

FROM GEOMETRIC TO FUNCTION-THEORETIC LANGLANDS (OR HOW TO INVENT SHTUKAS)

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ABSTRACT. This is an informal note that explains that the classical Langlands theory over function fields can be obtained from the geometric one by taking the trace of Frobenius. The operation of taking the trace of Frobenius takes place at the categorical level, and this we deduce that the *space* of automorphic functions is the trace of the Frobenius on the *category* of automorphic sheaves.

1. TRACE OF FROBENIUS ON A CATEGORY

1.1. The trace of an endomorphism.

1.1.1. The notion of a trace of an endomorphism of a dualizable object \mathfrak{o} in a symmetric monoidal category \mathbf{O} is well-known:

If $\mathfrak{o} \in \mathbf{O}$ is dualizable and $T : \mathbf{O} \rightarrow \mathbf{O}$ consider the corresponding map

$$Q_T : \mathbf{1}_{\mathbf{O}} \rightarrow \mathfrak{o} \otimes \mathfrak{o}^\vee$$

and $\mathrm{Tr}(T, \mathfrak{o}) \in \mathrm{End}_{\mathbf{O}}(\mathbf{1}_{\mathbf{O}})$ is by definition the composition

$$\mathbf{1}_{\mathbf{O}} \xrightarrow{Q_T} \mathfrak{o} \otimes \mathfrak{o}^\vee \xrightarrow{\mathrm{ev}} \mathbf{1}_{\mathbf{O}}.$$

1.1.2. Set $\mathbf{O} = \mathrm{DGCat}$, and let $\mathfrak{o} = \mathrm{D}\text{-mod}(\mathcal{Y})$, where \mathcal{Y} is a quasi-compact algebraic stack with affine diagonal. It is known that $\mathrm{D}\text{-mod}(\mathcal{Y})$ is dualizable (as a DG category), where the evaluation functor

$$\mathrm{D}\text{-mod}(\mathcal{Y}) \otimes \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{Vect}$$

is given by

$$\mathrm{D}\text{-mod}(\mathcal{Y}) \otimes \mathrm{D}\text{-mod}(\mathcal{Y}) \xrightarrow{\boxtimes} \mathrm{D}\text{-mod}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta_{\mathcal{Y}}^!} \mathrm{D}\text{-mod}(\mathcal{Y}) \xrightarrow{\mathrm{H}(\mathcal{Y}, \omega_{\mathcal{Y}})} \mathrm{Vect},$$

where $\mathrm{H}(\mathcal{Y}, -)$ is functor of sheaf cohomology (refined to the chain level).

Suppose that $T : \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y})$ is given by pullback with respect to an endomorphism $F : \mathcal{Y} \rightarrow \mathcal{Y}$.

Then the claim is that

$$\mathrm{Tr}(F^!, \mathrm{D}\text{-mod}(\mathcal{Y})) \simeq \mathrm{H}(\mathcal{Y}^F, \omega_{\mathcal{Y}^F}) = \mathrm{H}^{\mathrm{BM}}(\mathcal{Y}^F),$$

where \mathcal{Y}^F is the stack-theoretic fixed-point locus of F , i.e.,

$$\mathcal{Y} \times_{\Gamma_F, \mathcal{Y} \times \mathcal{Y}, \Delta_{\mathcal{Y}}} \mathcal{Y},$$

and where $\mathrm{H}^{\mathrm{BM}}(-)$ stands for Borel-Moore chains.

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Indeed, this follows by base change from the fact that the corresponding to Q_T is given by

$$\mathrm{Vect} \xrightarrow{\omega_{\mathcal{Y}}} \mathrm{D}\text{-mod}(\mathcal{Y}) \xrightarrow{(\mathrm{Graph}_F)^\bullet} \mathrm{D}\text{-mod}(\mathcal{Y} \times \mathcal{Y}) \simeq \mathrm{D}\text{-mod}(\mathcal{Y}) \otimes \mathrm{D}\text{-mod}(\mathcal{Y}).$$

1.1.3. Let us now change the context, where \mathcal{Y} is a stack over $\overline{\mathbb{F}}_q$. We let $D(\mathcal{Y})$ be the ind-completion usual bounded constructible derived category.

The problem is that in the constructible setting, the functor

$$D(\mathcal{Y}_1) \otimes D(\mathcal{Y}_2) \rightarrow D(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

is *no longer* an equivalence. However, we will pretend that it is. That said, Drinfeld had an idea how to provide a framework for this; I think this amounts to tweaking the definition of DGCat.

When considering functors $D(\mathcal{Y}_1) \rightarrow D(\mathcal{Y}_2)$, we will restrict ourselves to functors given by kernels: i.e., to an object $Q \in D(\mathcal{Y}_1 \times \mathcal{Y}_2)$ we attach a functor

$$\mathcal{F} \mapsto (\mathrm{pr}_{\mathcal{Y}_2})_\bullet((\mathrm{pr}_{\mathcal{Y}_1})^! \otimes Q).$$

When talking about a functor admitting a right adjoint, we will also mean that this functor is given by a kernel.

1.1.4. Assume now that \mathcal{Y} is defined over \mathbb{F}_q , and F is the corresponding geometric Frobenius endomorphism $\mathrm{Frob}_{\mathcal{Y}}$ of \mathcal{Y} . Then the calculation of Sect. 1.1.2 implies that

$$(1.1) \quad \mathrm{Tr}(\mathrm{Frob}_{\mathcal{Y}}^!, D(\mathcal{Y})) \simeq \mathrm{Func}(\mathcal{Y}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

1.2. Functoriality.

1.2.1. Suppose that in the context of Sect. 1.1.1, \mathbf{O} is actually a monoidal 2-category. Let

$$(\mathfrak{o}, T_{\mathfrak{o}} : \mathfrak{o} \rightarrow \mathfrak{o}) \text{ and } (\mathfrak{o}', T_{\mathfrak{o}'} : \mathfrak{o}' \rightarrow \mathfrak{o}')$$

be two pairs of objects, each equipped with an endomorphism. Let

$$S : \mathfrak{o} \rightarrow \mathfrak{o}'$$

be a 1-morphism, equipped with a 2-morphism

$$(1.2) \quad S \circ T_{\mathfrak{o}} \rightarrow T_{\mathfrak{o}'} \circ S.$$

Assume also that S admits a right adjoint in \mathbf{O} (this is an intrinsic 2-categorical condition).

