

CONTRACTIBILITY OF THE SPACE OF RATIONAL MAPS

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ABSTRACT. We define an algebro-geometric model for the space of rational maps from a smooth curve X to an algebraic group G , and show that this space is homologically contractible. As a consequence, we deduce that the moduli space Bun_G of G -bundles on X is uniformized by the appropriate rational version of the affine Grassmannian, where the uniformizing map has contractible fibers.

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Date: January 17, 2012.

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INTRODUCTION

0.1. **The origins of the problem.** Let X be a smooth, connected and complete curve, and G a reductive group (over an algebraically closed field k of characteristic 0). A fundamental object of study in the Geometric Langlands program is the moduli stack of G -bundles on X , denoted Bun_G .

This paper arose from the desire to approximate Bun_G by its local cousin, namely the *adelic* Grassmannian Gr .

0.1.1. On the first pass, we shall loosely understand Gr as the moduli space of G -bundles on X equipped with a *rational* trivialization. The local nature of Gr expresses itself in that if we specify the locus $\overset{\circ}{X} \subset X$ over which our trivialization is regular, the corresponding subspace of Gr is a product of copies of the affine Grassmannians Gr_x over the missing point $x \in X - \overset{\circ}{X}$, see [BD2], 4.3.14. So, one perceives Gr as an inherently simpler object than Bun_G .

In fact, one can think of Bun_G as the quotient of Gr by the group of rational maps from X to G , denoted $\mathbf{Maps}(X, G)^{\text{rat}}$.

So, for example, line bundles on Bun_G can be thought of as line bundles on Gr equipped with the data of equivariance with respect to the group $\mathbf{Maps}(X, G)^{\text{rat}}$. However, the following crucial observation was made in [BD2]:

If \mathcal{Y} is a space acted on by $\mathbf{Maps}(X, G)^{\text{rat}}$, and \mathcal{L} is a line bundle on \mathcal{Y} , then \mathcal{L} has a unique equivariant structure with respect to $\mathbf{Maps}(X, G)^{\text{rat}}$.

The last observation leads one to think that the group $\mathbf{Maps}(X, G)^{\text{rat}}$, although wildly infinite-dimensional, for some purposes behaves like the point-scheme.¹ E.g., in [BD2], 4.3.13 it is shown that any function on $\mathbf{Maps}(X, G)^{\text{rat}}$ is constant.

¹The author learned this idea from conversations with A. Beilinson, who in turn attributed it to Drinfeld.

0.1.2. In this paper we take up the task of establishing some of the point-like properties of $\mathbf{Maps}(X, G)^{\text{rat}}$. But we will go in a direction, slightly different than the one mentioned above. Namely, we will be interested not so much in line bundles or other quasi-coherent sheaves on $\mathbf{Maps}(X, G)^{\text{rat}}$ (and, respectively, Bun_G and Gr), but in D-modules.

Namely, we will show that $\mathbf{Maps}(X, G)^{\text{rat}}$ is *contractible*. A concise way to formulate this is by saying that the cohomology with compact supports of the dualizing sheaf on $\mathbf{Maps}(X, G)^{\text{rat}}$ is isomorphic to scalars.

(The word “contractibility” is somewhat of a misnomer: in this paper we only establish that $\mathbf{Maps}(X, G)^{\text{rat}}$ is homologically contractible. Full contractibility should also include the statement that every local system on $\mathbf{Maps}(X, G)^{\text{rat}}$ is trivial, see Remark 0.5.6. The latter can also be proved, but we will not need it for the applications we have in mind in this paper.)

0.2. **Uniformization of Bun_G .** We can regard the map $\pi : \text{Gr} \rightarrow \text{Bun}_G$ as an instance of uniformization of a “global” object by a “local” one. Thus, we obtain that not only does Gr uniformize Bun_G , but it does so with contractible fibers.

As an almost immediate consequence of the contractibility of the fibers of π , we obtain that the pullback functor from the category of D-modules on Bun_G to that on Gr is fully faithful. The latter fact itself has multiple consequences.

0.2.1. First, we recall that Gr is ind-proper. We will show that this implies that Bun_G also exhibits properties of proper schemes with respect to cohomology of D-modules and quasi-coherent sheaves: in either context we will show that coherent objects get sent by the cohomology functor to finite-dimensional vector spaces.

Moreover, we will show that this is true not only for Bun_G itself, but for a certain family of its open substacks, introduced in [DrGal].

0.2.2. Second, we will show that if one considers the category of D-modules on Bun_G , *equivariant* with respect to the Hecke groupoid, then this category is equivalent to Vect .

0.2.3. Third, the contractibility of the fibers of π implies that the two spaces have isomorphic cohomology. Using this fact, in Sect. 5 we will show how this allows one to re-derive the Atiyah-Bott formula for the cohomology of Bun_G .

Furthermore, in the forthcoming work [GaLu] it will be shown that the same game can be played when X is a curve over a finite field and D-modules are replaced by ℓ -adic sheaves. Then the expression for the trace of Frobenius on the homology of Bun_G in terms of that on Gr leads to a geometric proof of Weil’s conjecture on the Tamagawa number of the automorphic space corresponding to G and the global field of rational functions on X .

0.3. Uniformization of Bun_G in differential geometry and topology.

0.3.1. We should mention a curious discrepancy between the above way to calculate the cohomology of Bun_G , and the classical differential-geometric method (see, however, Sect. 0.3.2 below).

Namely, in differential geometry (for G semi-simple and simply-connected), one uniformizes the analytic space underlying Bun_G by the space $\text{Conn}_{\bar{\gamma}}(\mathcal{P}_{C^\infty}^0)$ of complex structures on the trivial (C^∞) G -bundle $\mathcal{P}_{C^\infty}^0$ on X . The space Bun_G is the quotient of $\text{Conn}_{\bar{\gamma}}(\mathcal{P}_{C^\infty}^0)$ by the group of gauge transformations $\mathbf{Maps}_{C^\infty}(X, G)$.

The space $\text{Conn}_{\bar{\gamma}}(\mathcal{P}_{C^\infty}^0)$ is contractible, and hence the cohomology of Bun_G is isomorphic to that of the classifying space $B\mathbf{Maps}_{C^\infty}(X, G)$ of $\mathbf{Maps}_{C^\infty}(X, G)$.

So, in differential geometry, the uniformizing space is topologically trivial, and the homotopy type of Bun_G is encoded by the structure group.

By contrast, in algebraic geometry, the structure group is contractible, and the homotopy type of Bun_G is encoded by the uniformizing space.

0.3.2. A closer analogy to the uniformization of Bun_G by Gr is provided by Lurie’s non-abelian Poincaré duality, see [Lu1], Sect. 5.3.6.

Taking our source to be the Riemann surface corresponding to X , and our target the topological space BG , the assertion of *loc.cit.* says that the homotopy type of $\mathbf{Maps}_{C^\infty}(X, BG)$ is isomorphic to the “chiral homology” of a certain factorization algebra on X , whose fiber at a finite collection of points $x_1, \dots, x_n \in X$ is the space of maps from $\mathbf{B}_{x_1} \times \dots \times \mathbf{B}_{x_n}$ to BG , trivialized on $\mathbf{S}_{x_1} \times \dots \times \mathbf{S}_{x_n}$, where \mathbf{B}_x denoted a ball in X around x , and \mathbf{S}_x is its boundary.

Now, by Sect. 0.3.1, the homotopy type of Bun_G is isomorphic to that of $B\mathbf{Maps}_{C^\infty}(X, G)$, which can be shown to be isomorphic to

$$\mathbf{Maps}_{C^\infty}(X, BG).$$

Further, the homotopy type of $\mathbf{Maps}_{C^\infty}((\mathbf{B}_x; \mathbf{S}_x), BG)$ is isomorphic to the fiber of Gr over x , denoted Gr_x .

Thus, non-abelian Poincaré duality gives another way to obtain an expression for the homology of Bun_G . In a sense, what we do in this paper is to show that this picture goes through also in the context of algebraic geometry.

Remark 0.3.3. As we have just seen, the homotopy type of (the analytic space underlying) Bun_G can be realized as the space of C^∞ maps from X to BG . We should emphasize that this is not a general fact, but rather specific to our target space being BG . One cannot expect a close relation between spaces of algebraic and C^∞ maps from X to general targets.

0.4. **D-modules on “spaces”.** In order to proceed to a more rigorous discussion, we need to address the following question: what do we mean by “the space of G -bundles equipped with a rational trivialization” and “the space of rational maps from X to G ”?

0.4.1. First, what do we mean by a “space”? By definition, a “space” is a prestack, which in this paper is an arbitrary functor from the category affine schemes of finite type over k to the category of ∞ -groupoids, denoted $\infty\text{-Grpd}$.² We shall denote this category by PreStk .

0.4.2. As is explained in [GL:Crystals], there is a canonically defined functor

$$\mathfrak{D}_{\text{Sch}^{\text{aff}}}^! : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow \text{DGCat},$$

that attaches to an affine scheme S of finite type the DG category $\mathfrak{D}(S)$ of D-modules on S , and to a map $f : S_1 \rightarrow S_2$ the functor

$$f^! : \mathfrak{D}(S_2) \rightarrow \mathfrak{D}(S_1).$$

This functor automatically gives rise to a functor $\mathfrak{D}_{\text{PreStk}}^! : \text{PreStk}^{\text{op}} \rightarrow \text{DGCat}$, by the procedure of right Kan extension, see [GL:Crystals, Sect. 2.3]. Namely,

$$\mathfrak{D}(\mathcal{Y}) := \lim_{S \in (\text{Sch}_{\mathcal{Y}}^{\text{aff}})^{\text{op}}} \mathfrak{D}(S),$$

where the limit is taken with respect to the $!$ -pullback functors.

²By a prestack we shall always understand a “higher” prestack, i.e., one taking values in ∞ -groupoids, rather than ordinary groupoids.

Informally, for $\mathcal{Y} \in \text{PreStk}$, the data of an object $\mathcal{M} \in \mathfrak{D}(\mathcal{Y})$ is that of a family of objects $\mathcal{M}_S \in \mathfrak{D}(S)$ for every affine scheme S endowed with a map $S \xrightarrow{\phi} \mathcal{Y}$, and a homotopy-coherent system of isomorphisms $f^!(\mathcal{M}_S) \simeq \mathcal{M}_{S'}$ for every $f : S' \rightarrow S$ and an isomorphism between the resulting maps

$$S' \xrightarrow{\phi'} \mathcal{Y} \text{ and } S' \xrightarrow{f} S \xrightarrow{\phi} \mathcal{Y}.$$

In particular, for a morphism $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ there is a well-defined functor

$$f^! : \mathfrak{D}(\mathcal{Y}_2) \rightarrow \mathfrak{D}(\mathcal{Y}_1).$$

When \mathcal{Y} is a non-affine scheme, the Zariski descent property of the category of D-modules implies that the category $\mathfrak{D}(\mathcal{Y})$ defines above recovers the usual DG category of D-modules on \mathcal{Y} .

0.4.3. For example, for any \mathcal{Y} , the category $\mathfrak{D}(\mathcal{Y})$ contains a canonical object, denoted $\omega_{\mathcal{Y}}$, referred to as the dualizing sheaf on \mathcal{Y} , and defined as $p_{\mathcal{Y}}^!(k)$, where $p_{\mathcal{Y}}$ is the map $\mathcal{Y} \rightarrow \text{pt}$, and k is the generator of $\mathfrak{D}(\text{pt}) \simeq \text{Vect}$.

We shall say that \mathcal{Y} is homologically contractible if the functor

$$(0.1) \quad \text{Vect} \rightarrow \mathfrak{D}(\mathcal{Y}), \quad V \mapsto V \otimes \omega_{\mathcal{Y}}$$

is fully faithful.

The category $\mathfrak{D}(\mathcal{Y})$ contains a full subcategory $\mathfrak{D}(\mathcal{Y})_{\text{loc.const}}$ consisting of objects, whose pullback on any affine scheme S mapping to \mathcal{Y} belongs to the full subcategory of $\mathfrak{D}(S)$ generated by Verdier duals of lisse D-modules with regular singularities.

We shall say that \mathcal{Y} is contractible if the functor (0.1) is an equivalence.

0.4.4. When our ground field k is \mathbb{C} , there exists a canonically defined functor

$$\mathcal{Y} \mapsto \mathcal{Y}(\mathbb{C})^{\text{top}} : \text{PreStk} \rightarrow \infty\text{-Grpd},$$

where $\infty\text{-Grpd}$ is thought of as the category of homotopy types.

Namely, the above functor is the *left* Kan extension of the functor

$$\text{Sch} \rightarrow \infty\text{-Grpd},$$

that assigns to S the homotopy type of the analytic space underlying $S(\mathbb{C})$.

One can show that \mathcal{Y} is homologically contractible (resp., contractible) in the sense of Sect. 0.4.3 if and only if $H_{\bullet}(\mathcal{Y}(\mathbb{C})^{\text{top}}, \mathbb{Q}) \simeq \mathbb{Q}$ (resp., if the rational homotopy type of $\mathcal{Y}(\mathbb{C})^{\text{top}}$ is trivial).

0.4.5. However, it is no surprise that unless we impose some additional conditions on \mathcal{Y} , the category $\mathfrak{D}(\mathcal{Y})$ will be rather intractable:

The closer a prestack \mathcal{Y} is to being a scheme, the more manageable the category $\mathfrak{D}(\mathcal{Y})$ is. What is even more important is the functorial properties of $\mathfrak{D}(-)$ with respect to morphisms $g : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$. I.e., the closer a given morphism is given to being schematic, the more we can say about the behavior of the direct image functor.

One class of prestacks for which the category $\mathfrak{D}(\mathcal{Y})$ is really close to the case of schemes is that of indschemes. By definition, an indscheme is a prestack that can be exhibited as a *filtered* direct limit of prestacks representable by schemes with transition maps being closed embeddings.

A wider class is that of pseudo-indchemes, where the essential difference is that we omit the filteredness condition. We discuss this notion in some detail in Sect. 1.2.

On a technical note, we should remark that “filtered” vs. “non-filtered” makes a huge difference. For example, an indscheme perceived as a functor $\text{Sch}^{\text{aff}} \rightarrow \infty\text{-Grpd}$ takes values in 0-groupoids, i.e., $\text{Set} \subset \infty\text{-Grpd}$, whereas this is no longer true for pseudo-indchemes. We should also add it is crucial that we consider our functors with values in $\infty\text{-Grpd}$ and not truncate them to ordinary groupoids or sets: this is necessary to obtain reasonably behaved (derived) categories of D-modules.

0.5. Spaces of rational maps and the Ran space.

0.5.1. The first, and perhaps, geometrically the most natural, definition of the prestack corresponding to Gr and $\mathbf{Maps}(X, G)^{\text{rat}}$ is the following:

For Gr , to an affine scheme S we attach the set of pairs (\mathcal{P}, α) , where \mathcal{P} is a G -bundle on $S \times X$, and α is a trivialization of \mathcal{P} defined over an open subset $U \subset S \times X$, whose intersection with every geometric fiber of $S \times X$ over S is non-empty.

For $\mathbf{Maps}(X, G)^{\text{rat}}$, to an affine scheme S we attach the set of maps $m : U \rightarrow G$, where U is as above.

The problem is, however, that the above prestacks are not indchemes. Neither is it clear that they are pseudo-indchemes. So, although the categories of D-modules on the above spaces are well-defined, we do not *a priori* know how to make calculations in them, and in particular, how to prove the contractibility of $\mathbf{Maps}(X, G)^{\text{rat}}$ (see, however, Sect. 0.6.3 below).

0.5.2. An alternative device to handle spaces such as Gr or $\mathbf{Maps}(X, G)^{\text{rat}}$ has been suggested in [BD1], and is known as the Ran space.

The underlying idea is that instead of just talking about *something* being rational, we specify the finite collection of points of X , outside of which this *something* is regular.

0.5.3. The basic object is the Ran space itself, denoted $\text{Ran } X$, which is defined as follows. By definition, $\text{Ran } X$ is the colimit (a.k.a. direct limit) in PreStk of the diagram of schemes $I \mapsto X^I$, where I runs through the category $(\text{fSet})^{\text{op}}$ opposite to that of finite sets and surjective maps between them.

Similarly, for an arbitrary target scheme Y , one defines the space $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ as the colimit, over the same index category, of the spaces $\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}$, where the latter is the prestack that assigns to a test scheme S the data of an S -point x^I of X^I and a map

$$m : (S \times X - \{x^I\}) \rightarrow Y,$$

where $\{x^I\}$ denotes the incidence divisor in $S \times X$.

The point is that if Y is an affine scheme, each $\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}$ is an indscheme, and hence $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ is a pseudo-indscheme.³

In a similar fashion, we define the space $\text{Gr}_{\text{Ran } X}$ as the colimit over the same index category $(\text{fSet})^{\text{op}}$ of the indchemes Gr_{X^I} , where the latter is the moduli space of the data $(x^I, \mathcal{P}, \alpha)$ where x^I is as above, \mathcal{P} is a principal G -bundle on $S \times X$, and α is a trivialization of \mathcal{P} on $S \times X - \{x^I\}$.

³We are grateful to J. Barlev for explaining this point of view to us: that instead of the functor $I \mapsto \mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}$, for most practical purposes it is more efficient to just remember its colimit, i.e., $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$.

Although, as we emphasized above, in the formation of the spaces $\text{Ran } X$, $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$, and $\text{Gr}_{\text{Ran } X}$, we must take the colimit in the category $\infty\text{-Grpd}$, i.e., a priori, the value of our functor on a test affine scheme will be an infinity-groupoid, it will turn out that in the above cases, our functors take values in $\text{Set} \subset \infty\text{-Grpd}$.

0.5.4. It is with the above version of the space of rational maps given by $\mathbf{Maps}(X, G)_{\text{Ran } X}^{\text{rat}}$ that we prove its homological contractibility. The main result of this paper reads:

Theorem 0.5.5. *Let Y be an affine scheme, which is connected and can be covered by open subsets, each which is isomorphic to an open subset of the affine space \mathbb{A}^n for some fixed n . Then the space $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ is homologically contractible.*

Remark 0.5.6. Although we do not prove it in this paper, as was mentioned earlier, one can show that under the same assumptions on Y , the space $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ is actually contractible in the sense of Sect. 0.4.3. Moreover, when $k = \mathbb{C}$, one can show that the resulting homotopy is actually trivial, and not just rationally trivial.

It is tempting to conjecture that the corresponding facts (homological contractibility, contractibility, and the triviality of homotopy type for $k = \mathbb{C}$) hold more generally: i.e., that it sufficient to require that Y be smooth and birationally equivalent to \mathbb{A}^n .

0.5.7. Let us briefly indicate the strategy of the proof of Theorem 0.5.5.

First, we consider the case when $Y = \mathbb{A}^n$. Then the theorem is proved by a direct calculation: the corresponding space $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ is essentially comprised of affine spaces.

We then consider the case when Y can be realized as an open subset of \mathbb{A}^n . We show that the corresponding map

$$\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, \mathbb{A}^n)_{\text{Ran } X}^{\text{rat}}$$

is a homological equivalence. The idea is that the complement is “of infinite codimension”.

Finally, we show that if U_α is a Zariski cover of Y , and U_\bullet is its Čech nerve, then the map

$$|\mathbf{Maps}(X, U_\bullet)_{\text{Ran } X}^{\text{rat}}| \rightarrow \mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$$

is also a homological equivalence (here $|-|$ denotes the functor of geometric realization of a simplicial object).

Since, by the above, each term in $\mathbf{Maps}(X, U_\bullet)_{\text{Ran } X}^{\text{rat}}$ is homologically contractible, we deduce the corresponding fact for $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$.

0.6. Should $\text{Ran } X$ appear in the story?

0.6.1. Let us note that the appearance of the Ran space allows us to connect spaces such as $\text{Gr}_{\text{Ran } X}$ to chiral/factorization algebras of [BD1] (see also [FrGa], where the derived version of chiral/factorization algebras is discussed in detail).

For example, the direct image of the dualizing sheaf under the forgetful map

$$\text{Gr}_{\text{Ran } X} \rightarrow \text{Ran } X$$

is a factorization algebra. It is the latter fact that allows one to connect the cohomology of $\text{Gr}_{\text{Ran } X}$, and hence of Bun_G , with the Atiyah-Bott formula.

And in general, the factorization property of Gr provides a convenient tool of interpreting various cohomological questions on Bun_G to that of calculation of chiral homology of chiral algebras.

In particular, in a subsequent paper we will show how this approach allows to prove that chiral homology of the integrable quotient of the Kac-Moody chiral algebra is isomorphic to the dual vector space to that of the cohomology of the corresponding line bundle on Bun_G .

The (dual vector space of the) 0-th chiral homology of the above chiral algebra is known under the name of “conformal blocks of the WZW model”. The fact that WZW conformal blocks are isomorphic to the space of global sections of the corresponding line bundle on Bun_G is well-known. However, the fact that the same isomorphism holds at the derived level has been a conjecture proposed in [BD1], Sect. 4.9.10; it was proved in *loc.cit.* for G being a torus.

So, in a sense it is “a good thing” to have the Ran space appearing in the picture.

0.6.2. However, the Ran space also brings something that can be perceived as a handicap of our approach.

Namely, consider for example the space of rational maps $X \rightarrow Y$, realized $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$. We see that along with the the data of a rational map, this space retains also the data of the locus outside of which it is defined. For example, for $Y = \text{pt}$, whereas we would like the space of rational maps $X \rightarrow \text{pt}$ to be isomorphic to pt , instead we obtain $\text{Ran } X$.

But the above problem can be remedied. Namely, $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ comes with an additional structure given by the action of $\text{Ran } X$, viewed as a semi-group in PreStk , given by enlarging the allowed locus of irregularity. Following [GL:Ran], we call it “the unital structure”.

The action of $\text{Ran } X$ on $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ gives rise to a semi-simplicial object

$$\dots \text{Ran } X \times \mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} \rightrightarrows \mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$$

of PreStk , and we let $\mathbf{Maps}(X, Y)_{\text{Ran } X, \text{indep}}^{\text{rat}}$ denote its geometric realization.

The effect of the passage

$$\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} \rightsquigarrow \mathbf{Maps}(X, Y)_{\text{Ran } X, \text{indep}}^{\text{rat}}$$

is that one gets rid of the extra data of remembering the locus of irregularity.⁴

In addition, in Sect. 2 we will show the forgetful functor

$$\mathfrak{D}(\mathbf{Maps}(X, Y)_{\text{Ran } X, \text{indep}}^{\text{rat}}) \rightarrow \mathfrak{D}(\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}})$$

is fully faithful, so questions such as contractibility for the two models of the space of rational maps are equivalent.

0.6.3. Finally, recall the space $\mathbf{Maps}(X, Y)^{\text{rat}}$ introduced in Sect. 0.5.1. There exists a natural map

$$(0.2) \quad \mathbf{Maps}(X, Y)_{\text{Ran } X, \text{indep}}^{\text{rat}} \rightarrow \mathbf{Maps}(X, Y)^{\text{rat}}.$$

The following will be shown in [Ba]:

Theorem 0.6.4. *The pullback functor*

$$\mathfrak{D}(\mathbf{Maps}(X, Y)^{\text{rat}}) \rightarrow \mathfrak{D}(\mathbf{Maps}(X, Y)_{\text{Ran } X, \text{indep}}^{\text{rat}})$$

is an equivalence. In particular, if $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ is homologically contractible, then so is $\mathbf{Maps}(X, Y)^{\text{rat}}$.

⁴Hence the notation “indep” to express the fact the resulting notion of rational map is independent of the specification of the locus of regularity.

In fact, the assertion of Theorem 0.6.4 is applicable not just to $\mathbf{Maps}(X, Y)^{\text{rat}}$, but for a large class of similar problems.

So, although $\mathbf{Maps}(X, Y)^{\text{rat}}$ is not a priori a pseudo-indscheme, the category of D-modules on it is, after all, manageable, and in particular, assuming Theorem 0.6.4, one can prove its homological contractibility as well.

0.7. Conventions and notation.

0.7.1. Our conventions regarding ∞ -categories follow those of [FrGa]. Throughout the text, whenever we say “category”, by default we shall mean an ∞ -category.

0.7.2. The conventions regarding DG categories follow those of [GL:DG]. (Since [GL:DG] is only a survey, for a better documented theory, one can replace the ∞ -category of DG categories by an equivalent one of stable ∞ -categories tensored over k , which has been developed by J. Lurie in [Lu1].)

Throughout this paper we shall be working in the category denoted in [GL:DG] by $\text{DGCat}_{\text{cont}}$: namely, all our DG categories will be assumed presentable, and in particular, cocomplete (i.e., closed under arbitrary direct sums). Unless explicitly said otherwise, all our functors will be assumed continuous, i.e., commute with arbitrary direct sums (or, equivalently, all colimits).

By Vect we shall denote the DG category of complexes of k -vector spaces.

0.7.3. *Schemes and prestacks.* Our general reference for prestacks is [GL:Stacks].

