A GENERALIZED VANISHING CONJECTURE

ABSTRACT. The following argument was indicated in the course of a conversation between D. Arinkin, V. Drinfeld, D. Gaitsgory and D. Kazhdan on 7.29.2010.

1. FORMULATION

1.1. **Classical localization.** Let $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$ be the Ran version of the category $\mathrm{Rep}(\check{G})$. The (symmetric) monoidal structure on $\mathrm{Rep}(\check{G})$ coupled with convolution on the Ran space makes $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$ into a (symmetric) monoidal category (without a unit).

Consider the category $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$. We have a naturally defined symmetric monoidal functor

$$\mathrm{Loc}_{\check{G},cl} : \mathrm{Rep}(\check{G})[\mathrm{Ran}] \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}),$$

which admits a right adjoint, that we denote by $\Gamma_{\check{G},cl}$.

Theorem 1.2. *The functor $\Gamma_{\check{G},cl}$ is fully faithful.*

Another way to phrase this theorem is that the functor $\mathrm{Loc}_{\check{G},cl}$ realizes $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ as a localization of $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$ with respect to the subcategory $\ker(\mathrm{Loc}_{\check{G},cl})$. In particular, at the level of triangulated categories, the former is a Verdier quotient of the latter by the corresponding triangulated subcategory.

1.3. **Remarks.** Explicitly, the functor $\mathrm{Loc}_{\check{G},cl}$ can be viewed as follows. For points $x_1, ..., x_n \in X$ and $\check{G}$-representations $V_1, ..., V_n$, we view

$$(V_1)_{x_1} \otimes ... \otimes (V_n)_{x_n}$$

as an object of $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$, and its image under $\mathrm{Loc}_{\check{G},cl}$ is the vector bundle on $\mathrm{LocSys}_{\check{G}}$ equal to

$$\bigotimes_i \mathcal{E}_{V_i,x_i},$$

where $\mathcal{E}_{V_i,x_i}$ is the corresponding "evaluation" vector bundle.

The functor $\Gamma_{\check{G},cl}$ can in turn be described as follows. By definition, to define it we need to define a compatible collection of functors

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})^{\otimes n}$$

for each finite collection of points $x_1, ..., x_n$. The corresponding functor is given by direct image with respect to

$$\mathrm{LocSys}_{\check{G}} \to B\check{G}^n,$$

corresponding to $\{x_1\} \cup ... \cup \{x_n\} \hookrightarrow X$.

1.4. **Remarks.** In fact, the above situation isn't specific to the stack $B\check{G}$. Given any Artin stack $\mathcal{Z}$ over X equipped with a connection, we can consider the corresponding category $\mathsf{QCoh}(\mathcal{Z}_{\mathrm{Ran}})$ and the functors

$$\mathrm{Loc}_{\mathcal{Z}} : \mathsf{QCoh}(\mathcal{Z}_{\mathrm{Ran}}) \rightleftarrows \mathsf{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Z})),$$

where $\mathrm{Sect}_\nabla(X, \mathcal{Z})$ is the stack of horizontal sections of $\mathcal{Z}$.

It is likely that Theorem 1.2 holds, as is, in this more general context.

1.5. **Action on** Bun_G**.** Hecke functors give rise to a monoidal action of the category $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$ on $D(\mathrm{Bun}_G)$, denoted

$$\mathcal{F} \in \mathrm{Rep}(\check{G})[\mathrm{Ran}], \mathcal{M} \in D(\mathrm{Bun}_G) \mapsto \mathcal{F} \star \mathcal{M}.$$

The generalized vanishing conjecture reads as follows:

Conjecture 1.6. *The action of* $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$ *on* $D(\mathrm{Bun}_G)$ *factors through*

$$\mathrm{Loc}_{\check{G},cl} : \mathrm{Rep}(\check{G})[\mathrm{Ran}] \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

In particular, Conjecture 1.6 endows $D(\mathrm{Bun}_G)$ with a structure of category *over* the stack $\mathrm{LocSys}_{\check{G}}$.

1.7. **Reformulations.** Tautologically, Conjecture 1.6 can be reformulated as a statement that for $\mathcal{F} \in \mathrm{Rep}(\check{G})[\mathrm{Ran}]$ with $\mathrm{Loc}_{\check{G},cl}(\mathcal{F}) = 0$, we have

$$\mathcal{F} \star \mathcal{M} = 0$$

for all $\mathcal{M} \in D(\mathrm{Bun}_G)$.

