CHAPTER II.1. IND-COHERENT SHEAVES ON SCHEMES

Contents

Introduction

In this Chapter we initiate the study of ind-coherent sheaves, which is the main object of this book. Here we will define and study the category $\text{IndCoh}(X)$ for X being a scheme (assumed almost of finite type), and its basic functorality properties for maps of schemes.

In subsequent Chapters will be extend the definition IndCoh to a much wider class of algebro-geometric objects, namely, prestacks locally almost of finite type. The latter will allow us to create a paradigm that contains both D-modules and $\mathcal{O}$-modules.

Date: November 21, 2021.

0.1. **Why** IndCoh**?** The basic question is: why bother with IndCoh? I.e., why is the usual QCoh not good enough?

0.1.1. There are multiple reasons for why one would like to have the theory of IndCoh. Here are two mutually related reasons that can be spelled out already for schemes.

(i) For a proper morphism $f : X \to Y$ between schemes, the functor of !-pullback, *right adjoint* to the *-direct image, is *not necessarily continuous* when viewed as a functor

$$\mathrm{QCoh}(Y) \to \mathrm{QCoh}(X).$$

But it is continuous, when viewed as a functor $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$. Since for various reasons, explained elsewhere in the book, we wanted to stay within the world of cocomplete categories and continuous functors, the above phenomenon was, for us, the main reason to introduce and study IndCoh.

(ii) Many categories that naturally arise in geometric representation theory are $\mathrm{IndCoh}(X)$ (for some scheme X), and not $\mathrm{QCoh}(X)$. A remarkable set of examples of this are the categories appearing on the spectral side of the geometric Langlands theory (see, e.g., [Bezr] ot [AG]). A baby example of this would be the Koszul duality that says that the category A-mod for

$$A = k[\xi], \quad \deg(\xi) = 2$$

is equivalent to $\mathrm{IndCoh}(X)$, where $X = \mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt}$. This is while $\mathrm{QCoh}(X)$ is the subcategory of A-mod consisting of objects on which the generator ξ acts locally nilpotently.

0.1.2. We should emphasize, however, that one should not be tempted to think that IndCoh is a 'better object' than QCoh. In fact, both categories are needed and they interact in interesting ways, see [Chapter II.3].

0.1.3. We would also like to mention that the category $\mathrm{IndCoh}(X)$ has appeared significantly before the present book (and its predecessor [Ga1]). Namely, if X is classical, it was introduced in the work of H. Krause [Kr], and it was subsequently studied by him and his collaborators.

Specifically, in *loc.cit.*, $\mathrm{IndCoh}(X)$ appeared as the category of *injective complexes* on X.

0.1.4. In this chapter we start from (i) mentioned above, and develop the theory of IndCoh for schemes so that the !-pullback for a proper morphism is continuous.

In [Chapter II.2, Sect. 2.1] we will expand the functoriality of IndCoh by showing that it admits !-pullbacks for arbitrary (i.e., not necessarily proper) morphisms, and that these !-pullbacks satisfy *base change* against *-push forwards. The difficulty here is that base change is not a property but an extra piece of structure, and one needs to introduce a new categorical device, the *category of correspondences* to account for it[1].

[1] The idea of the category of correspondences was suggested to us by J. Lurie.

0.1.5. Having defined !-pullbacks for arbitrary morphisms, we will be able to define $\text{IndCoh}(\mathcal{X})$, where $\mathcal{X}$ is now an object of $\text{PreStk}_{\text{laft}}$, see [Chapter II.2, Sect. 3.4]. We should emphasize that, whereas in the case of schemes IndCoh, can be thought of as a small modification of QCoh, for general prestacks the two categories are very different. The former is functorial with respect to the !-pullback, and the latter is functorial with respect to the *-pullback.

For a map $f : \mathcal{X} \to \mathcal{Y}$ between prestacks we will have the !-pullback functor

$$f^! : \text{IndCoh}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{X}).$$

However, for a general f there is no conceivable way to define the *-push forward functor from $\text{IndCoh}(\mathcal{X})$ to $\text{IndCoh}(\mathcal{Y})$ so that it satisfies base change against the !-pullback.

That said, in Part III of the book we will single a class of morphsisms, called *inf-schematic*, for which the push-forward functor f_*^{IndCoh} is defined and has the desired base change property.

This will allow to extend the formalism of IndCoh as a functor out of the category of correspondences from schemes to *inf-schemes* (these are algebro-geometric objects that include formal schemes as well as de Rham prestacks of schemes). In this way we will obtain a convenient formalism that allows to treat D-modules and $\mathcal{O}$-modules on an equal footing.

0.2. **What is done in this chapter?**

0.2.1. In Sect. 1 we introduce $\text{IndCoh}(X)$ for a scheme X. We show that it is endowed with a t-structure and a t-exact functor

$$\Psi_X : \text{IndCoh}(X) \to \text{QCoh}(X),$$

which induces an equivalence on the eventually coconnective subcategories, i.e., the induced functor $\text{IndCoh}(X)^+ \to \text{QCoh}(X)^+$ is an equivalence.

Thus, $\text{IndCoh}(X)$ is only different from QCoh 'at $-\infty$'. So, one can say that the whole point here is *convergence*, i.e., convergence of spectral sequences.

0.2.2. In Sect. 2 we introduce the direct image functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(X) \to \text{IndCoh}(Y)$$

for a morphism $f : X \to Y$ between schemes.

This functor is 'inherited' from QCoh via the equivalence $\Psi : \text{IndCoh}(-)^+ \to \text{QCoh}(-)^+$.

We then extend the assignment

$$X \rightsquigarrow \text{IndCoh}(X), \quad f \rightsquigarrow f_*^{\text{IndCoh}}$$

to a functor

$$\text{Sch} \to \text{DGCat}_{\text{cont}}.$$

0.2.3. In Sect. 3 we study the functor of (the usual) *-pullback

$$f^{\text{IndCoh},*} : \text{IndCoh}(Y) \to \text{IndCoh}(X)$$

for a morphism $f : X \to Y$. This functor is supposed to be the *left* adjoint of f_*^{IndCoh}.

However, there is a caveat: the functor $f^{\text{IndCoh},*}$ is only defined for morphisms f that are *of finite Tor amplitude*. A functor that is defined for all morphisms is introduced in Sect. 5: this is the !-pullback.

It is fair to say that QCoh is well-adapted to the *-pullback and IndCoh is well-adapted to the !-pullback. (But if f is of finite Tor amplitude, both functors exist and are continuous for both categories.)

0.2.4. In Sect. 4 we study the behavior of IndCoh under open embeddings. In particular, we show that it satisfies Zariski descent.

0.2.5. Beyond the definition of IndCoh, Sect. 5 is the central in this chapter. In this section we show that if $f : X \to Y$ is a proper morphism, then the functor f_*^{IndCoh} admits a continuous right adjoint, denoted

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X).$$

We show that the !-pullback (so far only defined for proper maps) satisfies base change against the *-push forward (unless the general base change, this instance of base change is a *property* and *not an extra piece of structure*.

Finally, we establish the following crucial piece of compatibility that will eventually imply that the !-pullback is defined for all maps. Let

$$\begin{array}{ccc} X' & \xrightarrow{g_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g_Y} & Y \end{array}$$

be a Cartesian diagram, where the vertical arrows are proper and the horizontal ones are open embeddings.

In this case, we have a canonically defined natural transformation

$$g_X^{\mathrm{IndCoh},*} \circ f^! \to f'^! \circ g_Y^{\mathrm{IndCoh},*}.$$

We show that this natural transformation is an isomorphism.

0.2.6. In Sect. 6 we study several additional properties of the assignment

$$X \rightsquigarrow \mathrm{IndCoh}(X).$$

We show:

(i) For a closed subscheme $Y \subset X$, the subcategory $\mathrm{IndCoh}_Y(X)$ of $\mathrm{IndCoh}(X)$ consisting of objects that vanish when restricted to $X - Y$, is compactly generated by $\mathrm{Coh}_Y(X)$.

(ii) If $f : X \to Y$ is a proper and point-wise surjective map, then the functor $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ is conservative.

(iii) For two schemes X_1 and X_2, the external tensor product functor

$$\mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2) \to \mathrm{IndCoh}(X_1 \times X_2)$$

is an equivalence.

(iv) The assignment $X \rightsquigarrow \mathrm{IndCoh}(X)$ is *convergent* in the sense of [Chapter I.2, Sect. 1.4], i.e., the functor

$$\mathrm{IndCoh}(X) \to \lim_n \mathrm{IndCoh}({}^{\leq n}X)$$

is an equivalence, where ${}^{\leq n}X$ denotes the n-coconnective truncation of X. Note that the corresponding assertion is *false* for QCoh.

0.2.7. Finally, in Sect. 7 we establish the proper descent for IndCoh: if $X \to Y$ is a proper map, which is surjective at the level of k-points, then the functor

$$\mathrm{IndCoh}(Y) \to \mathrm{Tot}(\mathrm{IndCoh}(X^\bullet))$$

is an equivalence, where $X^\bullet$ is the simplicial scheme equal to the Čech nerve of $X \to Y$. In [Chapter II.2] we will strengthen this, and show that IndCoh satisfies h-descent (and in particular, ppf descent).

1. Ind-coherent sheaves on a scheme

In this section we introduce the category $\mathrm{IndCoh}(X)$ and study its basic properties. The material here repeats [Ga1, Sect. 1].

1.1. **Definition of the category.** In this subsection we define $\mathrm{IndCoh}(X)$ and the functors that connect it to the usual category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves.

1.1.1. For $X \in \mathrm{Sch}_{\mathrm{aft}}$ we consider the category $\mathrm{QCoh}(X)$ and its full (but not cocomplete) subcategory $\mathrm{Coh}(X)$, consisting of bounded complexes with coherent cohomologies.

We define the category $\mathrm{IndCoh}(X)$ by

$$\mathrm{IndCoh}(X) := \mathrm{Ind}(\mathrm{Coh}(X)).$$

1.1.2. By construction, we have a naturally defined functor

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$

obtained by ind-extension of the tautological inlcusion $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$.

We have:

Lemma 1.1.3. *Assume that X is a smooth classical scheme. Then Ψ_X is an equivalence.*

Proof. It is known by [TT] (see also [Ne]) that for X classical, $\mathrm{QCoh}(X) \simeq \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{perf}})$. Now, for X a regular classical scheme, we have

$$\mathrm{QCoh}(X)^{\mathrm{perf}} = \mathrm{Coh}(X),$$

as subcategories of $\mathrm{QCoh}(X)$. □

Remark 1.1.4. It is shown in [Ga1, Proposition 1.5.4] that the assertion of the above lemma is in fact 'if and only if'.

1.1.5. We give the following definition:

Definition 1.1.6. *We shall say that $X \in \mathrm{Sch}_{\mathrm{aft}}$ is* eventually coconnective *if the structure sheaf $\mathcal{O}_X$ belongs to* $\mathrm{Coh}(X)$.

I.e., X is eventually coconnective if, Zariski locally, the structure sheaf had non-zero cohomologies in finitely many degrees.

We have:

Lemma 1.1.7. *If X is eventually coconnective, the functor Ψ_X admits a left adjoint, to be denoted Ξ_X, and this left adjoint is fully faithful.*

Proof. If X is eventually coconnective, we have

$$\mathrm{QCoh}(X)^{\mathrm{perf}} \subset \mathrm{Coh}(X),$$

and, using the fact that $\mathrm{QCoh}(X) \simeq \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{perf}})$, the functor Ξ_X is obtained as the ind-extension of the above embedding.

The composition $\Psi_X \circ \Xi_X$ is the ind-extension of the functor

$$\mathrm{QCoh}(X)^{\mathrm{perf}} \hookrightarrow \mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X),$$

and it is manifest that its map to $\mathrm{Id}_{\mathrm{QCoh}(X)}$ is an isomorphism. □

Remark 1.1.8. In [Ga1, Proposition 1.5.2] it is shown that Ψ_X admits a left adjoint *if and only if* X is eventually coconnective.

1.2. **t-structure.** Some of the most basic operations on the IndCoh category (such as the functor of direct image studied in the next section) are inherited from those on QCoh using the t-structures on both categories. The crucial fact is that the *eventually coconnective* (a.k.a., bounded below) parts of the two categories are equivalent.

The goal of this subsection is to define the t-structure on $\mathrm{IndCoh}(X)$ and establish its basic properties.

1.2.1. We claim:

Proposition 1.2.2. *The category* $\mathrm{IndCoh}(X)$ *carries a unique t-structure that satisfies*

$$\mathrm{IndCoh}(X)^{\leq 0} = \{\mathcal{F} \in \mathrm{IndCoh}(X) \,|\, \Psi_X(\mathcal{F}) \in \mathrm{QCoh}(X)^{\leq 0}\}.$$

Moreover:

(a) *The functor* Ψ_X *is t-exact.*

(b) *This t-structure is compatible with filtered colimits (i.e., the subcategory* $\mathrm{IndCoh}(X)^{\geq 0}$ *is closed under filtered colimits).*

(c) *The induced functor*

$$\Psi_X : \mathrm{IndCoh}(X)^{\geq n} \to \mathrm{QCoh}(X)^{\geq n}$$

is an equivalence for any n.