However, the “simplifying good news” for this paper is that, since we will only be interested in D-modules, we can stay within the world of classical, i.e., non-derived, algebraic geometry.

Throughout the paper, we shall only be working with schemes of finite type over k (in particular, quasi-compact). We denote this category by Sch . By Sch^{aff} we denote the full subcategory of affine schemes.

Thus, the category of prestacks, denoted in this paper by PreStk is what is denoted in [GL:Stacks], Sect. 1.3 by ${}^{\leq 0}\text{PreStk}_{\text{laft}}$.

0.7.4. *D-modules.* In [FrGa], Sect. 1.4, it was explained what formalism of D-modules on schemes is required to have a theory suitable for applications. Namely, we needed $\mathfrak{D}(-)$ to be a functor $\text{Sch}_{\text{corr}} \rightarrow \text{DGCat}$, where Sch_{corr} is the category whose objects are schemes and morphisms are correspondences.

Fortunately, for this paper, a more restricted formalism will suffice. Namely, we will only need the functor

$$\mathfrak{D}_{\text{Sch}}^! : \text{Sch}^{\text{op}} \rightarrow \text{DGCat}.$$

I.e., we will only need to consider the $!$ -pullback functor for morphisms. This formalism is developed in the paper [GL:Crystals]. As was mentioned in Sect. 0.4.2, the functor $\mathfrak{D}_{\text{Sch}}^!$ extends to a functor

$$\mathfrak{D}_{\text{PreStk}}^! : \text{PreStk} \rightarrow \text{DGCat}.$$

An important technical observation is the following: let \mathcal{Y}_1 and \mathcal{Y}_2 be prestacks, such that the category $\mathfrak{D}(\mathcal{Y}_1)$ is dualizable. Then the natural functor

$$(0.3) \quad \mathfrak{D}(\mathcal{Y}_1) \otimes \mathfrak{D}(\mathcal{Y}_2) \rightarrow \mathfrak{D}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

is an equivalence. The proof is a word-for-word repetition of the argument in [DrGa0], Proposition 1.4.4.

0.8. Acknowledgements. The idea of contractibility established in this paper, as well as that the homology of rational maps should be insensitive to removing from the target subvarieties of positive codimension is due to V. Drinfeld.

The author would like to express his gratitude to A. Beilinson, for teaching him about most of the mathematics that went into this paper (the above ideas of Drinfeld’s, the affine Grassmannian, the Ran space, chiral homology, etc.)

This paper is part of a joint project with J. Lurie, which would hopefully see light soon in the form of [GaLu]. The author is grateful to Jacob for offering his insight in the many discussions on the subject that we have had.

Special thanks are due to S. Raskin for extensive comments on the first draft of the paper, which have greatly helped to improve the exposition.

We would also like to thank J. Barlev and N. Rozenblyum for numerous helpful conversations related to the subject of the paper.

The author’s research is supported by NSF grant DMS-1063470.

1. THE “SPACE” OF RATIONAL MAPS AS A PSEUDO-INDSCHEME

1.1. Indschemes.

1.1.1. Before we define pseudo-indschemes, let us recall the notion of usual indscheme. By definition, an indscheme is an object $\mathcal{Y} \in \text{PreStk}$ which can be written as a

$$(1.1) \quad \text{colim}_{I \in \mathcal{S}} Z(I),$$

for some functor $Z : \mathcal{S} \rightarrow \text{Sch}$, such that

- For $(I_1 \rightarrow I_2)$, the map $Z(I_1) \rightarrow Z(I_2)$ is a closed embedding,
- The category \mathcal{S} of indices is *filtered*.

In the above formula the colimit is taken in PreStk . Recall that colimits in PreStk are computed object-wise, i.e., for $S \in \text{Sch}^{\text{aff}}$,

$$(1.2) \quad \text{Maps}(S, \mathcal{Y}) \simeq \text{colim}_{I \in \mathcal{S}} \text{Maps}(S, Z(I)),$$

where the latter colimit is taken in $\infty\text{-Grpd}$.

We let IndSch denote the full subcategory of PreStk spanned by indschemes.

Remark 1.1.2. The condition that \mathcal{S} be filtered is really crucial, as it insures some of the key properties of indschemes. For example:

(i) As long as we stay in the realm of classical (i.e., non-derived) algebraic geometry, the functor $S \mapsto \text{Maps}(S, \mathcal{Y})$ takes value in sets, rather than $\infty\text{-Grpd}$. Indeed, a filtered colimit of k -truncated groupoids is k -truncated (and we take $k = 0$).

(ii) An indscheme belongs to Stk , i.e., it satisfies fppf descent. This is so because all $Z(I)$ satisfy descent, and the fact that finite limits (that are involved in the formulation of descent) commute with filtered colimits. Therefore, formula (1.2) holds also for $S \in \text{Sch}$.

These two properties will fail for pseudo-indschemes.

⁵Although this will be of no practical consequence, let us note that since we are working with classical schemes, rather than derived ones, Sch is an ordinary category, and so, we can take the category of indices \mathcal{S} to be an ordinary category as well.

1.1.3. Recall that a map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ in PreStk is called a closed embedding if for any $S \in \text{Sch}_{/\mathcal{Y}_2}^{\text{aff}}$, the resulting map $S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \rightarrow S$ is a closed embedding; in particular, the left hand side is a scheme.⁶

We have:

Lemma 1.1.4. *Let \mathcal{Y} be an indscheme written as in (1.1). For $S \in \text{Sch}$, a map $S \rightarrow \mathcal{Y}$ is a closed embedding if and only if for some/any index I , for which the above map factors through $Z(I)$, the resulting map $S \rightarrow Z(I)$ is a closed embedding.*

Proof. Let T be an affine scheme mapping to \mathcal{Y} . Let I be an index such that both maps $S, T \rightarrow \mathcal{Y}$ factor through $Z(I)$. The filteredness assumption on \mathcal{S} implies that

$$T \times_{\mathcal{Y}} S \simeq \text{colim}_{I' \in \mathcal{S}_{I'}} (S \times_{Z(I')} T).$$

However (in classical algebraic geometry), since $Z(I) \rightarrow Z(I')$ is a closed embedding, the map

$$S \times_{Z(I)} T \rightarrow S \times_{Z(I')} T$$

is an isomorphism, so $T \times_{\mathcal{Y}} S \simeq S \times_{Z(I)} T$, and the assertion is manifest. \square

Thus, we obtain that the notion of closed embedding of a scheme into an indscheme is invariantly defined. In particular, the tautological maps $\mathbf{e}(I) : Z(I) \rightarrow \mathcal{Y}$ are closed embeddings. (None of that will be the case for pseudo-indchemes.)

Additionally, one obtains that the category $(S \in \text{Sch}, S \xrightarrow{\text{cl-emb.}} \mathcal{Y})$ is filtered, i.e., an indscheme \mathcal{Y} has a canonical presentation as in (1.1), where the category of indices is that of *all* schemes equipped with a closed embedding into \mathcal{Y} .

1.1.5. For future reference, let us give the following definitions.

Let \mathcal{Y} be an indscheme mapping to a scheme S . We shall say that \mathcal{Y} is relatively ind-proper (resp., ind-closed subscheme) over S if for any closed embedding $T \rightarrow \mathcal{Y}$, the composed map $T \rightarrow S$ is proper (resp., closed embedding).

It is easy to see that this is equivalent to requiring that for a given presentation of \mathcal{Y} as in (1.1), the composed maps $Z(I) \rightarrow S$ are proper (resp., closed embeddings).

A map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ in PreStk is called ind-schematic, if for any $S \in \text{Sch}_{/\mathcal{Y}_2}$, the base change $S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ is an ind-scheme.

It is easy to see that a morphism between two indschemes is ind-schematic (again, this relies on the filteredness assumption of the index category).

A map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ in PreStk is called ind-proper (resp., ind-closed embedding), if it is ind-schematic and for every $S \in \text{Sch}_{/\mathcal{Y}_2}$ as above, the indscheme $S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ is relatively ind-proper (resp., ind-closed subscheme) over S .

1.2. Pseudo-indchemes.

⁶We emphasize that the above definition of closed embedding is only suitable for classical, i.e., non-derived, algebraic geometry. By contrast, in the setting of DAG, “closed embedding” means by definition “a closed embedding at the level of the underlying classical prestacks”, so $S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ does not have to be a derived scheme, but its underlying classical prestack must be a classical scheme.

1.2.1. By a pseudo-indscheme we shall mean an object $\mathcal{Y} \in \text{PreStk}$ that can be written as

$$(1.3) \quad \mathcal{Y} \simeq \text{colim}_{I \in \mathcal{S}} Z(I)$$

for some functor $Z : \mathcal{S} \rightarrow \text{IndSch}$, such that

- For $(I_1 \rightarrow I_2)$, the map $Z(I_1) \rightarrow Z(I_2)$ is ind-proper,

where \mathcal{S} is an arbitrary category of indices. (As was remarked before, with no restriction of generality, we can take \mathcal{S} to be an ordinary category, rather than ∞ -category.)

We let $\mathbf{e}(I)$ denote the tautological map $Z(I) \rightarrow \mathcal{Y}$.

In the above expression, the colimit is taken in the category PreStk . Again, the value of \mathcal{Y} on $S \in \text{Sch}^{\text{aff}}$ is an object of $\infty\text{-Grpd}$ isomorphic to

$$\text{colim}_{I \in \mathcal{S}} \text{Maps}(S, Z(I)),$$

where the latter colimit is taken in $\infty\text{-Grpd}$.

1.2.2. Several remarks are in order:

(i) It is crucial for what follows that in the definition of pseudo-indchemes, the colimit is taken in $\infty\text{-Grpd}$ and not, naively, in the category of sets.

(ii) Non-filtered colimits in $\infty\text{-Grpd}$ are somewhat unwieldy objects: we cannot algorithmically describe the space $\text{Maps}(S, \mathcal{Y})$ for $S \in \text{Sch}^{\text{aff}}$. One manifestation of this phenomenon is that for a pair of affine schemes S, T mapping to \mathcal{Y} , we cannot describe their Cartesian product $S \times_{\mathcal{Y}} T$. So, many of the properties enjoyed by indchemes will fail for pseudo-indchemes.

(iii) It is easy to see that in the definition of pseudo-indchemes, one can require $Z(I)$ be schemes rather than indchemes. Indeed, given a presentation as in (1.3), we can define a new category of indices $\tilde{\mathcal{S}}$ that consists of pairs $(I \in \mathcal{S}, T \xrightarrow{\text{cl.emb.}} Z(I))$, and we will have

$$\mathcal{Y} \simeq \text{colim}_{(I, T) \in \tilde{\mathcal{S}}} T.$$

(iv) The reason for singling out pseudo-indchemes among all prestacks is that the category of D-modules on an pseudo-indscheme is more manageable than on an arbitrary prestack, see Sect. 1.4.

1.2.3. *The Ran space.* Let us consider an example of a pseudo-indscheme that will play a prominent role in this paper.

We take \mathcal{S} to be the category $(\text{fSet})^{\text{op}}$, where fSet denotes the category of non-empty finite sets and surjective maps. Let X be a separated scheme. We define a functor

$$X^{\text{fSet}} : (\text{fSet})^{\text{op}} \rightarrow \text{Sch}$$

by assigning to a finite set I the scheme X^I , and to a surjective map $\phi : J \twoheadrightarrow I$ the corresponding diagonal map $\Delta(\phi) : X^I \rightarrow X^J$.

We shall denote the resulting pseudo-indscheme

$$\text{colim}_{(\text{fSet})^{\text{op}}} X^{\text{fSet}} \in \text{PreStk}$$

by $\text{Ran } X$, and called it “the Ran space of X ”.

For a finite set I , we let Δ^I denote the corresponding map $X^I \rightarrow \text{Ran } X$.

Remark 1.2.4. Although the colimit in the formation of $\text{Ran } X$ was taken in PreStk , it is easy to see that the functor $\text{Ran } X : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow \infty\text{-Grpd}$ takes values in 0-truncated groupoids, i.e., in $\text{Set} \subset \infty\text{-Grpd}$.

In fact, $\text{Maps}(S, \text{Ran } X)$ is the set of non-empty finite subsets of $\text{Maps}(S, X)$.

Indeed, for $S \in \text{Sch}^{\text{aff}}$, we have:

$$\text{Maps}(S, \text{Ran } X) = \underset{(\text{fSet})^{\text{op}}}{\text{colim}} (\text{Maps}(S, X))^I,$$

but it is easy to see that for a set A , the colimit $\underset{(\text{fSet})^{\text{op}}}{\text{colim}} A^I$ is discrete, and is isomorphic to set of all finite non-empty subsets of A .

1.2.5. For future reference, let us give the following definitions. (The reader can skip this subsection and return to it when necessary.)

Let \mathcal{Y} be a pseudo-indscheme mapping to a scheme S . We shall say that \mathcal{Y} is pseudo ind-proper over S if *there exists* a presentation of \mathcal{Y} as in (1.3), such that the resulting maps $Z(I) \rightarrow S$ are ind-proper.

We shall say that a map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ in PreStk is *pseudo ind-schematic* if for any $S \in \text{Sch}^{\text{aff}}_{/\mathcal{Y}_2}$, the object $S \times_{\mathcal{Y}_1} \mathcal{Y}_2$ is a pseudo-indscheme.

Unlike indschemes, it is not true that any map between pseudo-indschemes is pseudo ind-schematic. It is not even true that a map from an affine scheme to a pseudo-indscheme is pseudo ind-schematic.

We shall say that a map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ in PreStk is *pseudo ind-proper* if for every $S \in \text{Sch}^{\text{aff}}_{/\mathcal{Y}_2}$ as above, the resulting pseudo-indscheme $S \times_{\mathcal{Y}_1} \mathcal{Y}_2$ is pseudo ind-proper over S .

1.3. D-modules on pseudo-indschemes.

1.3.1. Recall that for any object $\mathcal{Y} \in \text{PreStk}$ we define the category $\mathfrak{D}(\mathcal{Y})$ as

$$\lim_{S \in (\text{Sch}^{\text{aff}}/\mathcal{Y})^{\text{op}}} \mathfrak{D}(S).$$

In other words, we define the functor

$$\mathfrak{D}_{\text{PreStk}}^! : \text{PreStk}^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}},$$

as the right Kan extension of the functor

$$\mathfrak{D}_{\text{Sch}^{\text{aff}}}^! : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}},$$

along the tautological embedding

$$(\text{Sch}^{\text{aff}})^{\text{op}} \hookrightarrow \text{PreStk}^{\text{op}},$$

see Sect. 0.4.2.⁷

In what follows, when no confusion is likely to occur, we will suppress the subscript “!” and the subscript “PreStk” from the notation, i.e., we shall simply write $\mathfrak{D}(\mathcal{Y})$ rather than $\mathfrak{D}_{\text{PreStk}}^!(\mathcal{Y})$.

⁷This is the same as the right Kan extension of the functor $\mathfrak{D}_{\text{Sch}}^!$ along $(\text{Sch})^{\text{op}} \hookrightarrow \text{PreStk}^{\text{op}}$.

1.3.2. The main object of study of this section is the category $\mathfrak{D}(\mathcal{Y})$, where \mathcal{Y} is a pseudo-indscheme.

The above interpretation of the functor $\mathfrak{D}_{\text{PreStk}}$ as the right Kan extension implies that it takes colimits in PreStk to limits in $\text{DGCat}_{\text{cont}}$.

Hence, for \mathcal{Y} written as in (1.3), we have:

$$(1.4) \quad \mathfrak{D}(\mathcal{Y}) \simeq \lim_{I \in \mathcal{S}^{op}} \mathfrak{D}(Z(I)),$$

where the limit is formed using the $!$ -pullback functors.

1.3.3. Let $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ be a map between pseudo-indschemes. By construction, we have a pullback functor $f^! : \mathfrak{D}(\mathcal{Y}') \rightarrow \mathfrak{D}(\mathcal{Y})$.

For example, taking $\mathcal{Y}' = \text{pt}$, and the tautological map $p_{\mathcal{Y}} : \mathcal{Y} \rightarrow \text{pt}$, we obtain a functor

$$p_{\mathcal{Y}}^! : \text{Vect} \rightarrow \mathfrak{D}(\mathcal{Y}).$$

In particular, the category $\mathfrak{D}(\mathcal{Y})$ contains the canonical object $\omega_{\mathcal{Y}} := p_{\mathcal{Y}}^!(k)$, which we shall refer to as “the dualizing sheaf”.

1.4. The category of D-modules as a colimit. A distinctive feature of pseudo-indschemes among arbitrary objects of PreStk is that the category $\mathfrak{D}(\mathcal{Y})$ can be alternatively described as a *colimit* in $\text{DGCat}_{\text{cont}}$.

1.4.1. Let us recall the following general construction. Let \mathbf{C} be an ∞ -category, and let

$$\Phi : \mathbf{C} \rightarrow \text{DGCat}_{\text{cont}}$$

be a functor. Suppose that for every arrow $g : \mathbf{c}_1 \rightarrow \mathbf{c}_2$, the resulting functor

$$\Phi(g) : \Phi(\mathbf{c}_1) \rightarrow \Phi(\mathbf{c}_2)$$

admits a left adjoint, ${}^L\Phi(g)$. Then the assignment $g \rightsquigarrow {}^L\Phi(g)$ canonically extends to a functor

$${}^L\Phi : \mathbf{C}^{op} \rightarrow \text{DGCat}.$$

1.4.2. Let \mathcal{Y} be a pseudo-indscheme, written as in (1.3). Consider the functor

$$\mathfrak{D}^!(Z) : \mathcal{S}^{op} \rightarrow \text{DGCat}$$

equal to the composition of $Z^{op} : \mathcal{S}^{op} \rightarrow \text{PreStk}^{op}$ with the functor

$$\mathfrak{D}_{\text{PreStk}}^! : \text{PreStk}^{op} \rightarrow \text{DGCat}.$$

The properness assumption on the maps $Z(I) \rightarrow Z(J)$ implies that the functor $\mathfrak{D}^!(Z)$ satisfies the assumption of Sect. 1.4.1. Let $\mathfrak{D}_!(Z)$ denote the resulting functor $\mathcal{S} \rightarrow \text{DGCat}$.

By [GL:DG], Lemma 1.3.3, we have:

$$(1.5) \quad \lim_{\mathcal{S}^{op}} \mathfrak{D}^!(Z) \simeq \text{colim}_{\mathcal{S}} \mathfrak{D}_!(Z).$$

In particular, this implies that the category $\mathfrak{D}(\mathcal{Y})$ is compactly generated, see [GL:DG], Sect. 2.2.1. The latter observation, combined with the isomorphism (0.3), implies that for any $\mathcal{Y}' \in \text{PreStk}$, the functor

$$(1.6) \quad \mathfrak{D}(\mathcal{Y}) \otimes \mathfrak{D}(\mathcal{Y}') \rightarrow \mathfrak{D}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence.

Note also, that by [GL:DG], Sect. 2.2.1, a choice of presentation of \mathcal{Y} as in (1.3), defines an equivalence $\mathfrak{D}(\mathcal{Y})^\vee \simeq \mathfrak{D}(\mathcal{Y})$, i.e., an anti self-equivalence

$$(\mathfrak{D}(\mathcal{Y})^c)^{op} \simeq \mathfrak{D}(\mathcal{Y})^c.$$

1.4.3. For $I \in \mathcal{S}$, we shall denote by $\mathbf{e}(I)^\dagger$ the tautological forgetful functor

$$\mathfrak{D}(\mathcal{Y}) = \lim_{\mathcal{S}^{op}} \mathfrak{D}^\dagger(Z) \rightarrow \mathfrak{D}(Z(I))$$

and by $\mathbf{e}(I)_!$ its left adjoint, which in terms of the equivalence (1.5) corresponds to the tautological functor

$$\mathfrak{D}(Z(I)) \rightarrow \operatorname{colim}_{\mathcal{S}} \mathfrak{D}_!(Z).$$

1.4.4. The two functors in (1.5) can be explicitly described as follows:

The functor

$$\operatorname{colim}_{I \in \mathcal{S}} \mathfrak{D}(Z(I)) \rightarrow \mathfrak{D}(\mathcal{Y})$$

corresponds to the compatible family of functors $\mathfrak{D}(Z(I)) \rightarrow \mathfrak{D}(\mathcal{Y})$ given by $\mathbf{e}(I)_!$.

The functor

$$\mathfrak{D}(\mathcal{Y}) \rightarrow \operatorname{colim}_{I \in \mathcal{S}} \mathfrak{D}(Z(I))$$

sends

$$(1.7) \quad (\mathcal{F} \in \mathfrak{D}(\mathcal{Y})) \rightsquigarrow \operatorname{colim}_{I \in \mathcal{S}} \mathbf{e}(I)^\dagger(\mathcal{F}) \in \operatorname{colim}_{I \in \mathcal{S}} \mathfrak{D}(Z(I)).$$

1.5. **Direct images with compact supports.** Let us return to the situation of a morphism $f : \mathcal{Y} \rightarrow \mathcal{Y}'$. It is not, in general, true that the functor $f^! : \mathfrak{D}(\mathcal{Y}') \rightarrow \mathfrak{D}(\mathcal{Y})$ admits a left adjoint.

1.5.1. In general, if $G : \mathbf{C}' \rightarrow \mathbf{C}$ is a functor between ∞ -categories, one can consider the full subcategory of \mathbf{C} , denoted \mathbf{C}_{good} , consisting of objects $\mathbf{c} \in \mathbf{C}$, for which the functor

$$\mathbf{C}' \rightarrow \infty\text{-Grpd}, \quad \mathbf{c}' \mapsto \operatorname{Maps}_{\mathbf{C}}(\mathbf{c}, G(\mathbf{c}'))$$

is co-representable.

In this case, there exists a canonically defined functor $F : \mathbf{C}_{\text{good}} \rightarrow \mathbf{C}'$, equipped with an isomorphism of functors

$$(\mathbf{C}_{\text{good}})^{op} \times \mathbf{C}' \rightrightarrows \infty\text{-Grpd}$$

that send $\mathbf{c} \in \mathbf{C}_{\text{good}}$ and $\mathbf{c}' \in \mathbf{C}'$ to

$$\operatorname{Maps}_{\mathbf{C}'}(F(\mathbf{c}), \mathbf{c}') \text{ and } \operatorname{Maps}_{\mathbf{C}}(\mathbf{c}, G(\mathbf{c}')),$$

respectively.

In this case, we shall refer to F as “the partially defined left adjoint of G ”. For an object $\mathbf{c} \in \mathbf{C}$ we shall say that “the partially defined left adjoint of G is defined on \mathbf{c} ” if $\mathbf{c} \in \mathbf{C}_{\text{good}}$.

1.5.2. Let

$$\mathfrak{D}(\mathcal{Y})_{\text{good for } f} \subset \mathfrak{D}(\mathcal{Y})$$

be the full subcategory that consists of objects, on which the partially defined left adjoint $f_!$ to $f^!$ is defined.

For a general map f , it is not clear how to construct objects in $\mathfrak{D}(\mathcal{Y})_{\text{good for } f}$. However, below we shall describe a situation when can generate “many” objects of this category.

Note, however, that for any map $f : \mathcal{Y} \rightarrow \mathcal{Y}'$, for which $\omega_{\mathcal{Y}}$ belongs to $\mathfrak{D}(\mathcal{Y})_{\text{good for } f}$, we have a canonical map

$$(1.8) \quad \text{Tr}_{\omega}(f) : f_!(\omega_{\mathcal{Y}}) \rightarrow \omega_{\mathcal{Y}'},$$

coming by adjunction from

$$\omega_{\mathcal{Y}} \simeq f^!(\omega_{\mathcal{Y}'}).$$

1.5.3. First, let us recall the following general paradigm of constructing maps between colimits.

Let $F : \mathbf{C}_1 \rightarrow \mathbf{C}_2$ is a functor between ∞ -categories, and let

$$\Phi_1 : \mathbf{C}_1 \rightarrow \mathbf{D} \text{ and } \Phi_2 : \mathbf{C}_2 \rightarrow \mathbf{D}$$

be functors, where \mathbf{D} is another ∞ -category. Let us be given a natural transformation of functors $\mathbf{C}_1 \rightrightarrows \mathbf{D}$:

$$\Phi_1 \rightrightarrows \Phi_2 \circ F.$$

Then we obtain a map in \mathbf{D} :

$$(1.9) \quad \text{colim}_{\mathbf{C}_1} \Phi_1 \rightarrow \text{colim}_{\mathbf{C}_2} \Phi_2.$$

1.5.4. Let \mathcal{S} and \mathcal{S}' be two categories of indices and $f_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{S}'$ a functor. Let $Z : \mathcal{S} \rightarrow \text{IndSch}$ and $Z' : \mathcal{S}' \rightarrow \text{IndSch}$ be two functors as in Sect. 1.2.1, and let f_Z be a natural transformation between the resulting two functors $\mathcal{S} \rightrightarrows \text{IndSch}$:

$$Z \rightrightarrows Z' \circ f_{\mathcal{S}}.$$

For $I \in \mathcal{S}$ we let $f(I)$ denote the resulting map of indschemes $Z(I) \rightarrow Z'(I')$, where $I' = f_{\mathcal{S}}(I)$.