In addition, according to [BoD], the statement of Conjecture 1.6 is equivalent to just one isomorphism, namely, that the natural morphism

$$\mathbf{1}_{\mathrm{Rep}(\check{G})[\mathrm{Ran}]} \to \Gamma_{\check{G},cl}(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}})$$

induces an isomorphism when acting on $D(\mathrm{Bun}_G)$ by Hecke functors.

Conjecture 1.6 also implies (and is equivalent to) the "spectral projector" conjecture of A. Beilinson:

Corollary 1.8. *The assignment*

$$\mathcal{F} \in \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}), \mathcal{M} \in D(\mathrm{Bun}_G) \mapsto \Gamma_{\check{G},cl}(\mathcal{F}) \star \mathcal{M}$$

defines a monoidal action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ *on* $D(\mathrm{Bun}_G)$*.*

1.9. **Relation to the vanishing conjecture for** GL_n**.** Let $G = GL_n$, in which case $\check{G} \simeq \mathrm{GL}_n$, and let σ be an irreducible m-dimensional local system on X. Let V_n be the standard n-dimensional representation of GL_n. For an integer d we can consider $(V_n \otimes \sigma)^{(d)}$ as an object of $\mathrm{Rep}(\check{G})[\mathrm{Ran}]$.

The main theorem of [Ga] asserts that the Hecke action of $(V_n \otimes \sigma)^{(n)}$ on $D(\mathrm{Bun}_G)$ denoted in *loc.cit.* by Av_σ^d, vanishes, provided that $d > (2g - 2) \cdot m \cdot n$.

We note, however, that the above result follows immediately from Conjecture 1.6. Indeed, the assumptions on σ and d imply that

$$\mathrm{Loc}_{\check{G},cl}((V_n \otimes \sigma)^{(d)}) = 0.$$

As was pointed out by S. Lysenko, there are similar statements that one can make for an arbitrary G. Indeed, let V be a $\check{G}$-representation, and σ an irreducible local system of rank

$> \dim(V)$. Then Conjecture 1.6 implies that the Hecke functor $(V_n \otimes \sigma)^{(d)}$ acting on $D(\mathrm{Bun}_G)$ will again vanish for $d > (2g - 2) \cdot \mathrm{rk}(\sigma) \cdot \dim(V)$. We do not know how to prove this statement in the context of ℓ-adic sheaves.

1.10. **The idea of proof.** The main idea suggested by Drinfeld is that there are "many" D-modules on Bun_G, whose behavior with respect to Hecke functors we can control. Namely, these are D-modules obtained by localization from critical level representations.

Of course, the construction of a D-module by localization ultimately boils down to defining it by "generators and relation". It is this ability that makes the D-module case easier.

2. LOCALIZATION AT THE CRITICAL LEVEL

2.1. **Kazhdan-Lusztig category.** Let $\mathrm{KL}_{G,crit}$ be the Kazhdan-Lusztig category at the critical level, i.e., the category of $G[[t]]$-equivariant representations of the Kac-Moody algebra $\widehat{\mathfrak{g}}_{crit}$. We denote by $\mathrm{KL}_{G,crit}[\mathrm{Ran}]$ its Ran version.

We have the localization functor

$$\mathrm{Loc}_{G,crit} : \mathrm{KL}_{G,crit}[\mathrm{Ran}] \to D(\mathrm{Bun}_G).$$

We'll deduce Conjecture 1.6 from the following statement:

Main Lemma 2.2. *For* $\mathcal{F} \in \ker(\mathrm{Loc}_{\check{G},cl})$ *and* $\mathcal{M} \in \mathrm{KL}_{G,crit}[\mathrm{Ran}]$ *we have*

$$\mathcal{F} \star \mathrm{Loc}_{G,crit}(\mathcal{M}) = 0.$$

As we shall see, Main Lemma 2.2 is an easy corollary of (a generalization of) the main theorem in [BD].

2.3. **Proof of Main Lemma 2.2.** To simplify the notation, let us fix points $x_1, ..., x_n \in X$ and consider the corresponding functor

$$\mathrm{Loc}_{G,crit,x_1,...,x_n} : \mathrm{KL}_{G,crit,x_1} \otimes ... \otimes \mathrm{KL}_{G,crit,x_n} \to D(\mathrm{Bun}_G).$$

For each i let $\mathrm{Op}_{\check{G},x_i}^{\mathrm{m.f.}}$ denote the ind-scheme of monodromy-free $\check{G}$-opers on the formal disc around x_i. According to [FG1], the category $\mathrm{KL}_{G,crit,x_i}$ is a category *over* $\mathrm{Op}_{\check{G},x_i}^{\mathrm{m.f.}}$.