As a corollary we obtain:

Corollary 1.2.3. *The functor* Ψ_X *defines an equivalence* $\mathrm{IndCoh}(X)^+ \to \mathrm{QCoh}(X)^+$.

Proof of Proposition 1.2.2. It is clear that the condition of the proposition determines the t-structure uniquely. To establish its properties we will use the following general assertion:

Lemma 1.2.4. *Let* $\mathbf{C}_0$ *be a (non-cocomplete) DG category, endowed with a t-structure. Then* $\mathbf{C} := \mathrm{Ind}(\mathbf{C}_0)$ *carries a unique t-structure, which is compatible with filtered colimits, and for which the tautological inclusion* $\mathbf{C}_0 \hookrightarrow \mathbf{C}$ *is t-exact. Moreover:*

(1) *The subcategory* $\mathbf{C}^{\leq 0}$ *(resp.,* $\mathbf{C}^{\geq 0}$*) is compactly generated under filtered colimits by* $\mathbf{C}_0^{\leq 0}$ *(resp.,* $\mathbf{C}_0^{\geq 0}$*).*

(2) *Let* $\mathbf{D}$ *be another DG category endowed with a t-structure which is compatible with filtered colimits, and let* $F : \mathbf{C} \to \mathbf{D}$ *a continuous functor. Then* F *is t-exact (resp., left t-exact, right t-exact) if and only if* $F|_{\mathbf{C}_0}$ *is.*

We apply Lemma 1.2.4(1) to $\mathbf{C}_0 = \mathrm{Coh}(X)$ and obtain a (a priori different) t-structure on $\mathrm{IndCoh}(X)$, which satisfies point (b) of the proposition. It also satisfies point (a) of the proposition, by Lemma 1.2.4(2) applied to $\mathbf{D} = \mathrm{QCoh}(X)$ and $F = \Psi_X$.

To show that this t-structure coincides with the one introduced earlier, it suffices to show that Ψ_X is conservative when restricted to $\mathrm{IndCoh}(X)^{\geq 0}$. Hence, it remains to show that the t-structure, given by Lemma 1.2.4, satisfies points (c) of the proposition.

To prove point (c), it is sufficient to consider the case of $n = 0$. Using Lemma 1.2.4(1), the required assertion follows from the next statement:

Lemma 1.2.5. *The category* $\mathrm{QCoh}(X)^{\geq 0}$ *is compactly generated under filtered colimits by* $\mathrm{Coh}(X)^{\geq 0}$.

□

1.2.6. Note that Proposition 1.2.2 implies that, as long as the functor Ψ_X is *not* an equivalence (i.e., X is not a classical smooth scheme), the category $\mathrm{IndCoh}(X)$ is *not* left-complete in its t-structure. The latter means that for $\mathcal{F} \in \mathrm{IndCoh}(X)$, the canonical arrow

$$\mathcal{F} \to \lim_{n} \tau^{\geq -n}(\mathcal{F})$$

is not necessarily an isomorphism.

Furthermore, we see that for any X, the functor Ψ_X realizes $\mathrm{QCoh}(X)$ as the left completion of $\mathrm{IndCoh}(X)$.

1.2.7. From Proposition 1.2.2 we also obtain the following:

Corollary 1.2.8. *The inclusion* $\mathrm{Coh}(X) \subset \mathrm{IndCoh}(X)^c$ *is an equality.*

Proof. Since the category $\mathrm{Coh}(X)$ compactly generates $\mathrm{IndCoh}(X)$, the category $\mathrm{IndCoh}(X)^c$ is the Karoubi-completion of $\mathrm{Coh}(X)$. I.e., every object $\mathcal{F} \in \mathrm{IndCoh}(X)^c$ can be realized as a direct summand of an object $\mathcal{F}' \in \mathrm{Coh}(X)$. In particular, $\mathcal{F} \in \mathrm{IndCoh}(X)^+$.

The object $\Psi_X(\mathcal{F})$ is a direct summand of $\Psi_X(\mathcal{F}')$. Hence, $\Psi_X(\mathcal{F})$, regarded as an object of $\mathrm{QCoh}(X)$, belongs to $\mathrm{Coh}(X)$. Let us denote this object of $\mathrm{Coh}(X)$ by $\widetilde{\mathcal{F}}$.

Thus, we can regard $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ as objects of $\mathrm{IndCoh}(X)^+$ such that

$$\Psi_X(\mathcal{F}) \simeq \Psi_X(\widetilde{\mathcal{F}}).$$

Applying Proposition 1.2.2(c), we obtain that $\mathcal{F} \simeq \widetilde{\mathcal{F}}$ as objects of $\mathrm{IndCoh}(X)$.

□

1.2.9. *The monoidal action of* QCoh. We claim:

Proposition 1.2.10. *There exists a uniquely defined monoidal action of* $\mathrm{QCoh}(X)$*, viewed as a monoidal category, on* $\mathrm{IndCoh}(X)$*, such that the functor* Ψ_X *is compatible with the* $\mathrm{QCoh}(X)$*-actions.*

Proof. The action in question is obtained by ind-extension of the action of the non-cocomplete monoidal category $\mathrm{QCoh}(X)^{\mathrm{perf}}$ on $\mathrm{Coh}(X)$.

To prove uniqueness, by Corollaries 1.2.3 and 1.2.8, it suffices to show that, given an action of $\mathrm{QCoh}(X)$ on $\mathrm{IndCoh}(X)$, the objects of $\mathrm{QCoh}(X)^{\mathrm{perf}} \subset \mathrm{QCoh}(X)$ map compact objects of $\mathrm{IndCoh}(X)$ to compact ones. However, this follows from the fact that objects in $\mathrm{QCoh}(X)^{\mathrm{perf}}$ are dualizable in the monoidal category $\mathrm{QCoh}(X)$.

□

2. The direct image functor

The assignment

$$X \rightsquigarrow \mathrm{IndCoh}(X)$$

is 'very functorial'. However, all of this functoriality is born from a single cource: the operation of direct image, defined in this section.

2.1. **Direct image for an individual morphism.** In this subsection we perform the first step in developing the formalism of direct image for IndCoh: we define the corresponding functor for one given morphism between schemes.

2.1.1. Let $f : X \to Y$ be a morphism in $\mathrm{Sch}_{\mathrm{aft}}$. We claim:

Proposition 2.1.2. *There exists a uniquely defined functor*

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

that is left t-exact and makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\Psi_X} & \mathrm{QCoh}(X) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \mathrm{QCoh}(Y) \end{array}$$

commute.

Proof. By continuity, the functor f_*^{IndCoh} is the ind-extension of its retstriction to

$$\mathrm{Coh}(X) \subset \mathrm{IndCoh}(X).$$

The commutative diagram in the proposition implies that

$$\Psi_Y \circ f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(X)} = f_*|_{\mathrm{Coh}(X)},$$

as functors $\mathrm{Coh}(X) \to \mathrm{QCoh}(Y)$. Furthermore, $f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(X)}$ is a functor that takes values in $\mathrm{QCoh}(Y)^+$.

Note that $\Psi_Y|_{\mathrm{IndCoh}(Y)^+}$ is invertible by Proposition 1.2.2(c). Hence, $f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(X)}$ is recovered as

$$(\Psi_Y|_{\mathrm{IndCoh}(Y)^+})^{-1} \circ (f_*|_{\mathrm{Coh}(X)}).$$

□

2.1.3. Recall (see [Chapter I.3, Sects. 3.5.1 and 3.7.4]) that for $X \in \mathrm{Sch}_{\mathrm{aft}}$, the monoidal category $\mathrm{QCoh}(X)$ is *rigid* (see [Chapter I.1, Sect. 9.1] for what this means). Hence, by [Chapter I.1, Lemma 9.3.6], for a morphism $f : X \to Y$, the functor

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$$

has a canonical structure of morphism in $\mathrm{QCoh}(Y)$-**mod**, where $\mathrm{QCoh}(Y)$ acts on $\mathrm{QCoh}(X)$ via f^*.

As in Proposition 2.1.2 and Proposition 1.2.10, one shows:

Proposition 2.1.4. *For a morphism $f : X \to Y$, the functor*

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

has a unique structure of 1*-morphism in* $\mathrm{QCoh}(Y)$-**mod** *which makes the square*

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\Psi_X} & \mathrm{QCoh}(X) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \mathrm{QCoh}(Y) \end{array}$$

commute in $\mathrm{QCoh}(Y)$-**mod**.

2.2. **Upgrading to a functor.** We now claim that the assigment

$$X \rightsquigarrow \mathrm{IndCoh}(X), \quad f \rightsquigarrow f_*^{\mathrm{IndCoh}}$$

upgrades to a functor

$$\mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{2.1}$$

to be denoted $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$.

Such an extension is not altogether automatic because we live in the world of higher categories. But constructing it will not be very difficult.

2.2.1. First, we consider the functor

$$\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained by restriction from the functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

og [Chapter I.3, Sect. 1.1.3].

Applying [Chapter I.1, Sect. 8.4.2], we obtain a functor

$$\mathrm{QCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained from $\mathrm{QCoh}^*_{\mathrm{Sch}_{\mathrm{aft}}}$ by passage to *right adjoints*.

2.2.2. Now, we claim:

Proposition 2.2.3. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

equipped with a natural transformation

$$\Psi_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{QCoh}_{\mathrm{Sch}_{\mathrm{aft}}},$$

which at the level of objects and 1*-morphisms is given by the assignment*

$$X \rightsquigarrow \mathrm{IndCoh}(X), \quad f \rightsquigarrow f_*^{\mathrm{IndCoh}}.$$

The rest of this subsection is devoted to the proof of this proposition.

2.2.4. Consider the following $(\infty,1)$-categories:

$$\mathrm{DGCat}_{+}^{\mathrm{cont}} \text{ and } \mathrm{DGCat}_{\mathrm{cont}}^{t}:$$

The category $\mathrm{DGCat}_{+}^{\mathrm{cont}}$ consists if of non-cocomplete DG categories $\mathbf{C}$, endowed with a t-structure, such that $\mathbf{C}=\mathbf{C}^{+}$. We also require that $\mathbf{C}^{\geq 0}$ contains filtered colimits and that the embedding $\mathbf{C}^{\geq 0}\hookrightarrow \mathbf{C}$ commutes with filtered colimits. As 1-morphisms we take those exact functors $F:\mathbf{C}_1\to\mathbf{C}_2$ that are *left t-exact up to a finite shift*, and such that $F|_{\mathbf{C}_1^{\geq 0}}$ commutes with filtered colimits. The higher categorical structure is uniquely determined by the requirement that the forgetful functor

$$\mathrm{DGCat}_{+}^{\mathrm{cont}}\to \mathrm{DGCat}$$

be 1-fully faithful (see [Chapter I.1, Sect. 1.2.4] for what this means).

The category $\mathrm{DGCat}_{\mathrm{cont}}^{t}$ consists of cocomplete DG categories $\mathbf{C}$, endowed with a t-structure, such that $\mathbf{C}^{\geq 0}$ is closed under filtered colimits, and such that $\mathbf{C}$ is compactly generated by objects from $\mathbf{C}^{+}$. As 1-morphisms we allow those exact functors $F:\mathbf{C}_1\to\mathbf{C}_2$ that are continuous and *left t-exact up to a finite shift*. The higher categorical structure is uniquely determined by the requirement that the forgetful functor

$$\mathrm{DGCat}_{\mathrm{cont}}^{t}\to \mathrm{DGCat}_{\mathrm{cont}}$$

be 1-fully faithful.

We have a naturally defined functor

$$\mathrm{DGCat}_{\mathrm{cont}}^{t}\to \mathrm{DGCat}_{+}^{\mathrm{cont}},\quad \mathbf{C}\mapsto\mathbf{C}^{+}. \tag{2.2}$$

Lemma 2.2.5. *The functor* (2.2) *is 1-fully faihful.*

2.2.6. We will use the following general assertion. Let $T:\mathbf{D}'\to\mathbf{D}$ be a functor between $(\infty,1)$-categories, which is 1-fully faithful. Let $\mathbf{I}$ be another $(\infty,1)$-category, and let

$$\mathbf{i}\rightsquigarrow F'(\mathbf{i}), \tag{2.3}$$

be an assignment, such that the assignment

$$\mathbf{i}\mapsto T\circ F'(\mathbf{i})$$

has been extended to a functor $F:\mathbf{I}\to\mathbf{D}$.