By the above, we obtain a map $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ between the corresponding two pseudo-indschemes.

1.5.5. Keeping the notation of the previous subsection, for each $I \in \mathcal{S}$, let

$$\mathfrak{D}(Z(I))_{\text{good for } f(I)}$$

be the full subcategory of $\mathfrak{D}(Z(I))$ that consists of objects, on which the partially defined left adjoint $f(I)_!$ to $f(I)^!$ is defined.

By adjunction, we obtain that in the next diagram the upper horizontal arrow is well-defined (i.e., its image belongs to the indicated subcategory) and that the diagram commutes:

$$\begin{array}{ccc} \mathfrak{D}(Z(I))_{\text{good for } f(I)} & \xrightarrow{e(I)_!} & \mathfrak{D}(\mathcal{Y})_{\text{good for } f} \\ f(I)_! \downarrow & & \downarrow f_! \\ \mathfrak{D}(Z'(I')) & \xrightarrow{e(I')_!} & \mathfrak{D}(\mathcal{Y}'). \end{array}$$

As a consequence, from (1.7), we obtain that if $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})$ is such that for each I ,

$$e(I)_!(\mathcal{F}) \in \mathfrak{D}(Z(I))_{\text{good for } f(I)},$$

then $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } f}$, and we have:

$$(1.10) \quad f_!(\mathcal{F}) \simeq \operatorname{colim}_{I \in \mathcal{S}} \mathbf{e}(I)! \circ f(I)! \circ \mathbf{e}(I)^!(\mathcal{F}) \in \mathfrak{D}(\mathcal{Y}').$$

1.5.6. The above description shows:

- (i) If all the maps $f(I)$ happen to be proper, then $\mathfrak{D}(\mathcal{Y})_{\text{good for } f}$ equals all of $\mathfrak{D}(\mathcal{Y})$.
- (ii) If $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})$ is such that for all I , the object $\mathbf{e}(I)^!(\mathcal{F}) \in \mathfrak{D}(Z(I))$ has holonomic cohomologies, then $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } f}$.
- (iii) The object $\omega_{\mathcal{Y}}$ always belongs to $\mathfrak{F} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } f}$.

1.6. (Co)homology.

1.6.1. Let us take $\mathcal{Y}' = \text{pt}$ and $f = p_{\mathcal{Y}} : \mathcal{Y} \rightarrow \text{pt}$. It is clear that the map $p_{\mathcal{Y}}$ falls into the paradigm described in Sect. 1.5.4.

For $\mathcal{Y} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } p_{\mathcal{Y}}}$, we shall also use the notation

$$\Gamma_{\text{dR},c}(\mathcal{Y}, -) := (p_{\mathcal{Y}})!(-).$$

From (1.10), we obtain that if $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})$ is such that for each I ,

$$\mathbf{e}(I)^!(\mathcal{F}) \in \mathfrak{D}(Z(I))_{\text{good for } p_{Z(I)}},$$

we have $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } p_{\mathcal{Y}}}$, and

$$(1.11) \quad \Gamma_{\text{dR},c}(\mathcal{Y}, \mathcal{F}) \simeq \operatorname{colim}_{I \in \mathcal{S}} \Gamma_{\text{dR},c}(Z(I), \mathbf{e}(I)^!(\mathcal{F})).$$

1.6.2. Let us use the notation

$$H_{\bullet}(\mathcal{Y}) := \Gamma_{\text{dR},c}(\mathcal{Y}, \omega_{\mathcal{Y}}).$$

From (1.8), we obtain a canonical map

$$(1.12) \quad \operatorname{Tr}_{H_{\bullet}} : H_{\bullet}(\mathcal{Y}) \rightarrow k.$$

Moreover, for any map $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ for which $\omega_{\mathcal{Y}} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } f}$, by applying the partially defined functor $\Gamma_{\text{dR},c}(\mathcal{Y}', -)$ to (1.8), we obtain a map

$$(1.13) \quad \operatorname{Tr}_{H_{\bullet}}(f) : H_{\bullet}(\mathcal{Y}) \rightarrow H_{\bullet}(\mathcal{Y}').$$

1.6.3. Let \mathcal{Y} be written as in (1.3). From (1.11) we obtain:

$$(1.14) \quad H_{\bullet}(\mathcal{Y}) \simeq \operatorname{colim}_{I \in \mathcal{S}} H_{\bullet}(Z(I)).$$

In particular, we conclude that the object $H_{\bullet}(\mathcal{Y}) \in \text{Vect}$ is always connective, i.e., lives in non-positive cohomological degrees. Also, we conclude that the trace map

$$\operatorname{Tr}_{H_{\bullet}} : H_{\bullet}(\mathcal{Y}) \rightarrow k$$

has the property that the map

$$H_0(\mathcal{Y}) \rightarrow k$$

is non-zero whenever \mathcal{Y} is non-empty, and if all $Z(I)$ are connected, the map $H_0(\mathcal{Y}) \rightarrow k$ is an isomorphism.

1.6.4. Finally, let us recall the following basic result of [BD1], Proposition 4.3.3, that will be crucial for this paper:

Theorem 1.6.5. *Let X be a connected separated scheme. Then the above map*

$$\mathrm{Tr}_{H_\bullet} : H_\bullet(\mathrm{Ran} X) \rightarrow k$$

is an isomorphism.

Remark 1.6.6. Strictly speaking, in [BD1], the above result is only proved when X is a curve (which is also our main case of interest in this paper). However, the proof given in *loc.cit.* applies to the general case as well. For completeness, we shall include the argument in the general case in Sect. 6, by essentially repeating *loc.cit.*

1.7. The space of rational maps.

1.7.1. Let us recall the following construction. Let U be a scheme and Y an affine scheme (according to our conventions, assumed of finite type).

Consider the functor

$$\mathbf{Maps}(U, Y) : \mathrm{Sch}^{op} \rightarrow \mathrm{Sets}$$

that assigns to a test scheme S the set of maps $S \times U \rightarrow Y$.

It is easy to see that this functor is representable by an indscheme (of ind-finite type). Indeed, by representing Y as a Cartesian product

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^m \end{array}$$

we reduce the assertion to the case when $Y = \mathbb{A}^n$, and by taking products further to the case $Y = \mathbb{A}^1$.

In the latter case, $\mathbf{Maps}(U, Y)$ is representable by the (infinite-dimensional, unless U is proper) vector space $\Gamma(U, \mathcal{O}_U)$, viewed as an indscheme.

1.7.2. In what follows we shall need a version of the above construction in the relative situation, namely, when U is a scheme flat over a base T . In this case we define $\mathbf{Maps}_T(U, Y)$ as a functor on the category of schemes over T .

We do not know what conditions on the map $\pi : U \rightarrow T$ guarantee this in general. However, we have the following statement:

Lemma 1.7.3. *Suppose that the (derived) direct image $\pi_*(\mathcal{O}_U) \in \mathrm{QCoh}(T)$ can be written as a filtered colimit of objects $\mathcal{E}_\alpha \in \mathrm{QCoh}(T)^{\mathrm{perf}}$ such that for every index α , the \mathcal{O}_T -dual \mathcal{E}_α^\vee belongs to $\mathrm{QCoh}(T)^{\leq 0}$, and for every map of indices $\alpha_1 \rightarrow \alpha_2$, the corresponding map $\mathcal{E}_{\alpha_2}^\vee \rightarrow \mathcal{E}_{\alpha_1}^\vee$ is a surjection on $H^0(-)$. Then $\mathbf{Maps}_T(U, Y)$ is representable by an ind-scheme.*

Remark 1.7.4. If $\pi_*(\mathcal{O}_U) \in \mathrm{QCoh}(T)$ is concentrated in cohomological degree 0, then the condition of the lemma is equivalent to requiring that $\pi_*(\mathcal{O}_U)$ be a Mittag-Leffler module in the sense of [BD2, Sect. 7.12.1].

In the situation of the lemma, the indscheme representing $\mathbf{Maps}_T(U, Y)$ can be explicitly written as

$$\mathrm{colim}_\alpha \mathrm{Spec}_T(\mathrm{Sym}(H^0(\mathcal{E}_\alpha^\vee))).$$

The conditions of the lemma are satisfied, for example, when the map π admits a compactification

$$\bar{\pi} : \bar{U} \rightarrow T,$$

such that $\bar{\pi}$ is also flat, and the complement $\bar{U} - U$ is a relatively ample divisor D which is flat over T . In this case we take the category of indices to be \mathbb{N} , and for $n \gg 0$, we take

$$\mathcal{E}_n := \bar{\pi}_*(\mathcal{O}_{\bar{U}}(n \cdot D)).$$

1.7.5. Let X be a smooth, connected and complete curve, and let Y be an affine scheme. We define a functor

$$\mathbf{Maps}(X, Y)_{X^{\text{fSet}}}^{\text{rat}} : (\text{fSet})^{\text{op}} \rightarrow \text{IndSch}$$

as follows.

For $I \in \text{fSet}$ we consider the scheme

$$(X^I \times X) - \Gamma^I,$$

where $\Gamma^I \subset X^I \times X$ is the incidence divisor. We regard it as a scheme over X^I . For future reference, for an S -point x^I of X^I we will denote by $\{x^I\}$ the closed subscheme of $S \times X$ equal to $(x^I \times \text{id}_X)^{-1}(\Gamma^I)$.

We let

$$\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}} := \mathbf{Maps}_{X^I}((X^I \times X) - \Gamma^I, Y).$$

By Lemma 1.7.3, $\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}$ is an indscheme.

1.7.6. We define the object $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} \in \text{PreStk}$ as

$$\text{colim}_{(\text{fSet})^{\text{op}}} \mathbf{Maps}(X, Y)_{X^{\text{fSet}}}^{\text{rat}}.$$

It is a pseudo-indscheme, by construction.

Note that also by construction, the functor

$$\mathbf{Maps}(X, Y)_{X^{\text{fSet}}}^{\text{rat}} : (\text{fSet})^{\text{op}} \rightarrow \text{IndSch}$$

comes equipped with a natural transformation to X^{fSet} . We let f denote the resulting map

$$\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} \rightarrow \text{Ran } X,$$

see Sect. 1.5.4.

Remark 1.7.7. As in Remark 1.2.4, one can show that

$$\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow \infty\text{-Grpd}$$

takes values in $\text{Set} \subset \infty\text{-Grpd}$.

In fact, a data of an S -point of $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$ is equivalent to that of a non-empty finite subset $\bar{x} \subset \text{Maps}(S, X)$ plus a rational map $S \times X \rightarrow Y$, which is regular on the complement to the graph of \bar{x} .

1.7.8. We define “homology of the space of rational maps” as

$$H_{\bullet}(\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}).$$

Note that by transitivity,

$$H_{\bullet}(\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}) \simeq \Gamma_{\mathrm{dR}, c} \left(\mathrm{Ran} X, f_! (\omega_{\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}}) \right).$$

Remark 1.7.9. We emphasize that the functor $f_!$ is defined via the realization of the category $\mathfrak{D}(\mathrm{Ran} X)$ as in Sect. 1.4 as $\mathop{\mathrm{colim}}_{I \in (\mathrm{fSet})^{\mathrm{op}}} \mathfrak{D}(X^I)$. That is, if we want to “evaluate” the object

$f_! (\omega_{\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}})$ on a given finite set I , i.e., if we are interested in

$$(\Delta^I)_! \left(f_! (\omega_{\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}}) \right) \in \mathfrak{D}(X^I),$$

we will have to apply the equivalence (1.5), and the result will *not* be isomorphic to

$$f(I)_! \left(\omega_{\mathbf{Maps}(X, Y)_{X^I}^{\mathrm{rat}}} \right).$$

1.8. Statement of the the main result.

1.8.1. The main result of this paper is:

Consider the trace map of (1.12)

$$(1.15) \quad \mathrm{Tr}_{H_{\bullet}} : H_{\bullet}(\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}) \rightarrow k.$$

Theorem 1.8.2. *Suppose that Y is connected and can be covered by open subsets U_{α} , each of which is isomorphic to an open subset of the affine space \mathbb{A}^n (for some integer n). Then the map (1.15) is an isomorphism.*

Note that for $Y = \mathrm{pt}$, the statement of Theorem 1.8.2 coincides with that of Theorem 1.6.5.

Remark 1.8.3. The assumption that the curve X be complete is inessential: if $\overset{\circ}{X} \subset X$ is a non-empty open subset, the spaces $\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}$ and $\mathbf{Maps}(\overset{\circ}{X}, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}$ have isomorphic homology. This follows from Corollary 2.5.10 and Equation (3.21).

1.8.4. A typical example of a scheme Y satisfying the assumption of Theorem 1.8.2 is a connected affine algebraic group G . Then the required cover is provided by the Bruhat decomposition.

1.8.5. Finally, we propose:

Conjecture 1.8.6. *The assertion of Theorem 1.8.2 holds for any Y which is connected, smooth and birational to \mathbb{A}^n .*

2. THE UNITAL SETTING

In this section we shall introduce a unital structure on the space $\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}$, and the corresponding space $\mathbf{Maps}(X, Y)_{\mathrm{Ran} X, \mathrm{indep}}^{\mathrm{rat}}$. Its significance will be two-fold:⁸

At the conceptual level, the space $\mathbf{Maps}(X, Y)_{\mathrm{Ran} X, \mathrm{indep}}^{\mathrm{rat}}$ gets rid of the redundancy inherent in the definition of $\mathbf{Maps}(X, Y)_{\mathrm{Ran} X}^{\mathrm{rat}}$, namely, of specifying the locus where our rational map is defined.

⁸The terminology “unital” is motivated by the property possessed by the factorization algebra corresponding to a unital chiral algebra, see [BD1], Sect. 3.4.5.

Technically, certain calculations are easier to perform in the unital version; in particular, ones that appear in the proofs of Theorem 1.8.2 and Theorem 4.1.6.

However, the reader may skip this section on the first pass: we will explain an alternative (but equivalent) way to perform the above mentioned calculations involved in the proofs of the main theorems.

Throughout this section, X will be a separated connected scheme.

2.1. Spaces acted on by $\text{Ran } X$.

2.1.1. Observe that the category fSet (and, hence, $(\text{fSet})^{op}$) has a natural symmetric monoidal (but non-unital) structure given by disjoint union.

The functor $X^{\text{fSet}} : (\text{fSet})^{op} \rightarrow \text{Sch}$ also has a natural symmetric monoidal structure. This defines on $\text{Ran } X = \underset{(\text{fSet})^{op}}{\text{colim}} X^{\text{fSet}}$ a structure of commutative (but non-unital) semi-group in PreStk .

Concretely, the map

$$\text{union}_{\text{Ran}} : \text{Ran } X \times \text{Ran } X \rightarrow \text{Ran } X$$

can be described in terms of Sect. 1.5.4 as follows. It corresponds to the functor

$$\sqcup : (\text{fSet} \times \text{fSet})^{op} \rightarrow (\text{fSet})^{op}$$

and to the natural transformation (in fact, an isomorphism) of the resulting two functors $(\text{fSet} \times \text{fSet})^{op} \rightrightarrows \text{Sch}$:

$$X^{\text{fSet}} \times X^{\text{fSet}} \Rightarrow X^{\text{fSet}} \circ \sqcup,$$

given by

$$(I_1, I_2) \rightsquigarrow \left(X^{I_1} \times X^{I_2} \rightarrow X^{I_2 \sqcup I_1} \right).$$

2.1.2. Let \mathcal{S} be an index category, which is a module for $(\text{fSet})^{op}$. We shall denote by \sqcup the action functor

$$(J \in (\text{fSet})^{op}, I \in \mathcal{S}) \rightsquigarrow J \sqcup I \in \mathcal{S}.$$

Let $Z : \mathcal{S} \rightarrow \text{IndSch}$ be a functor with a structure of module for X^{fSet} . I.e., for every $J \in (\text{fSet})^{op}$ and $I \in \mathcal{S}$ we are given a map

$$\text{unit}_{J,I} : X^J \times Z(I) \rightarrow Z(J \sqcup I),$$

which are functorial and associative in a natural sense.

We shall refer to this structure as a *unital structure* on Z with respect to $\text{Ran } X$. In this case $\mathcal{Y} := \underset{\mathcal{S}}{\text{colim}} Z$ becomes a module over $\text{Ran } X$. We let unit_{Ran} denote the resulting map $\text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$.

2.1.3. We let \mathcal{Y}_{Δ_s} denote the corresponding semi-simplicial object⁹ of PreStk :

$$\dots \text{Ran } X \times \mathcal{Y} \rightrightarrows \mathcal{Y}.$$

We define $\mathcal{Y}_{\text{indep}}$ to be the geometric realization of \mathcal{Y}_{Δ_s} , i.e.,

$$\mathcal{Y}_{\text{indep}} := \underset{\Delta_s^{op}}{\text{colim}} \mathcal{Y}_{\Delta_s}.$$

⁹We denote by Δ_s the non-full subcategory of Δ obtained by restricting 1-morphisms to be injections of finite ordered sets.

Remark 2.1.4. Note that if the maps $\text{unit}_{J,I} : X^J \times Z(I) \rightarrow Z(J \sqcup I)$ are proper, then $\mathcal{Y}_{\text{indep}}$ is a pseudo-indscheme.

2.1.5. *Example.* Let X and Y be as in Sect. 1.7.5. Let $\mathcal{S} = (\text{fSet})^{op}$, with the natural action of $(\text{fSet})^{op}$ on itself. Let Z be the functor

$$\mathbf{Maps}(X, Y)_{X^{\text{fSet}}}^{\text{rat}} : (\text{fSet})^{op} \rightarrow \text{IndSch}.$$

It has a natural unital structure. Indeed for finite sets J and I the map

$$X^J \times \mathbf{Maps}(X, Y)_{X^I}^{\text{rat}} \subset \mathbf{Maps}(X, Y)_{X^{J \sqcup I}}^{\text{rat}}$$

is the closed embedding corresponding to restricting a map

$$m : (S \times X - \{x^I\}) \rightarrow Y$$

to a map

$$m' : (S \times X - (\{x^J\} \sqcup \{x^I\})) \rightarrow Y.$$

Consider the resulting object $\mathbf{Maps}(X, Y)_{\text{Ran } X, \text{indep}}^{\text{rat}}$. We can regard it as a version of $\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}$, where we have explicitly “modded out” by the dependence on the locus of singularity.

We shall see two more classes of examples in Sects. 2.5 and 2.7, respectively.

2.2. Strongly unital structures.

2.2.1. Let us now assume that we have the following additional pieces of structure on the category \mathcal{S} . Namely, let us be given a functor $f_{\mathcal{S}} : \mathcal{S} \rightarrow (\text{fSet})^{op}$, and a natural transformation

$$\mathbf{d} : \text{Id} \Rightarrow \sqcup \circ (f_{\mathcal{S}} \times \text{Id}),$$

i.e., map functorially assigned to every $I \in \mathcal{S}$:

$$\mathbf{d}(I) : I \rightarrow f_{\mathcal{S}}(I) \sqcup I.$$

2.2.2. *Example.* Assume that $\mathcal{S} = (\text{fSet})^{op}$, and let $f_{\mathcal{S}}$ be the identity functor. In this case we take \mathbf{d} to be the canonical map

$$I \sqcup I \rightarrow I$$

for $I \in \text{fSet}$.

2.2.3. Let $Z : \mathcal{S} \rightarrow \text{IndSch}$ be a unital functor. We shall say that Z is *strongly unital* with respect to $(f_{\mathcal{S}}, \mathbf{d})$ if we are given a natural transformation

$$f_Z : Z \Rightarrow X^{\text{fSet}} \circ f_{\mathcal{S}}$$

(i.e., we have a map $f(I) : Z(I) \rightarrow X^{f_{\mathcal{S}}(I)}$ that functorially depends on $I \in \mathcal{S}$), such that for every $I \in \mathcal{S}$ the composed map

$$(2.1) \quad Z(I) \xrightarrow{f(I) \times \text{id}} X^{f_{\mathcal{S}}(I)} \times Z(I) \xrightarrow{\text{unit}_{f_{\mathcal{S}}(I), I}} Z(f_{\mathcal{S}}(I) \sqcup I)$$

equals the map

$$Z(\mathbf{d}(I)) : Z(I) \rightarrow Z(f_{\mathcal{S}}(I) \sqcup I).$$

Remark 2.2.4. In Sect. 2.4.6 we shall explain the meaning of the above data in the example of Sect. 2.2.2.

2.2.5. *Example.* Let $(\mathcal{S}, f_{\mathcal{S}}, \mathbf{d})$ be as in Sect. 2.2.2, and let us take $Z = X^{\text{fSet}}$, where f_Z is the identity map. In this case the requirement of Sect. 2.2.3 holds tautologically.

2.2.6. *Example.* Let $(\mathcal{S}, f_{\mathcal{S}}, \mathbf{d})$ be again as in Sect. 2.2.2. Let us take $Z = \mathbf{Maps}(X, Y)_{X^{\mathbf{fSet}}}^{\text{rat}}$.

We let f_Z be the map that assigns to I the tautological projection

$$\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}} \rightarrow X^I.$$

It is easy to see that this functor satisfies the condition of Sect. 2.2.3.

2.2.7. Let Z be equipped with a strongly unital structure, and consider the corresponding prestack

$$\mathcal{Y} := \underset{\mathcal{S}}{\text{colim}} Z.$$

We claim that in this case there exists a canonically defined map

$$(2.2) \quad \mathcal{Y} \rightarrow \text{Ran } X \times \mathcal{Y},$$

such that its compositions with both the action map

$$\text{unit}_{\text{Ran}} : \text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$$

and the projection $\text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$ are the identity maps on \mathcal{Y} .

In terms of Sect. 1.5.4, the map (2.2) corresponds to the functor

$$\mathcal{S} \rightarrow (\mathbf{fSet})^{\text{op}} \times \mathcal{S}$$

given by $f_{\mathcal{S}} \times \text{Id}$, and the natural transformation

$$Z \Rightarrow X^{\mathbf{fSet}} \times Z$$

given by

$$Z(I) \xrightarrow{f(I) \times \text{id}} X^{f_{\mathcal{S}}(I)} \times Z(I).$$

The fact that the composition of the map (2.2) with the projection $\text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$ is the identity map on \mathcal{Y} is immediate.

To show that the composition

$$(2.3) \quad \mathcal{Y} \rightarrow \text{Ran } X \times \mathcal{Y} \xrightarrow{\text{unit}_{\text{Ran}}} \mathcal{Y}$$

is the identity map, we will use the following general observation.

2.2.8. Suppose that in the paradigm of Sect. 1.5.3, we are given another functor $F' : \mathbf{C}_1 \rightarrow \mathbf{C}_2$, and a natural transformation $\beta : F \Rightarrow F'$. Let α denote the original natural transformation $\Phi_1 \Rightarrow \Phi_2 \circ F$. Composing, we obtain a natural transformation $\alpha' : \Phi_1 \rightarrow \Phi_2 \circ F'$, and hence *another* map

$$\underset{\mathbf{C}_1}{\text{colim}} \Phi_1 \rightarrow \underset{\mathbf{C}_2}{\text{colim}} \Phi_2.$$

However, the resulting two maps, one coming from α , another from α' :

$$\underset{\mathbf{C}_1}{\text{colim}} \Phi_1 \rightrightarrows \underset{\mathbf{C}_2}{\text{colim}} \Phi_2$$

are canonically homotopic.

2.2.9. Returning to the composition (2.3), we apply the setting of Sect. 2.2.8 to $\mathbf{C}_1 = \mathbf{C}_2 = \mathcal{S}$, $\Phi_1 = \Phi_2 = Z$, with F being the identity functor, and F' being the functor $\text{unit}_{\text{Ran}} \circ (f_{\mathcal{S}} \times \text{Id})$, i.e.,

$$I \rightsquigarrow (f_{\mathcal{S}}(I) \sqcup I).$$

We claim that the resulting natural transformation

$$Z \Rightarrow Z \circ (\text{unit}_{\text{Ran}} \circ (f_{\mathcal{S}} \times \text{Id}))$$

comes from the natural transformation β

$$\text{Id} \rightarrow (\text{unit}_{\text{Ran}} \circ (f_{\mathcal{S}} \times \text{Id})),$$

supplied by \mathbf{d} . Indeed, this follows from the requirement on \mathbf{d} expressed by (2.1).

2.2.10. *Example.* Let us return to the example of Sect. 2.2.5, i.e., $Z = X^{\text{fSet}}$ as a functor $(\text{fSet})^{\text{op}} \rightarrow \text{Sch}$, with the data of f_Z being the identity map. Note that in this case, the map

$$\text{Ran } X \rightarrow \text{Ran } X \times \text{Ran } X$$

of (2.2) is the diagonal map. Thus, the semi-group $\text{Ran } X$ has the feature of square map is equal to the identity.