We claim that in this case there is a canonical map

$$\mathrm{Tr}(S) : \mathrm{Tr}(T_{\mathfrak{o}}, \mathfrak{o}) \rightarrow \mathrm{Tr}(T_{\mathfrak{o}'}, \mathfrak{o}')$$

in the 1-category $\mathrm{End}(\mathbf{1}_{\mathbf{O}})$.

Indeed, $\mathrm{Tr}(S)$ is given as a composition

$$\begin{aligned} \mathrm{ev}_{\mathfrak{o}} \circ Q_{T_{\mathfrak{o}}} &\rightarrow \mathrm{ev}_{\mathfrak{o}'} \circ (S \otimes (S^R)^\vee) \circ Q_{T_{\mathfrak{o}}} \simeq \mathrm{ev}_{\mathfrak{o}'} \circ (\mathrm{id}_{\mathfrak{o}'} \otimes (S^R)^\vee) \circ Q_{S \circ T_{\mathfrak{o}}} \rightarrow \\ &\rightarrow \mathrm{ev}_{\mathfrak{o}'} \circ (\mathrm{id}_{\mathfrak{o}'} \otimes (S^R)^\vee) \circ Q_{T_{\mathfrak{o}'}, \circ S} \simeq \mathrm{ev}_{\mathfrak{o}'} \circ Q_{T_{\mathfrak{o}'}, \circ S \circ S^R} \rightarrow \mathrm{ev}_{\mathfrak{o}'} \circ Q_{T_{\mathfrak{o}'}}. \end{aligned}$$

1.2.2. One checks directly that the above construction is compatible with compositions. I.e., if we have three pairs $(\mathfrak{o}, T_{\mathfrak{o}} : \mathfrak{o} \rightarrow \mathfrak{o})$, $(\mathfrak{o}', T_{\mathfrak{o}'} : \mathfrak{o}' \rightarrow \mathfrak{o}')$ and $(\mathfrak{o}'', T_{\mathfrak{o}''} : \mathfrak{o}'' \rightarrow \mathfrak{o}'')$ and the 1-morphisms

$$S' : \mathfrak{o} \rightarrow \mathfrak{o}', \quad S'' : \mathfrak{o}' \rightarrow \mathfrak{o}'',$$

and also the corresponding 2-morphisms (1.2), then

$$\mathrm{Tr}(S'' \circ S') \simeq \mathrm{Tr}(S'') \circ \mathrm{Tr}(S'),$$

as morphisms in $\mathrm{End}(\mathbf{1}_{\mathbf{O}})$.

1.2.3. For example, take $\mathbf{O} = \mathrm{DGCat}$ and let (\mathfrak{o}, T) be an object with an endomorphism, i.e., a DG category \mathcal{C} with an endo-functor T . Let $c \in \mathcal{C}$ be a *compact* object, equipped with a morphism

$$c \rightarrow T(c).$$

We can view such c as a datum of 1-morphism

$$\mathrm{Vect} = \mathbf{1}_{\mathrm{DGCat}} \rightarrow \mathcal{C}$$

that admits a (continuous!) right adjoint and a 2-morphism as in (1.2).

The construction in Sect. 1.2.1 yields a map in Vect

$$k \rightarrow \mathrm{Tr}(T, \mathcal{C}),$$

where k is the ground field (i.e., the unit object in Vect). I.e., we obtain an element in the vector space $\mathrm{Tr}(T, \mathcal{C})$; we denote this element by $\mathrm{Tr}(T, c)$.

1.2.4. Let us return to the example of Sect. 1.1.4. Let \mathcal{M} be a compact object in $D(\mathcal{Y})$, equipped with a map

$$(1.3) \quad \mathcal{M} \rightarrow \mathrm{Frob}_{\mathcal{Y}}^!(\mathcal{M}).$$

One checks that

$$\mathrm{Tr}(\mathrm{Frob}_{\mathcal{Y}}^!, \mathcal{M}) \in \mathrm{Tr}(\mathrm{Frob}_{\mathcal{Y}}^!, D(\mathcal{Y})) \simeq \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q), \overline{\mathbb{Q}}_{\ell})$$

is the same as the function obtained from \mathcal{M} by the usual *faisceaux-fonctions*.

Remark 1.2.5. Here we are using the following version of *faisceaux-fonctions*: to $\mathcal{M} \in D(\mathcal{Y})$, equipped with a map (1.3), we associate the function of $\mathcal{Y}(\mathbb{F}_q)$ equal to traces of the Frobenius on the $!$ -fibers of \mathcal{M} . This equals the usual *faisceaux-fonctions* (i.e., for $*$ -fibers) on the Verdier dual of \mathcal{M} .

1.3. Action of local systems.

1.3.1. In the general context of Sect. 1.1.1 let us again take $\mathbf{O} = \mathrm{DGCat}$ but $\mathfrak{o} = \mathrm{QCoh}(\mathcal{Y})$. Note that $\mathrm{QCoh}(\mathcal{Y})$ is again self-dual with the evaluation functor being

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta^*} \mathrm{QCoh}(\mathcal{Y}) \xrightarrow{\Gamma(\mathcal{Y}, -)} \mathrm{Vect}.$$

Let T be again given by pullback along an endomorphism F . Then the same calculation as in Sect. 1.1.2 shows that

$$\mathrm{Tr}(F^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \Gamma(\mathcal{Y}^T, \mathcal{O}_{\mathcal{Y}^T}).$$

By the functoriality developed in Sect. 1.2.1, the structure of symmetric monoidal category on $\mathrm{QCoh}(\mathcal{Y})$ defines a structure of commutative algebra on $\Gamma(\mathcal{Y}^T, \mathcal{O}_{\mathcal{Y}^T})$. It is straightforward to check that this is the usual structure of commutative algebra on $\Gamma(\mathcal{Y}^T, \mathcal{O}_{\mathcal{Y}^T})$.

1.3.2. We want to take \mathcal{Y} to be the stack LocSys of étale local systems on a curve over $\overline{\mathbb{F}}_q$. Of course, it does not exist as an algebraic stack. But we will pretend that it does. There is a hope that the corresponding monoidal category $\text{QCoh}(\text{LocSys})$, or whatever we need from it, could actually be defined, as an algebra object in DGCat .

If our curve is defined over \mathbb{F}_q , then LocSys acquires an automorphism, given by Frobenius. By Sect. 1.3.1, the corresponding category

$$\text{Tr}(\text{Frob}_{\text{LocSys}}^*, \text{QCoh}(\text{LocSys}))$$

identifies with the vector space $\Gamma(\text{LocSys}^{\text{arithm}}, \mathcal{O}_{\text{LocSys}^{\text{arithm}}})$, where $\text{LocSys}^{\text{arithm}}$ is the stack of *arithmetic* local systems.