Lemma 2.2.7. *Suppose that for every $\alpha\in\mathrm{Maps}_{\mathbf{I}}(\mathbf{i}_1,\mathbf{i}_2)$, the point $F(\alpha)\in\mathrm{Maps}_{\mathbf{D}}(F(\mathbf{i}_1),F(\mathbf{i}_2))$ lies in the connected component corresponding to the image of*

$$\mathrm{Maps}_{\mathbf{D}'}(F'(\mathbf{i}_1),F'(\mathbf{i}_2))\to\mathrm{Maps}_{\mathbf{D}}(F(\mathbf{i}_1),F(\mathbf{i}_2)).$$

Then there exists a unique extension of (2.3) *to a functor $F':\mathbf{I}\to\mathbf{D}'$ equipped with an isomorphism $T\circ F'\simeq F$.*

Let now F_1' and F_2' be two assignments as in (2.3), satisfying the assumption of Lemma 2.2.7. Let us be given an assignment

$$\mathbf{i}\rightsquigarrow\psi'_{\mathbf{i}}\in\mathrm{Maps}_{\mathbf{D}'}(F_1'(\mathbf{i}),F_2'(\mathbf{i})). \tag{2.4}$$

Lemma 2.2.8. *Suppose that the assignment*

$$\mathbf{i}\rightsquigarrow T(\psi'_{\mathbf{i}})\in\mathrm{Maps}_{\mathbf{D}}(F_1(\mathbf{i}),F_2(\mathbf{i}))$$

has been extended to a natural transformation $\psi:F_1\to F_2$. Then there exists a unique extension of (2.4) *to a natural transformation $\psi:F_1'\to F_2'$ equipped with an isomorphism $T\circ\psi'\simeq\psi$.*

2.2.9. We are now ready to prove Proposition 2.2.3:

Step 1. We start with the functor

$$\mathrm{QCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and consider

$$\mathbf{I} = \mathrm{Sch}_{\mathrm{aft}},\ \mathbf{D} = \mathrm{DGCat}_{\mathrm{cont}},\ \mathbf{D}' := \mathrm{DGCat}^t_{\mathrm{cont}},\ F = \mathrm{QCoh}_{\mathrm{Sch}_{\mathrm{aft}}},$$

and the assignment

$$(X \in \mathrm{Sch}_{\mathrm{aft}}) \rightsquigarrow (\mathrm{QCoh}(X) \in \mathrm{DGCat}^t_{\mathrm{cont}}).$$

Apping Lemma 2.2.7, we obtain a functor

$$\mathrm{QCoh}^t_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}^t_{\mathrm{cont}}\,. \tag{2.5}$$

Step 2. Note that Proposition 2.1.2 defines a functor

$$\mathrm{IndCoh}^t_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}^t_{\mathrm{cont}},$$

and the natural transformation

$$\Psi^t_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{IndCoh}^t_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{QCoh}^t_{\mathrm{Sch}_{\mathrm{aft}}}$$

at the level of objects and 1*-morphisms.*

Since the functor $\mathrm{DGCat}^t_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$ is 1-fully faithful, by Lemmas 2.2.7 and 2.2.8, the existence and uniqueness of the pair $(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}, \Psi_{\mathrm{Sch}_{\mathrm{aft}}})$ with a fixed behavior on objects and 1-morphisms, is equivalent to that of $(\mathrm{IndCoh}^t_{\mathrm{Sch}_{\mathrm{aft}}}, \Psi^t_{\mathrm{Sch}_{\mathrm{aft}}})$.

Step 3. By Lemma 2.2.5, combined with Lemmas 2.2.7 and 2.2.8, we obtain that the existence and uniqueness of the pair $(\mathrm{IndCoh}^t_{\mathrm{Sch}_{\mathrm{aft}}}, \Psi^t_{\mathrm{Sch}_{\mathrm{aft}}})$, with a fixed behavior on objects and 1-morphisms is equivalent to the existence and uniqueness of the pair

$$(\mathrm{IndCoh}^+_{\mathrm{Sch}_{\mathrm{aft}}}, \Psi^+_{\mathrm{Sch}_{\mathrm{aft}}}),$$

obtained by composing with the functor (2.2).

The latter, however, is given by

$$\mathrm{IndCoh}^+_{\mathrm{Sch}_{\mathrm{aft}}} := \mathrm{QCoh}^+_{\mathrm{Sch}_{\mathrm{aft}}} \text{ and } \Psi^+_{\mathrm{Sch}_{\mathrm{aft}}} := \mathrm{Id}\,.$$

□

3. The functor of 'usual' inverse image

We now construct another piece of functoriality in the assignment $X \rightsquigarrow \mathrm{IndCoh}(X)$, namely, the functor of *-pullback.

Unlike the case of QCoh, its role in the theory is rather limited–a 'more important' functor is that of !-pullback. However, the *-pullback is a necessary step in the construction of the !-pullback, which is why we discuss it.

3.1. **Inverse image with respect to eventually coconnective morphisms.** Unlike the case of QCoh, the functor of *-pullback on IndCoh is not defined for all maps of schemes, but only for *eventually coconnective* ones. In this subsection we give the corresponding construction.

3.1.1. Let $f : X \to Y$ be a morphism in $\mathrm{Sch}_{\mathrm{aft}}$.

Definition 3.1.2. *We shall say that f is* eventually coconnective *if the functor*

$$f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$$

sends $\mathrm{Coh}(Y) \subset \mathrm{QCoh}(Y)$ *to* $\mathrm{QCoh}(X)^+$.

It is easy to see that if f is eventually coconnective, then it sends $\mathrm{Coh}(Y)$ to $\mathrm{Coh}(X)$: indeed, for any morphism f, the functor f^* sends objects of $\mathrm{QCoh}(Y)^-$ *with coherent cohomologies* to objects with a similar property on X.

In addition, we have the following fact, established in [Ga1, Lemma 3.4.2]:

Lemma 3.1.3. *The following conditions are equivalent:*

(a) *f is eventually coconnective;*

(b) *f is of finite Tor amplitude, i.e., is left t-exact up to a finite cohomological shift.*

Corollary 3.1.4. *The class of eventually coconnective morphisms is stable under base change.*

3.1.5. Let $f : X \to Y$ be eventually coconnective. Ind-extending the functor

$$f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$$

we obtain a functor

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X),$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\Psi_X} & \mathrm{QCoh}(X) \\ {\scriptstyle f^{\mathrm{IndCoh},*}}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \mathrm{QCoh}(Y) \end{array}$$

commute.

We have:

Proposition 3.1.6. *The functor $f^{\mathrm{IndCoh},*}$ is a left adjoint to f_*^{IndCoh}.*

Proof. It is sufficient to construct a functorial isomorphism

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}(f^{\mathrm{IndCoh},*}(\mathcal{F}_Y), \mathcal{F}_X) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(Y)}(\mathcal{F}_Y, f_*^{\mathrm{IndCoh}}(\mathcal{F}_X)) \tag{3.1}$$

for $\mathcal{F}_Y \in \mathrm{Coh}(Y)$ and $\mathcal{F}_X \in \mathrm{Coh}(X)$. However, by construction, the left-hand side in (3.1) is

$$\mathrm{Maps}_{\mathrm{Coh}(X)}(f^*(\mathcal{F}_Y), \mathcal{F}_X)) \simeq \mathrm{Maps}_{\mathrm{QCoh}(X)}(f^*(\mathcal{F}_Y), \mathcal{F}_X)),$$

while the right-hand side maps isomorphically by the functor Ψ_Y to

$$\mathrm{Maps}_{\mathrm{QCoh}(Y)}(\mathcal{F}_Y, f_*(\mathcal{F}_X)).$$

Now, (3.1) follows from the (f^*, f_*)-adjunction on QCoh.

□

Remark 3.1.7. It is shown in [Ga1], that the functor f_*^{IndCoh} admits a left adjoint *if and only if* the morphism f is eventually coconnective.

3.1.8. Note that by [Chapter I.1, Lemma 9.3.6], the functor $f^{\mathrm{IndCoh},*}$ carries a canonical structure of morphism in $\mathrm{QCoh}(Y)$-**mod**. It is easy to see that this is the same structure as obtained by ind-extending the structure of compatibility with the action of $\mathrm{QCoh}(Y)^{\mathrm{perf}}$ on

$$f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X).$$

3.1.9. Let $(\mathrm{Sch}_{\mathrm{aft}})_{\text{event-coconn}} \subset \mathrm{Sch}_{\mathrm{aft}}$ be the 1-full subcategory, where we restrict 1-morphisms to maps that are eventually coconnective.

By [Chapter I.1, Sect. 8.4.2], combining Propositions 2.2.3 and 3.1.6, we obtain:

Corollary 3.1.10. *The assignment*

$$X \rightsquigarrow \mathrm{IndCoh}(X), \quad f \rightsquigarrow f^{\mathrm{IndCoh},*}$$

canonically extends to a functor, to be denoted $\mathrm{IndCoh}^*_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{event-coconn}}}$,

$$((\mathrm{Sch}_{\mathrm{aft}})_{\text{event-coconn}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained from

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{event-coconn}}}$$

by adjunction.

3.2. **Base change for eventually coconnective morphisms.** An important property of the *-pullback (and one which is crucial for the construction of the !-pullback) is *base change.* It closely mimics the corresponding phenomenon for QCoh.

3.2.1. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{g_X} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g_Y} & Y_2 \end{array}$$

be a Cartesian diagram in $\mathrm{Sch}_{\mathrm{aft}}$.

Suppose that f_2 is eventually coconnective. By Corollary 3.1.4, the morphism f_1 is also eventually coconnective. Then the isomorphism of functors

$$(g_Y)^{\mathrm{IndCoh}}_* \circ (f_1)^{\mathrm{IndCoh}}_* \simeq (f_2)^{\mathrm{IndCoh}}_* \circ (g_X)^{\mathrm{IndCoh}}_*$$

gives rise to a natural transformation.

$$(f_2)^{\mathrm{IndCoh},*} \circ (g_Y)^{\mathrm{IndCoh}}_* \to (g_X)^{\mathrm{IndCoh}}_* \circ (f_1)^{\mathrm{IndCoh},*}. \tag{3.2}$$

Proposition 3.2.2. *The map* (3.2) *is an isomorphism.*

Proof. It is enough to show that

$$(f_2)^{\mathrm{IndCoh},*} \circ (g_Y)^{\mathrm{IndCoh}}_*(\mathcal{F}) \to (g_X)^{\mathrm{IndCoh}}_* \circ (f_1)^{\mathrm{IndCoh},*}(\mathcal{F}) \tag{3.3}$$

is an isomorphism for $\mathcal{F} \in \mathrm{Coh}(Y_1)$.

In this case both sides of (3.3) belong to $\mathrm{IndCoh}(X_2)^+$. By Proposition 1.2.2, it is therefore sufficient to show that the map

$$\Psi_{X_2} \circ (f_2)^{\mathrm{IndCoh},*} \circ (g_Y)^{\mathrm{IndCoh}}_* \to \Psi_{X_2} \circ (g_X)^{\mathrm{IndCoh}}_* \circ (f_1)^{\mathrm{IndCoh},*} \tag{3.4}$$

is an isomorphism.

We have:

$$\Psi_{X_2} \circ (f_2)^{\mathrm{IndCoh},*} \circ (g_Y)^{\mathrm{IndCoh}}_* \simeq (f_2)^* \circ \Psi_{Y_2} \circ (g_Y)^{\mathrm{IndCoh}}_* \simeq (f_2)^* \circ (g_Y)_* \circ \Psi_{Y_1}$$

and

$$\Psi_{X_2} \circ (g_X)^{\mathrm{IndCoh}}_* \circ (f_1)^{\mathrm{IndCoh},*} \simeq (g_X)_* \circ \Psi_{X_1} \circ (f_1)^{\mathrm{IndCoh},*} \simeq (g_X)_* \circ (f_1)^* \circ \Psi_{Y_1}.$$

Now, it follows from the construction of the $(f^{\mathrm{IndCoh},*}, f_*^{\mathrm{IndCoh}})$-adjunction, that the map in (3.4) corresponds to the map

$$(f_2)^* \circ (g_Y)_* \circ \Psi_{Y_1} \to (g_X)_* \circ (f_1)^* \circ \Psi_{Y_1},$$

obtained from the (f^*, f_*)-adjunction.

Hence, (3.4) is an isomorphism by base change for QCoh.

□

3.3. **Tensoring up.** In this section we study the following question: given a map $f : X \to Y$, how closely can we approximate $\mathrm{IndCoh}(X)$ from knowing $\mathrm{IndCoh}(Y)$ and the QCoh categories on both schemes.

In the process we will come across several *convergence-type* assertions, that are of significant technical importance: some maps that are isomorphisms in QCoh are much less obviously so in the IndCoh context.

3.3.1. Let $f : X \to Y$ be an eventually coconnective map. Regarding the functor $f^{\mathrm{IndCoh},*}$ as a map in $\mathrm{QCoh}(Y)$-**mod**, we obtain a functor

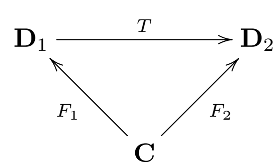
$$(\mathrm{Id}_{\mathrm{QCoh}(X)} \otimes f^{\mathrm{IndCoh},*}) : \mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X). \tag{3.5}$$

We claim:

Proposition 3.3.2. *The functor* (3.5) *is fully faithful.*

Remark 3.3.3. It is shown in [Ga1, Proposition 4.4.9] that the functor (3.5) is an equivalence when f is smooth. In Proposition 4.1.6, we will prove this in the case when f is an open embedding.

3.3.4. Note that for a diagram of $(\infty, 1)$-categories



if the functors F_1 and F_2 admit right adjoints, we have a natural transformation of the resulting endo-functors of $\mathbf{C}$:

$$F_1^R \circ F_1 \to F_2^R \circ F_2.$$

Furthermore, if T is fully faithful, then the above natural transformation is an isomorphism.