2.3. The “independent” category of D-modules. Throughout this subsection we let Z be as in Sect. 2.1.2.

2.3.1. Consider the resulting category $\mathfrak{D}(\mathcal{Y}_{\text{indep}})$. By definition

$$\mathfrak{D}(\mathcal{Y}_{\text{indep}}) \simeq \lim_{\Delta_s} \mathfrak{D}^!(\mathcal{Y}_{\Delta_s}),$$

where $\mathfrak{D}^!(\mathcal{Y}_{\Delta_s})$ is the functor

$$\Delta_s \xrightarrow{\mathcal{Y}_{\Delta_s}} \text{PreStk}^{\text{op}} \xrightarrow{\mathfrak{D}^!_{\text{PreStk}}} \text{DGCat}_{\text{cont}}.$$

Consider the forgetful functor

$$(2.4) \quad \mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y}).$$

It turns out that the functor (2.4) is often fully faithful.

2.3.2. Assume that \mathcal{Y} is such that there exists a map $\mathcal{Y} \rightarrow \text{Ran } X \times \mathcal{Y}$, such that its compositions with both the action map

$$\text{unit}_{\text{Ran}} : \text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$$

and the projection $\text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$ are the identity maps on \mathcal{Y} . As we have seen in Sect. 2.2.7, this happens if the unital structure on Z can be upgraded to a strongly unital one.

Proposition 2.3.3. *Under the above circumstances, the forgetful functor*

$$\mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y})$$

is fully faithful.

Proof.

Step 1. Consider the functor

$$\text{unit}_{\text{Ran}}^! : \mathfrak{D}(\mathcal{Y}) \rightarrow \mathfrak{D}(\text{Ran } X \times \mathcal{Y}) \simeq \mathfrak{D}(\text{Ran } X) \otimes \mathfrak{D}(\mathcal{Y})$$

(the last isomorphism is due to (1.6)).

Consider also the map $(p_{\text{Ran } X} \times \text{id}) : \text{Ran } X \times \mathcal{Y} \rightarrow \mathcal{Y}$ and the corresponding functor

$$(p_{\text{Ran } X} \times \text{id})^! : \mathfrak{D}(\mathcal{Y}) \simeq \text{Vect} \otimes \mathfrak{D}(\mathcal{Y}) \rightarrow \mathfrak{D}(\text{Ran } X \times \mathcal{Y}) \simeq \mathfrak{D}(\text{Ran } X) \otimes \mathfrak{D}(\mathcal{Y}).$$

Note, however, that by Theorem 1.6.5, the latter functor is fully faithful.

Let $\mathfrak{D}(\mathcal{Y})'$ be the *full subcategory* of $\mathfrak{D}(\mathcal{Y})$ spanned by objects, whose image under $\text{unit}_{\text{Ran}}^!$ lies in the essential image of $(p_{\text{Ran } X} \times \text{id})^!$.

It is clear that the forgetful functor $\mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y})$ factors as

$$\mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y})' \rightarrow \mathfrak{D}(\mathcal{Y}).$$

We will show that the above functor $\mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y})'$ is an equivalence.

Step 2. It is clear that the assignment

$$[n] \mapsto \mathfrak{D}(\mathcal{Y})' = \text{Vect}^{\otimes n} \otimes \mathfrak{D}(\mathcal{Y})' \subset \mathfrak{D}(\text{Ran } X)^{\otimes n} \otimes \mathfrak{D}(\mathcal{Y})$$

extends to a functor

$$\mathfrak{D}^!(\mathcal{Y}_{\Delta_s})' : \Delta_s \rightarrow \text{DGCat},$$

equipped with a natural transformation

$$\mathfrak{D}^!(\mathcal{Y}_{\Delta_s})' \Rightarrow \mathfrak{D}^!(\mathcal{Y}_{\Delta_s}),$$

which is fully faithful for every $[n] \in \Delta_s$.

In particular, the resulting functor

$$(2.5) \quad \mathfrak{D}(\mathcal{Y}_{\text{indep}})' := \lim_{\Delta_s} \mathfrak{D}^!(\mathcal{Y}_{\Delta_s})' \rightarrow \lim_{\Delta_s} \mathfrak{D}^!(\mathcal{Y}_{\Delta_s}) = \mathfrak{D}(\mathcal{Y}_{\text{indep}}).$$

is also fully faithful.

However, it is easy to see that for every n and every object $\mathfrak{D}(\mathcal{Y}_{\text{indep}})$, its evaluation on $[n] \in \Delta_s$ belongs to $\mathfrak{D}^!(\mathcal{Y}_{[n]})'$. So, the functor (2.5) is an equivalence.

The composition

$$\mathfrak{D}(\mathcal{Y}_{\text{indep}})' \rightarrow \mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y})'$$

is functor of evaluation on $[0] \in \Delta_s$. Thus, we obtain that is sufficient to show that the above evaluation functor is an equivalence.

Step 3. We claim that for any map $\phi : [0] \rightarrow [n]$ in Δ_s , and the corresponding map $\phi_{\mathcal{Y}} : (\text{Ran } X)^{\times n} \times \mathcal{Y} \rightarrow \mathcal{Y}$, the functor

$$\phi_{\mathcal{Y}}^! : \mathfrak{D}^!(\mathcal{Y}_{[0]})' \rightarrow \mathfrak{D}^!(\mathcal{Y}_{[n]})',$$

is an equivalence. In fact, both categories in question are identified with $\mathfrak{D}(\mathcal{Y})$, and we claim that the above functor is canonically isomorphic to the identity functor.

There are two cases: if ϕ is the map $0 \mapsto 0 \in \{0, 1, \dots, n\}$, then $\phi_{\mathcal{Y}}$ is the projection map $(\text{Ran } X)^{\times n} \times \mathcal{Y} \rightarrow \mathcal{Y}$, and there is nothing to prove.

If ϕ is any other map, we claim that the map $\phi_{\mathcal{Y}}$ admits a canonical right inverse, denoted $\psi_{\mathcal{Y}}$. Namely, $\psi_{\mathcal{Y}}$ is composition of the map $\mathcal{Y} \rightarrow \text{Ran } X \times \mathcal{Y}$ of (2.2) with the n -fold diagonal map

$$\text{Ran } X \times \mathcal{Y} \rightarrow (\text{Ran } X)^{\times n} \times \mathcal{Y}.$$

By construction, the composition of $\psi_{\mathcal{Y}}$ with the projection of $(\text{Ran } X)^{\times n} \times \mathcal{Y}$ onto \mathcal{Y} is the identity map on \mathcal{Y} .¹⁰

The latter property implies that the functor $\psi_{\mathcal{Y}}^! : \mathfrak{D}^!(\mathcal{Y}_{[n]}) \rightarrow \mathfrak{D}(\mathcal{Y})$ induces the identity functor

$$\mathfrak{D}(\mathcal{Y}) = \text{Vect}^{\otimes n} \otimes \mathfrak{D}(\mathcal{Y}) \hookrightarrow \mathfrak{D}^!(\mathcal{Y}_{[n]}) \xrightarrow{\psi_{\mathcal{Y}}^!} \mathfrak{D}(\mathcal{Y}),$$

¹⁰Despite this fact, the structure of semi-simplicial object on \mathcal{Y}_{Δ_s} does not upgrade to a simplicial one.

and hence, the composition

$$\mathfrak{D}^!(\mathcal{Y}_{[n]})' \hookrightarrow \mathfrak{D}^!(\mathcal{Y}_{[n]}) \xrightarrow{\psi_{\mathcal{Y}}^!} \mathfrak{D}(\mathcal{Y})$$

is also the identity map of $\mathfrak{D}^!(\mathcal{Y}_{[n]})'$ onto $\mathfrak{D}(\mathcal{Y})' \subset \mathfrak{D}(\mathcal{Y})$.

Hence, it is enough to show that the composition

$$\mathfrak{D}(\mathcal{Y})' = \mathfrak{D}^!(\mathcal{Y}_{[0]})' \xrightarrow{\phi_{\mathcal{Y}}^!} \mathfrak{D}^!(\mathcal{Y}_{[n]})' \hookrightarrow \mathfrak{D}^!(\mathcal{Y}_{[n]}) \xrightarrow{\psi_{\mathcal{Y}}^!} \mathfrak{D}(\mathcal{Y})$$

is also isomorphic to the identity map onto $\mathfrak{D}(\mathcal{Y})' \subset \mathfrak{D}(\mathcal{Y})$. However, this follows from the fact that $\psi_{\mathcal{Y}}^! \circ \phi_{\mathcal{Y}}^! = \text{Id}_{\mathfrak{D}(\mathcal{Y})}$.

Thus, we obtain that the semi-simplicial category $\mathfrak{D}^!(\mathcal{Y}_{\Delta_s})'$ consists of equivalences, and since the index category Δ_s is contractible, we obtain that evaluation on $[0]$ is an equivalence. \square

2.3.4. As a corollary of Proposition 2.3.3 we obtain the following:

Let $\mathcal{F}_{\text{indep}}$ be an object of $\mathfrak{D}(\mathcal{Y}_{\text{indep}})$, and let $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})$ be its image under the forgetful functor $\mathfrak{D}(\mathcal{Y}_{\text{indep}}) \rightarrow \mathfrak{D}(\mathcal{Y})$. Assume that $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})_{\text{good for } p_{\mathcal{Y}}}$, see Sect. 1.5.1 for the notation.

Under these circumstances we have:

Corollary 2.3.5. *The natural map*

$$\Gamma_{\text{dR},c}(\mathcal{Y}, \mathcal{F}) \rightarrow \Gamma_{\text{dR},c}(\mathcal{Y}_{\text{indep}}, \mathcal{F}_{\text{indep}})$$

is an isomorphism; in particular, the right-hand side is defined.

As a particular case, we obtain:

Corollary 2.3.6. *For X and Z as in Proposition 2.3.3, the trace map*

$$H_{\bullet}(\mathcal{Y}) \rightarrow H_{\bullet}(\mathcal{Y}_{\text{indep}})$$

is an isomorphism.

2.3.7. *Example.* Let us calculate the category $\mathfrak{D}(Z_{\text{Ran } X, \text{indep}})$ for $Z = X^{\text{fSet}}$; we shall denote it by $\mathfrak{D}(\text{Ran } X, \text{indep})$.

Note that we expect this category to be Vect : the idea of the “indep” category was to get rid of the dependence on the Ran space, so we are supposed to be dealing with D-modules on the Ran space that “do not depend on the Ran space variable”. So, let us see that this is indeed the case.

We have an obvious functor

$$(2.6) \quad \text{Vect} \rightarrow \mathfrak{D}(\text{Ran } X, \text{indep})$$

that sends $k \mapsto \omega_{\text{Ran } X}$.

By Theorem 1.6.5, the composed functor

$$\text{Vect} \rightarrow \mathfrak{D}(\text{Ran } X, \text{indep}) \rightarrow \mathfrak{D}(\text{Ran } X)$$

is fully faithful. Hence, by Proposition 2.3.3, the above functor (2.6) is also fully faithful. Thus, it remains to see that it is essentially surjective.

For $\mathcal{F} \in \mathfrak{D}(\text{Ran } X)$ consider the object

$$\text{union}^!(\mathcal{F}) \in \mathfrak{D}(\text{Ran } X \times \text{Ran } X).$$

If \mathcal{F} lies in the essential image of the forgetful functor $\mathfrak{D}(\text{Ran } X, \text{indep}) \rightarrow \mathfrak{D}(\text{Ran } X)$, then

$$\text{union}^!(\mathcal{F}) \simeq \omega_{\text{Ran } X} \boxtimes \mathcal{F}.$$

However, since the multiplication on $\text{Ran } X$ is commutative, we also have

$$\text{union}^!(\mathcal{F}) \simeq \mathcal{F} \boxtimes \omega_{\text{Ran } X},$$

so we obtain an isomorphism

$$(2.7) \quad \omega_{\text{Ran } X} \boxtimes \mathcal{F} \simeq \mathcal{F} \boxtimes \omega_{\text{Ran } X}.$$

By Theorem 1.6.5, for an object $\mathcal{F}_1 \in \mathfrak{D}(\text{Ran } X \times \text{Ran } X)$ in the essential image of the functor

$$\begin{aligned} (\text{id} \times p_{\text{Ran } X})^! : \mathfrak{D}(\text{Ran } X) &\simeq \mathfrak{D}(\text{Ran } X) \otimes \text{Vect} \xrightarrow{\text{Id} \boxtimes p_{\text{Ran } X}^!} \mathfrak{D}(\text{Ran } X) \otimes \mathfrak{D}(\text{Ran } X) \simeq \\ &\simeq \mathfrak{D}(\text{Ran } X \times \text{Ran } X), \end{aligned}$$

we have

$$\mathcal{F}_1 \simeq (\text{id} \times p_{\text{Ran } X})^!(\mathcal{F}_1) \boxtimes \omega_{\text{Ran } X}.$$

Hence, from (2.7), we obtain that

$$\omega_{\text{Ran } X} \boxtimes \mathcal{F} \simeq \omega_{\text{Ran } X \times \text{Ran } X} \otimes \Gamma_{\text{dR},c}(\text{Ran } X, \mathcal{F}).$$

Since the functor $(p_{\text{Ran } X} \times \text{id})^!$ is fully faithful, the latter isomorphism implies that

$$\mathcal{F} \simeq \omega_{\text{Ran } X} \otimes \Gamma_{\text{dR},c}(\text{Ran } X, \mathcal{F}),$$

as required.

2.4. Spaces over $\text{Ran } X$.

In this subsection we specialize to the setting of the Example in Sect. 2.2.2. Namely, $\mathcal{S} = (\text{fSet})^{op}$, $f_{\mathcal{S}} = \text{Id}$ and the natural transformation \mathbf{d} being the canonical map

$$(\text{id} \sqcup \text{id}) : I \sqcup I \rightarrow I$$

for $I \in \text{fSet}$.

As we have seen above, X^{fSet} and $\mathbf{Maps}(X, Y)_{X^{\text{fSet}}}^{\text{rat}}$ provide examples of functors Z equipped with a strong unital structure.

2.4.1. We shall consider functors $Z : (\text{fSet})^{op} \rightarrow \text{IndSch}$ equipped with a strongly unital structure with respect to $(f_{\mathcal{S}}, \mathbf{d})$ specified above. I.e., these are functors

$$I \rightsquigarrow Z(I),$$

equipped with a functorial assignment to every pair of non-empty finite sets J and I of a map

$$\text{unit}_{J,I} : X^J \times Z(I) \rightarrow Z(J \sqcup I),$$

and a natural transformation

$$I \rightsquigarrow (Z(I) \xrightarrow{f^{(I)}} X^I),$$

such that the condition from Sect. 2.2.3 holds.

We shall impose the following two additional conditions:

(1) For $I, J \in \mathbf{fSet}$, the diagram

$$\begin{array}{ccc} X^J \times Z(I) & \xrightarrow{\text{unit}_{J,I}} & Z(J \sqcup I) \\ \text{id} \times f(I) \downarrow & & \downarrow f(J \sqcup I) \\ X^J \times X^I & \xrightarrow{\sim} & X^{J \sqcup I} \end{array}$$

commutes.

(2) For every arrow in $(\mathbf{fSet})^{op}$, i.e., a surjective map of finite sets $I_1 \twoheadrightarrow I_2$, the resulting map

$$Z(I_2) \rightarrow X^{I_2} \times_{X^{I_1}} Z(I_1)$$

is an isomorphism.

The examples of Z being $X^{\mathbf{fSet}}$ and $\mathbf{Maps}(X, Y)_{X^{\mathbf{fSet}}}^{\text{rat}}$ satisfy these conditions.

We let $Z_{\text{Ran } X}$ denote the corresponding space

$$\text{colim}_{(\mathbf{fSet})^{op}} Z.$$

By Sect. 2.1.2, the space $Z_{\text{Ran } X}$ is acted on by the semi-group $\text{Ran } X$.

Remark 2.4.2. As in Remark 1.7.7, the prestack $Z_{\text{Ran } X}$, considered as a functor $(\text{Sch}^{\text{aff}})^{op} \rightarrow \infty\text{-Grpd}$, actually takes values in $\text{Set} \subset \infty\text{-Grpd}$. Indeed, for $S \in \text{Sch}^{\text{aff}}$, the infinity-groupoid $\text{Maps}(S, Z_{\text{Ran } X})$ maps to the *set* of non-empty finite subsets of $\text{Maps}(S, X)$, and for $\bar{x} \subset \text{Maps}(S, X)$, the fiber over it is $Z(\bar{x}) \times_{X^{\bar{x}}} \text{pt}$, where $\text{pt} \rightarrow X^{\bar{x}}$ is the tautological point of $X^{\bar{x}}$ corresponding to \bar{x} .

For this property it was not necessary that the functor Z take values in indschemes, we only need that each $Z(I)$, considered as a functor $(\text{Sch}^{\text{aff}})^{op} \rightarrow \infty\text{-Grpd}$, take values in $\text{Set} \subset \infty\text{-Grpd}$.

2.4.3. Note that the condition to be a strongly unital structure in Sect. 2.2.3 says that for every $I \in \mathbf{fSet}$, the diagram

$$\begin{array}{ccc} Z(I) & \xrightarrow{f(I) \times \text{id}} & X^I \times Z(I) \\ \text{id} \downarrow & & \downarrow \text{unit}_{I,I} \\ Z(I) & \longrightarrow & Z(I \sqcup I) \end{array}$$

commutes, where the bottom arrow corresponds to the map $(\text{id} \sqcup \text{id}) : I \sqcup I \rightarrow I$ in \mathbf{fSet} .

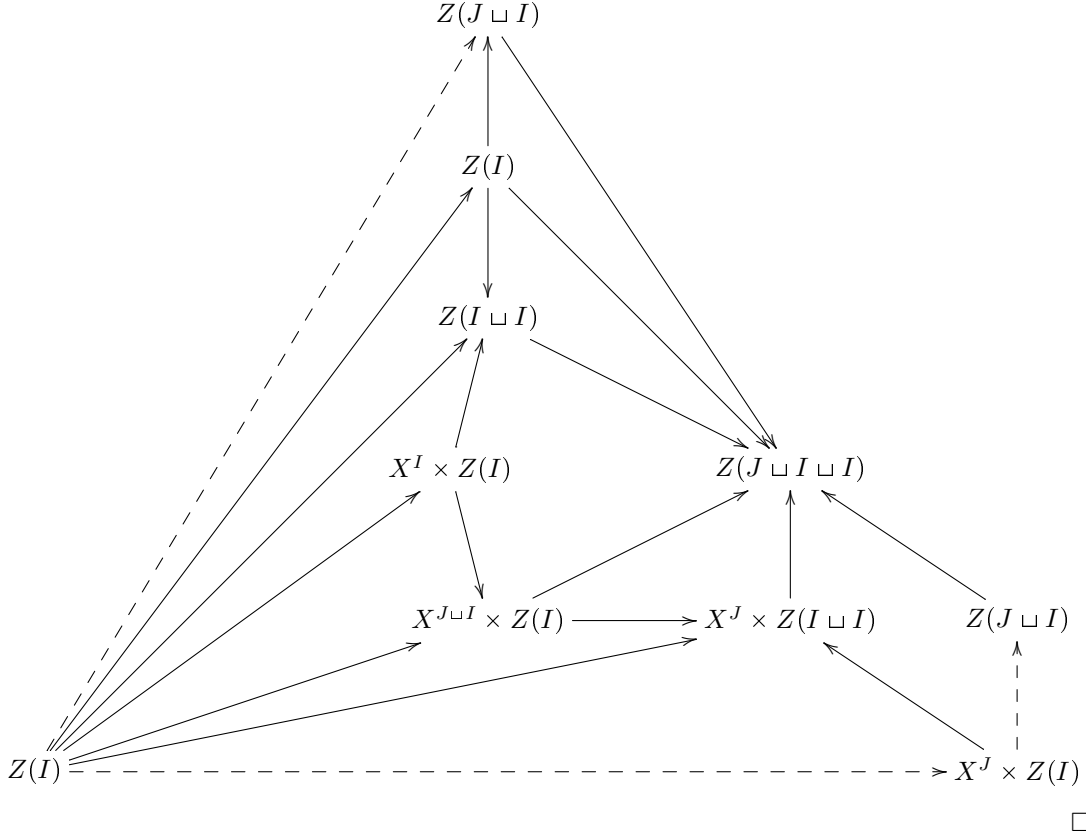
Lemma 2.4.4. *For any (not necessarily surjective) map of finite sets $\phi : J \rightarrow I$, the diagram*

$$\begin{array}{ccc} Z(I) & \xrightarrow{\Delta(\phi) \circ f(I) \times \text{id}} & X^J \times Z(I) \\ \text{id} \downarrow & & \downarrow \text{unit}_{J,I} \\ Z(I) & \longrightarrow & Z(J \sqcup I) \end{array}$$

commutes as well, where the bottom arrow corresponds to the map $(\phi \sqcup \text{id}) : J \sqcup I \rightarrow I$ in $(\mathbf{fSet})^{op}$.

Proof. It is enough to show that the two maps $Z(I) \rightrightarrows Z(J \sqcup I)$ in question, composed with the map $Z(J \sqcup I) \rightarrow Z(J \sqcup I \sqcup I)$, corresponding to the obvious surjection $J \sqcup I \sqcup I \rightarrow J \sqcup I$, coincide. The latter follows by chasing through the following diagram, in which every quadrangle and triangle are commutative (the dotted arrows are the original arrows in the lemma):

(2.8)



Remark 2.4.5. The complexity of the above diagram leads one to wonder, in the situation when $Z(I)$'s are no longer indschemes, but more general prestacks (so, we can no longer talk about equality of maps but homotopy equivalences), whether the constructed identification of the two maps $Z(I) \rightrightarrows Z(J \sqcup I)$ is canonical. I.e., whether a different diagram would not produce a different identification.

The answer is that this isomorphism is canonical. Indeed, consider the diagram

$$\begin{array}{ccc}
 X^I \times_{X^{J \sqcup I}} (X^J \times Z(I)) & \xrightarrow{\text{unit}_{J,I}} & X^I \times_{X^{J \sqcup I}} Z(J \sqcup I) \\
 \uparrow & & \uparrow \\
 Z(I) & \xrightarrow{\alpha(I)} & Z(I),
 \end{array}$$

where the left vertical is an isomorphism, and the right vertical arrows is an isomorphism by the assumption on Z (see Sect. 2.4.1).

The assertion of Lemma 2.4.4 says that the bottom map is the identity map on $Z(I)$. However, it was enough to show that $\alpha(I)$ is an isomorphism. Indeed, associativity of the action of X^{fSet} on Z would then imply that $\alpha(I) \circ \alpha(I)$ is canonically homotopic to $\alpha(I)$, implying that $\alpha(I) \sim \text{id}$.

2.4.6. One can interpret the assertion of Lemma 2.4.4 as follows. Let $Z : (\mathbf{fSet})^{op} \rightarrow \mathbf{IndSch}$ be as in Sect. 2.4.1. We claim that for a pair of non-empty finite sets I and J and *any* (i.e., not necessarily surjective) map $I \rightarrow J$, we have a well-defined map

$$X^J \times_{X^I} Z(I) \rightarrow Z(J),$$

compatible with the projection to X^J , and compatible with compositions.

Indeed, when $I \rightarrow J$ is surjective, the map in question is the isomorphism of Condition (2) in Sect. 2.4.1. When $I \rightarrow J$ is an injection $I \hookrightarrow I \sqcup K \simeq J$, the map in question is

$$X^J \times_{X^I} Z(I) \simeq X^K \times Z(I) \xrightarrow{\text{unit}_{K,I}} Z(K \sqcup I).$$

The compatibility with compositions is assured by Lemma 2.4.4.

2.5. Ran space with marked points. In this subsection we will consider a certain class of statements that a strongly unital structure allows one to prove.

2.5.1. Let A be a finite set. Let \mathbf{fSet}_A be the category whose objects are finite sets I , equipped with a map $A \rightarrow I$, and where the morphisms are surjective maps $I_1 \twoheadrightarrow I_2$, for which the diagram

$$\begin{array}{ccc} I_1 & \longrightarrow & I_2 \\ \uparrow & & \uparrow \\ A & \xrightarrow{\text{id}} & A \end{array}$$

commutes.

The category \mathbf{fSet}_A is naturally a module over \mathbf{fSet} via operation of disjoint union. The corresponding functor f_S is the forgetful functor $\mathbf{oblv}_A : \mathbf{fSet}_A \rightarrow \mathbf{fSet}$. The natural transformation \mathbf{d} is given by the canonical map $(\text{id} \sqcup \text{id}) : I \sqcup I \rightarrow I$.

2.5.2. The first example of a functor $Z : (\mathbf{fSet}_A)^{op} \rightarrow \mathbf{IndSch}$ satisfying the requirements of Sect. 2.2.3 is constructed as follows:

Let x^A be a k -point of X^A . Let $X^{\mathbf{fSet}_A}$ be the functor $(\mathbf{fSet}_A)^{op} \rightarrow \mathbf{Sch}$ given by

$$I \mapsto X_A^I := x^A \times_{X^A} X^I.$$

Let

$$\mathbf{Ran} X_A := \mathop{\text{colim}}_{(\mathbf{fSet}_A)^{op}} X^{\mathbf{fSet}_A}.$$

We shall refer to $\mathbf{Ran} X_A$ as the *relative version of Ran X with marked points*.