1.3.3. We will now assume the *geometric spectral decomposition*, i.e., the action of the monoidal category $\text{QCoh}(\text{LocSys})$ on $D(\text{Bun}_G)$. Recall that such an action does indeed exist in the context of D-modules, by the “generalized vanishing theorem”, see [Ga, Corollary 4.5.5].

Applying the functoriality construction from Sect. 1.2.1, we obtain an action of the algebra $\Gamma(\text{LocSys}^{\text{arithm}}, \mathcal{O}_{\text{LocSys}^{\text{arithm}}})$ on $\text{Funct}(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$.

We claim that this is the action constructed in Vincent Lafforgue’s work [Laf], using shtukas. A convenient way to do this is to first reinterpret Lafforgue’s construction à la Drinfeld, i.e., organize the cohomologies of shtukas into an object of $\text{QCoh}(\text{LocSys}^{\text{arithm}})$. This will be done in the next section.

1.4. Hecke action.

1.4.1. Fix a rational point $x \in X$. For a representation V of the dual group \check{G} , we have the Hecke functor

$$H_{x,V} : D(\text{Bun}_G) \rightarrow D(\text{Bun}_G),$$

which is naturally compatible with the Frobenius endo-functor $\text{Frob}_{\text{Bun}_G}^!$ on both sides.

By Sect. 1.2.1, it gives rise to an endomorphism

$$\text{Tr}(H_{x,V}) : \text{Funct}(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \rightarrow \text{Funct}(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

It should be possible to see (but I haven’t done that yet) that this endomorphism is the usual Hecke functor corresponding to V at x .

1.4.2. Restriction to the formal disc around x defines a map

$$\text{LocSys}^{\text{arithm}} \rightarrow \check{G}/\text{Ad}(\check{G}),$$

where we think of $\check{G}/\text{Ad}(\check{G})$ as the stack of unramified arithmetic local systems on the disc around x .

In particular, we obtain a map

$$\mathcal{H}^{\text{cl}} \simeq \Gamma(\check{G}/\text{Ad}(\check{G}), \mathcal{O}_{\check{G}/\text{Ad}(\check{G})}) \rightarrow \Gamma(\text{LocSys}^{\text{arithm}}, \mathcal{O}_{\text{LocSys}^{\text{arithm}}}),$$

where \mathcal{H}^{cl} is the classical spherical Hecke algebra.

Combining with Sect. 1.3.2 we obtain an action of \mathcal{H}^{cl} on $\text{Funct}(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$.

It should be possible to see (but I haven’t done that either) that for a representation V of the dual group, the action of the corresponding element $H_{x,V}^{\text{cl}} \in \mathcal{H}^{\text{cl}}$ on $\text{Funct}(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ equals $\text{Tr}(H_{x,V})$.

If this is the case, this gives a “conceptual” explanation of Vincent’s main formula that expresses the Hecke operators as particular excursion operators.

2. ENHANCED TRACE

2.1. Trace on 2-categories.

2.1.1. We now want to take our monoidal 2-category \mathbf{O} to be that of *2-categories tensored over* DGCat , denoted 2-DGCat . We do not quite know how to give a completely satisfactory definitions, but modulo the questions of 1-affineness, the following will do:

We let the objects of 2-DGCat be monoidal DG categories (i.e., associative algebra objects in DGCat). For two such, denoted \mathcal{A}_0 and \mathcal{A}_1 , we let the category of 1-morphisms $\mathcal{A}_0 \rightarrow \mathcal{A}_1$ to be that of $(\mathcal{A}_1, \mathcal{A}_0)$ -bimodules. Note that the latter is naturally a 2-category, but for our purposes we will not need to consider non-invertible 2-morphisms in it.

In other words, we are thinking of a monoidal DG category \mathcal{A} in terms of the 2-category $\mathcal{A}\text{-mod}$. And the category of $(\mathcal{A}_1, \mathcal{A}_0)$ -bimodules is naturally that of functors

$$\mathcal{A}_0\text{-mod} \rightarrow \mathcal{A}_1\text{-mod}.$$

So, we will denote objects of 2-DGCat by $\mathcal{A}\text{-mod}$ (rather than \mathcal{A}).

2.1.2. The symmetric monoidal structure on 2-DGCat is given by

$$\mathcal{A}, \mathcal{B} \mapsto \mathcal{A} \otimes \mathcal{B},$$

i.e.,

$$\mathcal{A}\text{-mod} \otimes \mathcal{B}\text{-mod} := (\mathcal{A} \otimes \mathcal{B})\text{-mod}.$$

The unit object in 2-DGCat is $\mathcal{A} = \mathrm{Vect}$ so that

$$\mathcal{A}\text{-mod} = \mathrm{DGCat}.$$

Note that

$$\mathrm{End}_{2\text{-DGCat}}(\mathbf{1}_{2\text{-DGCat}}) = \mathrm{DGCat}$$

as a symmetric monoidal category.

2.1.3. Let $\mathcal{A}\text{-mod}$ be an object of 2-DGCat , and let \mathcal{T} be its endofunctor, i.e., an \mathcal{A} -bimodule category. Then

$$\mathrm{Tr}(\mathcal{T}, \mathcal{A}\text{-mod}) = \mathcal{T} \underset{\mathcal{A}^{\mathrm{rev-mult}} \otimes \mathcal{A}}{\otimes} \mathcal{A} \in \mathrm{DGCat},$$

I.e., this Hochschild homology of \mathcal{A} with coefficients in \mathcal{T} , which we will also denote by $\mathrm{HH}(\mathcal{A}, \mathcal{T})$.

Let \mathcal{C} be an object of $\mathcal{A}\text{-mod}$. Assume that \mathcal{C} is *dualizable as an \mathcal{A} -module category*. This is equivalent to the condition that the 1-morphism

$$\mathrm{DGCat} \rightarrow \mathcal{A}\text{-mod}$$

in 2-DGCat admits a right adjoint.