From here we obtain that the functor $(\mathrm{Id}_{\mathrm{QCoh}(X)} \otimes f^{\mathrm{IndCoh},*})$ gives rise to a map of endo-functors of $\mathrm{IndCoh}(Y)$:

$$(f_*(\mathcal{O}_X) \otimes -) \simeq ((f_* \circ f^*) \otimes \mathrm{Id}_{\mathrm{IndCoh}(Y)}) \to f_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh},*}, \tag{3.6}$$

where $f_*(\mathcal{O}_X) \otimes -$ denotes the functor of action of $f_*(\mathcal{O}_X) \in \mathrm{QCoh}(Y)$ on $\mathrm{IndCoh}(Y)$.

Thus, from Proposition 3.3.2 we obtain:

Corollary 3.3.5. *The map* (3.6) *is an isomorphism.*

3.3.6. The rest of the subsection is devoted to the proof of Proposition 3.3.2. We note that the left-hand side in (3.5) is compactly generated by objects of the form

$$\mathcal{E}_X \otimes \mathcal{F}_Y \in \mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y),$$

where $\mathcal{E}_X \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ and $\mathcal{F}_Y \in \mathrm{Coh}(Y)$. Moreover, the functor $(\mathrm{Id}_{\mathrm{QCoh}(X)} \otimes f^{\mathrm{IndCoh},*})$ sends these objects to compact objects in $\mathrm{IndCoh}(X)$.

Hence, it is enough to show that for $\mathcal{E}^1_X, \mathcal{E}^2_X$ and $\mathcal{F}^1_Y, \mathcal{F}^2_Y$ as above, the map

$$\text{(3.7)} \quad \mathrm{Maps}_{\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y)}(\mathcal{E}^1_X \otimes \mathcal{F}^1_Y, \mathcal{E}^2_X \otimes \mathcal{F}^2_Y) \to$$
$$\to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{E}^1_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}^1), \mathcal{E}^2_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}^2))$$

is an isomorphism, where in the right-hand side $\otimes$ denotes the action of QCoh on IndCoh.

We can rewrite the map in (3.7) as

$$\text{(3.8)} \quad \mathrm{Maps}_{\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y)}(\mathcal{O}_X \otimes \mathcal{F}^1_Y, \mathcal{E}_X \otimes \mathcal{F}^2_Y) \to$$
$$\to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{O}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}^1), \mathcal{E}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}^2)),$$

where $\mathcal{E}_X \simeq \mathcal{E}^2_X \otimes (\mathcal{E}^1_X)^\vee$.

Furthermore, we rewrite the map in (3.8) as

$$\mathrm{Maps}_{\mathrm{IndCoh}(Y)}(\mathcal{F}^1_Y, f_*(\mathcal{E}) \otimes \mathcal{F}^2_Y) \to \mathrm{Maps}_{\mathrm{IndCoh}(Y)}(\mathcal{F}^1_Y, f^{\mathrm{IndCoh}}_*(\mathcal{E} \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}^2_Y))).$$

I.e., we are reduced to showing that the following version of the projection formula:

Proposition 3.3.7. *For an eventally coconnective map, the natural transformation between the functors*

$$\mathrm{QCoh}(X) \times \mathrm{IndCoh}(Y) \rightrightarrows \mathrm{IndCoh}(Y),$$

that sends $\mathcal{E}_X \in \mathrm{QCoh}(X)$ *and* $\mathcal{F}_Y \in \mathrm{IndCoh}(\mathcal{Y})$ *to the map*

$$\text{(3.9)} \qquad f_*(\mathcal{E}_X) \otimes \mathcal{F}_Y \to f^{\mathrm{IndCoh}}_*(\mathcal{E}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y)),$$

is an isomorphism.

Remark 3.3.8. Note that there is another kind of projection formula, that encodes the compatibility of f^{IndCoh}_* with the monoidal action of $\mathrm{QCoh}(Y)$, and which holds tautologically for any morphism f, see Proposition 2.1.4:

For $\mathcal{E}_Y \in \mathrm{QCoh}(Y)$ and $\mathcal{F}_X \in \mathrm{IndCoh}(X)$ we have:

$$f^{\mathrm{IndCoh}}_*(f^*(\mathcal{E}_Y) \otimes \mathcal{F}_X) \simeq \mathcal{E}_Y \otimes f^{\mathrm{IndCoh}}_*(\mathcal{F}_X).$$

3.3.9. *Proof of Proposition 3.3.7.* It is enough to prove the isomorphism (3.9) holds for $\mathcal{E}_X \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ and $\mathcal{F}_Y \in \mathrm{Coh}(Y)$.

We also note that the map (3.9) becomes an isomorphism after applying the functor Ψ_Y, by the usual projection formula for QCoh. For $\mathcal{E}_X \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ and $\mathcal{F}_Y \in \mathrm{Coh}(Y)$ we have

$$f^{\mathrm{IndCoh}}_*(\mathcal{E}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y)) \in \mathrm{IndCoh}(Y)^+.$$

Hence, by Proposition 1.2.2, it suffices to show that in this case

$$f_*(\mathcal{E}_X) \otimes \mathcal{F}_Y \in \mathrm{IndCoh}(Y)^+.$$

We note that the object $f_*(\mathcal{E}_X) \in \mathrm{QCoh}(Y)^b$ is of bounded Tor dimension. The required fact follows from the next general observation:

Lemma 3.3.10. *For $X \in \mathrm{Sch}_{\mathrm{aft}}$ and $\mathcal{E} \in \mathrm{QCoh}(X)^b$, whose Tor dimension is bounded on the left by an integer n, the functor*

$$\mathcal{E} \otimes - : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$$

has a cohomological amplitude bounded on the left by n.

□

3.3.11. *Proof of Lemma 3.3.10.* We need to show that the functor $\mathcal{E} \otimes -$ sends $\mathrm{IndCoh}(X)^{\geq 0}$ to $\mathrm{IndCoh}(X)^{\geq -n}$. By Lemma 1.2.4(1), it is sufficient to show that this functor sends $\mathrm{Coh}(X)^{\geq 0}$ to $\mathrm{IndCoh}(X)^{\geq -n}$. By cohomological devissage, the latter is equivalent to sending $\mathrm{Coh}(X)^{\heartsuit}$ to $\mathrm{IndCoh}(X)^{\geq -n}$.

Let i denote the closed embedding ${}^{\mathrm{cl}}X =: X' \to X$. The functor i_*^{IndCoh} induces an equivalence $\mathrm{Coh}(X')^{\heartsuit} \to \mathrm{Coh}(X)^{\heartsuit}$. So, it is enough to show that for $\mathcal{F}' \in \mathrm{Coh}(X')^{\heartsuit}$, we have

$$\mathcal{E} \otimes i_*^{\mathrm{IndCoh}}(\mathcal{F}') \in \mathrm{IndCoh}(X)^{\geq -n}.$$

We have:

$$\mathcal{E} \otimes i_*^{\mathrm{IndCoh}}(\mathcal{F}') \simeq i_*^{\mathrm{IndCoh}}(i^*(\mathcal{E}) \otimes \mathcal{F}').$$

Note that the functor i_*^{IndCoh} is t-exact (since i_* is), and $i^*(\mathcal{E})$ has Tor dimension bounded by the same integer n.

This reduces the assertion of the lemma to the case when X is classical. Further, by Proposition 4.2.4 (which will be proved independently later), the statement is Zariski local, so we can assume that X is affine.

In the latter case, the assumption on $\mathcal{E}$ implies that it can be represented by a complex of flat $\mathcal{O}_X$-modules that lives in the cohomological degrees $\geq -n$. This reduces the assertion further to the case when $\mathcal{E}$ is a flat $\mathcal{O}_X$-module in degree 0. In this case we claim that the functor

$$\mathcal{E} \otimes - : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$$

is t-exact.

The latter follows from Lazard's lemma: such an $\mathcal{E}$ is a filtered colimit of locally free $\mathcal{O}_X$-modules $\mathcal{E}'$, while for each such $\mathcal{E}'$, the functor $\mathcal{E}' \otimes - : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$ is by definition the ind-extension of the functor

$$\mathcal{E}' \otimes - : \mathrm{Coh}(X) \to \mathrm{Coh}(X),$$

and the latter is t-exact.

□

4. Open embeddings

The behavior of direct and inverse image functors for open embeddings is, obviously, an important piece of information about IndCoh.

4.1. **Restriction to an open.** In this subsection we show that the behavior of IndCoh with respect to open embeddings is 'exactly the same' as that of QCoh.

4.1.1. Let now $j : \overset{\circ}{X} \hookrightarrow X$ be an open embedding. We claim:

Proposition 4.1.2. *The functor* $j_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\overset{\circ}{X}) \to \mathrm{IndCoh}(X)$ *is fully faithful.*

Proof. We need to show that the co-unit of the adjunction

$$j^{\mathrm{IndCoh},*} \circ j_*^{\mathrm{IndCoh}} \to \mathrm{Id}_{\mathrm{IndCoh}(\overset{\circ}{X})}$$

is an isomorphism.

Since the functors in question are continuous, it is enough to check that

$$j^{\mathrm{IndCoh},*} \circ j_*^{\mathrm{IndCoh}}(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism for $\mathcal{F} \in \mathrm{Coh}(X)$. However, in this case both $j^{\mathrm{IndCoh},*} \circ j_*^{\mathrm{IndCoh}}(\mathcal{F})$ and $\mathcal{F}$ belong to $\mathrm{IndCoh}(X)^+$, so by Proposition 1.2.2, it is sufficient to check that

$$\Psi_{\overset{\circ}{X}} \circ j^{\mathrm{IndCoh},*} \circ j_*^{\mathrm{IndCoh}} \to \Psi_{\overset{\circ}{X}}$$

is an isomorphism.

However,

$$\Psi_{\overset{\circ}{X}} \circ j^{\mathrm{IndCoh},*} \circ j_*^{\mathrm{IndCoh}} \simeq j^* \circ \Psi_X \circ j_*^{\mathrm{IndCoh}} \simeq j^* \circ j_* \circ \Psi_{\overset{\circ}{X}},$$

and it follows from the construction of the $(f^{\mathrm{IndCoh},*}, f_*^{\mathrm{IndCoh}})$-adjunction that the resulting map

$$j^* \circ j_* \circ \Psi_{\overset{\circ}{X}} \to \Psi_{\overset{\circ}{X}}$$

comes from the co-unit of the (j^*, j_*)-adjunction. Therefore, it is an isomorphism, as

$$j^* \circ j_* \to \mathrm{Id}_{\mathrm{QCoh}(X)}$$

is an isomorphism.

□

4.1.3. The next assertion follows immediately from Lemma 1.2.4:

Lemma 4.1.4. *For an open embedding j, the functor $j^{\mathrm{IndCoh},*}$ is t-exact.*

4.1.5. Finally, let us recall the functor (3.5). We claim:

Proposition 4.1.6. *Assume that f is an open embedding. Then the functor* (3.5) *is an equivalence.*

Proof. We already know that the functor in question is fully faithful. Hence, it remains to show that its essential image generates the target category. But this follows from Proposition 4.1.2. □

4.2. **Zariski descent.** In this subsection we will show that IndCoh can be glued locally from a Zariski cover, in a way completely parallel to QCoh.

4.2.1. Let now $f : U \to X$ be a Zariski cover, i.e., U is the disjoint union of open subschemes of X, whose union is all of X. Let $U^\bullet$ be the Čech nerve of f. The functors of $(\text{IndCoh}, *)$-pullback define a cosimplicial category

$$\text{IndCoh}(U^\bullet),$$

which is augmented by $\text{IndCoh}(X)$.

We claim:

Proposition 4.2.2. *The functor*

$$\text{IndCoh}(X) \to \text{Tot}(\text{IndCoh}(U^\bullet))$$

is an equivalence.

Proof. The usual argument reduces the assertion of the proposition to the following. Let $X = U_1 \cup U_2$; $U_{12} = U_1 \cap U_2$. Let

$$U_1 \overset{j_1}{\hookrightarrow} X,\ U_2 \overset{j_2}{\hookrightarrow} X,\ U_{12} \overset{j_{12}}{\hookrightarrow} X,\ U_{12} \overset{j_{12,1}}{\hookrightarrow} U_1,\ U_{12} \overset{j_{12,2}}{\hookrightarrow} U_2$$

denote the corresponding open embeddings.

We need to show that the functor

$$\text{IndCoh}(X) \to \text{IndCoh}(U_1) \underset{\text{IndCoh}(U_{12})}{\times} \text{IndCoh}(U_1)$$

that sends $\mathcal{F} \in \text{IndCoh}(X)$ to the datum of

$$\{j_1^{\text{IndCoh},*}(\mathcal{F}_1), j_2^{\text{IndCoh},*}(\mathcal{F}_2), j_{12,1}^{\text{IndCoh},*}(j_1^{\text{IndCoh},*}(\mathcal{F})) \simeq j_{12}^{\text{IndCoh},*}(\mathcal{F}) \simeq j_{12,2}^{\text{IndCoh},*}(j_2^{\text{IndCoh},*}(\mathcal{F}))\}$$

is an equivalence.