Remark 2.5.3. For $S \in \mathbf{Sch}^{\text{aff}}$, the ∞ -groupoid $\mathbf{Maps}(S, \mathbf{Ran} X_A)$ is in fact a set isomorphic to that of non-empty finite subsets $\bar{x} \subset \mathbf{Maps}(S, X)$ that contain as a subset the image of A under

$$A \rightarrow \mathbf{Maps}(\text{pt}, X) \rightarrow \mathbf{Maps}(S, X).$$

2.5.4. Let Z be a functor $(\mathbf{fSet})^{op} \rightarrow \mathbf{IndSch}$ as in Sect. 2.4.1. For (A, x^A) as above, let Z_A be a functor $(\mathbf{fSet}_A)^{op} \rightarrow \mathbf{IndSch}$ given by

$$I \mapsto Z(I)_A := X_A^I \times_{X^I} Z(I).$$

The strongly unital structure on Z induces one on Z_A . Set

$$Z_{\mathbf{Ran} X_A} := \mathop{\text{colim}}_{(\mathbf{fSet}_A)^{op}} Z_A \in \mathbf{PreStk}.$$

We obtain that $Z_{\mathbf{Ran} X_A}$ becomes a module over $\mathbf{Ran} X$.

2.5.5. By Sect. 1.5.4, the forgetful functor $\mathbf{obl}_A : \mathbf{fSet}_A \rightarrow \mathbf{fSet}$ and the natural transformation $Z_A \Rightarrow Z \circ (\mathbf{obl}_A)^{op}$, given by $Z(I)_A \rightarrow Z(I)$, define a map

$$(2.9) \quad Z_{\mathbf{Ran} X_A} \rightarrow Z_{\mathbf{Ran} X}.$$

Moreover, it is easy to see that the map (2.9) is compatible with the $\mathbf{Ran} X$ -actions. Therefore, it gives rise to a map of the corresponding semi-simplicial objects

$$Z_{\mathbf{Ran} X_A, \Delta_s} \rightarrow Z_{\mathbf{Ran} X, \Delta_s}$$

and

$$(2.10) \quad Z_{\mathbf{Ran} X_A, \text{indep}} \rightarrow Z_{\mathbf{Ran} X, \text{indep}}.$$

We will prove:

Proposition 2.5.6. *The map (2.10) is an isomorphism.*

Remark 2.5.7. The meaning of this proposition is that in the unital context, constraining our finite subset of points of X to contain a given set x^A does not alter the resulting space.

2.5.8. Before we prove the proposition, let us discuss some corollaries:

Let Z be as in Sect. 2.4.1, and let \mathcal{F} be an object of $\mathfrak{D}(Z_{\mathbf{Ran} X})$, which lies in the essential image of the forgetful functor $\mathfrak{D}(Z_{\mathbf{Ran} X, \text{indep}}) \rightarrow \mathfrak{D}(Z_{\mathbf{Ran} X})$, and which also lies in $\mathfrak{D}(Z_{\mathbf{Ran} X})_{\text{good for } p_{Z_{\mathbf{Ran} X}}}$.

Let (A, x^A) be as in Sect. 2.5.1. Let \mathcal{F}_A be the pullback of \mathcal{F} under the map $Z_{\mathbf{Ran} X_A} \rightarrow Z_{\mathbf{Ran} X}$ of (2.9). Combining Corollary 2.3.5 with Proposition 2.5.6, we obtain:

Corollary 2.5.9. *The trace map*

$$\Gamma_{\text{dR}, c}(Z_{\mathbf{Ran} X_A}, \mathcal{F}_A) \rightarrow \Gamma_{\text{dR}, c}(Z_{\mathbf{Ran} X}, \mathcal{F})$$

is an isomorphism.

As a particular case, we have:

Corollary 2.5.10. *Then the trace map*

$$H_\bullet(Z_{\mathbf{Ran} X_A}) \rightarrow H_\bullet(Z_{\mathbf{Ran} X})$$

is an isomorphism.

2.5.11. *Proof of Proposition 2.5.6.*

Step 1. Consider now the functor $(A \sqcup -) : \mathbf{fSet} \rightarrow \mathbf{fSet}_A$:

$$I \mapsto A \sqcup I.$$

The unital structure on Z defines a natural transformation

$$Z \Rightarrow Z_A \circ (A \sqcup -)$$

between the resulting two functors $(\mathbf{fSet})^{op} \rightrightarrows \mathbf{IndSch}$:

$$Z(I) \simeq x^A \times Z(I) \xrightarrow{\text{unit}_{A, I}} Z(A \sqcup I)_A.$$

By Sect. 1.5.4, we obtain a map

$$(2.11) \quad Z_{\mathbf{Ran} X} \rightarrow Z_{\mathbf{Ran} X_A}.$$

It is easy to see from the construction that this map is also compatible with the action of the semi-group $\mathbf{Ran} X$. In particular, we obtain the map of semi-simplicial objects

$$Z_{\mathbf{Ran} X, \Delta_s} \rightarrow Z_{\mathbf{Ran} X_A, \Delta_s},$$

and a map

$$(2.12) \quad Z_{\text{Ran } X, \text{indep}} \rightarrow Z_{\text{Ran } X_A, \text{indep}}.$$

We will show that composition of the maps (2.9) and (2.11):

$$(2.13) \quad Z_{\text{Ran } X_A} \rightarrow Z_{\text{Ran } X} \rightarrow Z_{\text{Ran } X_A}$$

is (canonically homotopic to) the identity map on $Z_{\text{Ran } X_A}$, in a way compatible with the $\text{Ran } X$ -action. The latter compatibility will imply that the composition

$$Z_{\text{Ran } X_A, \text{indep}} \rightarrow Z_{\text{Ran } X, \text{indep}} \rightarrow Z_{\text{Ran } X_A, \text{indep}}$$

is also (canonically homotopic to) the identity map.

We will then show that the composition

$$(2.14) \quad Z_{\text{Ran } X, \text{indep}} \rightarrow Z_{\text{Ran } X_A, \text{indep}} \rightarrow Z_{\text{Ran } X, \text{indep}}$$

is (canonically homotopic to) the identity map as well.

Step 2. The assertion regarding the composition (2.13) follows from the paradigm described in Sect. 2.2.8:

We apply it to $\mathbf{C}_1 = \mathbf{C}_2 = (\text{fSet}_A)^{op}$, $\Phi_1 = \Phi_2 = Z_A$, F being the identity functor and F' being the functor

$$(A \rightarrow I) \rightsquigarrow (A \rightarrow A \sqcup I),$$

where $A \xrightarrow{\text{id}} A \hookrightarrow A \sqcup I$.

The natural transformation β is the given by the map $A \sqcup I \rightarrow I$ in fSet_A . The fact that the natural transformation

$$Z_A \Rightarrow Z_A \circ (A \sqcup -) \circ (\mathbf{oblv}_A)^{op}$$

is isomorphic to $Z_A \circ \beta$ follows from Lemma 2.4.4.

The naturality of the construction implies the compatibility with the action of the semi-group $\text{Ran } X$.

Step 3. In order to prove that the composed map (2.14) is (canonically homotopic to) the identity map, since the semi-group $\text{Ran } X$ is commutative, it suffices to show that the composition

$$Z_{\text{Ran } X} \xrightarrow{(2.11)} Z_{\text{Ran } X_A} \xrightarrow{(2.9)} Z_{\text{Ran } X}$$

factors as

$$Z_{\text{Ran } X} \xrightarrow{\gamma} \text{Ran } X \times Z_{\text{Ran } X} \xrightarrow{\text{unit}_{\text{Ran } X}} Z_{\text{Ran } X}$$

for some map $\gamma : Z_{\text{Ran } X} \rightarrow \text{Ran } X \times Z_{\text{Ran } X}$, such that the composition

$$Z_{\text{Ran } X} \xrightarrow{\gamma} \text{Ran } X \times Z_{\text{Ran } X} \xrightarrow{p_{\text{Ran } X} \times \text{id}} Z_{\text{Ran } X}$$

is the identity map.

The required map γ is given in terms of the paradigm of Sect. 1.5.4 as follows. Namely, the functor

$$\gamma_{\text{fSet}} : (\text{fSet})^{op} \rightarrow (\text{fSet} \times \text{fSet})^{op}$$

is

$$I \rightsquigarrow (A, I),$$

and the natural transformation between the two functors $(\text{fSet})^{op} \rightrightarrows \text{PreStk}$

$$Z \Rightarrow (X^{\text{fSet}} \times Z) \circ \gamma_{\text{fSet}}$$

sends $I \in \text{fSet}$ to the map $(x^A \times \text{id}) : Z(I) \rightarrow X^A \times Z(I)$.

□

2.5.12. We conclude this subsection with the observation that the assertion of Proposition 2.5.6 (with the same proof) holds also in the relative situation:

Let S be a scheme and let x^A be an S -point of X^A . Then we can consider a relative version of the functor X^{fSet_A} , which now maps $(\text{fSet}_A)^{op}$ to $\text{Sch}/_S$:

$$I \rightsquigarrow X_A^I := S \times_{X^A} X^I,$$

and the corresponding prestack $\text{Ran } X_A$ over S .

Given a functor $Z : (\text{fSet})^{op} \rightarrow \text{IndSch}/_S$ satisfying the conditions of Sect. 2.4.1 (with the target category IndSch replaced by $\text{IndSch}/_S$), we let Z_A be the functor $(\text{fSet}_A)^{op} \rightarrow \text{IndSch}/_S$ given by

$$I \rightsquigarrow S \times_{S \times X^A} Z(I) \simeq X_A^I \times_{S \times X^I} Z(I).$$

Set

$$Z_{\text{Ran } X_A} := \text{colim}_{(\text{fSet}_A)^{op}} Z_A.$$

Let p_A (resp., p) denote the map $Z_{\text{Ran } X_A} \rightarrow S$ (resp., $Z_{\text{Ran } X} \rightarrow S$).

We have the following relative version of Corollary 2.5.10:

Corollary 2.5.13. *For $\mathcal{F}_S \in \mathfrak{D}(S)$, the map*

$$(p_A)! \circ (p_A)^!(\mathcal{F}_S) \rightarrow p! \circ p^!(\mathcal{F}_S)$$

is an isomorphism.

2.6. **Products over $\text{Ran } X$.** In this subsection we will give another example of a statement that a unital structure allows to prove.

2.6.1. Let Z^1 and Z^2 be two functors $(\text{fSet})^{op} \rightarrow \text{IndSch}$ as in Sect. 2.4.1. Let Z be yet another functor defined by

$$Z(I) := Z^1(I) \times_{X^I} Z^2(I).$$

Then Z is also a functor satisfying the requirements of Sect. 2.4.1.

Consider the corresponding spaces $Z_{\text{Ran } X}^1$, $Z_{\text{Ran } X}^2$, and $Z_{\text{Ran } X}$.

2.6.2. Note that we have a naturally defined map

$$(2.15) \quad Z_{\text{Ran } X} \rightarrow Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2.$$

In terms of Sect. 1.5.4, the map in question corresponds to the diagonal functor

$$\mathbf{diag}^{op} : (\text{fSet})^{op} \rightarrow (\text{fSet} \times \text{fSet})^{op},$$

and the natural transformation between the resulting two functors $(\text{fSet})^{op} \rightrightarrows \text{IndSch}$:

$$Z \rightarrow (Z^1 \times Z^2) \circ \mathbf{diag}^{op},$$

given by

$$I \rightsquigarrow \left(Z(I) = Z^1(I) \times_{X^I} Z^2(I) \rightarrow Z^1(I) \times Z^2(I) \right).$$

It is easy to see that the map (2.15) is compatible with the action of $\text{Ran } X$ via the diagonal homomorphism $\text{Ran } X \rightarrow \text{Ran}_X \times \text{Ran}_X$ and the action of $\text{Ran}_X \times \text{Ran}_X$ on $Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2$.

2.6.3. We claim now that there exists a canonically defined map

$$(2.16) \quad Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2 \rightarrow Z_{\text{Ran } X}.$$

Namely, in terms of Sect. 1.5.4, the map in question corresponds to the functor

$$\text{union}^{op} : (\text{fSet} \times \text{fSet})^{op} \rightarrow (\text{fSet})^{op},$$

and the natural transformation between the resulting two functors $(\text{fSet} \times \text{fSet})^{op} \rightrightarrows \text{IndSch}$:

$$(Z^1 \times Z^2) \rightarrow Z \circ \text{union}^{op},$$

given by sending $(I_1, I_2) \in \text{fSet} \times \text{fSet}$ to the map

$$\begin{aligned} Z^1(I_1) \times Z^2(I_2) &\simeq (Z^1(I_1) \times X^{I_2}) \times_{X^{I_1} \times X^{I_2}} (X^{I_1} \times Z(I_2)) \rightarrow \\ &\rightarrow Z^1(I_1 \sqcup I_2) \times_{X^{I_1} \times X^{I_2}} Z^2(I_1 \sqcup I_2) = Z(I_1 \sqcup I_2). \end{aligned}$$

It is easy to see that the map (2.16) is compatible with the action of $\text{Ran } X$ via the homomorphism

$$\text{union} : \text{Ran } X \times \text{Ran}_X \rightarrow \text{Ran}_X$$

and the action of $\text{Ran}_X \times \text{Ran}_X$ on $Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2$.

Thus, the maps (2.15) and (2.16) give rise to maps

$$(2.17) \quad Z_{\text{Ran } X, \text{indep}} \xleftrightarrow{\cong} Z_{\text{Ran } X, \text{indep}}^1 \times Z_{\text{Ran } X, \text{indep}}^2.$$

Proposition 2.6.4. *The maps (2.17) are mutually inverse isomorphisms.*

We omit the proof, because it is completely analogous to the one of Proposition 2.5.6.

2.6.5. Proposition 2.6.4 admits corollaries analogous to those of Proposition 2.5.6:

Let \mathcal{F} be an object of $\mathfrak{D}(Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2)$ that lies in the essential image of the forgetful functor

$$\mathfrak{D}(Z_{\text{Ran } X, \text{indep}}^1 \times Z_{\text{Ran } X, \text{indep}}^2) \rightarrow \mathfrak{D}(Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2).$$

Assume also that \mathcal{F} belongs to $\mathfrak{D}(Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2)_{\text{good}}$ for $p_{Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2}$.

Let $\mathcal{F}^{\text{diag}}$ denote the pullback of \mathcal{F} to $Z_{\text{Ran } X}$ under the map (2.16). We have:

Corollary 2.6.6. *The trace map*

$$\Gamma_{\text{dR}, c}(Z_{\text{Ran } X}, \mathcal{F}^{\text{diag}}) \rightarrow \Gamma_{\text{dR}, c}(Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2, \mathcal{F})$$

is an isomorphism.

In particular,

Corollary 2.6.7. *The trace map*

$$H_{\bullet}(Z_{\text{Ran } X}) \rightarrow H_{\bullet}(Z_{\text{Ran } X}^1 \times Z_{\text{Ran } X}^2)$$

is an isomorphism.

2.7. **Pairs of finite sets.** In this subsection we shall consider another family of examples of the situation described in Sect. 2.2.

2.7.1. We consider the category $\mathbf{fSet}^{\rightarrow}$ whose objects can be depicted as $(J \rightarrow I)$, i.e., pairs of finite sets, equipped with a map between them (the map $J \rightarrow I$ is arbitrary, i.e., it is not required to be either injective or surjective), and morphisms are commutative diagrams

$$\begin{array}{ccc} J_1 & \longrightarrow & I_1 \\ \downarrow & & \downarrow \\ J_2 & \longrightarrow & I_2, \end{array}$$

where vertical maps are surjective.

This category is acted on by \mathbf{fSet} via

$$K \sqcup (J \rightarrow I) := (K \sqcup J \rightarrow K \sqcup I).$$

We have two naturally defined functors $\mathbf{pr}_{\text{source}}$ and $\mathbf{pr}_{\text{target}}$ from $\mathbf{fSet}^{\rightarrow}$ to \mathbf{fSet} , both compatible with the monoidal action.

2.7.2. In this subsection we shall take the index category \mathcal{S} to be $(\mathbf{fSet}^{\rightarrow})^{op}$. We will take the functor

$$f_{\mathcal{S}} : (\mathbf{fSet}^{\rightarrow})^{op} \rightarrow (\mathbf{fSet})^{op}$$

to be $(\mathbf{pr}_{\text{source}})^{op}$.

The natural transformation \mathbf{d} is given by the map in $\mathbf{fSet}^{\rightarrow}$:

$$(J \sqcup J \rightarrow J \sqcup I) \rightarrow (J \rightarrow I).$$

2.7.3. We shall consider functors $Z : (\mathbf{fSet}^{\rightarrow})^{op} \rightarrow \mathbf{IndSch}$ equipped with a strongly unital structure with respect to $(f_{\mathcal{S}}, \mathbf{d})$ specified above. I.e., these are functors

$$(J \rightarrow I) \rightsquigarrow Z(J \rightarrow I),$$

equipped with a functorial assignment

$$\text{unit}_{K, J \rightarrow I} : X^K \times Z(J \rightarrow I) \rightarrow Z(K \sqcup J \rightarrow K \sqcup I),$$

and a natural transformation

$$(2.18) \quad (J \rightarrow I) \rightsquigarrow (Z(J \rightarrow I) \xrightarrow{f(J \rightarrow I)} X^J),$$

such that the condition from Sect. 2.2.3 holds.

Let $Z_{\mathbf{Ran}^{\rightarrow} X}$ denote the object of \mathbf{PreStk} equal to

$$\text{colim}_{(\mathbf{fSet}^{\rightarrow})} Z_{\mathbf{Ran}^{\rightarrow} X}.$$

The space $Z_{\mathbf{Ran}^{\rightarrow} X}$ is acted on by the semi-group $\mathbf{Ran} X$. We shall consider the corresponding semi-simplicial object $Z_{\mathbf{Ran}^{\rightarrow} X, \Delta_s}$ and its geometric realization $Z_{\mathbf{Ran}^{\rightarrow} X, \text{indep}}$.

2.7.4. The prime example of such functor is $X^{\mathbf{fSet}^{\rightarrow}}$ given by

$$(J \rightarrow I) \rightsquigarrow X^{J \rightarrow I} := X^I,$$

i.e., $X^{\mathbf{fSet}^{\rightarrow}} = X^{\mathbf{fSet}} \circ (\mathbf{pr}_{\text{target}})^{op}$.

Set

$$\mathbf{Ran}^{\rightarrow} X := \text{colim}_{(\mathbf{fSet}^{\rightarrow})^{op}} X^{\mathbf{fSet}^{\rightarrow}}.$$

2.7.5. We shall impose the following additional structure on our functors

$$Z : (\mathbf{fSet}^\rightarrow)^{op} \rightarrow \mathbf{IndSch} : (J \rightarrow I) \rightsquigarrow Z(J \rightarrow I).$$

Namely, we assume that Z be equipped with a natural transformation $f_Z^\rightarrow : Z \rightarrow X^{\mathbf{fSet}^\rightarrow}$. I.e., for every $(J \rightarrow I)$ we have a map

$$f^\rightarrow(J \rightarrow I) : Z(J \rightarrow I) \rightarrow X^I,$$

such that the map $f(J \rightarrow I)$ of (2.18) equals the composition

$$Z(J \rightarrow I) \xrightarrow{f^\rightarrow(J \rightarrow I)} X^I \rightarrow X^J.$$

We require that the following additional conditions, analogous to those of Sect. 2.4.1 hold:

(1) For $I, J, K \in \mathbf{fSet}$, the diagram

$$\begin{array}{ccc} X^K \times Z(J \rightarrow I) & \xrightarrow{\text{unit}_{K, J \rightarrow I}} & Z(K \sqcup J \rightarrow K \sqcup I) \\ \text{id} \times f^\rightarrow(J \rightarrow I) \downarrow & & \downarrow f^\rightarrow(K \sqcup J \rightarrow K \sqcup I) \\ X^K \times X^I & \xrightarrow{\sim} & X^{K \sqcup I} \end{array}$$

commutes.

(2) For every arrow $(J_1 \rightarrow I_1) \rightarrow (J_2 \rightarrow I_2)$ in $\mathbf{fSet}^\rightarrow$, the resulting map

$$Z(J_2 \rightarrow I_2) \rightarrow X^{I_2} \times_{X^{I_1}} Z(J_1 \rightarrow I_1)$$

be an isomorphism. in $(\mathbf{fSet}^\rightarrow)^{op}$,

2.7.6. Note now that we have a functor $\mathbf{diag} : \mathbf{fSet} \rightarrow \mathbf{fSet}^\rightarrow$, given by

$$I \rightsquigarrow (I \xrightarrow{\text{id}} I).$$

For Z as in Sect. 2.7.5, let $Z^{\mathbf{diag}}$ denote the functor

$$Z \circ \mathbf{diag}^{op} : (\mathbf{fSet})^{op} \rightarrow \mathbf{IndSch}.$$

Note that if Z is strongly unital and satisfies the conditions of Sect. 2.7.5, then $Z^{\mathbf{diag}}$ is also unital and satisfies the conditions of Sect. 2.4.1.

By Sect. 1.5.4, we have a map

$$(2.19) \quad Z_{\mathbf{Ran} X}^{\mathbf{diag}} \rightarrow Z_{\mathbf{Ran}^\rightarrow X},$$

which is easily seen to be compatible with the action of the semi-group $\mathbf{Ran} X$. Hence, it gives rise to a map

$$(2.20) \quad Z_{\mathbf{Ran} X, \text{indep}}^{\mathbf{diag}} \rightarrow Z_{\mathbf{Ran}^\rightarrow X, \text{indep}}.$$

We shall prove:

Proposition 2.7.7. *The map (2.20) is an isomorphism.*

2.7.8. Proposition 2.7.7 has corollaries analogous to those of Proposition 2.5.6:

Let \mathcal{F} be an object of $\mathfrak{D}(Z_{\text{Ran}^\rightarrow X})$, which lies in the essential image of the forgetful functor $\mathfrak{D}(Z_{\text{Ran}^\rightarrow X, \text{indep}}) \rightarrow \mathfrak{D}(Z_{\text{Ran}^\rightarrow X})$. Assume also that \mathcal{F} belongs to $\mathfrak{D}(Z_{\text{Ran}^\rightarrow X})_{\text{good for } p_{Z_{\text{Ran}^\rightarrow X}}}$.

Let $\mathcal{F}^{\text{diag}}$ be the pullback of \mathcal{F} under the map (2.19). Combining Corollary 2.3.5 with Proposition 2.7.7, we obtain:

Corollary 2.7.9. *The trace map*

$$\Gamma_{\text{dR},c}(Z_{\text{Ran}^\rightarrow X}^{\text{diag}}, \mathcal{F}^{\text{diag}}) \rightarrow \Gamma_{\text{dR},c}(Z_{\text{Ran}^\rightarrow X}, \mathcal{F})$$

is an isomorphism.

As a particular case, we have:

Corollary 2.7.10. *Then the trace map*

$$H_\bullet(Z_{\text{Ran}^\rightarrow X}^{\text{diag}}) \rightarrow H_\bullet(Z_{\text{Ran}^\rightarrow X})$$

is an isomorphism.

2.7.11. *Proof of Proposition 2.7.7.*

Step 1. Consider the functor $\mathbf{pr}_{\text{target}} : \text{fSet}^\rightarrow \rightarrow \text{fSet}$. We shall now construct a natural transformation

$$Z \Rightarrow Z^{\text{diag}} \circ (\mathbf{pr}_{\text{target}})^{\text{op}}.$$

It is given by sending a pair $(J \rightarrow I)$ to the map

$$Z(J \rightarrow I) \simeq X^I \times_{X^{I \sqcup I}} (X^I \times Z(J \rightarrow I)) \xrightarrow{\text{unit}_{I,J \rightarrow I}} X^I \times_{X^{I \sqcup I}} Z(I \sqcup J \rightarrow I \sqcup I) \simeq Z^{\text{diag}}(I),$$

where the last isomorphism corresponds to the morphism $(I \sqcup J \rightarrow I \sqcup I) \rightarrow (I \rightarrow I)$ in fSet^\rightarrow .

