin}}$. Then the category $\mathrm{IndCoh}(\mathcal{Y})$ admits a unique t-structure, characterized by the property that for $S \in \mathrm{DGSch}_{\mathrm{aft}}$ with a flat map $f : S \to \mathcal{Y}$, the functor*

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(S)$$

is t-exact. Moreover:

(a) *The above t-structure is compatible with filtered colimits.*

(b) *The functor $\Psi_{\mathcal{Y}}$ is t-exact, induces an equivalence*

$$\mathrm{IndCoh}(\mathcal{Y})^+ \to \mathrm{QCoh}(\mathcal{Y})^+,$$

and identifies $\mathrm{QCoh}(\mathcal{Y})$ with the left completion of $\mathrm{IndCoh}(\mathcal{Y})$ in its t-structure.

Proof. The proof follows from Corollary 11.2.3 combined with Proposition 1.2.4, using the following general observation:

Let

$$A \to \mathrm{DGCat}_{\mathrm{cont}},\ \alpha \mapsto \mathbf{C}_\alpha$$

be a functor, and set

$$\mathbf{C} := \underset{\alpha \in A}{lim}\, \mathbf{C}_\alpha.$$

Assume now that each $\mathbf{C}_\alpha$ is endowed with a t-structure, and that for each arrow $\alpha_1 \to \alpha_2$ in A, the functor

$$\mathbf{C}_{\alpha_1} \to \mathbf{C}_{\alpha_2} \tag{11.2}$$

is t-exact.

Then the category $\mathbf{C}$ has a unique t-structure characterized by the property that the evaluation functors $\mathbf{C} \to \mathbf{C}_\alpha$ are t-exact.

Moreover, if the t-structure on each $\mathbf{C}_\alpha$ is compatible with filtered colimits, then the t-structure on $\mathbf{C}$ has the same property.

Point (b) of the proposition follows similarly.

□

References

[AG] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, arXiv:1201.6343.

[BFN] D. Benzvi, J. Francis and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909966.

[DrGa1] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, arXiv:1108.5351.

[DrGa2] V. Drinfeld and D. Gaitsgory, *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, arXiv:0805:1112.2402.

[FrGa] J. Francis and D. Gaitsgory, *Chiral Koszul Duality*, Selecta Math (New Series) **18** (2012), no. 1, 27-87.

[FrenGa] E. Frenkel and D. Gaitsgory, *D-modules on affine flag variety and representations of affine Kac-Moody algebras*, Jour. of Representation Theory **13** (2009), 470-608.

[GL:DG] Notes on Geometric Langlands, *Generalities on DG categories*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:Stacks] Notes on Geometric Langlands, *Stacks*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:QCoh] Notes on Geometric Langlands, *Quasi-coherent sheaves on stacks*, newline available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GR1] D. Gaitsgory and N. Rozenblyum, *DG Indschemes*, arXiv:1108.1738

[GR2] D. Gaitsgory and N. Rozenblyum, *Crystals and D-modules*, arXiv:1111.2087.

[GR3] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry*, in preparation, preliminary version will gradually become available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GoLi] T. G. Goodwillie and S. Lichtenbaum, *A cohomological bound for the h-topology*, American J. of Math, **123**, (2001), pp. 425-443

[Kr] H. Krause, *The stable derived category of a Noetherian scheme*, Compositio Mathematica **141**, (2005), 1128-1162.

[Lu0] J. Lurie, *Higher Topos Theory*, Princeton Univ. Press (2009).

[Ne] A. Neeman, *The Grothendieck duality theorem via Bouselds techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205236.

[TT] R. Thomason and T. Trobaugh, *Higher algebraic K -theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247435, Progr. Math., **88** (1990).