Assume that we are given a functor of \mathcal{A} -module categories

$$T : \mathcal{C} \rightarrow \mathcal{T} \underset{\mathcal{A}}{\otimes} \mathcal{C}.$$

The the construction of Sect. 1.2.1 produces an object

$$\mathrm{Tr}_{\mathcal{T}, \mathcal{A}}(T, \mathcal{C}) \in \mathrm{Tr}(\mathcal{T}, \mathcal{A}\text{-mod}).$$

2.2. Drinfeld's \mathcal{O} -module.

2.2.1. Let us take $\mathcal{A} = \mathrm{QCoh}(\mathrm{LocSys})$, with \mathcal{J} given by the pullback with respect to $\mathrm{Frob}_{\mathrm{LocSys}}$. Note that the DG category

$$\mathrm{Tr}(\mathrm{Frob}_{\mathrm{LocSys}}^*, \mathrm{QCoh}(\mathrm{LocSys}))$$

is by definition

$$\mathrm{QCoh}(\mathrm{LocSys})_{\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{LocSys}}}, \mathrm{QCoh}(\mathrm{LocSys}) \otimes \mathrm{QCoh}(\mathrm{LocSys}), \Delta_{\mathrm{LocSys}}} \otimes \mathrm{QCoh}(\mathrm{LocSys}),$$

while the latter identifies with

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}),$$

i.e.,

$$\mathrm{Tr}(\mathrm{Frob}_{\mathrm{LocSys}}^*, \mathrm{QCoh}(\mathrm{LocSys})) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}).$$

2.2.2. Let us now take $\mathcal{C} = D(\mathrm{Bun}_G)$, regarded as an $\mathrm{QCoh}(\mathrm{LocSys})$ -module category. We let the datum of T be given by $\mathrm{Frob}_{\mathrm{Bun}_G}^!$. Applying the construction of Sect. 2.1.3, we obtain an object

$$\mathrm{Tr}_{\mathrm{Frob}_{\mathrm{LocSys}}^*, \mathrm{QCoh}(\mathrm{LocSys})}(\mathrm{Frob}_{\mathrm{Bun}_G}^!, D(\mathrm{Bun}_G)) =: \mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}).$$

2.2.3. Note that the compatibility with compositions in Sect. 1.2.2 implies that

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathrm{Drinf}) \simeq \mathrm{Tr}(\mathrm{Frob}_{\mathrm{Bun}_G}^!, D(\mathrm{Bun}_G)),$$

while the latter identifies

$$\mathrm{Funct}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

2.2.4. Let us now take $\mathcal{C} = D(\mathrm{Bun}_G)_{\mathrm{temp}}$, and denote

$$\mathrm{Tr}_{\mathrm{Frob}_{\mathrm{LocSys}}^*, \mathrm{QCoh}(\mathrm{LocSys})}(\mathrm{Frob}_{\mathrm{Bun}_G}^!, D(\mathrm{Bun}_G)_{\mathrm{temp}}) =: \mathrm{Drinf}_{\mathrm{temp}}.$$

If we believe that the action of $\mathrm{QCoh}(\mathrm{LocSys})$ on the Whittaker object defines an equivalence

$$\mathrm{QCoh}(\mathrm{LocSys}) \rightarrow D(\mathrm{Bun}_G)_{\mathrm{temp}},$$

we obtain an equivalence

$$\mathrm{Drinf}_{\mathrm{temp}} \simeq \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}}.$$

2.3. Relation to shtukas.

2.3.1. We will now show that the object $\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$ is isomorphic to the one arising from cohomologies of shtukas, denoted $\mathrm{Drinf}\text{-Sht}$.

The object $\mathrm{Drinf}\text{-Sht}$ was characterized by the following property. Fix a rational point $x \in X$. Let $\mathcal{E}_{V,x}$ be the vector bundle on LocSys associated to x and a representation V of \check{G} .

Then

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{V,x}|_{\mathrm{LocSys}^{\mathrm{arithm}}} \otimes \mathrm{Drinf}\text{-Sht})$$

is by definition

$$\mathrm{H}(\mathrm{Bun}_G \times_{\mathrm{Frob}_{\mathrm{Bun}_G}, \mathrm{Bun}_G \times \mathrm{Bun}_G} \mathcal{H}_x, \mathcal{S}_V),$$

where \mathcal{H}_x is the Hecke stack at x and $\mathcal{S}_V \in D(\mathcal{H}_x)$ is the object corresponding to V by geometric Satake.

In order to establish the desired isomorphism $\mathrm{Drinf} \simeq \mathrm{Drinf}\text{-Sht}$, we will construct an isomorphism

$$(2.1) \quad \Gamma(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{V,x}|_{\mathrm{LocSys}^{\mathrm{arithm}}} \otimes \mathrm{Drinf}) \simeq \mathrm{H}(\mathrm{Bun}_G \times_{\mathrm{Frob}_{\mathrm{Bun}_G}, \mathrm{Bun}_G \times \mathrm{Bun}_G} \mathcal{H}_x, \mathcal{S}_V)$$

Remark 2.3.2. By making x move along X , the isomorphism

$$\Gamma(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{V,x}|_{\mathrm{LocSys}^{\mathrm{arithm}}} \otimes \mathrm{Drinf}\text{-}\mathrm{Sht}) \simeq \mathrm{H}(\mathrm{Bun}_G \times_{\mathrm{Frob}_{\mathrm{Bun}_G}, \mathrm{Bun}_G \times \mathrm{Bun}_G} \mathcal{H}_x, \mathcal{S}_V)$$

extends to one between Weil sheaves on X .

Hopefully, the same will be the case for the isomorphism (2.1) that we are about to construct.

2.3.3. In order to establish (2.1) let us unravel the definition of the left-hand side. It is the composition

$$\begin{aligned} (2.2) \quad \mathrm{Vect} &\rightarrow D(\mathrm{Bun}_G) \otimes_{\mathrm{QCoh}(\mathrm{LocSys})} D(\mathrm{Bun}_G) \simeq \\ &\simeq (D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G)) \otimes_{\mathrm{QCoh}(\mathrm{LocSys} \times \mathrm{LocSys})} \mathrm{QCoh}(\mathrm{LocSys}) \rightarrow \\ &\mathrm{QCoh}(\mathrm{LocSys}) \otimes_{\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{LocSys}}}, \mathrm{QCoh}(\mathrm{LocSys} \times \mathrm{LocSys}), \Delta_{\mathrm{LocSys}}} \mathrm{QCoh}(\mathrm{LocSys}) \simeq \\ &\simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}) \otimes_{\mathcal{E}_{V,x}|_{\mathrm{LocSys}^{\mathrm{arithm}}}} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}) \xrightarrow{\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, -)} \mathrm{Vect}, \end{aligned}$$

where the first arrow

$$\mathrm{Vect} \rightarrow D(\mathrm{Bun}_G) \otimes_{\mathrm{QCoh}(\mathrm{LocSys})} D(\mathrm{Bun}_G),$$

corresponds to the unit of the self-duality datum for $D(\mathrm{Bun}_G)$ as a module category over $\mathrm{QCoh}(\mathrm{LocSys})$, and the third arrow is induced by the arrow

$$\Phi : D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \rightarrow \mathrm{QCoh}(\mathrm{LocSys}),$$

defined as follows: for $\mathcal{E} \in \mathrm{QCoh}(\mathrm{LocSys})$ and $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, we have

$$(2.3) \quad \Gamma(\mathrm{LocSys}, \mathcal{E} \otimes \Phi(\mathcal{M}_1, \mathcal{M}_2)) = \mathrm{H} \left(\mathrm{Bun}_G, \mathrm{Frob}_{\mathrm{Bun}_G}^! (\mathcal{M}_1) \overset{!}{\otimes} (\mathcal{E} \star \mathcal{M}_2) \right).$$