Hence, by Sect. 1.5.4, we obtain a map

$$(2.21) \quad Z_{\text{Ran}^\rightarrow X} \rightarrow Z_{\text{Ran}^\rightarrow X}^{\text{diag}},$$

which is also easily seen to be compatible with the action of the semi-group $\text{Ran } X$. Hence, we obtain a map

$$(2.22) \quad Z_{\text{Ran}^\rightarrow X, \text{indep}} \rightarrow Z_{\text{Ran}^\rightarrow X, \text{indep}}^{\text{diag}}.$$

However, it follows from the condition of Sect. 2.2.3 that the composition

$$Z_{\text{Ran}^\rightarrow X}^{\text{diag}} \rightarrow Z_{\text{Ran}^\rightarrow X} \rightarrow Z_{\text{Ran}^\rightarrow X}^{\text{diag}}$$

equals the identity map. Hence, so does the composition

$$Z_{\text{Ran}^\rightarrow X, \text{indep}}^{\text{diag}} \rightarrow Z_{\text{Ran}^\rightarrow X, \text{indep}} \rightarrow Z_{\text{Ran}^\rightarrow X, \text{indep}}^{\text{diag}}.$$

Step 2. Thus, it remains to show that the composition

$$Z_{\text{Ran}^\rightarrow X, \text{indep}} \rightarrow Z_{\text{Ran}^\rightarrow X, \text{indep}}^{\text{diag}} \rightarrow Z_{\text{Ran}^\rightarrow X, \text{indep}}$$

is canonically homotopic to the identity map. For that, as in the proof of Proposition 2.5.6, it is enough to show that the composition

$$(2.23) \quad Z_{\text{Ran}^\rightarrow X} \rightarrow Z_{\text{Ran}^\rightarrow X}^{\text{diag}} \rightarrow Z_{\text{Ran}^\rightarrow X}$$

can be factored as

$$Z_{\text{Ran}^\rightarrow X} \xrightarrow{\gamma} \text{Ran } X \times Z_{\text{Ran}^\rightarrow X} \xrightarrow{\text{unit}_{\text{Ran } X}} Z_{\text{Ran}^\rightarrow X}$$

for some map $\gamma : Z_{\text{Ran}^\rightarrow X} \xrightarrow{\gamma} \text{Ran } X \times Z_{\text{Ran}^\rightarrow X}$, such that the composition

$$Z_{\text{Ran}^\rightarrow X} \xrightarrow{\gamma} \text{Ran } X \times Z_{\text{Ran}^\rightarrow X} \xrightarrow{p_{\text{Ran } X} \times \text{id}} Z_{\text{Ran}^\rightarrow X}$$

is the identity map.

We define the map γ as follows. It is defined in terms of Sect. 1.5.4 by the functor

$$\gamma_{\text{fSet}^\rightarrow} : (\text{fSet}^\rightarrow)^{op} \rightarrow (\text{fSet} \times \text{fSet}^\rightarrow)^{op},$$

given by

$$(J \rightarrow I) \rightsquigarrow (I, (J \rightarrow I)),$$

and the natural transformation between the two functors $(\text{fSet}^\rightarrow)^{op} \rightrightarrows \text{IndSch}$

$$Z \Rightarrow (X^{\text{fSet}} \times Z) \circ \gamma_{\text{fSet}}$$

given by

$$Z(J \rightarrow I) \xrightarrow{f^\rightarrow (J \rightarrow I) \times \text{id}} X^I \times Z(J \rightarrow I).$$

Step 3. Let us show that the resulting map

$$(2.24) \quad \text{unit}_{\text{Ran } X} \circ \gamma : Z_{\text{Ran}^\rightarrow X} \rightarrow Z_{\text{Ran}^\rightarrow X}$$

is indeed homotopic to the map (2.23). We shall do so using the paradigm of Sect. 2.2.8:

We take $\mathbf{C}_1 = \mathbf{C}_2 = (\text{fSet}^\rightarrow)^{op}$ and $\Phi_1 = \Phi_2 = Z$. We take the functor F to be

$$(J \rightarrow I) \rightsquigarrow (I \sqcup J \rightarrow I \sqcup I),$$

and the functor F' to be

$$(J \rightarrow I) \rightsquigarrow (I \rightarrow I).$$

The map (2.24) corresponds to the natural transformation $\alpha : \Phi_1 \Rightarrow \Phi_2 \circ F$ that sends $(J \rightarrow I)$ to the map

$$Z(J \rightarrow I) \xrightarrow{f^\rightarrow (J \rightarrow I) \times \text{id}} X^I \times Z(J \rightarrow I) \xrightarrow{\text{unit}_{I, (J \rightarrow I)}} Z(I \sqcup J, I \sqcup I).$$

The map (2.23) corresponds to the natural transformation $\alpha' : \Phi_1 \Rightarrow \Phi_2 \circ F'$ that sends $(J \rightarrow I)$ to the map

$$Z(J \rightarrow I) \simeq X^I \times_{X^{I \sqcup I}} (X^I \times Z(J \rightarrow I)) \xrightarrow{\text{unit}_{I, J \rightarrow I}} X^I \times_{X^{I \sqcup I}} Z(I \sqcup J \rightarrow I \sqcup I) \simeq Z(I \rightarrow I).$$

Now, the required natural transformation $\beta : F \rightarrow F'$ sends $(J \rightarrow I)$ to the map in $(\text{fSet}^\rightarrow)^{op}$

$$(I \rightarrow I) \rightarrow (I \sqcup J \rightarrow I \sqcup I)$$

opposite to the natural map in fSet^\rightarrow . □

3. PROOF OF THE MAIN THEOREM

3.1. The case of $Y = \mathbb{A}^n$. We will show directly that the assertion of Theorem 1.8.2 holds in this case, which is a triviality modulo Theorem 1.6.5.

3.1.1. By Sect. 1.5.4, the natural transformation of functors $(\mathbf{fSet})^{op} \rightrightarrows \mathbf{IndSch}$

$$\mathbf{Maps}(X, Y)_{X^{\mathbf{fSet}}}^{\text{rat}} \Rightarrow X^{\mathbf{fSet}}$$

induces a map

$$f : \mathbf{Maps}(X, Y)_{\mathbf{Ran} X}^{\text{rat}} \rightarrow \mathbf{Ran} X,$$

and by (1.8), combined with Sect. 1.5.6(iii), a map

$$(3.1) \quad \mathrm{Tr}_\omega(f) : f_!(\omega_{\mathbf{Maps}(X, Y)_{\mathbf{Ran} X}^{\text{rat}}}) \rightarrow \omega_{\mathbf{Ran} X}.$$

Applying $\Gamma_{\mathrm{dR},c}(\mathbf{Ran} X, -)$, we obtain a map

$$(3.2) \quad \mathrm{Tr}_{H_\bullet}(f) : H_\bullet(\mathbf{Maps}(X, Y)_{\mathbf{Ran} X}^{\text{rat}}) \rightarrow H_\bullet(\mathbf{Ran} X).$$

We claim that the map (3.1) is an isomorphism. The map (3.1) is given by a compatible system of maps

$$(3.3) \quad f(I)_! \left(\omega_{\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}} \right) \rightarrow \omega_{X^I}.$$

We will show that for every I , the map (3.3) is an isomorphism. This follows from the following general lemma:

Lemma 3.1.2. *Let T be a scheme and \mathcal{E} a locally projective \mathcal{O}_T -module, viewed as an indscheme over T . Then the map $f_!(\omega_{\mathcal{E}}) \rightarrow \omega_T$ in $\mathfrak{D}(T)$ is an isomorphism, where f denotes the map $\mathcal{E} \rightarrow T$.*

Proof. The question is Zariski local, so we can assume that \mathcal{E} is projective, and hence a direct summand of a free \mathcal{O}_T -module \mathcal{E}_0 . I.e., \mathcal{E} , viewed as an indscheme over T is a retract of \mathcal{E}_0 . Therefore $f_!(\omega_{\mathcal{E}})$ is a retract of the corresponding object for \mathcal{E}_0 . However, since ω_T is also a retract of $f_!(\omega_{\mathcal{E}})$ via the zero-section, we obtain that it suffices to assume that \mathcal{E} is free.

In the latter case we are dealing with the product situation, so we can assume that $T = \mathrm{pt}$, and \mathcal{E} corresponds to a vector space $E \simeq \mathrm{colim}_\alpha E_\alpha$, where E_α are finite-dimensional subspaces of E . In this case

$$H_\bullet(E) \simeq \mathrm{colim}_\alpha H_\bullet(E_\alpha),$$

by (1.11).

However, for each α , the canonical map

$$H_\bullet(E_\alpha) \rightarrow k$$

is an isomorphism. □

3.1.3. Thus, we obtain that the map (3.1) is an isomorphism. Hence, (3.2) is an isomorphism as well. This implies the assertion of Theorem 1.8.2 for $Y = \mathbb{A}^n$ in view of Theorem 1.6.5.

3.2. Subsequent strategy. The idea of the proof is to deduce the assertion for Y from that for the affine space. We will do so using an intermediate object,

$$\mathbf{Maps}(X, U \xrightarrow{\text{gen.}} Y)_{\mathbf{Ran} X}^{\text{rat}},$$

introduced below, where $U \subset Y$ is a dense open subset.

3.2.1. For $I \in \mathbf{fSet}$ we let

$$j(I) : \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}} \hookrightarrow \mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}$$

to be the open subfunctor that assigns to $x^I : S \rightarrow X^I$ the subset of those maps

$$m : (S \times X - \{x^I\}) \rightarrow \mathbb{A}^n$$

for which for every geometric point $s \in S$, the resulting map $(X - \{x_s^I\}) \rightarrow \mathbb{A}^n$ lands generically in $U \subset \mathbb{A}^n$.

Set

$$\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}} := \text{colim}_{(\mathbf{fSet})^{op}} \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}} \in \text{PreStk}.$$

By construction, $\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}$ is a pseudo-indscheme.

3.2.2. The open embeddings $j(I)$ induce a natural transformation of functors $(\mathbf{fSet})^{op} \rightrightarrows \text{IndSch}$

$$\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^{\mathbf{fSet}}}^{\text{rat}} \rightrightarrows \mathbf{Maps}(X, Y)_{X^{\mathbf{fSet}}}^{\text{rat}}.$$

Hence, by Sect. 1.5.4, we obtain a map

$$j : \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}.$$

Therefore, by (1.13), we obtain a map

$$(3.4) \quad \text{Tr}_{H_\bullet}(j) : H_\bullet \left(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}} \right) \rightarrow H_\bullet \left(\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}} \right).$$

The following is the only step in the proof of Theorem 1.8.2 that involves some algebraic geometry:

Proposition 3.2.3. *If $Y \simeq \mathbb{A}^n$, the map (3.4) is an isomorphism.*

We do not know whether the assertion of Proposition 3.2.3 hold for more general targets Y .

3.2.4. The open embedding $U \hookrightarrow Y$ induces also a natural transformation of functors

$$(\mathbf{fSet})^{op} \rightrightarrows \text{IndSch},$$

namely,

$$\mathbf{Maps}(X, U)_{X^{\mathbf{fSet}}}^{\text{rat}} \rightrightarrows \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^{\mathbf{fSet}}}^{\text{rat}},$$

Hence, by Sect. 1.5.4, we obtain a map

$$(3.5) \quad \mathbf{Maps}(X, U)_{\text{Ran } X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}.$$

Therefore, by (1.13), we obtain a map

$$(3.6) \quad H_\bullet \left(\mathbf{Maps}(X, U)_{\text{Ran } X}^{\text{rat}} \right) \rightarrow H_\bullet \left(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}} \right).$$

The next step in the proof will essentially be a formal manipulation with the Ran space:

Proposition 3.2.5. *For any $U \subset Y$, the map (3.6) is an isomorphism.*

3.3. Conclusion of the proof. Let us accept the assertions of the above Propositions 3.2.3 and 3.2.5 and conclude the proof of Theorem 1.8.2.

3.3.1. *Step 1.* Let U be an open subset of \mathbb{A}^n . Combining the assertions of Propositions 3.2.3 and 3.2.5. and that of Theorem 1.8.2 for \mathbb{A}^n , we obtain that the trace map

$$H_{\bullet}(\mathbf{Maps}(X, U)_{\text{Ran } X}^{\text{rat}}) \rightarrow k$$

is an isomorphism.

I.e., we obtain that the assertion of Theorem 1.8.2 holds for targets U that are isomorphic to open subsets of the affine space.

3.3.2. *Step 2.* Let Y be as in Theorem 1.8.2, and let $U_{\alpha} \subset Y$ be the corresponding open subsets.

For any finite collection of indices $\underline{\alpha} := \alpha_1, \dots, \alpha_k$, consider the open subset

$$U_{\underline{\alpha}} := \bigcap_j U_{\alpha_j}.$$

Consider the composition

$$H_{\bullet}(\mathbf{Maps}(X, U_{\underline{\alpha}})_{\text{Ran } X}^{\text{rat}}) \rightarrow H_{\bullet}(\mathbf{Maps}(X, U_{\underline{\alpha}} \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}) \rightarrow k.$$

From Step 1 we obtain that the composed arrow is an isomorphism. From Proposition 3.2.5 we obtain that the first arrow is an isomorphism. Hence, we conclude that the trace map

$$H_{\bullet}(\mathbf{Maps}(X, U_{\underline{\alpha}} \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}) \rightarrow k$$

is an isomorphism as well.

3.3.3. *Step 3.* For each finite set I we have a Zariski cover

$$(3.7) \quad \bigcup_{\alpha} \mathbf{Maps}(X, U_{\alpha} \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}} \rightarrow \mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}.$$

Let U_{\bullet} denote the Čech nerve of the cover $\bigcup_{\alpha} U_{\alpha} \rightarrow Y$. It is easy to see that the Čech nerve of the cover (3.7) identifies with the simplicial indscheme

$$\mathbf{Maps}(X, U_{\bullet} \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}}.$$

Since the category of D-modules satisfies Zariski descent, we obtain that the canonical map

$$|H_{\bullet}(\mathbf{Maps}(X, U_{\bullet} \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}})| \rightarrow H_{\bullet}(\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}})$$

is an isomorphism for each I , where $|-|$ denotes the functor of geometric realization

$$\mathbf{Vect}^{\Delta^{op}} \rightarrow \mathbf{Vect}.$$

Consider now the corresponding simplicial object

$$\mathbf{Maps}(X, U_{\bullet} \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}$$

in PreStk , obtained by taking the colimit over $(\text{fSet})^{op}$.

By taking the colimit over the category $\Delta^{op} \times (\text{fSet})^{op}$, we obtain that the resulting map

$$|H_{\bullet}(\mathbf{Maps}(X, U_{\bullet} \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}})| \rightarrow H_{\bullet}(\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}})$$

is an isomorphism as well.

We have a commutative diagram

$$\begin{array}{ccc} |H_{\bullet}(\mathbf{Maps}(X, U \xrightarrow{\text{gen.}}_{\mathbb{C}} Y)_{\text{Ran } X}^{\text{rat}})| & \longrightarrow & H_{\bullet}(\mathbf{Maps}(X, Y)_{\text{Ran } X}^{\text{rat}}) \\ \downarrow & & \downarrow \\ |k_{\bullet}| & \longrightarrow & k, \end{array}$$

where k_{\bullet} is the constant simplicial object of \mathbf{Vect} with value k , and where the vertical arrows are given by the trace maps.

As was shown above, the upper horizontal arrow is an isomorphism. The lower horizontal arrow is an isomorphism since the category $\mathbf{\Delta}^{op}$ is contractible. The left vertical arrow is an isomorphism by Step 2. Hence, we conclude that the right vertical arrow is also an isomorphism, as desired.

□(Theorem 1.8.2)

3.4. Proof of Proposition 3.2.3. The idea of the proof is to show that the complement of $\mathbf{Maps}(X, U \xrightarrow{\text{gen.}}_{\mathbb{C}} \mathbb{A}^n)_{\text{Ran } X}^{\text{rat}}$ inside $\mathbf{Maps}(X, \mathbb{A}^n)_{\text{Ran } X}^{\text{rat}}$ is “of infinite codimension”, thereby implying that the two spaces have the same homology.

3.4.1. We need to show that the map (3.4) is an isomorphism. In fact, we will show that it is a term-wise isomorphism, i.e., that for every $I \in \mathbf{fSet}$, the corresponding map

$$\text{Tr}_{H_{\bullet}}(j(I)) : H_{\bullet}(\mathbf{Maps}(X, U \xrightarrow{\text{gen.}}_{\mathbb{C}} \mathbb{A}^n)_{X^I}^{\text{rat}}) \rightarrow H_{\bullet}(\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}})$$

is an isomorphism.

In fact, we claim that for every $I \in \mathbf{fSet}$, the map

$$(3.8) \quad (f(I) \circ j(I))! (\omega_{\mathbf{Maps}(X, U \xrightarrow{\text{gen.}}_{\mathbb{C}} \mathbb{A}^n)_{X^I}^{\text{rat}}}) \rightarrow f(I)! (\omega_{\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}}})$$

is an isomorphism in $\mathfrak{D}(X^I)$, where $f(I)$ is the projection $\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}} \rightarrow X^I$.

3.4.2. Let us denote by V the complement $\mathbb{A}^n - U$ (with any scheme structure). Let us denote by $\iota(I)$ the closed embedding

$$\mathbf{Maps}(X, V)_{X^I}^{\text{rat}} \hookrightarrow \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}}.$$

We have:

$$\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}} - \mathbf{Maps}(X, V)_{X^I}^{\text{rat}} = \mathbf{Maps}(X, U \xrightarrow{\text{gen.}}_{\mathbb{C}} \mathbb{A}^n)_{X^I}^{\text{rat}},$$

as open sub-prestacks of $\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}}$.

Our current goal is to represent the above indschemes explicitly as union of schemes, whose dimensions we can control.

3.4.3. Since Γ^I is a Cartier divisor in $X^I \times X$, for an integer d , we can consider a closed subscheme

$$\mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat}, d} \subset \mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat}} = \mathbf{Maps}_{X^I}((X^I \times X) - \Gamma^I, \mathbb{A}^1)$$

that consists of maps, whose order of pole along Γ^I is of order $\leq d$.

By Riemann-Roch, for $d > (2g - 2)$, where g is the genus of X , $\mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat}, d}$ is a vector bundle over X^I of rank $d + 1 - g$.

For an integer n , let

$$\begin{aligned} \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d} &:= \underbrace{\left(\mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat},d} \right) \times \dots \times \left(\mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat},d} \right)}_n \subset \\ &\subset \underbrace{\left(\mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat}} \right) \times \dots \times \left(\mathbf{Maps}(X, \mathbb{A}^1)_{X^I}^{\text{rat}} \right)}_n = \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}} \end{aligned}$$

be the corresponding closed subscheme of $\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}}$.

Let $f(I)^d$ denote the restriction of the map $f(I)$ to $\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d}$.

3.4.4. Set also

$$\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} \mathbb{A}^n)_{X^I}^{\text{rat},d} := \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} \mathbb{A}^n)_{X^I}^{\text{rat}} \cap \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d} \text{ and}$$

$$\mathbf{Maps}(X, V)_{X^I}^{\text{rat},d} := \mathbf{Maps}(X, V)_{X^I}^{\text{rat}} \cap \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d}.$$

Denote by $j(I)^d$ and $\iota(I)^d$ the open and closed embeddings

$$\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} \mathbb{A}^n)_{X^I}^{\text{rat},d} \hookrightarrow \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d} \text{ and } \mathbf{Maps}(X, V)_{X^I}^{\text{rat},d} \hookrightarrow \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d},$$

respectively, which are complementary to each other.

Consider the projection

$$f(I)^d \circ \iota(I)^d : \mathbf{Maps}(X, V)_{X^I}^{\text{rat},d} \rightarrow X^I.$$

A key observation is that the codimension of

$$\mathbf{Maps}(X, V)_{X^I}^{\text{rat},d} \subset \mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d}$$

in the fibers of $f(I)$ uniformly tends to ∞ as $d \rightarrow \infty$. More precisely, we will show:

Lemma 3.4.5. *There exists a constant C , independent of d , such that the dimension of the fibers of the map $f(I)^d \circ \iota(I)^d$ is $\leq (n-1) \cdot d + C$.*

Let us show how this lemma implies the isomorphism in (3.8).

Proof. Note that the terms in (3.8) identify with

$$\text{colim}_d (f(I)^d \circ j(I)^d)_! (\omega_{\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} \mathbb{A}^n)_{X^I}^{\text{rat},d}})$$

and

$$\text{colim}_d (f(I)^d)_! (\omega_{\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d}}),$$

respectively.

Hence, the cone of the map in (3.8) is given by

$$\text{colim}_d (f(I)^d \circ \iota(I)^d)_! \circ (\iota(I)^d)^* (\omega_{\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d}}).$$

We claim that for every l there exists d_0 large enough so that for all $d \geq d_0$, the object

$$(3.9) \quad (f(I)^d \circ \iota(I)^d)_! \circ (\iota(I)^d)^* (\omega_{\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat},d}}) \in \mathfrak{D}(X^I)$$

lives in the cohomological degrees $\leq -l$.

Note that we are dealing with the subcategory of D-modules with holonomic cohomologies on a finite-dimensional scheme. To perform the required estimate we shall work in the usual (i.e., non-perverse) t-structure, which makes sense on the holonomic subcategory.¹¹

By Lemma 3.4.5, the fibers of the map $(f(I)^d \circ \iota(I)^d)_!$ are of dimension $\leq (n-1) \cdot d + C$ for some constant C independent of d . Hence, it is sufficient to show that

$$(\iota(I)^d)^*(\omega_{\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}}^{\text{rat}, d}) \in \mathfrak{D}(\mathbf{Maps}(X, V)_{X^I}^{\text{rat}, d})$$

lives in cohomological degrees $\leq -2 \cdot n \cdot d + C'$ for some other constant C' independent of d , with respect to the usual t-structure.

For the latter, it is sufficient to show that

$$\omega_{\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}}^{\text{rat}, d} \in \mathfrak{D}(\mathbf{Maps}(X, U)_{X^I}^{\text{rat}, d})$$

lives in cohomological degrees $\leq -2 \cdot n \cdot d + C'$, again for the usual t-structure.

However, the latter is evident: by Riemann-Roch, for d large enough, $\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}, d}$ is a vector bundle over X^I of rank

$$n \cdot (d - (1 + g)).$$

In particular, it is smooth, and hence its dualizing complex lives in cohomological degree

$$-2(|I| + n \cdot (d - (1 + g))),$$

as required. □

3.4.6. *Proof of Lemma 3.4.5.* By Noether normalization, we can choose a linear map

$$\pi : \mathbb{A}^n \rightarrow \mathbb{A}^{n-1},$$

such that $\pi|_V$ is finite. The map π induces a map

$$\mathbf{Maps}(X, \mathbb{A}^n)_{X^I}^{\text{rat}, d} \rightarrow \mathbf{Maps}(X, \mathbb{A}^{n-1})_{X^I}^{\text{rat}, d},$$

and hence a map

$$(3.10) \quad \mathbf{Maps}(X, V)_{X^I}^{\text{rat}, d} \rightarrow \mathbf{Maps}(X, \mathbb{A}^{n-1})_{X^I}^{\text{rat}, d}.$$

As the dimensions of the fibers of $\mathbf{Maps}(X, \mathbb{A}^{n-1})_{X^I}^{\text{rat}, d}$ over X^I grow as

$$(n-1) \cdot d + (n-1) \cdot (1-g),$$

it suffices to show that the map (3.10) is finite.

Since all the schemes involved are affine, it suffices to check that the map in question is proper. We shall show that the map (3.10) satisfies the valuative criterion of properness.

Since $\mathbf{Maps}(X, V)_{X^I}^{\text{rat}, d}$ is closed in $\mathbf{Maps}(X, V)_{X^I}^{\text{rat}}$, it is enough to show that the map

$$\mathbf{Maps}(X, V)_{X^I}^{\text{rat}} \rightarrow \mathbf{Maps}(X, \mathbb{A}^{n-1})_{X^I}^{\text{rat}}$$

satisfies the valuative criterion. We shall show this for any finite map $Y_1 \rightarrow Y_2$ of affine schemes.

¹¹I.e., this is the t-structure that for $k = \mathbb{C}$ corresponds under Riemann-Hilbert to the usual t-structure on the derived category of sheaves with constructible cohomology.

Thus, let

$$\begin{array}{ccc} \mathring{\mathbf{C}} & \longrightarrow & \mathbf{Maps}_{X^I}((X^I \times X) - \Gamma^I, Y_1) \\ \downarrow & & \downarrow \\ \mathbf{C} & \longrightarrow & \mathbf{Maps}_{X^I}((X^I \times X) - \Gamma^I, Y_2) \end{array}$$

be a commutative diagram, where \mathbf{C} is an affine regular curve, and $\mathring{\mathbf{C}}$ is the complement of a point $\mathbf{c} \in \mathbf{C}$. We would like to fill in the diagonal arrow from \mathbf{C} to $\mathbf{Maps}_{X^I}((X^I \times X) - \Gamma^I, Y_1)$.

Part of the data of a map

$$\mathbf{C} \rightarrow \mathbf{Maps}_{X^I}((X^I \times X) - \Gamma^I, Y_2)$$

is a map $x^I : \mathbf{C} \rightarrow X^I$.