(In fact, by [FG2], $\mathrm{KL}_{G,crit,x_i}$ is actually equivalent to $\mathsf{QCoh}(\mathrm{Op}_{\check{G},x_i}^{\mathrm{m.f.}})$, but we won't need this fact here.)

Let now $\mathrm{Op}_{\check{G},x_1,...,x_n}^{\mathrm{m.f.},glob}$ be the ind-scheme of opers on the punctured curve $X - \{x_1, ..., x_n\}$ which extend as local systems on the entire X. We have a natural closed embedding:

$$\mathrm{Op}_{\check{G},x_1,...,x_n}^{\mathrm{m.f.},glob} \hookrightarrow \mathrm{Op}_{\check{G},x_1}^{\mathrm{m.f.}} \times ... \times \mathrm{Op}_{\check{G},x_n}^{\mathrm{m.f.}},$$

and a forgetful map

$$\pi : \mathrm{Op}_{\check{G},x_1,...,x_n}^{\mathrm{m.f.},glob} \to \mathrm{LocSys}_{\check{G}}.$$

The main global Hecke eigen-property result of [BD] (with the corresponding local statement generalized to $\mathrm{Op}_{\check{G}}^{\mathrm{m.f.}}$ in [FG1]) reads as follows:

Theorem 2.4.

(1) *The functor*

$$\mathrm{Loc}_{G,crit,x_1,\ldots,x_n} : \mathrm{KL}_{G,crit,x_1} \otimes \ldots \mathrm{KL}_{G,crit,x_n} \to D(\mathrm{Bun}_G)$$

canonically factors as

$$\mathrm{KL}_{G,crit,x_1} \otimes \ldots \otimes \mathrm{KL}_{G,crit,x_n} \to$$

$$\to (\mathrm{KL}_{G,crit,x_1} \otimes \ldots \otimes \mathrm{KL}_{G,crit,x_n}) \underset{\mathrm{Op}_{\check{G},x_1}^{\mathrm{m.f.}} \times \ldots \times \mathrm{Op}_{\check{G},x_n}^{\mathrm{m.f.}}}{\times} \mathrm{Op}_{\check{G},x_1,\ldots,x_n}^{\mathrm{m.f.},glob} \to D(\mathrm{Bun}_G).$$

(2) *The resulting functor*

$$\mathrm{Loc}_{G,crit,x_1,\ldots,x_n}^{glob} : (\mathrm{KL}_{G,crit,x_1} \otimes \ldots \otimes \mathrm{KL}_{G,crit,x_n}) \underset{\mathrm{Op}_{\check{G},x_1}^{\mathrm{m.f.}} \times \ldots \times \mathrm{Op}_{\check{G},x_n}^{\mathrm{m.f.}}}{\times} \mathrm{Op}_{\check{G},x_1,\ldots,x_n}^{\mathrm{m.f.},glob} \to D(\mathrm{Bun}_G)$$

satisfies

$$\mathcal{F} \star \mathrm{Loc}_{G,crit,x_1,\ldots,x_n}^{glob}(\mathcal{M}) \simeq \mathrm{Loc}_{G,crit,x_1,\ldots,x_n}^{glob} \left(\pi^*(\mathrm{Loc}_{\check{G},cl}(\mathcal{F})) \underset{\mathcal{O}_{\mathrm{Op}_{\check{G},x_1,\ldots,x_n}^{\mathrm{m.f.},glob}}}{\otimes} \mathcal{M} \right).$$

It is clear that Theorem 2.4 implies the statelemt of Main Lemma 2.2.

2.5. **The rest of the argument.** The rest of this section is devoted to showing how Main Lemma 2.2 implies the statement of Conjecture 1.6.

If it was the case that the essential image of the functor $\mathrm{Loc}_{G,crit}$ generated $D(\mathrm{Bun}_G)$, the conclusion would be immediate. Unfortunately, the above generation statement is false because of the non-quasi-comactness of $\mathrm{Bun}(G)$. However, the degree of its failure is controllable.

2.6. **Eisenstein series.** For a parabolic $P \subset G$ with a Levi M, let $\mathrm{Eis}_!$ denote the corresponding (non-compactified) Eisenstein series functor

$$D(\mathrm{Bun}_M) \to D(\mathrm{Bun}_G).$$

The following piece of theory has been developed in [BG2] for $B = T$, but we expect that no major difficulties will arise in its generalization to the case of an arbitrary parabolic.