2.3.4. Note that the last three lines in (2.2) can be replaced by

$$\begin{aligned} &(D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G)) \otimes_{\mathrm{QCoh}(\mathrm{LocSys} \times \mathrm{LocSys})} \mathrm{QCoh}(\mathrm{LocSys}) \rightarrow \\ &\rightarrow D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \xrightarrow{\Phi} \mathrm{QCoh}(\mathrm{LocSys}) \xrightarrow{\otimes_{\mathcal{E}_{V,x}}} \mathrm{QCoh}(\mathrm{LocSys}) \xrightarrow{\Gamma(\mathrm{LocSys}, -)} \mathrm{Vect}, \end{aligned}$$

while the composition

$$\begin{aligned} \mathrm{Vect} &\rightarrow D(\mathrm{Bun}_G) \otimes_{\mathrm{QCoh}(\mathrm{LocSys})} D(\mathrm{Bun}_G) \simeq \\ &\simeq (D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G)) \otimes_{\mathrm{QCoh}(\mathrm{LocSys} \times \mathrm{LocSys})} \mathrm{QCoh}(\mathrm{LocSys}) \rightarrow D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \end{aligned}$$

is the unit of the absolute self-duality of $D(\mathrm{Bun}_G)$, i.e.,

$$\mathrm{Vect} \xrightarrow{\overline{\mathbb{Q}}_\ell \mapsto \omega_{\mathrm{Bun}_G}} D(\mathrm{Bun}_G) \xrightarrow{(\Delta_{\mathrm{Bun}_G})^\bullet} D(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G).$$

Hence, the map in (2.2) identifies with

$$\begin{aligned} (2.4) \quad \mathrm{Vect} &\xrightarrow{\overline{\mathbb{Q}}_\ell \mapsto \omega_{\mathrm{Bun}_G}} D(\mathrm{Bun}_G) \xrightarrow{(\Delta_{\mathrm{Bun}_G})^\bullet} D(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq \\ &\simeq D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \xrightarrow{\Phi} \mathrm{QCoh}(\mathrm{LocSys}) \xrightarrow{\otimes_{\mathcal{E}_{V,x}}} \mathrm{QCoh}(\mathrm{LocSys}) \xrightarrow{\Gamma(\mathrm{LocSys}, -)} \mathrm{Vect}. \end{aligned}$$

2.3.5. Using (2.3) and using the fact that the functor $\mathcal{E}_{V,x} \star -$ on $D(\mathrm{Bun}_G)$ is the Hecke functor $H_{V,x}$, we rewrite the map in (2.4) as

$$\begin{aligned} \mathrm{Vect}^{\overline{\mathbb{Q}}_\ell \mapsto \omega_{\mathrm{Bun}_G}} &\xrightarrow{\quad} D(\mathrm{Bun}_G)^{(\Delta_{\mathrm{Bun}_G})^\bullet} D(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq \\ &\simeq D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \xrightarrow{\mathrm{Id} \otimes H_{V,x}} D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \xrightarrow{\mathrm{Frob}_{\mathrm{Bun}_G}^! \otimes \mathrm{Id}} D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \rightarrow \\ &\hspace{15em} D(\mathrm{Bun}_G)^{(\Delta_{\mathrm{Bun}_G})^!} \xrightarrow{H(\mathrm{Bun}_G, -)} \mathrm{Vect}. \end{aligned}$$

However, by base change, the latter is isomorphic to

$$H(\mathrm{Bun}_G \times_{\mathrm{Frob}_{\mathrm{Bun}_G}, \mathrm{Bun}_G \times \mathrm{Bun}_G} \mathcal{H}_x, \mathcal{S}_V),$$

as required.

3. HIGHER REPRESENTATIONS VS. USUAL REPRESENTATIONS

3.1. Representations of groups on categories.

3.1.1. Let H be a group (ind-scheme) over $\overline{\mathbb{F}}_q$. We consider the 2-category

$$H\text{-}\mathbf{mod} := D(H)\text{-}\mathbf{mod}$$

as an object of 2-DGCat.

3.1.2. Assume that H is defined over \mathbb{F}_q . Then the Frobenius endomorphism of H defines an endo-functor $\mathrm{Frob}_H^!$ of $H\text{-}\mathbf{mod}$.

We claim that

$$(3.1) \quad \mathrm{Tr}(\mathrm{Frob}_H^!, H\text{-}\mathbf{mod}) = D(H)_{\mathrm{Ad}_{\mathrm{Frob}(H)}},$$

where the subscript $\mathrm{Ad}_{\mathrm{Frob}(H)}$ means convariants with respect to the Frobenius-twisted conjugation.

Indeed,

$$\mathrm{Tr}(\mathrm{Frob}_H^!, H\text{-}\mathbf{mod}) \simeq D(H) \otimes_{\mathrm{Graph}_{\mathrm{Frob}_H}, D(H) \otimes D(H), \Delta_H} D(H),$$

which can be rewritten as

$$D(H) \otimes_{D(H)} \mathrm{Vect},$$

where $D(H)$ acts on itself by Frobenius-twisted conjugation:

$$h(h') = \mathrm{Frob}_H(h) \cdot h' \cdot h^{-1}.$$

3.2. Representations of Chevalley groups.

3.2.1. There are two main cases that we want to consider: one is when H is an algebraic group G , considered in this section, and the other is when H is loop group $G(\mathcal{K})$, where \mathcal{K} is a local field and G is reductive (considered in the next section).

For $H = G$, we will pretend that $G\text{-}\mathbf{mod}$ behaves as in the case of D-modules (and hopefully, Drinfeld's formalism will justify that).