We construct a right adjoint functor

$$\text{IndCoh}(U_1) \underset{\text{IndCoh}(U_{12})}{\times} \text{IndCoh}(U_1) \to \text{IndCoh}(X)$$

by sending

$$\{\mathcal{F}_1 \in \text{IndCoh}(U_1), \mathcal{F}_2 \in \text{IndCoh}(U_2), \mathcal{F}_{12} \in \text{IndCoh}(U_{12}), j_{12,1}^{\text{IndCoh},*}(\mathcal{F}_1) \simeq \mathcal{F}_{12} \simeq j_{12,2}^{\text{IndCoh},*}(\mathcal{F}_2)\}$$

to

$$\ker\Big(\big((j_1)_*^{\text{IndCoh}}(\mathcal{F}_1) \oplus (j_2)_*^{\text{IndCoh}}(\mathcal{F}_1)\big) \to (j_{12})_*^{\text{IndCoh}}(\mathcal{F}_{12})\Big),$$

where the maps $(j_i)_*^{\text{IndCoh}}(\mathcal{F}_i) \to (j_{12})_*^{\text{IndCoh}}(\mathcal{F}_{12})$ are

$$(j_i)_*^{\text{IndCoh}}(\mathcal{F}_i) \to (j_i)_*^{\text{IndCoh}} \circ (j_{12,i})_*^{\text{IndCoh}} \circ (j_{12,i})^{\text{IndCoh},*}(\mathcal{F}_i) =$$
$$= (j_{12})_*^{\text{IndCoh}} \circ (j_{12,i})^{\text{IndCoh},*}(\mathcal{F}_i) \simeq (j_{12})_*^{\text{IndCoh}}(\mathcal{F}_{12}).$$

It is strightforward to see from Propositions 4.1.2 and 3.2.2 that the composition

$$\text{IndCoh}(U_1) \underset{\text{IndCoh}(U_{12})}{\times} \text{IndCoh}(U_1) \to \text{IndCoh}(X) \to \text{IndCoh}(U_1) \underset{\text{IndCoh}(U_{12})}{\times} \text{IndCoh}(U_1)$$

is canonically isomorphic to the identity functor.

To prove that the composition

$$\text{IndCoh}(X) \to \text{IndCoh}(U_1) \underset{\text{IndCoh}(U_{12})}{\times} \text{IndCoh}(U_1) \to \text{IndCoh}(X)$$

is also isomorphic to the identity functor, it is sufficient to show that for $\mathcal{F} \in \mathrm{IndCoh}(X)$, the canonical map from it to

$$\ker\Big(\big((j_1)_*^{\mathrm{IndCoh}} \circ (j_1)^{\mathrm{IndCoh},*}(\mathcal{F}) \oplus (j_2)_*^{\mathrm{IndCoh}} \circ (j_2)^{\mathrm{IndCoh},*}(\mathcal{F})\big) \to \to (j_{12})_*^{\mathrm{IndCoh}} \circ j_{12}^{\mathrm{IndCoh},*}(\mathcal{F}) \Big) \tag{4.1}$$

is an isomorphism.

Since all functors in question are continuous, it is sufficient to do so for $\mathcal{F} \in \mathrm{Coh}(X)$. In this case, both sides of (4.1) belong to $\mathrm{IndCoh}(X)^+$. So, it is enough to prove that the map in question becomes an isomorphism after applying the functor Ψ_X. However, in this case we are dealing with the map

$$\Psi_X(\mathcal{F}) \to \ker\Big(((j_1)_* \circ (j_1)^*(\Psi_X(\mathcal{F})) \oplus (j_2)_* \circ (j_2)^*(\Psi_X(\mathcal{F}))) \to (j_{12})_* \circ j_{12}^*(\Psi_X(\mathcal{F})) \Big),$$

which is known to be an isomorphism.

□

4.2.3. We also have

Proposition 4.2.4. *Let $f : U \to X$ be a Zariski cover. Then $\mathcal{F} \in \mathrm{IndCoh}(X)$ belongs to $\mathrm{IndCoh}(X)^{\leq 0}$ (resp., $\mathrm{IndCoh}(X)^{\geq 0}$) if and only if $f^{\mathrm{IndCoh},*}(\mathcal{F})$ does.*

Proof. The 'only if' direction for both statetements follows from Lemma 4.1.4.

For the 'if' direction, assuming that $f^*(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\leq 0}$, it is sufficient to show that $\Psi_X(\mathcal{F}) \in \mathrm{QCoh}(X)^{\leq 0}$, and the assertion follows from the corresponding assertion for QCoh.

If $f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\geq 0}$, the assertion follows from the construction of the inverse functor

$$\mathrm{Tot}(\mathrm{IndCoh}(U^\bullet)) \to \mathrm{IndCoh}(X).$$

□

5. Proper maps

If until now the theory of IndCoh has run in parallel to (and was inherited from that of) QCoh, in this section we will come across to the main point of difference between the two: the functor of !-pullback, studied in this section.

5.1. **The !-pullback.** In this subsection we introduce the functor of !-pullback for proper maps. Its extension for arbitrary maps between schemes is the subject of [Chapter II.2, Sect. 3].

5.1.1. Let $f : X \to Y$ be a map in $\mathrm{Sch}_{\mathrm{aft}}$. We recall the following definition:

Definition 5.1.2. *The map f is said to be proper (resp., closed embedding) if the corresponding map ${}^{\mathrm{cl}}X \to {}^{\mathrm{cl}}Y$ has this property.*

5.1.3. Let $f : X \to Y$ be a proper map. We claim:

Lemma 5.1.4. *The functor*

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

sends $\mathrm{Coh}(X) \subset \mathrm{IndCoh}(X)$ *to* $\mathrm{Coh}(Y) \subset \mathrm{IndCoh}(Y)$.

Proof. By the construction of f_*^{IndCoh}, it is sufficient to show that the functor

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$$

$\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ to $\mathrm{Coh}(Y) \subset \mathrm{QCoh}(Y)$.

First, we note that the assertion holds when f is a closed embedding.

In general, by the devissage with respect to the t-structure, it is sufficient to show that for $\mathcal{F} \in \mathrm{Coh}(X)^\heartsuit$, we have

$$f_*(\mathcal{F}) \in \mathrm{Coh}(Y).$$

Let i denote the canonical closed embedding ${}^{\mathrm{cl}}X \hookrightarrow X$. The functor i_* is an equivalence $\mathrm{Coh}({}^{\mathrm{cl}}X)^\heartsuit \to \mathrm{Coh}(X)^\heartsuit$. Hence, $\mathcal{F} = i_*(\mathcal{F}')$ for $\mathcal{F}' \in \mathrm{Coh}({}^{\mathrm{cl}}X)$. This reduces the assertion to the case when X is classical.

We factor the map $f : X \to Y$ as $X \to {}^{\mathrm{cl}}Y \overset{i}{\hookrightarrow} Y$. Since i is a closed embedding, we have reduced the assertion to the case when Y is classical as well. In the latter case, the assertion is well-known.

□

5.1.5. The above lemma implies that the functor f_*^{IndCoh} sends $\mathrm{IndCoh}(X)^c$ to $\mathrm{IndCoh}(Y)^c$.

Hence, f_*^{IndCoh} admits a continuous right adjoint, to be denoted

$$f^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y).$$

Remark 5.1.6. The continuity of the functor $f^!$ is the *raison d'etre* of the category IndCoh, and its main difference from QCoh.

5.1.7. By [Chapter I.1, Lemma 9.3.6], we obtain that the functor $f^!$ has a natural structure of 1-morphism in $\mathrm{QCoh}(Y)$-**mod**.

5.1.8. Note that the functor f_*^{IndCoh} is *right t-exact, up to a finite shift.* Hence, the functor $f^!$ is left t-exact up to a finite shift. In particular, $f^!$ maps $\mathrm{IndCoh}(Y)^+$ to $\mathrm{IndCoh}(X)^+$.

Let $f^{\mathrm{QCoh},!}$ denote the *not necessarily continuous* right adjoint to f_*. It also has the property that it maps $\mathrm{QCoh}(Y)^+$ to $\mathrm{QCoh}(X)^+$.

Lemma 5.1.9. *The diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}(X)^+ & \xleftarrow{f^!} & \mathrm{IndCoh}(Y)^+ \\ \Psi_X \downarrow & & \downarrow \Psi_Y \\ \mathrm{QCoh}(X)^+ & \xleftarrow{f^{\mathrm{QCoh},!}} & \mathrm{QCoh}(Y)^+ \end{array}$$

obtained by passing to right adjoints along the horizontal arrows in

$$\begin{array}{ccc} \mathrm{IndCoh}(X)^+ & \xrightarrow{f_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(Y)^+ \\ \Psi_X \downarrow & & \downarrow \Psi_Y \\ \mathrm{QCoh}(X)^+ & \xrightarrow{f_*} & \mathrm{QCoh}(Y)^+, \end{array}$$

commutes.

Proof. Follows from the fact that the vertical arrows are equivalences, by Proposition 1.2.2. □

Remark 5.1.10. It is not in general true that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xleftarrow{f^!} & \mathrm{IndCoh}(Y) \\ {\scriptstyle \Psi_X}\downarrow & & \downarrow{\scriptstyle \Psi_Y} \\ \mathrm{QCoh}(X) & \xleftarrow{f^{\mathrm{QCoh},!}} & \mathrm{QCoh}(Y) \end{array}$$

obtained by passing to right adjoints along the horizontal arrows in

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{f_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(Y) \\ {\scriptstyle \Psi_X}\downarrow & & \downarrow{\scriptstyle \Psi_Y} \\ \mathrm{QCoh}(X) & \xrightarrow{f_*} & \mathrm{QCoh}(Y), \end{array}$$

commutes.

For example, take $X = \mathrm{pt} = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(k[t]/t^2)$ and $0 \neq \mathcal{F} \in \mathrm{IndCoh}(Y)$ be in the kernel of the functor Ψ_Y. Then $\Psi_X \circ f^!(\mathcal{F}) \neq 0$. Indeed, Ψ_X is an equivalence, and $f^!$ is conservative, see Corollary 6.1.5.

5.1.11. Let $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}$ be a 1-full subcategory of $\mathrm{Sch}_{\mathrm{aft}}$ when we restrict 1-morphisms to be proper maps. By [Chapter I.1, Sect. 8.4.2], we obtain:

Corollary 5.1.12. *There exists a canonically defined functor*

$$\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} : ((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained from

$$\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} := \mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})}|_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}}}$$

by passing to right adjoints.

5.2. **Base change for proper maps.** A crucial property of the !-pullback is *base change* against the *-direct image. We establish it in this subsection.

5.2.1. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{g_X} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g_Y} & Y_2 \end{array}$$

be a Cartesian diagram in $\mathrm{Sch}_{\mathrm{aft}}$, with the vertical maps being proper.

The isomorphism of functors

$$(f_2)_*^{\mathrm{IndCoh}} \circ (g_X)_*^{\mathrm{IndCoh}} \simeq (g_Y)_*^{\mathrm{IndCoh}} \circ (f_1)_*^{\mathrm{IndCoh}}$$

gives rise to a natural transformation:

$$(g_X)_*^{\mathrm{IndCoh}} \circ f_1^! \to f_2^! \circ (g_Y)_*^{\mathrm{IndCoh}}. \tag{5.1}$$

We will prove:

Proposition 5.2.2. *The map* (5.1) *is an isomorphism.*

Proof. Since all functors involved are continuous, it is enough to show that the map

$$(g_X)_*^{\mathrm{IndCoh}} \circ f_1^!(\mathcal{F}) \to f_2^! \circ (g_Y)_*^{\mathrm{IndCoh}}(\mathcal{F})$$

is an isomorphism for $\mathcal{F} \in \mathrm{Coh}(Y_1)$. Hence, it is enough to show that (5.1) is an isomorphism when restricted to $\mathrm{IndCoh}(Y_1)^+$.

By Lemma 5.1.9 and Proposition 1.2.2, this reduces the assertion to showing that the natural transformation

$$(g_X)_* \circ f_1^{\mathrm{QCoh},!} \to f_2^{\mathrm{QCoh},!} \circ (g_Y)_* \tag{5.2}$$

is an isomorphism for the functors

$$\begin{array}{ccc} \mathrm{QCoh}(X_1)^+ & \xrightarrow{(g_X)_*} & \mathrm{QCoh}(X_2)^+ \\ {\scriptstyle f_1^{\mathrm{QCoh},!}}\uparrow & & \uparrow{\scriptstyle f_2^{\mathrm{QCoh},!}} \\ \mathrm{QCoh}(Y_1)^+ & \xrightarrow{(g_Y)_*} & \mathrm{QCoh}(Y_2)^+, \end{array}$$

where the natural transformation comes from the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X_1)^+ & \xrightarrow{(g_X)_*} & \mathrm{QCoh}(X_2)^+ \\ {\scriptstyle (f_1)_*}\downarrow & & \downarrow{\scriptstyle (f_2)_*} \\ \mathrm{QCoh}(Y_1)^+ & \xrightarrow{(g_Y)_*} & \mathrm{QCoh}(Y_2)^+ \end{array}$$

by passing to right adjoint along the vertical arrows.