So, we are given a map

$$m_2 : (\mathbf{C} \times X - \{x^I\}) \rightarrow Y_2,$$

and its lift to a map

$$\mathring{m}_1 : (\mathbf{C} \times X - \{x^I\}) \cap (\mathring{\mathbf{C}} \times X) \rightarrow Y_1.$$

We wish to extend \mathring{m}_1 to a map

$$m_1 : (\mathbf{C} \times X - \{x^I\}) \rightarrow Y_1.$$

However, since $Y_1 \rightarrow Y_2$ is proper, the map \mathring{m}_1 extends to an open subscheme of $\mathbf{C} \times X - \{x^I\}$ whose complement is of codimension 2. Now, since $\mathbf{C} \times X - \{x^I\}$ is normal and Y_1 is affine, the above map further extends to all of $\mathbf{C} \times X - \{x^I\}$. \square

3.5. Proof of Proposition 3.2.5. In order to prove that the map

$$\mathbf{Maps}(X, U)_{\text{Ran } X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}$$

induces an isomorphism on homology, we shall introduce an intermediate space, denoted

$$\mathbf{Maps}(X, U \subset Y)_{\text{Ran} \rightarrow X}^{\text{rat}},$$

along with the maps

$$\mathbf{Maps}(X, U)_{\text{Ran } X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \subset Y)_{\text{Ran} \rightarrow X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}},$$

and we will show that both these maps induce an isomorphism on homology.

The idea of $\mathbf{Maps}(X, U \subset Y)_{\text{Ran} \rightarrow X}^{\text{rat}}$ vs $\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\text{Ran } X}^{\text{rat}}$ is that instead of asking for a map $(X - \{x^I\}) \rightarrow Y$ to generically land in U , we will specify the locus outside of which it is defined as a regular map to U .

3.5.1. Consider the category $\text{fSet}^{\rightarrow}$ introduced in Sect. 2.7.1. I.e., its objects are pairs $(J \rightarrow I)$ of finite sets with an arbitrary map between them, and morphisms are commutative diagrams with surjective arrows.

Recall also the functor

$$X^{\text{fSet}^{\rightarrow}} : (\text{fSet}^{\rightarrow})^{op} \rightarrow \text{Sch}$$

given by

$$(J \rightarrow I) \rightsquigarrow X^{J \rightarrow I} := X^I,$$

and the space

$$\text{Ran}^{\rightarrow} X := \underset{(\text{fSet}^{\rightarrow})}{\text{colim}} X^{\text{fSet}^{\rightarrow}}.$$

3.5.2. We shall now introduce a functor

$$\mathbf{Maps}(X, U \subset Y)_{X^{\text{fSet}^\rightarrow}}^{\text{rat}} : (\text{fSet}^\rightarrow)^{\text{op}} \rightarrow \text{IndSch},$$

satisfying the assumptions of Sect. 2.7.5.

For a test scheme S mapping to $X^{J \rightarrow I}$ we have two incidence divisors in $S \times X$: one is $\{x^I\}$ corresponding to $\Gamma^I \subset X^I \times X$, and the other is $\{x^J\}$ corresponding to $\Gamma^J \subset X^J \times X$ and the composed map

$$S \rightarrow X^{J \rightarrow I} = X^I \rightarrow X^J.$$

For $(J \rightarrow I) \in \text{fSet}^\rightarrow$ we let $\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}}$ be the indscheme over $X^{J \rightarrow I} \simeq X^I$ equal to the following subfunctor of $X^{J \rightarrow I} \times_{X^J} \mathbf{Maps}(X, Y)_{X^J}^{\text{rat}}$:

We define

$$\text{Hom}_{X^I}(S, \mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}})$$

to consist of those maps

$$m : (S \times X - \{x^J\}) \rightarrow Y$$

for which the restriction of m to the open subset

$$(S \times X - \{x^I\}) \subset (S \times X - \{x^J\})$$

factors through U .

Diagrammatically, we can write

$$(3.11) \quad \mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} = \left(X^I \times_{X^J} \mathbf{Maps}(X, Y)_{X^J}^{\text{rat}} \right)_{\mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}} \times \mathbf{Maps}(X, U)_{X^I}^{\text{rat}},$$

where

$$X^I \times_{X^J} \mathbf{Maps}(X, Y)_{X^J}^{\text{rat}} \rightarrow \mathbf{Maps}(X, Y)_{X^I}^{\text{rat}}$$

is the natural map, obtained by restricting a map $m : S \times X - \{x^J\} \rightarrow Y$ to a map

$$m' : S \times X - \{x^J\} \rightarrow Y.$$

3.5.3. Set

$$\mathbf{Maps}(X, U \subset Y)_{\text{Ran}^\rightarrow X}^{\text{rat}} := \text{colim}_{(\text{fSet}^\rightarrow)^{\text{op}}} \mathbf{Maps}(X, U \subset Y)_{X^{\text{fSet}^\rightarrow}}^{\text{rat}}.$$

By construction, $\mathbf{Maps}(X, U \subset Y)_{\text{Ran}^\rightarrow X}^{\text{rat}}$ is a pseudo-indscheme that comes equipped with a map

$$f^\rightarrow : \mathbf{Maps}(X, U \subset Y)_{\text{Ran}^\rightarrow X}^{\text{rat}} \rightarrow \text{Ran}^\rightarrow X.$$

Remark 3.5.4. For $S \in \text{Sch}^{\text{aff}}$, the ∞ -groupoid $\text{Maps}(S, \mathbf{Maps}(X, U \subset Y)_{\text{Ran}^\rightarrow X}^{\text{rat}})$ is in fact a set described as follows. Its elements are pairs of non-empty finite subsets $\bar{x}' \subset \bar{x}$ of $\text{Maps}(S, X)$, plus a rational map $S \times X \rightarrow Y$, which is regular on the complement to the graph of \bar{x}' , and is regular as a map to U on the complement to the graph of \bar{x} .

3.5.5. Recall (see Sect. 2.7.1) that there are two natural forgetful functors

$$\mathbf{pr}_{\text{source}}, \mathbf{pr}_{\text{target}} : \mathbf{fSet}^{\rightarrow} \rightarrow \mathbf{fSet},$$

that send $J \rightarrow I$ to I and J , respectively.

Note also that we have a natural transformation between the two functors

$$(\mathbf{fSet}^{\rightarrow})^{op} \rightrightarrows \mathbf{IndSch} :$$

$$\mathbf{Maps}(X, U \subset Y)_{X^{\mathbf{fSet}^{\rightarrow}}}^{\text{rat}} \Rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^{\mathbf{fSet}^{\rightarrow}}}^{\text{rat}} \circ (\mathbf{pr}_{\text{source}})^{op}.$$

I.e., for every $(J \rightarrow I)$ we have a naturally defined map

$$(3.12) \quad f_{\text{source}}(J \rightarrow I) : \mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}}.$$

Thus, by Sect. 1.5.4, we obtain a map

$$f_{\text{source}} : \mathbf{Maps}(X, U \subset Y)_{\mathbf{Ran}^{\rightarrow} X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\mathbf{Ran} X}^{\text{rat}},$$

and by (1.13) a map

$$(3.13) \quad \text{Tr}_{H_{\bullet}}(f_{\text{source}}) : H_{\bullet}(\mathbf{Maps}(X, U \subset Y)_{\mathbf{Ran}^{\rightarrow} X}^{\text{rat}}) \rightarrow H_{\bullet}(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\mathbf{Ran} X}^{\text{rat}}),$$

We will prove:

Lemma 3.5.6. *The map (3.13) is an isomorphism.*

3.5.7. As was remarked above, the functor

$$\mathbf{Maps}(X, U \subset Y)_{X^{\mathbf{fSet}^{\rightarrow}}}^{\text{rat}} : (\mathbf{fSet}^{\rightarrow})^{op} \rightarrow \mathbf{IndSch}$$

falls into the paradigm of Sect. 2.7.5.

Note also that the resulting functor

$$\mathbf{Maps}(X, U \subset Y)_{X^{\mathbf{fSet}^{\rightarrow}}}^{\text{rat}} \circ (\mathbf{diag})^{op} : (\mathbf{fSet}^{\rightarrow})^{op} \rightarrow \mathbf{IndSch}$$

identifies with $\mathbf{Maps}(X, U)_{X^{\mathbf{fSet}^{\rightarrow}}}^{\text{rat}}$.

In particular, (2.19) defines a map

$$\mathbf{Maps}(X, U)_{\mathbf{Ran} X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \subset Y)_{\mathbf{Ran}^{\rightarrow} X}^{\text{rat}}.$$

Note that the composed map

$$\mathbf{Maps}(X, U)_{\mathbf{Ran} X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \subset Y)_{\mathbf{Ran}^{\rightarrow} X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\mathbf{Ran} X}^{\text{rat}}$$

is the map

$$\mathbf{Maps}(X, U)_{\mathbf{Ran} X}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\mathbf{Ran} X}^{\text{rat}},$$

of (3.5).

3.5.8. Now, it follows from Corollary 2.7.10, that the induced map

$$(3.14) \quad H_{\bullet}(\mathbf{Maps}(X, U)_{\mathbf{Ran} X}^{\text{rat}}) \rightarrow H_{\bullet}(\mathbf{Maps}(X, U \subset Y)_{\mathbf{Ran}^{\rightarrow} X}^{\text{rat}})$$

is an isomorphism. Combined with Lemma 3.5.6, this implies that the composed map

$$H_{\bullet}(\mathbf{Maps}(X, U)_{\mathbf{Ran} X}^{\text{rat}}) \rightarrow H_{\bullet}(\mathbf{Maps}(X, U \subset Y)_{\mathbf{Ran}^{\rightarrow} X}^{\text{rat}}) \rightarrow H_{\bullet}(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{\mathbf{Ran} X}^{\text{rat}})$$

is an isomorphism, as required.

□(Proposition 3.2.5).

3.5.9. *An alternative argument.* For the reader who chose to skip Sect. 2, an alternative (but essentially equivalent) way to deduce the fact that the map (3.14) is an isomorphism is to apply [BD1], Proposition 4.4.9 to the assignment

$$(J \rightarrow I) \mapsto (f^\rightarrow(J \rightarrow I))! \left(\omega_{\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}}} \right) \in \mathfrak{D}(X^I).$$

3.6. Proof of Lemma 3.5.6.

3.6.1. Note that the map (3.13) can be interpreted as follows. Namely, it is obtained by applying the paradigm of Sect. 1.5.3 to

$$\mathbf{C}_1 := (\text{fSet}^\rightarrow)^{op}, \quad \mathbf{C}_2 := (\text{fSet})^{op}, \quad F := (\mathbf{pr}_{\text{source}})^{op}, \quad \mathbf{D} = \text{Vect}$$

and

$$\Phi_1(J \rightarrow I) := H_\bullet \left(\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}} \right), \quad \Phi_2(J) := H_\bullet \left(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}} \right),$$

and the natural transformation whose value on $(J \rightarrow I) \in (\text{fSet}^\rightarrow)$ is

$$(3.15) \quad \text{Tr}_{H_\bullet}(f_{\text{source}}(J \rightarrow I)) : H_\bullet \left(\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}} \right) \rightarrow H_\bullet \left(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}} \right).$$

3.6.2. The proof that (3.13) is an isomorphism is based on the following observation.

Suppose that in the situation of Sect. 1.5.3 the functor F is a co-Cartesian fibration. Then in order to check that (1.9) is an isomorphism, it suffices to show that for every $\mathbf{c}_2 \in \mathbf{C}_2$, the map

$$(3.16) \quad \text{colim}_{\mathbf{C}_1 \times_{\mathbf{C}_2} \mathbf{c}_2} \Phi_1|_{\mathbf{C}_1 \times_{\mathbf{C}_2} \mathbf{c}_2} \rightarrow \Phi_2(\mathbf{c}_2)$$

is an isomorphism.

3.6.3. We note that the functor $(\mathbf{pr}_{\text{source}})^{op} : (\text{fSet}^\rightarrow)^{op} \rightarrow (\text{fSet})^{op}$ is a co-Cartesian fibration, i.e., the functor $\mathbf{pr}_{\text{source}} : \text{fSet}^\rightarrow \rightarrow \text{fSet}$ is Cartesian fibration. Indeed for a map $J_1 \rightarrow J_2$ in fSet and an object

$$(J_2 \rightarrow I_2) \in \text{fSet}^\rightarrow \times_{\text{fSet}} J_2,$$

its pullback to $\text{fSet}^\rightarrow \times_{\text{fSet}} J_1$ is given by composition $J_1 \rightarrow J_2 \rightarrow I_2$.

Thus, in order to show that (3.13) is an isomorphism, it suffices to check that the map (3.16) is an isomorphism in our situation, i.e., that for every finite set J , the map

$$\text{colim}_{I \in (\text{fSet}_{J_1})^{op}} H_\bullet \left(\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}} \right) \rightarrow H_\bullet \left(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}} \right)$$

is an isomorphism.

In fact, we will show that the isomorphism is taking place “upstairs”, i.e., at the level of objects of $\mathfrak{D} \left(\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}} \right)$. Namely, we will show that the trace map

$$(3.17) \quad \text{colim}_{I \in (\text{fSet}_{J_1})^{op}} f_{\text{source}}(J \rightarrow I)! \left(\omega_{\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}}} \right) \rightarrow \omega_{\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}}}$$

is an isomorphism.

3.6.4. First, we claim that the map $f_{\text{source}}(J \rightarrow I)$ is ind-proper; in fact for every $J \rightarrow I$ the map

$$\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \hookrightarrow X^I \times_{X^J} \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}}$$

is an ind-closed embedding.

Indeed, it follows from (3.11) that

$$\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \simeq \left(X^I \times_{X^J} \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}} \right) \times_{\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}}} \mathbf{Maps}(X, U)_{X^I}^{\text{rat}},$$

so it is sufficient to show that the map

$$\mathbf{Maps}(X, U)_{X^I}^{\text{rat}} \rightarrow \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^I}^{\text{rat}}$$

is an ind-closed embedding. It is easy to see that it is sufficient to do so in the universal case of $Y = \mathbb{A}^1$ and $U = \mathbb{A}^1 - \{0\}$.

The latter situation reduces to the following one: let S be a test scheme, and let D^1 and D^2 be two effective Cartier divisors on $S \times X$, both finite and flat over S . Consider the functor on $\text{Sch}/_S$ that sends $g : T \rightarrow S$ to the point-set if $(g \times \text{id})^{-1}(D_1)$ is *set-theoretically* contained in $(g \times \text{id})^{-1}(D_2)$, and to the empty set, otherwise. Then we claim that this functor is representable by a formal subscheme of S :

Indeed, the above functor is the colimit over $n \in \mathbb{N}$ of the functors, where for each n we require that $(g \times \text{id})^{-1}(D_1)$ be contained scheme-theoretically in $(g \times \text{id})^{-1}(n \cdot D_2)$. However, each of the latter functors is representable by a closed subscheme of S .

3.6.5. We stratify the indscheme $\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}}$ according to the pattern of collision of points in X corresponding to X^J and also according to the pattern of points of X , away from which the rational map $m : X \rightarrow U$ is regular.

We will show that the map (3.17) is an isomorphism after !-restriction to each stratum. However, in order to unburden the notation, we will do so only at the level of !-stalks at k -points of each stratum (the proof in the general case is the same). I.e., we will show that the map (3.17) is an isomorphism at the level of !-stalks at k -points of $\mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}}$.

For a fixed point $(x^J, m) \in \mathbf{Maps}(X, U \overset{\text{gen.}}{\subset} Y)_{X^J}^{\text{rat}}$ as above, let

$$\left(\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \right)_{(x^J, m)}$$

denote the fiber of the map $f_{\text{source}}(J \rightarrow I)$ over it.

Note that because $f_{\text{source}}(J \rightarrow I)$ is proper, the base-change formula applies, i.e., the !-stalk of

$$(f_{\text{source}}(J \rightarrow I))_! \left(\omega_{\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}}} \right)$$

at (x^J, m) is isomorphic to the cohomology of the !-restriction of $\omega_{\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}}}$ to $\left(\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \right)_{(x^J, m)}$, the latter being

$$\omega_{\left(\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \right)_{(x^J, m)}}.$$

I.e., we have to show that the trace map

$$(3.18) \quad \text{colim}_{I \in (\text{fSet}_J)_{\text{op}}} H_{\bullet} \left(\left(\mathbf{Maps}(X, U \subset Y)_{X^{J \rightarrow I}}^{\text{rat}} \right)_{(x^J, m)} \right) \rightarrow k$$

is an isomorphism. We will deduce this from a certain variant of Theorem 1.6.5.

3.6.6. Let us describe the indscheme

$$(3.19) \quad (\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}})_{(x^J, m)}$$

explicitly.

Let $\overset{\circ}{X} \subset X$ be the open subset over which the rational map $m : X \rightarrow U$ is regular. Let \mathbf{A} be the complementary finite set of points, and let $x^{\mathbf{A}}$ be the corresponding canonical point of $X^{\mathbf{A}}$. For a finite set I , let $X^{I, \mathbf{A}}$ be the following closed subscheme of X^I :

$$X^{I, \mathbf{A}} := \bigcup_{\psi: \mathbf{A} \rightarrow I} \Delta(\psi)^{-1}(x^{\mathbf{A}}).$$

I.e., $X^{I, \mathbf{A}}$ consists of those I -tuples of points of X , which *contain* $x^{\mathbf{A}}$ as a subset. (If $\mathbf{A} = \emptyset$, we have $X^{I, \mathbf{A}} = X^I$.)

Let $X_{x^J}^I$ denote the preimage in X^I of the point $x^J \in X^J$.

It follows that the indscheme (3.19), which according to Sect. 3.6.4, is ind-closed in X^I , equals

$$X_{x^J}^I \times_{X^I} \widehat{X}^{I, \mathbf{A}},$$

where $\widehat{X}^{I, \mathbf{A}}$ denotes the formal completion of $X^{I, \mathbf{A}}$ in X^I .

3.6.7. Thus, we obtain that the two functors $(\mathbf{fSet}/J)^{op} \rightarrow \text{IndSch}$

$$X_{x^J}^I \times_{X^I} X^{I, \mathbf{A}} \text{ and } (\mathbf{Maps}(X, U \subset Y)_{X^J \rightarrow I}^{\text{rat}})_{(x^J, m)}$$

become isomorphic after passing to the corresponding reduced indschemes.

Hence, we obtain that it is sufficient to show that the trace map

$$(3.20) \quad \text{colim}_{I \in (\mathbf{fSet}_J)^{op}} H_{\bullet} \left(X_{x^J}^I \times_{X^I} X^{I, \mathbf{A}} \right) \rightarrow k$$

is an isomorphism.

3.6.8. Thus, we consider the functor $X^{\mathbf{fSet}_J^{\mathbf{A}}} : (\mathbf{fSet}_J)^{op} \rightarrow \text{Sch}$ given by

$$I \rightsquigarrow X_{x^J}^I \times_{X^I} X^{I, \mathbf{A}}.$$

$$\text{Set Ran } X_J^{\mathbf{A}} := \text{colim}_{(\mathbf{fSet}_J)^{op}} X^{\mathbf{fSet}_J^{\mathbf{A}}}.$$

We need to show that the trace map defines an isomorphism

$$H_{\bullet}(\text{Ran } X_J^{\mathbf{A}}) \rightarrow k.$$

However, this follows by repeating the proof of Theorem 1.6.5, see Sect. 6.

3.6.9. Here is an alternative argument for the last statement. To simplify the notation, we will consider the space

$$\mathrm{Ran} X^A := \mathop{\mathrm{colim}}_{(\mathrm{fSet})^{op}} X^{\mathrm{fSet}^A},$$

where X^{fSet^A} is the functor $(\mathrm{fSet})^{op} \rightarrow \mathrm{Sch}$ that sends

$$I \rightsquigarrow X^{I,A}.$$

Let i^A denote the tautological map $\mathrm{Ran} X^A \rightarrow \mathrm{Ran} X$. We shall denote by i_A the map $\mathrm{Ran} X_A \rightarrow \mathrm{Ran} X$ of (2.9).

It is easy to see that for $\mathcal{F} \in \mathfrak{D}(\mathrm{Ran} X)$ we have:

$$(3.21) \quad (i_A)_! \circ (i_A)^!(\mathcal{F}) \simeq (i^A)_! \circ (i^A)^!(\mathcal{F}).$$

In particular,

$$H_\bullet(\mathrm{Ran} X_A) \simeq H_\bullet(\mathrm{Ran} X^A).$$

Now, the required statement follows from Corollary 2.5.10 which says that

$$H_\bullet(\mathrm{Ran} X_A) \simeq H_\bullet(\mathrm{Ran} X),$$

combined with Theorem 1.6.5, which says that $H_\bullet(\mathrm{Ran} X) \simeq k$.

4. APPLICATIONS TO D-MODULES ON Bun_G

4.1. The Beilinson-Drinfeld Grassmannian.

4.1.1. Let G be a connected linear algebraic group. Let $\mathrm{Gr}_{X^{\mathrm{fSet}}}$ denote the Beilinson-Drinfeld affine Grassmannian of G , viewed as a functor

$$(\mathrm{fSet})^{op} \rightarrow \mathrm{IndSch},$$

see e.g. [BD2], Sect. 5.3.11 or [MV], Sect. 5. For a finite set I , we shall denote by Gr_{X^I} the corresponding indscheme over X^I .

Let $\mathrm{Gr}_{\mathrm{Ran} X}$ denote the object $\mathop{\mathrm{colim}}_{(\mathrm{fSet})^{op}} \mathrm{Gr}_{X^{\mathrm{fSet}}} \in \mathrm{PreStk}$. By construction, $\mathrm{Gr}_{\mathrm{Ran} X}$ is a pseudo-indscheme.

We have a tautological natural transformation $\mathrm{Gr}_{X^{\mathrm{fSet}}} \Rightarrow X^{\mathrm{fSet}}$, which gives rise to map

$$f_{\mathrm{Gr}} : \mathrm{Gr}_{\mathrm{Ran} X} \rightarrow \mathrm{Ran} X.$$

In this section we will study the category $\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})$, i.e.,

$$\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X}) := \lim_{I \in \mathrm{fSet}} \mathfrak{D}(\mathrm{Gr}_{X^I}).$$

Remark 4.1.2. The functor $\mathrm{Gr}_{X^{\mathrm{fSet}}}$ is another example of a functor satisfying the assumptions of Sect. 2.4.1. Thus, we could also introduce the space $\mathrm{Gr}_{\mathrm{Ran} X, \mathrm{indep}}$. By Proposition 2.3.3, the forgetful functor

$$\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X, \mathrm{indep}}) \rightarrow \mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})$$

is fully faithful.

4.1.3. Let Bun_G denote the moduli stack of G -bundles on X . Note also that $\mathrm{Gr}_{\mathrm{Ran} X}$ is equipped with a forgetful map $\pi : \mathrm{Gr}_{\mathrm{Ran} X} \rightarrow \mathrm{Bun}_G$ in PreStk .

Note also that we have a Cartesian square in PreStk :

$$(4.1) \quad \begin{array}{ccc} \mathbf{Maps}(X, G)_{\mathrm{Ran} X}^{\mathrm{rat}} & \xrightarrow{\iota_1} & \mathrm{Gr}_{\mathrm{Ran} X} \\ \downarrow & & \downarrow \pi \\ \mathrm{pt} & \xrightarrow{\mathbf{1}} & \mathrm{Bun}_G, \end{array}$$

where $\mathbf{1}$ refers to the unit point of Bun_G corresponding to the trivial bundle.

Note that the map π is pseudo ind-schematic (see Sect. 1.2.5, where this notion is introduced).

4.1.4. Consider the category $\mathfrak{D}(\mathrm{Bun}_G)$. We remind that for any $\mathcal{Y} \in \mathrm{PreStk}$, the category $\mathfrak{D}(\mathcal{Y})$ is by definition

$$\lim_{S \in \mathrm{Sch}_{\mathcal{Y}}^{\mathrm{aff}}} \mathfrak{D}(S),$$

where the limit is taken $\mathrm{DGCat}_{\mathrm{cont}}$.

However, when \mathcal{Y} is an Artin stack, we can also replace the index category $S \in \mathrm{Sch}_{\mathcal{Y}}^{\mathrm{aff}}$ by its full subcategory consisting of those S that map smoothly to \mathcal{Y} , and, further, by the non-full subcategory of the latter, where we allow only smooth maps $S_1 \rightarrow S_2$. Further, writing \mathcal{Y} as a union of its quasi-compact open substacks \mathcal{Y}_α , we have

$$\mathfrak{D}(\mathcal{Y}) \simeq \lim_{\alpha} \mathfrak{D}(\mathcal{Y}_\alpha).$$

4.1.5. We are now ready to formulate the main result of this section. The morphism π defines a functor

$$\pi^! : \mathfrak{D}(\mathrm{Bun}_G) \rightarrow \mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X}).$$

We have:

Theorem 4.1.6. *The functor $\pi^! : \mathfrak{D}(\mathrm{Bun}_G) \rightarrow \mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})$ is fully faithful.*

Remark 4.1.7. In view of Diagram (4.1), the statement of Theorem 4.1.6 it is not surprising: once we show that the map π behaves like a fibration, which is what the proof of Theorem 4.1.6 will amount to, the assertion would follow from Theorem 1.8.2: the property of fully faithful pullback functor is enjoyed by fibrations with contractible fibers.