Namely, that for $\mathcal{C} \in G\text{-}\mathbf{mod}$, the *-averaging functor defines an equivalence

$$(3.2) \quad \mathcal{C}_G \rightarrow \mathcal{C}^G.$$

3.2.2. Thus, in particular, we identify

$$D(G)_{\mathrm{Ad}_{\mathrm{Frob}}(G)} \simeq D(G)^{\mathrm{Ad}_{\mathrm{Frob}}(G)}.$$

Assume now that G is connected. Then by Lang's theorem, the $\mathrm{Ad}_{\mathrm{Frob}}(G)$ -action of G on itself is transitive and the stabilizer of $1 \in G$ is the finite group $G(\mathbb{F}_q)$. Hence, we obtain

$$D(G)^{\mathrm{Ad}_{\mathrm{Frob}}(G)} \simeq \mathrm{Rep}(G(\mathbb{F}_q)).$$

Thus, we obtain a canonical equivalence of categories:

$$(3.3) \quad \mathrm{Tr}(\mathrm{Frob}_G^!, G\text{-}\mathbf{mod}) = \mathrm{Rep}(G(\mathbb{F}_q)).$$

I.e., the category of representations of the Chevalley group $G(\mathbb{F}_q)$ arises from the 2-category $G\text{-}\mathbf{mod}$ as the trace of Frobenius.

3.2.3. Let \mathcal{C} be a dualizable object of $G\text{-}\mathbf{mod}$. The equation (3.2) implies that this condition is equivalent to being dualizable as a plain DG-category. Moreover, the dual of \mathcal{C} , viewed as a category, equipped with an action of G , identifies canonically with \mathcal{C}^\vee , equipped with the natural G -action.

Let \mathcal{C} be equipped with a functor

$$(3.4) \quad \mathrm{Frob}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$$

compatible with the monoidal endomorphism $\mathrm{Frob}_G^!$ of $D(G)$. Then, following Sect. 2.1.3 we attach to $(\mathcal{C}, \mathrm{Frob}_{\mathcal{C}})$ an object

$$\mathrm{Tr}_{\mathrm{Frob}_G^!, D(G)}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C}) \in \mathrm{Rep}(G(\mathbb{F}_q)).$$

By Sect. 1.2.2, the vector space underlying $\mathrm{Tr}_{\mathrm{Frob}_G^!, D(G)}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C})$ is simply $\mathrm{Tr}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C})$.

3.3. An example: Deligne-Lusztig representations.

3.3.1. Let G be reductive, and take $\mathcal{C} = D(G/B)$. We let the datum of (3.4) be the composition of the usual Frobenius on G/B and the pull-push along the correspondence

$$(3.5) \quad G/B \leftarrow (G/B \times G/B)_w \rightarrow G/B,$$

where $(G/B \times G/B)_w \subset (G/B \times G/B)$ be the subvariety of pairs of Borels in relative position $w \in W$.

Then the corresponding object

$$\mathrm{Tr}_{\mathrm{Frob}_G^!, D(G)}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C}) \in \mathrm{Rep}(G(\mathbb{F}_q))$$

is by definition the Deligne-Lusztig representation corresponding to w .

3.3.2. Note that the pull-push functor in (3.5) can also be interpreted as the convolution on the right with the w -costandard object on $B \backslash G/B$. In particular, it is an equivalence, and admits a right adjoint, which is also given by a kernel (convolution on the right with the w^{-1} -standard).

3.4. Traces.

3.4.1. Consider the trace of the *identity* endomorphism of $G\text{-mod}$. In a way similar to Sect. 3.1.2, the resulting category identifies with

$$D(G)_{\text{Ad}(G)},$$

and using (3.2), further with

$$D(G)^{\text{Ad}(G)}.$$

To a dualizable $\mathcal{C} \in G\text{-mod}$, we can thus attach an object

$$\chi(\mathcal{C}) \in D(G)^{\text{Ad}(G)}.$$

By construction, the $!$ -fiber of $\chi(\mathcal{C})$ at $g \in G$ equals the trace of the endo-functor of \mathcal{C} , given by g .

3.4.2. For example, for $\mathcal{C} = D(G/B)$,

$$\chi(\mathcal{C}) = \text{Spr},$$

where Spr denotes the Springer sheaf.

3.4.3. Suppose again that \mathcal{C} is endowed with a datum of (3.4). Assume, moreover, that the functor $\text{Frob}_{\mathcal{C}}$ admits a right adjoint. Then the construction from Sect. 1.2.1 gives rise to a map

$$(3.6) \quad \chi(\mathcal{C}) \rightarrow \text{Frob}^!(\chi(\mathcal{C})).$$

3.4.4. *Example.* One can show that for $(\mathcal{C}, \text{Frob}_{\mathcal{C}})$ as in Sect. 3.3.1, the resulting map (3.6) is given by the composing the natural Weil structure on Spr and the automorphism of Spr , given by the element $w \in W$. This is so, because this is obviously so over the locus of G , consisting of regular semi-simple elements.

3.4.5. By imposing certain finiteness conditions on \mathcal{C} (in the spirit of *complete dualizability*) one can ensure that the objects

$$\chi(\mathcal{C}) \in D(G)^{\text{Ad}(G)} \text{ and } \text{Tr}_{\text{Frob}_G^!, D(G)}(\text{Frob}_{\mathcal{C}}, \mathcal{C}) \in \text{Rep}(G(\mathbb{F}_q))$$

are both compact.

In this case one can show that

$$\text{Tr}(\text{Frob}_G^!, \chi(\mathcal{C})),$$

which is an Ad-invariant function on $G(\mathbb{F}_q)$ equals the character of the representation $\text{Tr}_{\text{Frob}_G^!, D(G)}(\text{Frob}_{\mathcal{C}}, \mathcal{C})$.

3.4.6. In particular, this shows that the character of the Deligne-Lusztig representation corresponding to w is equal to the function obtained from Spr , by twisting its Weil structure by w .

4. THE LOCAL STORY

4.1. Categories acted on by the loop group.

4.1.1. We now consider the situation of Sect. 3.1 for $H = G(\mathcal{K})$.

In this case (at least in the case of D-modules), we still have an equivalence

$$(4.1) \quad \mathcal{C}_{G(\mathcal{X})} \rightarrow \mathcal{C}^{G(\mathcal{X})}.$$

But this equivalence depends on the choice of a point of a parahoric subgroup P . We make this choice to be $P = G(\mathcal{O})$.