We consider the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X_1) & \xrightarrow{(g_X)_*} & \mathrm{QCoh}(X_2) \\ {\scriptstyle (f_1)_*}\downarrow & & \downarrow{\scriptstyle (f_2)_*} \\ \mathrm{QCoh}(Y_1) & \xrightarrow{(g_Y)_*} & \mathrm{QCoh}(Y_2), \end{array}$$

and the the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X_1) & \xrightarrow{(g_X)_*} & \mathrm{QCoh}(X_2) \\ {\scriptstyle f_1^{\mathrm{QCoh},!}}\uparrow & & \uparrow{\scriptstyle f_2^{\mathrm{QCoh},!}} \\ \mathrm{QCoh}(Y_1) & \xrightarrow{(g_Y)_*} & \mathrm{QCoh}(Y_2), \end{array}$$

obtained by passing to right adjoints along the vertical arrows. (Note, however, that the functors involved are no longer continuous).

We claim that the resulting natural transformation

$$(g_X)_* \circ f_1^{\mathrm{QCoh},!} \to f_2^{\mathrm{QCoh},!} \circ (g_Y)_* \tag{5.3}$$

between the functors

$$\mathrm{QCoh}(Y_1) \rightrightarrows \mathrm{QCoh}(X_2)$$

is an isomorphism. This would imply that (5.2) is an isomorphism by restricting to the eventually coconnective subcategory.

To prove that (5.3) is an isomorphism, we note that this map is obtained by passing to right adjoints in the natural transformation

$$(g_Y)^* \circ (f_2)_* \to (f_1)_* \circ (g_X)^* \tag{5.4}$$

as functors

$$\mathrm{QCoh}(X_2) \rightrightarrows \mathrm{QCoh}(Y_1)$$

in the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X_1) & \xleftarrow{(g_X)^*} & \mathrm{QCoh}(X_2) \\ {\scriptstyle (f_1)_*}\downarrow & & \downarrow{\scriptstyle (f_2)_*} \\ \mathrm{QCoh}(Y_1) & \xleftarrow{(g_Y)^*} & \mathrm{QCoh}(Y_2). \end{array}$$

Now, (5.4) is an isomorphism by the usual base change for QCoh. Hence, (5.3) is an isomorphism as well.

□

5.3. **Pullback compatibilty.** The !-pullback for *arbitrary* maps between schemes will be defined in such a way that it is the !-pullback for proper morphisms, and the *-pullback for open embeddings.

Hence, if we want that !-pullback to be well-defined, a certain compatibility must take place, when we decompose a morphism in two different ways as a composition of a proper morphism and an open embedding. A basic case of such compatibility is established in this subsection.

5.3.1. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{g_X} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g_Y} & Y_2 \end{array}$$

be a Cartesian diagram in $\mathrm{Sch}_{\mathrm{aft}}$, with the vertical maps being proper, and horizontal maps being eventually coconnective.

We start with the base change isomorphism

$$(f_1)_*^{\mathrm{IndCoh}} \circ g_X^{\mathrm{IndCoh},*} \simeq g_Y^{\mathrm{IndCoh},*} \circ (f_2)_*^{\mathrm{IndCoh}}$$

of Proposition 3.2.2, and by the $(f_*^{\mathrm{IndCoh}}, f^!)$-adjunction obtain a map

$$g_X^{\mathrm{IndCoh},*} \circ f_2^! \to f_1^! \circ g_Y^{\mathrm{IndCoh},*}. \tag{5.5}$$

Remark 5.3.2. Note that one can get another map

$$g_X^{\mathrm{IndCoh},*} \circ f_2^! \to f_1^! \circ g_Y^{\mathrm{IndCoh},*}, \tag{5.6}$$

namely, via the $(g^{\mathrm{IndCoh},*}, g_*^{\mathrm{IndCoh}})$-adjunction from the isomorphism

$$f_2^! \circ (g_Y)_*^{\mathrm{IndCoh}} \simeq (g_X)_*^{\mathrm{IndCoh}} \circ f_1^!$$

of Proposition 5.2.2. A diagram chase shows that the map (5.6) is canonically the same as (5.5).

5.3.3. We are going to prove:

Proposition 5.3.4. *Suppose that g_Y (and hence g_X) are open embeddings. Then the map* (5.5) *is an isomorphism.*

Remark 5.3.5. It is shown in [Ga1, Proposition 7.1.6] that the map (5.5) is an isomorphism for any eventually coconnective g_Y.

Proof. By Proposition 4.1.2, it suffices to show that the induced map

$$(g_X)_*^{\mathrm{IndCoh}} \circ g_X^{\mathrm{IndCoh},*} \circ f_2^! \to (g_X)_*^{\mathrm{IndCoh}} \circ f_1^! \circ g_Y^{\mathrm{IndCoh},*}$$

is an isomorphism.

Using Proposition 5.2.2, we have

$$(g_X)_*^{\mathrm{IndCoh}} \circ f_1^! \circ g_Y^{\mathrm{IndCoh},*} \simeq f_2^! \circ (g_Y)_*^{\mathrm{IndCoh}} \circ g_Y^{\mathrm{IndCoh},*}.$$

Hence, we need to show that the map

$$(g_X)_*^{\mathrm{IndCoh}} \circ g_X^{\mathrm{IndCoh},*} \circ f_2^! \to f_2^! \circ (g_Y)_*^{\mathrm{IndCoh}} \circ g_Y^{\mathrm{IndCoh},*} \tag{5.7}$$

is an isomorphism.

By Corollary 3.3.5, for $\mathcal{F} \in \mathrm{IndCoh}(Y_2)$, we have canonical isomorphisms

$$(g_X)_*^{\mathrm{IndCoh}} \circ g_X^{\mathrm{IndCoh},*} \circ f_2^!(\mathcal{F}) \simeq (g_X)_*(\mathcal{O}_{X_1}) \otimes f_2^!(\mathcal{F})$$

and

$$(g_Y)_*^{\mathrm{IndCoh}} \circ g_Y^{\mathrm{IndCoh},*}(\mathcal{F}) \simeq (g_Y)_*(\mathcal{O}_{Y_1}) \otimes \mathcal{F},$$

where $\otimes$ denotes the action of QCoh on IndCoh.

By the compatibilty of the action of QCoh with the !-pullback, we have

$$f_2^! \circ (g_Y)_*^{\mathrm{IndCoh}} \circ g_Y^{\mathrm{IndCoh},*}(\mathcal{F}) \simeq f_2^*((g_Y)_*(\mathcal{O}_{Y_1})) \otimes f_2^!(\mathcal{F}) \simeq (g_X)_*(\mathcal{O}_{X_1}) \otimes f_2^!(\mathcal{F}).$$

Now, diagram chase shows that, under the above identifications, the map (5.7) is the identify map endomorphism of $(g_X)_*(\mathcal{O}_{X_1}) \otimes f_2^!(\mathcal{F})$.

□

6. Closed embeddings

The behavior of IndCoh with respect to closed embedding is 'better' than that of QCoh. The main point of difference is that the direct image functor under a closed embedding for IndCoh preserves compactness, which is not the case for QCoh.

6.1. **Category with support.** In this subsection we study the full subcategory of $\mathrm{IndCoh}(X)$ corresponding with objects 'with support' on a given closed subscheme.

6.1.1. Let X be an object of $\mathrm{Sch}_{\mathrm{aft}}$, and let $i : Z \to X$ be a closed embedding. Let $j : U \to X$ be the complementary open.

We let $\mathrm{IndCoh}(X)_Z$ denote the full subcategory of $\mathrm{IndCoh}(X)$ equal to

$$\ker\Big(j^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(U)\Big).$$

Note that the embedding

$$\mathrm{IndCoh}(X)_Z \hookrightarrow \mathrm{IndCoh}(X)$$

admits a right adjoint given by sending $\mathcal{F}$ to

$$\ker(\mathcal{F} \to j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*}(\mathcal{F})),$$

which by Corollary 3.3.5 is the same as

$$\ker(\mathcal{O}_X \to j_*(\mathcal{O}_U)) \otimes \mathcal{F}.$$

6.1.2. We claim:

Proposition 6.1.3.
(a) *The subcategory* $\mathrm{IndCoh}(X)_Z \subset \mathrm{IndCoh}(X)$ *is compatible with the t-structure (i.e., is preserved by the truncation functors).*
(b) *The subcategory* $\mathrm{IndCoh}(X)_Z \subset \mathrm{IndCoh}(X)$ *is generated by the essential image of the functor* $i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(X)$.
(c) *The functor* $i^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Z)$ *is conservative, when restricted to* $\mathrm{IndCoh}(X)_Z$.
(d) *The category* $\mathrm{IndCoh}(X)_Z$ *identifies with the ind-completion of*

$$\mathrm{Coh}(X)_Z := \ker\Big(j^* : \mathrm{Coh}(X) \to \mathrm{Coh}(U)\Big).$$

Proof. Point (a) follows from the fact that the functor $j^{\mathrm{IndCoh},*}$ is t-exact.

Let us prove point (b). The category $\mathrm{IndCoh}(X)$ is generated by $\mathrm{IndCoh}(X)^+$. The description of the right adjoint to $\mathrm{IndCoh}(X)_Z \hookrightarrow \mathrm{IndCoh}(Z)$ implies that the same is true for $\mathrm{IndCoh}(X)_Z$.

For every $\mathcal{F} \in \mathrm{IndCoh}(X)^+$, the map

$$\underset{n}{\mathrm{colim}}\, \tau^{\leq n}(\mathcal{F}) \to \mathcal{F},$$

is an isomorphism, by Proposition 1.2.2 and because the corresponding fact is true in QCoh^+.

Hence, by point (a), every $\mathcal{F} \in \mathrm{IndCoh}(X)_Z^+$ is a colimit of cohomologically bounded objects from $\mathrm{IndCoh}(X)_Z$. This implies that $\mathrm{IndCoh}(X)_Z$ is generated by

$$\mathrm{Coh}(X)_Z^\heartsuit \subset \mathrm{IndCoh}(X)_Z^\heartsuit.$$

Now, it is easy to see that every object $\mathcal{F} \in \mathrm{Coh}(X)_Z^\heartsuit$ has a filtration

$$\mathcal{F} = \underset{n}{\cup} \mathcal{F}_n$$

with $\mathcal{F}_n/\mathcal{F}_{n-1} \in i_*(\mathrm{Coh}(Z)^\heartsuit)$. This proves point (b).

Point (c) follows from point (b) by adjunction.

To prove point (d), it suffices to note that the objects from $\mathrm{Coh}(X)_Z$ are compact in $\mathrm{IndCoh}(X)_Z$ (because they are compact in $\mathrm{IndCoh}(X)$) and that they generate $\mathrm{IndCoh}(X)_Z$, by point (b).

□

6.1.4. In what follows we will use the following terminology: for a (derived) scheme X, we will denote by ${}^{\mathrm{red}}X$ the classical *reduced* scheme, underlying reduced scheme of the classical scheme ${}^{\mathrm{cl}}X$.

We shall say that a map of (derived) schemes $f : X' \to X$ is a *nil-isomorphism*, i.e., a map such that ${}^{\mathrm{red}}X' \to {}^{\mathrm{red}}X$ is an isomorphism.

As a corollary of Proposition 6.1.3 we obtain:

Corollary 6.1.5. *Let $f : X' \to X$ be a nil-isomorphism, i.e., a map such that ${}^{\mathrm{red}}X' \to {}^{\mathrm{red}}X$ is an isomorphism. Then the functor $f^!$ is conservative. Equivalently, the essential image of* $\mathrm{IndCoh}(X')$ *under* f_*^{IndCoh} *generates* $\mathrm{IndCoh}(X)$.

Proof. We can assume that $X' = {}^{\mathrm{red}}X$, so f is also a closed embedding. In this case the assertion of the corollary follows from Proposition 6.1.3(b). □

6.2. **A conservativeness result for proper maps.** The main result established in this subsection, Proposition 6.2.2, is of technical significance.

6.2.1. We shall now use Proposition 6.1.3 to prove the following:

Proposition 6.2.2. *Let $f : X \to Y$ be a proper map, which is surjective at the level of geometric points. Then the functor $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ is conservative.*

The rest of this subsection is devoted to the proof of the proposition.

6.2.3. By Corollary 6.1.5 we can assume that both X and Y are classical and reduced. We argue by Noetherian induction, assuming that the statement is true for all proper closed subschemes of Y. We need to show that the essential image of $\mathrm{IndCoh}(X)$ under f_*^{IndCoh} generates $\mathrm{IndCoh}(Y)$.

By Proposition 6.1.3(a), it is sufficient to show that Y contains an open subscheme $\overset{\circ}{Y} \subset Y$ such that for $\overset{\circ}{X} := f^{-1}(\overset{\circ}{Y})$, the essential image of $\mathrm{IndCoh}(\overset{\circ}{X})$ under $(f|_{\overset{\circ}{X}})_*^{\mathrm{IndCoh}}$ generates $\mathrm{IndCoh}(\overset{\circ}{Y})$.