By induction on semi-simple rank, we can assume that Conjecture 1.6 holds for M. Here is the summary of the main results of [BG2] (generalized to any parabolic) of how Hecke functors interact with Eisenstein series:

Theorem 2.7. *Let $\check{P}$ is the corresponding parabolic in $\check{G}$, and let*

$$\mathrm{LocSys}_{\check{G}} \overset{\mathbf{p}_{spec}}{\longleftarrow} \mathrm{LocSys}_{\check{P}} \overset{\mathbf{q}_{spec}}{\longrightarrow} \mathrm{LocSys}_{\check{M}}$$

denote the natural projections. We have:

(1) *The functor*

$$\mathrm{Eis}_! : D(\mathrm{Bun}_M) \to D(\mathrm{Bun}_G)$$

naturally factors through a functor

$$\mathrm{Eis}_!^{enh} : D(\mathrm{Bun}_M) \underset{\mathrm{LocSys}_{\check{M}}}{\times} \mathrm{LocSys}_{\check{P}} \to D(\mathrm{Bun}_G).$$

(2) *For $\mathcal{F} \in \mathrm{Rep}(\check{G})[\mathrm{Ran}]$ and $\mathcal{M} \in D(\mathrm{Bun}_M) \underset{\mathrm{LocSys}_{\check{M}}}{\times} \mathrm{LocSys}_{\check{P}}$ we have a natural isomorphism*

$$\mathcal{F} \star \mathrm{Eis}_!^{enh}(\mathcal{M}) \simeq \mathrm{Eis}_!^{enh}\left(\mathfrak{p}_{spec}^*(\mathrm{Loc}_{\check{G},cl}(\mathcal{F})) \underset{\mathcal{O}_{\mathrm{LocSys}_{\check{P}}}}{\otimes} \mathcal{M}\right).$$

2.8. **Generation of $D(\mathrm{Bun}_G)$.** Let $D(\mathrm{Bun}_G)^{\mathrm{Eis}} \subset D(\mathrm{Bun}_G)$ be the full subcategory generated by the essential images of the functors $\mathrm{Eis}_!$ for all parabolics. From Theorem 2.7 above, we obtain:

Corollary 2.9. *For $\mathcal{F} \in \ker(\mathrm{Loc}_{\check{G},cl})$, we have $\mathcal{F} \star \mathcal{M} = 0$ for $\mathcal{M} \in D(\mathrm{Bun}_G)^{\mathrm{Eis}}$.*

Thus, we obtain that Conjecture 1.6 follows from the next assertion:

Proposition 2.10. *The essential image of the functor $\mathrm{Loc}_{G,\kappa}$ together with $D(\mathrm{Bun}_G)^{\mathrm{Eis}}$ generate $D(\mathrm{Bun}_G)$.*

In the statement of the proposition, κ is any level, not necessarily critical. The rest of the section is devoted to the proof of Proposition 2.10.

2.11. **The cuspidal subcategory.** The functors $\mathrm{Eis}_!$ are easily seen to send compact objects to compact ones. In particular, the subcategory $D(\mathrm{Bun}_G)^{\mathrm{Eis}}$ is generated by objects that are compact in $D(\mathrm{Bun}_G)$. Let $D(\mathrm{Bun}_G)^{cusp}$ be the right orthogonal of $D(\mathrm{Bun}_G)^{\mathrm{Eis}}$. We obtain that $D(\mathrm{Bun}_G)^{cusp}$ is a full subcategory closed under colimits, and the embedding

$$D(\mathrm{Bun}_G) \hookleftarrow D(\mathrm{Bun}_G)^{cusp} : \mathbf{e}$$

admits a left adjoint, deboted $\mathbf{c}$, which realizes $D(\mathrm{Bun}_G)^{cusp}$ as a localization with respect to $D(\mathrm{Bun}_G)^{\mathrm{Eis}}$.

We have:

Lemma 2.12. *There exists an open substack of finite type $j : \mathbf{U} \hookrightarrow \mathrm{Bun}_G$, such that for every $\mathcal{F} \in D(\mathrm{Bun}_G)^{cusp}$, the canonical arrow*

$$\mathcal{F} \to j_* \circ j^*(\mathcal{F})$$

is an isomorphism.