4.1.2. Namely, the equivalence

$$(4.2) \quad \mathcal{C}_P \rightarrow \mathcal{C}^P$$

is the same as in (3.2). The category \mathcal{C}_P is acted by the Hecke category $\mathcal{H} := D(P \backslash G(\mathcal{K})/P)$. We interpret

$$\mathcal{C}^{G(\mathcal{K})} \simeq \text{Funct}_{\mathcal{H}}(D(\text{pt}/P), \mathcal{C}^P),$$

and using (4.2) also

$$\mathcal{C}_{G(\mathcal{K})} \simeq \mathcal{C}^P \otimes_{\mathcal{H}} D(\text{pt}/P).$$

Now, the *ind-properness* of $G(\mathcal{K})/P$ implies that for any $\mathcal{C}' \in \mathcal{H}\text{-mod}$, we have a canonical equivalence

$$\text{Funct}_{\mathcal{H}}(D(\text{pt}/P), \mathcal{C}') \simeq \mathcal{C}' \otimes_{\mathcal{H}} D(\text{pt}/P),$$

characterized by the fact that the natural projection

$$\mathcal{C}' \simeq \mathcal{C}' \otimes_{D(\text{pt}/P)} D(\text{pt}/P) \rightarrow \mathcal{C}' \otimes_{\mathcal{H}} D(\text{pt}/P)$$

becomes identified with the *left* adjoint of the forgetful functor

$$\text{Funct}_{\mathcal{H}}(D(\text{pt}/P), \mathcal{C}') \rightarrow \text{Funct}_{D(\text{pt}/P)}(D(\text{pt}/P), \mathcal{C}') \simeq \mathcal{C}'.$$

4.1.3. If we choose a different parabolic, say $P' \subset P$, the two identifications in (4.1) will differ by $\det(H_c(P/P', k))$. I.e., the choice in (4.1) is really that of a trivialization of the dimension torsor on $G(\mathcal{K})$.

4.2. The local category.

4.2.1. We now assume that the ground field is $\overline{\mathbb{F}}_q$ and that G is defined over \mathbb{F}_q . Combing (4.1) with (3.1), we obtain an identification

$$(4.3) \quad \text{Tr}(\text{Frob}_{G(\mathcal{K})}^!, G(\mathcal{K})\text{-mod}) \simeq D(G(\mathcal{K}))^{\text{Ad}_{\text{Frob}}(G(\mathcal{K}))}.$$

The category in the right-hand side in (4.3) is *the* category on the geometric side of Langlands that we propose to study in a joint project with Alain Genestier and Vincent Lafforgue.

This category is related to one appearing in Fargues' conjecture [Far].

4.2.2. Taking the $!$ -fiber at $1 \in G(\mathcal{K})$ we obtain a functor

$$\mathbf{R} : D(G(\mathcal{K}))^{\text{Ad}_{\text{Frob}}(G(\mathcal{K}))} \rightarrow \text{Rep}(G(\mathcal{K})(\mathbb{F}_q)).$$

However, since Lang's theorem fails for $G(\mathcal{K})$, the above functor \mathbf{R} is no longer an equivalence.

4.2.3. Let \mathcal{C} be a dualizable object of $G(\mathcal{K})\text{-mod}$. As in the case of a finite-dimensional G , the equivalence (4.1) implies that the above dualizability condition is equivalent to \mathcal{C} being dualizable as a plain object of DGCat .

Moreover, the dual of \mathcal{C} as a category acted on by $G(\mathcal{K})$ identifies with \mathcal{C}^\vee , equipped with the natural $G(\mathcal{K})$ -action (but this identification depends on the choice of a trivialization of the dimension torsor on $G(\mathcal{K})$).

4.2.4. Assume that \mathcal{C} is equipped with a functor

$$\mathrm{Frob}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$$

compatible with the monoidal endomorphism $\mathrm{Frob}_{G(\mathcal{K})}^!$ of $D(G(\mathcal{K}))$.

We obtain that to this datum one can canonically attach an object

$$\mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}^!, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C}) \in D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}}.$$

The $!$ -fiber of $\mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}^!, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C})$ at $1 \in G(\mathcal{K})$, viewed as a mere vector space identifies with

$$\mathrm{Tr}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C}).$$

This identification also depends on the choice of a trivialization of the dimension torsor on $G(\mathcal{K})$.

The group $G(\mathcal{K})(\mathbb{F}_q)$ acts on $\mathrm{Tr}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C})$ by transport of structure. This action coincides with one obtained by considering the object

$$\mathrm{R} \left(\mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}^!, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C}) \right).$$

4.3. Action of local systems.

4.3.1. Let $\mathrm{LocSys}_{\mathrm{loc}}$ be the stack of local systems on the punctured disc. The local version of Hecke action says that the algebra object $\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}})$ -**mod** in 2-DGCat acts on $G(\mathcal{K})$ -**mod**.

This structure can be also formulated as saying that the symmetric monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}})$ acts canonically on every object $\mathcal{C} \in G(\mathcal{K})$ -**mod**.

4.3.2. By Sect. 1.2.1 we obtain an action of the monoidal category

$$\mathrm{Tr}(\mathrm{Frob}_{\mathrm{LocSys}_{\mathrm{loc}}}^*, \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}})$$
-**mod**) $\simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})$

on

$$\mathrm{Tr}(\mathrm{Frob}_{G(\mathcal{K})}^!, G(\mathcal{K})$$
-**mod**) = $D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}}$.

This is the action that we aim to construct in a joint project with Alain and Vincent.

4.3.3. Let $(G(\mathcal{K})$ -**mod**)_{temp} be the tempered part of $G(\mathcal{K})$ -**mod**. We expect that the 2-category $(G(\mathcal{K})$ -**mod**)_{temp}, when viewed as a module over $\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}})$, is free on one generator.

Define

$$D(G(\mathcal{K}))_{\mathrm{temp}}^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}} := \mathrm{Tr}(\mathrm{Frob}_{G(\mathcal{K})}^!, G(\mathcal{K})$$
-**mod**)_{temp}.

We obtain that the category $D(G(\mathcal{K}))_{\mathrm{temp}}^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}}$ is a free module on one generator over $\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})$.

4.4. Imposing $G(\mathcal{O})$ -invariance.

4.4.1. It is an expectation of the local *geometric* Langlands that the 2-category

$$G(\mathcal{K})\text{-}\mathbf{mod} \otimes_{\mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc}})} \mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}})\text{-}\mathbf{mod}$$

is equivalent to $\mathcal{H}\text{-}\mathbf{mod}$, where

$$\mathcal{H} = D(G(\mathcal{O}) \backslash G(\mathcal{K}) / G(\mathcal{O}))$$

is the spherical Hecke category.

In the above formula $\mathrm{LocSys}_{\mathrm{loc-unr}}$ is the stack of unramified geometric local systems on the formal disc around x (i.e., the classifying stack pt/\check{G}).