Since Y is classical and reduced, it contains a non-empty open smooth subscheme, which we take to be $\overset{\circ}{Y}$. By Lemma 1.1.3 and Lemma 1.1.7, we are reduced to showing the following:

Lemma 6.2.4. *Let $f : X \to Y$ be a proper surjective morphism of classical schemes with Y smooth. Then the essential image of* $\mathrm{QCoh}(X)$ *under f_* generates* $\mathrm{QCoh}(Y)$.

□

Proof of Lemma 6.2.4. Let $\mathbf{C} \subset \mathrm{QCoh}(Y)$ be the full subcategory generated by the essential image of $\mathrm{QCoh}(X)$ under f_*. Note that $\mathbf{C}$ is a monoidal ideal, since the functor f_* respects the action of $\mathrm{QCoh}(Y)$. Consider $\mathcal{E} := f_*(\mathcal{O}_X) \in \mathbf{C}$. This is an object of $\mathrm{QCoh}(Y)$, whose fiber at every geometric point of Y is non-zero.

However, it is easy to see that for any Noetherian classical scheme Y and $\mathcal{E} \in \mathbf{C} \subset \mathrm{QCoh}(Y)$ with the above properties, we have

$$\mathbf{C} = \mathrm{QCoh}(Y).$$

□

6.3. **Products.** It is known that for a pair of quasi-compact schemes X_1 and X_2, tensor product defines an equivalence of DG categories

$$\mathrm{QCoh}(X_1) \otimes \mathrm{QCoh}(X_2) \to \mathrm{QCoh}(X_1 \times X_2).$$

In this subsection we will establish a similar assertion for IndCoh. This is not altogether tautological; for example the validity of this fact replies on the assumption that the ground field k be perfect.

6.3.1. Let X_1 and X_2 be two objects of $\mathrm{Sch}_{\mathrm{aft}}$. We claim:

Lemma 6.3.2. *There exists a uniquely defined functor*

$$\boxtimes : \mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2) \to \mathrm{IndCoh}(X_1 \times X_2) \tag{6.1}$$

that preserves compactness and makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2) & \xrightarrow{\boxtimes} & \mathrm{IndCoh}(X_1 \times X_2) \\ {\scriptstyle \Psi_{X_1} \times \Psi_{X_2}} \downarrow & & \downarrow {\scriptstyle \Psi_{X_1 \times X_2}} \\ \mathrm{QCoh}(X_1) \otimes \mathrm{QCoh}(X_2) & \xrightarrow{\boxtimes} & \mathrm{QCoh}(X_1 \times X_2) \end{array}$$

commute.

Proof. The anticlock-wise composition sends the compact generators of the category

$$\mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2)$$

(i.e., objects of the form $\mathcal{F}_1 \otimes \mathcal{F}_2$ for $\mathcal{F}_i \in \mathrm{Coh}(X_i)$) to $\mathrm{QCoh}(X_1 \times X_2)^+$. Hence, it sends all of $(\mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2))^c$ to $\mathrm{QCoh}(X_1 \times X_2)^+$.

The sought-for functor is the ind-extension of

$$(\mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2))^c \to \mathrm{QCoh}(X_1 \times X_2)^+ \overset{\Psi_{X_1 \times X_2}}{\simeq} \mathrm{IndCoh}(X_1 \times X_2)^+.$$

This functor preserves compactness since the objects

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \mathrm{QCoh}(X_1 \times X_2)^+, \quad \mathcal{F}_i \in \mathrm{Coh}(X_i)$$

belong to $\mathrm{Coh}(X_1 \times X_2)$.

□

6.3.3. We now claim:

Proposition 6.3.4.

(a) *The functor* (6.1) *is fully faithful.*

(b) *If the ground field k is perfect, then* (6.1) *is an equivalence.*

Proof. Since the functor (6.1) preserves compactness, for point (a) it is sufficient to show that for $\mathcal{F}'_i, \mathcal{F}''_i \in \mathrm{Coh}(X_i)$, $i = 1, 2$, the map

$$\mathcal{M}aps_{\mathrm{IndCoh}(X_1) \otimes \mathrm{IndCoh}(X_2)}(\mathcal{F}'_1 \otimes \mathcal{F}'_2, \mathcal{F}''_1 \otimes \mathcal{F}''_2) \to \mathcal{M}aps_{\mathrm{IndCoh}(X_1 \times X_2)}(\mathcal{F}'_1 \boxtimes \mathcal{F}'_2, \mathcal{F}''_1 \boxtimes \mathcal{F}''_2)$$

is an isomorphism.

We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}aps_{\mathrm{IndCoh}(X_1)\otimes\mathrm{IndCoh}(X_2)}(\mathcal{F}'_1\otimes\mathcal{F}'_2,\mathcal{F}''_1\otimes\mathcal{F}''_2) & \longrightarrow & \mathcal{M}aps_{\mathrm{IndCoh}(X_1\times X_2)}(\mathcal{F}'_1\boxtimes\mathcal{F}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2) \\
\sim\uparrow & & \\
\mathcal{M}aps_{\mathrm{Coh}(X_1)}(\mathcal{F}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{Coh}(X_2)}(\mathcal{F}'_2,\mathcal{F}''_2) & & \downarrow\sim \\
\sim\downarrow & & \\
\mathcal{M}aps_{\mathrm{QCoh}(X_1)}(\mathcal{F}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{QCoh}(X_2)}(\mathcal{F}'_2,\mathcal{F}''_2) & \longrightarrow & \mathcal{M}aps_{\mathrm{QCoh}(X_1\times X_2)}(\mathcal{F}'_1\boxtimes\mathcal{F}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2).
\end{array}$$

Hence, it remains to show that

$$\mathcal{M}aps_{\mathrm{QCoh}(X_1)}(\mathcal{F}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{QCoh}(X_2)}(\mathcal{F}'_2,\mathcal{F}''_2)\to\mathcal{M}aps_{\mathrm{QCoh}(X_1\times X_2)}(\mathcal{F}'_1\boxtimes\mathcal{F}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)$$

is an isomorphism. This is not immediate since the objects $\mathcal{F}'_i\in\mathrm{QCoh}(S_i)$ are not compact. To circumvent this, we proceed as follows.

It is enough to show that

$$\tau^{\leq n}\left(\mathcal{M}aps_{\mathrm{QCoh}(X_1)}(\mathcal{F}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{QCoh}(X_2)}(\mathcal{F}'_2,\mathcal{F}''_2)\right)\to$$
$$\to\tau^{\leq n}\left(\mathcal{M}aps_{\mathrm{QCoh}(X_1\times X_2)}(\mathcal{F}'_1\boxtimes\mathcal{F}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)\right)$$

is an isomorphism for any fixed n.

Choose $\alpha_1:\widetilde{\mathcal{F}}'_1\to\mathcal{F}'_1$ (resp., $\alpha_2:\widetilde{\mathcal{F}}'_2\to\mathcal{F}'_2$) with $\widetilde{\mathcal{F}}'_1$ (resp., $\widetilde{\mathcal{F}}'_2$) in $\mathrm{QCoh}(X_1)^{\mathrm{perf}}$ (resp., $\mathrm{QCoh}(X_2)^{\mathrm{perf}}$), such that

$$\mathrm{Cone}(\alpha_1)\in\mathrm{QCoh}(X_1)^{\leq -N}\text{ and }\mathrm{Cone}(\alpha_2)\in\mathrm{QCoh}(X_2)^{\leq -N}$$

for $N\gg 0$.

By choosing N large enough, we can ensure that

$$\tau^{\leq m}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_i)}(\mathcal{F}'_i,\mathcal{F}''_i)\right)\to\tau^{\leq m}\left(\mathcal{M}aps_{\mathrm{QCoh}(S_i)}(\widetilde{\mathcal{F}}'_i,\mathcal{F}''_i)\right)$$

is an isomorphism for a given integer m. This implies that for $N\gg 0$ and our fixed n, the maps

$$\tau^{\leq n}\left(\mathcal{M}aps_{\mathrm{QCoh}(X_1)}(\mathcal{F}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{QCoh}(X_2)}(\mathcal{F}'_2,\mathcal{F}''_2)\right)\to$$
$$\to\tau^{\leq n}\left(\mathcal{M}aps_{\mathrm{QCoh}(X_1)}(\widetilde{\mathcal{F}}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{QCoh}(X_2)}(\widetilde{\mathcal{F}}'_2,\mathcal{F}''_2)\right)$$

and

$$\tau^{\leq n}\left(\mathcal{M}aps_{\mathrm{QCoh}(X_1\times X_2)}(\mathcal{F}'_1\boxtimes\mathcal{F}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)\right)\to\tau^{\leq n}\left(\mathcal{M}aps_{\mathrm{QCoh}(X_1\times X_2)}(\widetilde{\mathcal{F}}'_1\boxtimes\widetilde{\mathcal{F}}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)\right)$$

are isomorphisms.

Hence, it is enough to show that

$$\mathcal{M}aps_{\mathrm{QCoh}(X_1)}(\widetilde{\mathcal{F}}'_1,\mathcal{F}''_1)\otimes\mathcal{M}aps_{\mathrm{QCoh}(X_2)}(\widetilde{\mathcal{F}}'_2,\mathcal{F}''_2)\to\mathcal{M}aps_{\mathrm{QCoh}(X_1\times X_2)}(\widetilde{\mathcal{F}}'_1\boxtimes\widetilde{\mathcal{F}}'_2,\mathcal{F}''_1\boxtimes\mathcal{F}''_2)$$

is an isomorphism. But this follows from the fact that the fuctor

$$\mathrm{QCoh}(X_1)\boxtimes\mathrm{QCoh}(X_2)\to\mathrm{QCoh}(X_1\times X_2)\tag{6.2}$$

is an equivalence.

This finishes the proof of point (a).

To prove point (b), we have to show that the essential image of the functor (6.1) generates $\text{IndCoh}(X_1 \times X_2)$. By Corollary 6.1.5 we can assume that both X_1 and X_2 are classical and reduced.

We argue by Noetherian induction, assuming that the statement is true for all proper closed subschemes $X'_i \subset X_i$.

By Proposition 6.1.3(b), it is sufficient to show that X_1 and X_2 contain non-empty open subschemes $\overset{\circ}{X}_i \subset X_i$, for which the statement of the proposition is true.

Since X_i are classical and reduced, we can take $\overset{\circ}{X}_i$ to be a non-empty open smooth subscheme of X_i. Note that the assumption that k be perfect implies that $\overset{\circ}{X}_1 \times \overset{\circ}{X}_2$ is also smooth. Now, the assertion follows from Lemma 1.1.3 and the fact that (6.2) is an equivalence.

□

6.3.5. *Upgrading to a functor.* Consider the category Sch_{aft} as endowed with a symmetric monoidal structure given by Cartesian product. We consider the category $\text{DGCat}_{\text{cont}}$ also as a symmetric monoidal ∞-category with respect to the operation of tensor product.

First we recall that the functor

$$\text{QCoh}^*_{\text{Sch}_{\text{aft}}} : (\text{Sch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}$$

has a natural symmetric monoidal structure.

Indeed, this follows from [Chapter I.3, Sect. 3.1.3 and Proposition 3.1.7].

6.3.6. Passing to adjoints, by [Chapter V.3, Sect. 3.1.3], we obtain that the functor

$$\text{QCoh}_{\text{Sch}_{\text{aft}}} : \text{Sch}_{\text{aft}} \to \text{DGCat}_{\text{cont}}$$

also has a natural symmetric monoidal structure.

Now, as in Proposition 2.2.3 one shows:

Proposition 6.3.7. *There exists a unique symmetric monoidal structure on the functor*

$$\text{IndCoh}_{\text{Sch}_{\text{aft}}} : \text{Sch}_{\text{aft}} \to \text{DGCat}_{\text{cont}}$$

and the natural transformation $\Psi_{\text{Sch}_{\text{aft}}}$ that at the level of objects is given by Lemma 6.3.2.

6.4. **Convergence.** In this subsection we will establish another crucial property of IndCoh that distinguishes it from QCoh. Namely, we will show that for a given scheme, its category IndCoh can be recovered from IndCoh on the n-coconnective truncations of this scheme.

6.4.1. Let X be an object of Sch_{aft}. For each n let i_n denote the closed embedding ${}^{\leq n}X \to X$, and for $n_1 \leq n_2$, let i_{n_1,n_2} denote the closed embedding

$${}^{\leq n_1}X \to {}^{\leq n_2}X.$$

By [Chapter I.1, Proposition 2.5.7], we have a canonical equivalence

$$\underset{n}{\text{colim}}\, \text{IndCoh}({}^{\leq n}X) \simeq \underset{n}{\lim}\, \text{IndCoh}({}^{\leq n}X),$$

where in the left-hand side the transition functors are $(i_{n_1,n_2})^{\text{IndCoh}}_*$, and in the right-hand side $(i_{n_1,n_2})^!$.

The functors $i^!_n$ define a functor

$$\text{IndCoh}(X) \to \underset{n}{\lim}\, \text{IndCoh}({}^{\leq n}X), \tag{6.3}$$

whose left adjoint

$$\underset{n}{\operatorname{colim}} \operatorname{IndCoh}({}^{\leq n}X) \to \operatorname{IndCoh}(X) \tag{6.4}$$

is given by the compatible family of functors $(i_n)_*^{\operatorname{IndCoh}}$.