In other words, the functor $\mathbf{e}$ can be canonically factored as

$$D(\mathrm{Bun}_G)^{cusp} \xrightarrow{\mathbf{e}'} D(\mathbf{U}) \xrightarrow{j_*} D(\mathrm{Bun}_G).$$

Corollary 2.13. *The functor $\mathbf{c}$ can be canonically factored as*

$$D(\mathrm{Bun}_G) \xrightarrow{j^*} D(\mathbf{U}) \xrightarrow{\mathbf{c}'} D(\mathrm{Bun}_G)^{cusp}.$$

Thus, we obtain that Proposition 2.10 follows from the next general statement:

Proposition 2.14. *For any open substack $\mathbf{U} \subset \mathrm{Bun}_G$ of finite type, the essential image of the composed functor*

$$\mathrm{KL}_{G,\kappa}[\mathrm{Ran}] \overset{\mathrm{Loc}_{G,\kappa}}{\to} D(\mathrm{Bun}_G) \xrightarrow{j^*} D(\mathbf{U})$$

generates $D(\mathbf{U})$. In fact, the above functor sends compact objects to compacts and realizes $D(\mathbf{U})$ as a localization of $\mathrm{KL}_{G,\kappa}[\mathrm{Ran}]$ (i.e., the right adjoint is fully faithful).

2.15. **Proof of Proposition 2.14.** To avoid clashing notation, let us denote $D(\mathrm{Bun}_G)$ by $D(\mathrm{Bun}_G)_\kappa$ to emphasize the dependence on the level.

The fact that $\mathrm{Loc}_{G,\kappa}$ sends compact objects from $\mathrm{KL}_{G,\kappa}[\mathrm{Ran}]$ to objects in $D(\mathrm{Bun}_G)_\kappa$ that are compact when restricted to any open of finite type is easy: $\mathrm{KL}_{G,\kappa}[\mathrm{Ran}]$ admits Weyl modules as compact generators, which under $\mathrm{Loc}_{G,\kappa}$ go to D-modules induced from vector bundles.

Consider now the following general set-up. Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between compactly generated categories. Assume that F sends compact objects to compact ones. Let $F^{op} : \mathbf{C}_1^\vee \to \mathbf{C}_2^\vee$ be the corresponding Verdier-dual functor, i.e., one obtained by ind-extension of the functor

$$F^{op} : ((\mathbf{C}_1^\vee)^c \simeq \mathbf{C}_1^{op}) \to ((\mathbf{C}_2^\vee)^c \simeq \mathbf{C}_2^{op}) .$$

Let $R_i \in \mathbf{C}_i^\vee \otimes \mathbf{C}_i$ be the object corresponding to the identity functor. We have a natural map

$$(F \otimes F^{op})(R_1) \to R_2.$$

It is easy to see that F realizes $\mathbf{C}_2$ as a localization of $\mathbf{C}_1$ if and only if the above map is an isomorphism.

In our situation

$\mathbf{C}_1 := \mathrm{KL}_{G,\kappa}[\mathrm{Ran}],\ \ \mathbf{C}_1^\vee = \mathrm{KL}_{G,-\kappa}[\mathrm{Ran}],\ \ \mathbf{C}_2 = D(\mathbf{U})_\kappa,\ \ \mathbf{C}_2^\vee = D(\mathbf{U})_{-\kappa},\ \ F = j^* \circ \mathrm{Loc}_{G,\kappa} .$

In this case F^{op} identifies with $j^* \circ \mathrm{Loc}_{G,-\kappa}$. The object R_1 is the factorization algebra corresponding to $\mathrm{CDO}(G)_{\kappa,-\kappa}$. The resulting isomorphism follows from the following:

Lemma 2.16. *In $D(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\kappa,-\kappa} \simeq D(\mathrm{Bun}_G)_\kappa \otimes D(\mathrm{Bun}_G)_{-\kappa}$ there is a canonical isomorphism:*

$$(\mathrm{Loc}_{G,\kappa} \otimes \mathrm{Loc}_{G,-\kappa})(\mathrm{CDO}(G)_{\kappa,-\kappa}[\mathrm{Ran}]) \simeq \Delta_*(\omega_{\mathrm{Bun}_G}).$$

References

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's Hamiltonians and Hecke eigensheaves.*

[BoD] M. Boyarchenko, V. Drinfeld, *Character sheaves on unipotent groups in positive characteristic: foundations*

[BG1] A. Braverman, D Gaitsgory, *Geometric Eisenstein Series.*

[BG2] A. Braverman, D Gaitsgory, *Eisenstein Series and deformations.*

[FG1] E. Frenkel, D. Gaitsgory, *Fusion and Convolution.*

[FG2] E. Frenkel, D. Gaitsgory, *The spherical case.*

[Ga] D. Gaitsgory, *On a vanishing conjecture arising in geometric Langlands correspondence.*