Remark 4.4.2. Similarly, we expect that

$$G(\mathcal{K})\text{-}\mathbf{mod} \otimes_{\mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc}})} \mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc-tame}})\text{-}\mathbf{mod}$$

is equivalent to the 2-category of module categories over the Iwahori-Hecke category.

4.4.3. Taking the traces of the Frobenius (here I'm assuming that the operation of taking the trace of an endo-functor is compatible with geometric realizations under certain hypothesis), we obtain that the category

$$\mathrm{Tr}(\mathrm{Frob}_{G(\mathcal{K})}^!, G(\mathcal{K})\text{-}\mathbf{mod}) \otimes_{\mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}}^{\mathrm{arithm}})$$

is canonically equivalent to the Hochschild homology of \mathcal{H} regarded as a bi-module category over itself, with the left action being the monoidal operation and the right action is twisted by $\mathrm{Frob}^!$.

One shows that the latter Hochschild homology identifies with $\mathbf{QCoh}(\check{G}/\mathrm{Ad}(\check{G}))$. Indeed, this would be obviously so if instead of \mathcal{H} we took the naive Hecke category, i.e., $\mathrm{Rep}(\check{G})$. Now, the quasi-coherent interpretation of \mathcal{H} shows that the derived stuff disappears under the Frobenius (note, however, that a parallel fact fails completely if instead of the Frobenius we consider the identity functor, i.e., when we consider the plain Hochschild homology of \mathcal{H}).

Combining with (4.3), we obtain an equivalence

$$(4.4) \quad D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}}(G(\mathcal{K}))} \otimes_{\mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}}^{\mathrm{arithm}}) \simeq \mathbf{QCoh}(\check{G}/\mathrm{Ad}(\check{G})).$$

4.4.4. Let $(\mathcal{C}, \mathrm{Frob}_{\mathcal{C}})$ be as in Sect. 4.2.4. On the one hand, consider the image of $\mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}^!, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C})$ under the functor

$$D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}}(G(\mathcal{K}))} \rightarrow D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}}(G(\mathcal{K}))} \otimes_{\mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathbf{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}}^{\mathrm{arithm}}).$$

On the other hand, consider $\mathcal{C}^{G(\mathcal{O})}$ as an object of $\mathcal{H}\text{-}\mathbf{mod}$, and consider

$$\mathrm{Tr}_{\mathrm{Frob}^!, \mathcal{H}}(\mathrm{Frob}_{\mathcal{C}}, \mathcal{C}^{G(\mathcal{O})}) \in \mathbf{QCoh}(\check{G}/\mathrm{Ad}(\check{G})).$$

It follows from the definitions, that the above two objects are identified under (4.4).

4.5. Local vs global compatilby.

4.5.1. Let X be again a complete curve, and $x \in X$ a rational point. Consider $\mathrm{Bun}_G^{\mathrm{level}_x}$ as a stack equipped with an action of $G(\mathcal{K})$, where \mathcal{K} is the local field at x . So $D(\mathrm{Bun}_G^{\mathrm{level}_x})$ is naturally an object of $G(\mathcal{K})$ -**mod**, equipped with a compatible Frobenius.

Hence, by Sect. 4.2.4, we have a canonically defined object

$$(4.5) \quad \mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathrm{Bun}_G^{\mathrm{level}_x}}^!, D(\mathrm{Bun}_G^{\mathrm{level}_x})) \in D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}}.$$

By construction, we have

$$\mathrm{R}\left(\mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathrm{Bun}_G^{\mathrm{level}_x}}^!, D(\mathrm{Bun}_G^{\mathrm{level}_x}))\right) \simeq \mathrm{Func}(\mathrm{Bun}_G^{\mathrm{level}_x}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell),$$

as representations of $G(\mathcal{K})(\mathbb{F}_q)$.

4.5.2. Let $\mathrm{LocSys}_{\mathrm{glob}}$ denote the stack of local systems on $X - x$. We have the geometric Hecke action of the monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}})$ on $D(\mathrm{Bun}_G^{\mathrm{level}_x})$.

As in Sect. 1.3, this action defines an action of the commutative algebra

$$\Gamma(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}, \mathcal{O}_{\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}})$$

on the object (4.5).

4.5.3. In particular, by functoriality, we obtain an action of $\Gamma(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}, \mathcal{O}_{\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}})$ on $\mathrm{Func}(\mathrm{Bun}_G^{\mathrm{level}_x}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$.

However, the above action of $\Gamma(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}, \mathcal{O}_{\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}})$ on $\mathrm{Func}(\mathrm{Bun}_G^{\mathrm{level}_x}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ produces nothing essentially new as compared to the situation of Sect. 1.3: this is the action obtained from the action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}})$ on $D(\mathrm{Bun}_G^{\mathrm{level}_x})$ by taking traces of the Frobenius endo-morphisms.

4.5.4. Here is, however, an object that did not appear previously: the procedure from Sect. 2.2 allows to refine the object (4.5) to an object

$$\mathrm{Tr}_{\mathrm{Frob}_{G(\mathcal{K})}, D(G(\mathcal{K}))}(\mathrm{Frob}_{\mathrm{Bun}_G^{\mathrm{level}_x}}^!, D(\mathrm{Bun}_G^{\mathrm{level}_x})) \in D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}}$$

to an object

$$\mathrm{Drinf}_x \in D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}} \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}).$$

4.5.5. Here is how the above object Drinf_x is related to

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}})$$

from Sect. 2.2.

Consider the image of Drinf_x under the functor

$$\begin{aligned} & D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}} \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}) \rightarrow \\ \rightarrow & D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}} \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}) \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}}). \end{aligned}$$

Note, however, that the latter category identifies with

$$D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}(G(\mathcal{K}))}} \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}}^{\mathrm{arithm}}) \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}}),$$

and further by (4.4) with

$$\begin{aligned} \mathrm{QCoh}(\check{G}/\mathrm{Ad}(\check{G})) &\otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc-unr}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}}) \simeq \\ &\simeq \mathrm{QCoh}(\check{G}/\mathrm{Ad}(\check{G})) \otimes_{\mathrm{QCoh}(\check{G}/\mathrm{Ad}(\check{G}))} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}}) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}}). \end{aligned}$$

Thus, we obtain a functor

$$D(G(\mathcal{K}))^{\mathrm{Ad}_{\mathrm{Frob}}(G(\mathcal{K}))} \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{loc}}^{\mathrm{arithm}})} \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob}}^{\mathrm{arithm}}) \rightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\mathrm{glob-unr}}^{\mathrm{arithm}}).$$

Now, it follows from Sect. 4.4.4 that the image of Drinf_x under the above functor identifies with Drinf .

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