6.4.2. We are going to establish the following property of the category IndCoh:

Proposition 6.4.3. *The functors* (6.3) *and* (6.4) *are mutually inverse equivalences.*

Proof. First, we note that Corollary 6.1.5 shows that the functor (6.3) is conservative. It is clear that the functor (6.4) sends compact objects to compact ones.

Hence, it remains to prove the following: for

$$\mathcal{F}_1, \mathcal{F}_2 \in \operatorname{Coh}({}^{\leq 0}X)^{\heartsuit} \simeq \operatorname{Coh}(X)^{\heartsuit}$$

and $k \in \mathbb{N}$, the map

$$\underset{n}{\operatorname{colim}} \operatorname{Maps}_{\operatorname{Coh}({}^{\leq n}X)}(\mathcal{F}_1, \mathcal{F}_2[k]) \to \operatorname{Maps}_{\operatorname{Coh}(X)}(\mathcal{F}_1, \mathcal{F}_2[k])$$

is an isomorphism.

We will prove more generally the following:

Lemma 6.4.4. *For*

$$\mathcal{F}_1 \in (\operatorname{QCoh}({}^{\leq n}X))^{\leq 0},\ \mathcal{F}_2 \in (\operatorname{QCoh}({}^{\leq n}X))^{\geq -k},$$

the map

$$\operatorname{Maps}_{\operatorname{QCoh}({}^{\leq n}X)}(\mathcal{F}_1, \mathcal{F}_2) \to \operatorname{Maps}_{\operatorname{QCoh}(X)}((i_n)_*(\mathcal{F}_1), (i_n)_*(\mathcal{F}_2)) \tag{6.5}$$

is an isomorphism for $n \geq k$.

□

6.4.5. *Proof of Lemma 6.4.4.* We rewrite the right-hand side as

$$\operatorname{Maps}_{\operatorname{QCoh}({}^{\leq n}X)}((i_n)^* \circ (i_n)_*(\mathcal{F}_1), \mathcal{F}_2),$$

and we claim that

$$\operatorname{Cone}((i_n)^* \circ (i_n)_*(\mathcal{F}_1) \to \mathcal{F}_1) \in (\operatorname{QCoh}({}^{\leq n}X))^{\leq -n-1},$$

which is equivalent to

$$\operatorname{Cone}((i_n)_* \circ (i_n)^* \circ (i_n)_*(\mathcal{F}_1) \to (i_n)_*(\mathcal{F}_1)) \in \operatorname{QCoh}(X)^{\leq -n-1},$$

and further equivalent to

$$\operatorname{Cone}((i_n)_*(\mathcal{F}_1) \to (i_n)_* \circ (i_n)^* \circ (i_n)_*(\mathcal{F}_1)) \in \operatorname{QCoh}(X)^{\leq -n}.$$

In fact, we claim that for $\mathcal{F} \in \operatorname{QCoh}(X)^{\leq 0}$,

$$\operatorname{Cone}(\mathcal{F} \to (i_n)_* \circ (i_n)^*(\mathcal{F})) \in \operatorname{QCoh}(X)^{\leq -n}.$$

Indeed,

$$\operatorname{Cone}(\mathcal{F} \to (i_n)_* \circ (i_n)^*(\mathcal{F}))) \simeq \operatorname{Cone}(\mathcal{O}_X \to (i_n)_*(\mathcal{O}_{\leq n X})) \otimes \mathcal{F},$$

and the assertion follows.

□

7. Groupoids and descent

In this section we will show that the category IndCoh satisfies descent with respect to proper surjective maps. We will later strengthen this to show that IndCoh satisfies h-descent.

7.1. **The Beck-Chevalley condition.** The Beck-Chevalley condition gives a sufficient condition for when the totalization of a given co-simplicial category can be described as co-modules over a co-monad acting on the category of 0-simplices.

7.1.1. Let us recall the following general framework.

Let $\mathbf{C}^\bullet$ be a co-simplicial ∞-category. Consider the corresponding category $\mathbf{C}^{\bullet+1}$, and the co-simplicial functor

$$\mathbf{C}^{\bullet+1} \leftarrow \mathbf{C}^\bullet : \mathfrak{s}^\bullet.$$

Note that the co-simplicial category $\mathbf{C}^{\bullet+1}$ is augmented and split by $\mathbf{C}^0$. Hence, we have a canonical equivalence

$$\mathbf{C}^0 \simeq \mathrm{Tot}(\mathbf{C}^{\bullet+1}),$$

so that the composed functor

$$\mathbf{C}^0 \simeq \mathrm{Tot}(\mathbf{C}^{\bullet+1}) \overset{\mathrm{Tot}(\mathfrak{s}^\bullet)}{\longleftarrow} \mathrm{Tot}(\mathbf{C}^\bullet)$$

identifies with $\mathrm{ev}^0_{\mathbf{C}^\bullet}$.

Furthermore, the functor

$$\mathbf{C}^0 \simeq \mathrm{Tot}(\mathbf{C}^{\bullet+1}) \overset{\mathrm{ev}^0_{\mathbf{C}^{\bullet+1}}}{\longrightarrow} \mathbf{C}^1$$

identifies with p_s, where p_s, p_t are the two functors

$$\mathbf{C}^0 \rightrightarrows \mathbf{C}^1.$$

7.1.2. Recall the following definition:

Definition 7.1.3. *We shall say that $\mathbf{C}^\bullet$ satisfies the Beck-Chevalley condition if for each n the functor*

$$\mathbf{C}^{n+1} \leftarrow \mathbf{C}^n : \mathfrak{s}^n$$

admits a left adjoint (to be denoted by $\mathfrak{t}^n$), and for every map $[m] \to [n]$ in $\mathbf{\Delta}$, the diagram

$$\begin{array}{ccc} \mathbf{C}^{n+1} & \xrightarrow{\mathfrak{t}^n} & \mathbf{C}^n \\ \uparrow & & \uparrow \\ \mathbf{C}^{m+1} & \xrightarrow{\mathfrak{t}^m} & \mathbf{C}^m, \end{array}$$

that a priori commutes up to a natural transformation, actually commutes.

We have:

Lemma 7.1.4. *Suppose that $\mathbf{C}^\bullet$ satisfies the Beck-Chevalley condition. Then:*

(a) *The functor*

$$\mathbf{C}^0 \leftarrow \mathrm{Tot}(\mathbf{C}^\bullet) : \mathrm{ev}^0_{\mathbf{C}}$$

admits a left adjoint.

(b) *The monad*

$$\mathrm{ev}^0_{\mathbf{C}} \circ (\mathrm{ev}^0_{\mathbf{C}})^L,$$

viewed as an endo-functor of $\mathbf{C}^0$, identifies with $(p_t)^L \circ p_s$, where $(p^t)^L$ is the left adjoint of p^t.

(c) *The adjoint pair*

$$(\mathrm{ev}^0_{\mathbf{C}^\bullet})^L : \mathbf{C}^0 \rightleftarrows \mathrm{Tot}(\mathbf{C}^\bullet) : \mathrm{ev}^0_{\mathbf{C}^\bullet}$$

is monadic.

Proof. The Beck-Chevalley condition implies that the simplex-wise left adjoints $\mathfrak{t}^n$ form a co-simplicial functor

$$\mathfrak{t}^\bullet : \mathbf{C}^{\bullet+1} \leftarrow \mathbf{C}^\bullet.$$

In particular, we obtain a pair of adjoint functors

$$\mathrm{Tot}(\mathfrak{t}^\bullet) : \mathrm{Tot}(\mathbf{C}^{\bullet+1}) \rightleftarrows \mathrm{Tot}(\mathbf{C}^\bullet) : \mathrm{Tot}(\mathfrak{s}^\bullet),$$

that commute with evaluation on n-simplicies for every n.

Note also that $\mathfrak{s}^0 \simeq p^t$ and so $\mathfrak{t}^0 \simeq (p^t)^L$. Now, the required assertion concering

$$\mathrm{ev}^0_{\mathbf{C}^\bullet} \circ (\mathrm{ev}^0_{\mathbf{C}^\bullet})^L$$

follows from the commutative diagram

$$\begin{array}{ccc} \mathbf{C}^1 & \xrightarrow{\mathfrak{t}^0} & \mathbf{C}^0 \\ {\scriptstyle \mathrm{ev}^0_{\mathbf{C}^{\bullet+1}}}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{ev}^0_{\mathbf{C}^\bullet}} \\ \mathbf{C}^0 \simeq \mathrm{Tot}(\mathbf{C}^{\bullet+1}) & \xrightarrow{\mathrm{Tot}(\mathfrak{t}^\bullet)} & \mathrm{Tot}(\mathbf{C}^\bullet). \end{array}$$

Finally, it is easy to see that the functor $\mathrm{ev}^0_{\mathbf{C}^\bullet}$ is conservative and commutes with $\mathrm{ev}^0_{\mathbf{C}^\bullet}$-split geometric realizations. Hence, it satisfies the conditions of the Barr-Beck-Lurie theorem, and therefore is monadic.

□

7.2. **Proper descent.** We will now prove proper descent for IndCoh.

7.2.1. Let $X^\bullet$ be a groupoid simplicial object in $\mathrm{Sch}_{\mathrm{aft}}$ (see [Lu1], Definition 6.1.2.7 for the notion of groupoid in the context of ∞-categories).

Denote by

$$p_s, p_t : X^1 \rightrightarrows X^0 \tag{7.1}$$

the corresponding maps. Let us assume that the map p_s (and hence also p_t) is proper.

We form a co-simplicial category $\mathrm{IndCoh}(X^\bullet)^!$ using the !-pullback functors, and consider its totalization $\mathrm{Tot}(\mathrm{IndCoh}(X^\bullet)^!)$. Consider the functor of evaluation on 0-simplices:

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathrm{IndCoh}(X^\bullet)) \to \mathrm{IndCoh}(X^0).$$

Proposition 7.2.2.
(a) *Then functor* ev^0 *admits a left adjoint. The resulting monad on* $\mathrm{IndCoh}(X^0)$*, viewed as an endo-functor, is canonically isomorpic to* $(p_t)_*^{\mathrm{IndCoh}} \circ (p_s)^!$*. The adjoint pair*

$$\mathrm{IndCoh}(X^0) \rightleftarrows \mathrm{Tot}(\mathrm{IndCoh}(X^\bullet)^!)$$

is monadic.
(b) *Suppose that* $X^\bullet$ *is the Čech nerve of a map* $f : X^0 \to Y$*, where* f *proper. Assume also that* f *is surjective at the level of geometric points. Then the resulting map*

$$\mathrm{IndCoh}(Y) \to \mathrm{Tot}(\mathrm{IndCoh}(X^\bullet)^!)$$

is an equivalence.

Remark 7.2.3. Note that the fact that ev^0 admits a left adjoint follows from [Chapter I.1, Proposition 2.5.7].

Indeed, the maps in $\mathrm{IndCoh}(X^\bullet)^!$ admit left adjoints, and we can interpret $\mathrm{Tot}(\mathrm{IndCoh}(X^\bullet)^!)$ as the geometric realization of the corresponding simplicial category $\mathrm{IndCoh}(X^\bullet)$, with the left adjoint to ev^0 being the corresponding tautological functor

$$\mathrm{IndCoh}(X^0) \to |\mathrm{IndCoh}(X^\bullet)|.$$

7.2.4. *Proof of Proposition 7.2.2(a).* By Lemma 7.1.4 we need to show that the co-simplicial category $\mathrm{IndCoh}(X^{\bullet+1})^!$ satisfies the Beck-Chevalley condition. However, this follows immediately from Proposition 5.2.2.

□

7.2.5. *Proof of Proposition 7.2.2(b).* Consider the adjoint pair

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X^0) \rightleftarrows \mathrm{IndCoh}(Y) : f^!.$$

By Proposition 6.2.2, the functor $f^!$ is conservative, and continuous. Hence, the above pair is monadic.

The composition

$$\mathrm{IndCoh}(Y) \to \mathrm{Tot}(\mathrm{IndCoh}(X^\bullet)) \to \mathrm{IndCoh}(X^0)$$

identifies with the functor $f^!$. Hence, it remains to show that the map of the corresponding monads

$$\mathrm{ev}^0 \circ (\mathrm{ev}^0)^L \to f^! \circ f_*^{\mathrm{IndCoh}}$$

induces an isomorphism at the level of the underlying endo-functors of $\mathrm{IndCoh}(X^0)$.

By Proposition 7.2.2(a), the left-hand side identifies with

$$(p_t)_*^{\mathrm{IndCoh}} \circ p_s^!.$$

Furthermore, it follows from the construction that the resulting map

$$(p_t)_*^{\mathrm{IndCoh}} \circ p_s^! \to f^! \circ f_*^{\mathrm{IndCoh}}$$

is the base change morphism of Proposition 5.2.2 for the Cartesian diagram

$$\begin{array}{ccc} X^1 & \xrightarrow{p_s} & X^0 \\ {\scriptstyle p^t}\downarrow & & \downarrow{\scriptstyle f} \\ X^0 & \xrightarrow{f} & Y. \end{array}$$

Hence, the required isomorphism follows from Proposition 5.2.2.

□