4.1.8. Note that Theorem 4.1.6 can be reformulated as follows:

Corollary 4.1.9. *The partially defined left adjoint $\pi_!$ to $\pi^!$ is defined on the essential image of $\pi^!$, and for $\mathcal{F} \in \mathfrak{D}(\mathrm{Bun}_G)$ the adjunction map*

$$(4.2) \quad \pi_! \circ \pi^!(\mathcal{F}) \rightarrow \mathcal{F}$$

is an isomorphism.

Applying (4.2) to $\mathcal{F} := \omega_{\mathrm{Bun}_G}$, we obtain:

Corollary 4.1.10. *The trace map*

$$\mathrm{Tr}_\omega(\pi) : \pi_!(\omega_{\mathrm{Gr}_{\mathrm{Ran} X}}) \rightarrow \omega_{\mathrm{Bun}_G}$$

is an isomorphism.

4.1.11. *A variant.* For a finite subscheme $D \subset X$ we can consider the stack $\mathrm{Bun}_G^{\mathrm{level}_D}$ that classifies G -bundles “with structure of level D ”, i.e., equipped with a trivialization when restricted to D .

Similarly, one can consider the functor

$$\mathrm{Gr}_{X^{\mathrm{fSet}}}^{\mathrm{level}_D} : (\mathrm{fSet})^{\mathrm{op}} \rightarrow \mathrm{IndSch}$$

equal by definition to

$$\mathrm{Bun}_G^{\mathrm{level}_D} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{X^{\mathrm{fSet}}},$$

and the corresponding object

$$\mathrm{Gr}_{\mathrm{Ran} X}^{\mathrm{level}_D} := \mathop{\mathrm{colim}}_{(\mathrm{fSet})^{\mathrm{op}}} \mathrm{Gr}_{X^{\mathrm{fSet}}}^{\mathrm{level}_D} \in \mathrm{PreStk}.$$

Denote by π_D the map $\mathrm{Gr}_{\mathrm{Ran} X}^{\mathrm{level}_D} \rightarrow \mathrm{Bun}_G^{\mathrm{level}_D}$.

The proof of Theorem 4.1.6 applies equally to the present situation, i.e., the functor

$$\pi_D^! : \mathfrak{D}(\mathrm{Bun}_G^{\mathrm{level}_D}) \rightarrow \mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X}^{\mathrm{level}_D})$$

is fully faithful.

The rest of this subsection is devoted to the proof of Theorem 4.1.6.

4.1.12. *Step 1.* We claim that it suffices to show that for every affine scheme S equipped with a map $g : S \rightarrow \mathrm{Bun}_G$, for the Cartesian diagram

$$\begin{array}{ccc} S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran} X} & \xrightarrow{g'} & \mathrm{Gr}_{X^{\mathrm{fSet}}} \\ \pi_S \downarrow & & \downarrow \pi \\ S & \xrightarrow{g} & \mathrm{Bun}_G, \end{array}$$

and $\mathcal{F}_S \in \mathfrak{D}(S)$, the trace map

$$(4.3) \quad (\pi_S)_! \circ (\pi_S)^!(\mathcal{F}_S) \rightarrow \mathcal{F}_S$$

is an isomorphism (and in particular, the left hand side is defined as an object of $\mathfrak{D}(S)$). Let us assume that (4.3) is an isomorphism, and deduce the assertion of Theorem 4.1.6.

First, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{D}(\mathrm{Bun}_G)$, the map

$$\mathrm{Hom}_{\mathfrak{D}(\mathrm{Bun}_G)}(\mathcal{F}_1, \mathcal{F}_2) \rightarrow \mathop{\mathrm{lim}}_{(S,g) \in \mathrm{Sch}_{\mathrm{Bun}_G}^{\mathrm{aff}}} \mathrm{Hom}_{\mathfrak{D}(S)}(g^!(\mathcal{F}_1), g^!(\mathcal{F}_2))$$

is an isomorphism.

However, for any map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ in PreStk , the restriction map

$$\mathfrak{D}(\mathcal{Y}_1) \rightarrow \mathop{\mathrm{lim}}_{S \in \mathrm{Sch}_{\mathcal{Y}_2}^{\mathrm{aff}}} \mathfrak{D}(S \times_{\mathcal{Y}_2})$$

is an equivalence.

Therefore, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{D}(\mathrm{Bun}_G)$, the map

$$\mathrm{Hom}_{\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})}(\pi^!(\mathcal{F}_1), \pi^!(\mathcal{F}_2)) \rightarrow \mathop{\mathrm{lim}}_{(S,g) \in \mathrm{Sch}_{\mathrm{Bun}_G}^{\mathrm{aff}}} \mathrm{Hom}_{\mathfrak{D}(S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran} X})}(g^! \circ \pi^!(\mathcal{F}_1), g^! \circ \pi^!(\mathcal{F}_2))$$

is an isomorphism.

Hence, in order to show that

$$\mathrm{Hom}_{\mathfrak{D}(\mathrm{Bun}_G)}(\mathcal{F}_1, \mathcal{F}_2) \rightarrow \mathrm{Hom}_{\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})}(\pi^!(\mathcal{F}_1), \pi^!(\mathcal{F}_2))$$

is an isomorphism, it suffices to show that for every S as above, the map

$$(4.4) \quad \mathrm{Hom}_{\mathfrak{D}(S)}(g^!(\mathcal{F}_1), g^!(\mathcal{F}_2)) \rightarrow \mathrm{Hom}_{\mathfrak{D}(S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran} X})}(g^! \circ \pi^!(\mathcal{F}_1), g^! \circ \pi^!(\mathcal{F}_2))$$

is an isomorphism.

However, noting that $g^! \circ \pi^!(\mathcal{F}_i) \simeq \pi^!_S \circ g^!(\mathcal{F}_i)$ for $i = 1, 2$ and taking $\mathcal{F}_S = g^!(\mathcal{F}_1)$, we obtain that the isomorphism (4.4) follows from (4.3).

4.1.13. *Step 2.* We fix a map $S \rightarrow \mathrm{Bun}_G$, and we wish to establish (4.3).

It is easy to see, however, that (possibly after passing to an étale cover of S), there exists a finite set A and a map $x^A : S \rightarrow X^A$, such that the pullback of the universal G -bundle under

$$S \times X \rightarrow \mathrm{Bun}_G \times X$$

admits a trivialization over $S \times X - \{x^A\}$. This follows, e.g., from Theorem 3 of [DS].

Recall the category fSet_A (see Sect. 2.5.1) and the functor $X^{\mathrm{fSet}_A} : (\mathrm{fSet}_A)^{op} \rightarrow \mathrm{Sch}/S$ (see Sect. 2.5.12).

Let $\mathrm{Gr}_{X^{\mathrm{fSet}_A}}$ be the functor $(\mathrm{fSet}_A)^{op} \rightarrow \mathrm{IndSch}$ defined by sending $I \in \mathrm{fSet}_A$ to

$$\mathrm{Gr}_{X^I_A} := X^I_A \times_{(S \times X^I)} \left(S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{X^I} \right) \simeq S \times_{X^A \times \mathrm{Bun}_G} \mathrm{Gr}_{X^I} \simeq S \times_{S \times X^A} \left(S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{X^I} \right).$$

Set

$$\mathrm{Gr}_{\mathrm{Ran} X_A} := \mathop{\mathrm{colim}}_{(\mathrm{fSet}_A)^{op}} \mathrm{Gr}_{X^{\mathrm{fSet}_A}} \in \mathrm{PreStk}.$$

Let $\pi_{S,A} : \mathrm{Gr}_{\mathrm{Ran} X_A} \rightarrow S$ denote the resulting map.

We claim that the map

$$\mathrm{Gr}_{\mathrm{Ran} X_A} \hookrightarrow S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran} X}$$

induces an isomorphism

$$(4.5) \quad (\pi_{S,A})_! \circ (\pi_{S,A})^!(\mathcal{F}_S) \simeq (\pi_S)_! \circ (\pi_S)^!(\mathcal{F}_S).$$

Indeed, this follows from Corollary 2.5.13.

Remark 4.1.14. For the reader who chose to skip Sect. 2, here is an alternative way to deduce the isomorphism (4.5): namely, one can repeat the argument of [BD1], Proposition 4.4.2.

4.1.15. *Step 3.* Consider now the functor

$$\mathbf{Maps}_S(X, G)_{X^{\mathrm{fSet}_A}}^{\mathrm{rat}} : (\mathrm{fSet}_A)^{op} \rightarrow \mathrm{IndSch}/S$$

defined by

$$I \rightsquigarrow X^I_A \times_{X^I} \mathbf{Maps}(X, G)_{X^I}^{\mathrm{rat}} \simeq S \times_{X^A} \mathbf{Maps}(X, G)_{X^I}^{\mathrm{rat}}.$$

Set

$$\mathbf{Maps}_S(X, G)_{\mathrm{Ran} X_A}^{\mathrm{rat}} := \mathop{\mathrm{colim}}_{(\mathrm{fSet}_A)^{op}} \mathbf{Maps}_S(X, G)_{X^{\mathrm{fSet}_A}}^{\mathrm{rat}}.$$

Note now that a choice of a trivialization of the pulled-back G -bundle on the open subscheme $S \times X - \{x^A\}$ defines an isomorphism

$$\mathrm{Gr}_{X^{\mathrm{fSet}_A}} \simeq \mathbf{Maps}_S(X, G)_{X^{\mathrm{fSet}_A}}^{\mathrm{rat}},$$

and hence

$$\mathrm{Gr}_{\mathrm{Ran} X_A} \simeq \mathbf{Maps}_S(X, G)_{\mathrm{Ran} X_A}^{\mathrm{rat}}.$$

Let p_A denote the projection

$$\mathbf{Maps}_S(X, G)_{\mathrm{Ran} X_A}^{\mathrm{rat}} \rightarrow S.$$

Thus, we have to show that for $\mathcal{F}_S \in \mathfrak{D}(S)$, the trace map

$$(p_A)_! \circ (p_A)^!(\mathcal{F}_S) \rightarrow \mathcal{F}_S$$

is an isomorphism.

However, by the same logic as in Step 2, i.e., by Corollary 2.5.13, we can replace the pair

$$(\mathbf{Maps}_S(X, G)_{\mathrm{Ran} X_A}^{\mathrm{rat}}, p_A)$$

by

$$(S \times \mathbf{Maps}(X, G)_{\mathrm{Ran} X}^{\mathrm{rat}}, \mathrm{id}_S \times p),$$

where $p : \mathbf{Maps}(X, G)_{\mathrm{Ran} X}^{\mathrm{rat}} \rightarrow \mathrm{pt}$.

Thus, we have to show that the trace map

$$(\mathrm{id}_S \times p)_! \circ (\mathrm{id}_S \times p)^!(\mathcal{F}_S) \rightarrow \mathcal{F}_S$$

is an isomorphism. However, the left-hand side is isomorphic to

$$H_\bullet(\mathbf{Maps}(X, G)_{\mathrm{Ran} X}^{\mathrm{rat}}) \otimes \mathcal{F}_S,$$

and the desired assertion follows from Theorem 1.8.2. \square

4.2. Recollections on Verdier duality and D-modules on stacks. In order to state the corollaries of Theorem 4.1.6 pertaining to cohomology of D-modules and quasi-coherent sheaves on Bun_G , let us recall some basics from the theory of D-modules. For a more detailed treatment, the reader is referred to [DrGa0, Sects. 5 and 6].

4.2.1. First, let us review Verdier duality on schemes. For $Z \in \mathrm{Sch}$ consider the category $\mathfrak{D}(Z)$. (We remind that according to our conventions, all schemes are assumed of finite type, and in particular, quasi-compact.) It is known that $\mathfrak{D}(Z)$ is compactly generated; its compact objects are bounded complexes, whose cohomologies are finitely generated.

Verdier duality is a canonical equivalence

$$(4.6) \quad \mathbb{D}_{\mathfrak{D}(Z)} : (\mathfrak{D}(Z)^c)^{op} \rightarrow \mathfrak{D}(Z)^c.$$

It is characterized by the property that for $\mathcal{F} \in \mathfrak{D}(Z)^c$ and any $\mathcal{F}_1 \in \mathfrak{D}(Z)$,

$$(4.7) \quad \mathrm{Hom}_{\mathfrak{D}(Z)}(\mathcal{F}_1, \mathbb{D}_{\mathfrak{D}(Z)}(\mathcal{F})) = \mathrm{Hom}_{\mathfrak{D}(Z \times Z)}(\mathcal{F}_1 \boxtimes \mathcal{F}, \Delta_{\mathrm{dR},*}(\omega_Z)),$$

where $\Delta_{\mathrm{dR},*}$ denotes the D-module direct image functor (in this case for the diagonal morphism).

By [GL:DG], Sect. 2.3, the equivalence (4.6) defines an equivalence

$$(4.8) \quad \mathfrak{D}(Z)^\vee \simeq \mathfrak{D}(Z),$$

where for $\mathbf{C} \in \mathrm{DGCat}$, we denote by \mathbf{C}^\vee the dual category, see, e.g., [GL:DG], Sect. 2.1.

Let now $g : Z_1 \rightarrow Z_2$ be a morphism, and consider the functor $g^! : \mathfrak{D}(Z_2) \rightarrow \mathfrak{D}(Z_1)$. The dual functor, $(g^!)^\vee$, which due to (4.8) can be thought of as a functor $\mathfrak{D}(Z_1) \rightarrow \mathfrak{D}(Z_2)$, identifies with the functor $g_{\mathrm{dR},*}$ of D-module (a.k.a. de Rham) direct image.

Note also that [GL:DG], Lemma 2.3.3, formally implies the relationship between $g_{\mathrm{dR},*}$, which is the *dual* of $g^!$, with $g_!$, which is the left adjoint of $g^!$, whenever the latter is defined on all of $\mathfrak{D}(Z_1)$:

Namely, for $g_!$ to be defined on all of $\mathfrak{D}(Z_1)$, it is necessary and sufficient that $g_{\mathrm{dR},*}$ send $\mathfrak{D}(Z_1)^c \rightarrow \mathfrak{D}(Z_2)^c$. For a given object $\mathcal{F} \in \mathfrak{D}(Z_1)^c$, the functor $g_!$ is defined on it if and only if $g_{\mathrm{dR},*}(\mathbb{D}_{\mathfrak{D}(Z_1)}(\mathcal{F}))$ belongs to $\mathfrak{D}(Z_2)^c$, and in the latter case we have:

$$g_!(\mathcal{F}) \simeq \mathbb{D}_{\mathfrak{D}(Z_2)} \circ g_{\mathrm{dR},*} \circ \mathbb{D}_{\mathfrak{D}(Z_1)}(\mathcal{F}).$$

4.2.2. In particular, it is easy to see that if g is not proper, the functor $g_!$ is not defined on all of $\mathfrak{D}(Z_1)$:

Indeed, for a scheme Z , let $'\mathbf{ind}_Z^{\mathfrak{D}} : \mathrm{QCoh}(Z) \rightarrow \mathfrak{D}(Z)$ denote the functor left adjoint to the forgetful functor

$$'\mathbf{oblv}_Z^{\mathfrak{D}} : \mathfrak{D}(Z) \rightarrow \mathrm{QCoh}(Z),$$

where we are thinking of $\mathfrak{D}(-)$ in the “right D-module realization”.¹²

The functor $'\mathbf{ind}_Z^{\mathfrak{D}}$ sends $\mathrm{Coh}(Z)$ to the subcategory $\mathfrak{D}(Z)^c$ of $\mathfrak{D}(Z)$. Furthermore, it is easy to see that for $\mathcal{F} \in \mathrm{QCoh}(Z)$, we have $'\mathbf{ind}_Z^{\mathfrak{D}}(\mathcal{F}) \in \mathfrak{D}(Z)^c$ if and only if $\mathcal{F} \in \mathrm{Coh}(Z)$.

For a morphism $g : Z_1 \rightarrow Z_2$ we have:

$$(4.9) \quad g_{\mathrm{dR},*} \circ '\mathbf{ind}_{Z_1}^{\mathfrak{D}} \simeq '\mathbf{ind}_{Z_2}^{\mathfrak{D}} \circ g_*.$$

However, it is known that unless g is proper, the functor $g_* : \mathrm{QCoh}(Z_1) \rightarrow \mathrm{QCoh}(Z_2)$ does not send $\mathrm{Coh}(Z_1)$ to $\mathrm{Coh}(Z_2)$, hence $\mathbf{ind}_{Z_2}^{\mathfrak{D}} \circ g_*(\mathrm{Coh}(Z_1)) \notin \mathfrak{D}(Z_2)^c$.

4.2.3. The above discussion is also applicable when a scheme Z is replaced by a quasi-compact Artin stack \mathcal{Y} , whose inertia stack, i.e., $\mathcal{Y} \times_{\mathcal{Y}} \mathcal{Y}$, is affine over \mathcal{Y} (these facts are established in [DrGa0, Sect. 7]):

(i) We have a pair of adjoint functors

$$'\mathbf{ind}_{\mathcal{Y}}^{\mathfrak{D}} : \mathrm{QCoh}(\mathcal{Y}) \rightleftarrows \mathfrak{D}(\mathcal{Y}) : '\mathbf{oblv}_{\mathcal{Y}}^{\mathfrak{D}},$$

with $'\mathbf{oblv}_{\mathcal{Y}}^{\mathfrak{D}}$ being conservative.

(ii) The category $\mathfrak{D}(\mathcal{Y})$ is compactly generated. In fact, a set of compact generators is obtained by applying the functor $'\mathbf{ind}_{\mathcal{Y}}^{\mathfrak{D}}$ to $\mathrm{Coh}(\mathcal{Y}) \subset \mathrm{QCoh}(\mathcal{Y})$.

(iii) Verdier duality, defined by formula (4.7), is an equivalence

$$\mathbb{D}_{\mathcal{Y}} : (\mathfrak{D}(\mathcal{Y})^c)^{op} \rightarrow \mathfrak{D}(\mathcal{Y})^c,$$

and hence defines an identification $\mathfrak{D}(\mathcal{Y})^{\vee} \simeq \mathfrak{D}(\mathcal{Y})$.

The substantial difference from the case of schemes is the following: for a non-necessarily schematic morphism $g : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$, where \mathcal{Y}_1 and \mathcal{Y}_2 are as above, the functor dual to $g^!$ is no longer the naively defined functor $g_{\mathrm{dR},*}$, but rather its renormalized version that we denote by $g_{\mathrm{ren-dR},*}$, which is the ind-extension of the functor $g_{\mathrm{dR},*}|_{\mathfrak{D}(\mathcal{Y}_1)^c}$. We refer the reader to [DrGa0, Sect. 8.3], where the basic properties of the functor $g_{\mathrm{ren-dR},*}$ are established.

In particular,

$$(4.10) \quad g_{\mathrm{ren-dR},*}|_{\mathfrak{D}(\mathcal{Y}_1)^c} \simeq g_{\mathrm{dR},*}|_{\mathfrak{D}(\mathcal{Y}_1)^c}.$$

¹²We use the notation $'\mathbf{ind}_Z^{\mathfrak{D}}$ and $'\mathbf{oblv}_Z^{\mathfrak{D}}$ instead of $\mathbf{ind}_Z^{\mathfrak{D}}$ and $\mathbf{oblv}_Z^{\mathfrak{D}}$, respectively, because the latter is reserved for the corresponding adjoint pair of functors $\mathrm{IndCoh}(Z) \rightleftarrows \mathfrak{D}(Z)$, where $\mathrm{IndCoh}(Z)$ is the category introduced in [GL:IndCoh]. The functor $'\mathbf{oblv}_Z^{\mathfrak{D}}$ is the composition of $\mathbf{oblv}_Z^{\mathfrak{D}}$ and the colocalization functor $\Psi_Z : \mathrm{IndCoh}(Z) \rightarrow \mathrm{QCoh}(Z)$. The functor $\mathbf{ind}_Z^{\mathfrak{D}}$ is isomorphic to the composition of Ψ_Z and $'\mathbf{ind}_Z^{\mathfrak{D}}$.

The original functor

$$g_{\mathrm{dR},*} : \mathfrak{D}(\mathcal{Y}_1) \rightarrow \mathfrak{D}(\mathcal{Y}_2)$$

may fail to be continuous, as can be seen in the example of $\mathcal{Y}_1 = \mathrm{pt}/B\mathbb{G}_m$ and $\mathcal{Y}_2 = \mathrm{pt}$. (Whenever $g_{\mathrm{dR},*}$ is continuous, it is canonically isomorphic to $g_{\mathrm{ren-dR},*}$.)

The relationship between g_l , whenever the latter is defined, and the functor $g_{\mathrm{ren-dR},*}$ introduced above, is the same as in the case of schemes: for $\mathcal{F} \in \mathfrak{D}(\mathcal{Y}_1)^c$, the functor g_l is defined on it if and only if $g_{\mathrm{ren-dR},*}(\mathbb{D}_{\mathfrak{D}(\mathcal{Y}_1)}(\mathcal{F})) \in \mathfrak{D}(\mathcal{Y}_2)^c$, and in the latter case we have:

$$(4.11) \quad g_l(\mathcal{F}) \simeq \mathbb{D}_{\mathfrak{D}(\mathcal{Y}_2)} \circ g_{\mathrm{ren-dR},*} \circ \mathbb{D}_{\mathfrak{D}(\mathcal{Y}_1)}(\mathcal{F}).$$

Note, however, that formula (4.9) still holds for the naive direct image $g_{\mathrm{dR},*}$, i.e.,

$$(4.12) \quad g_{\mathrm{dR},*} \circ \mathbf{ind}_{\mathcal{Y}_1}^{\mathfrak{D}} \simeq \mathbf{ind}_{\mathcal{Y}_2}^{\mathfrak{D}} \circ g_*.$$

In fact, one can show that for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)$, the canonical map

$$g_{\mathrm{ren-dR},*} \circ \mathbf{ind}_{\mathcal{Y}_1}^{\mathfrak{D}}(\mathcal{F}) \rightarrow g_{\mathrm{dR},*} \circ \mathbf{ind}_{\mathcal{Y}_1}^{\mathfrak{D}}(\mathcal{F})$$

is an isomorphism, see [DrGa0, Corollary 8.3.9 and Example 8.2.4].

We shall mostly apply the above discussion to $\mathcal{Y}_1 = \mathcal{Y}$, $\mathcal{Y}_2 = \mathrm{pt}$ and $g = p_{\mathcal{Y}}$. We shall use the notation

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, -) := (p_{\mathcal{Y}})_{\mathrm{dR},*}(-) \text{ and } \Gamma_{\mathrm{ren-dR}}(\mathcal{Y}, -) := (p_{\mathcal{Y}})_{\mathrm{ren-dR},*}(-).$$

4.3. (Co)homology of D-modules on Bun_G . For the rest of the paper we will assume that G is reductive. The main feature of this situation is that in this case the indschemes Gr_{X_I} are ind-proper. In particular, $\mathrm{Gr}_{\mathrm{Ran} X}$ is pseudo ind-proper (see Sect. 1.2.5, where the latter notion is introduced).

By Sect. 1.5.6, from the pseudo ind-properness of $\mathrm{Gr}_{\mathrm{Ran} X}$, the functor

$$\Gamma_{\mathrm{dR},c}(\mathrm{Gr}_{\mathrm{Ran} X}, -) : \mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X}) \rightarrow \mathrm{Vect},$$

left adjoint to $p_{\mathrm{Gr}_{\mathrm{Ran} X}}^!$, is defined on all of $\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})$.

4.3.1. We claim:

Corollary 4.3.2. *The functor*

$$\Gamma_{\mathrm{dR},c}(\mathrm{Bun}_G, -) : \mathfrak{D}(\mathrm{Bun}_G) \rightarrow \mathrm{Vect},$$

left adjoint to $p_{\mathrm{Bun}_G}^! : \mathrm{Vect} \rightarrow \mathfrak{D}(\mathrm{Bun}_G)$, is defined on all of $\mathfrak{D}(\mathrm{Bun}_G)$. Moreover, we have a canonical isomorphism

$$\Gamma_{\mathrm{dR},c}(\mathrm{Bun}_G, -) \simeq \Gamma_{\mathrm{dR},c}(\mathrm{Gr}_{\mathrm{Ran} X}, \pi^!(-)).$$

In light of the discussion in Sect. 4.2.1, the meaning of this corollary is that, with respect to the functor $p_{\mathrm{Bun}_G}^!$, the stack Bun_G exhibits features of a proper scheme.

Warning: However, it is not true that the functor $\Gamma_{\mathrm{dR},c}(\mathrm{Bun}_G, -)$ is isomorphic to either $(p_{\mathrm{Bun}_G})_{\mathrm{dR},*}$ or its renormalized version.

Proof. We wish to prove that for $\mathcal{F} \in \mathfrak{D}(\mathrm{Bun}_G)$ and $V \in \mathrm{Vect}$ there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathfrak{D}(\mathrm{Bun}_G)}(\mathcal{F}, p_{\mathrm{Bun}_G}^!(V)) \simeq \mathrm{Hom}_{\mathrm{Vect}}(\Gamma_{\mathrm{dR},c}(\mathrm{Gr}_{\mathrm{Ran} X}, \pi^!(\mathcal{F})), V).$$

By Theorem 4.1.6, the left-hand side maps isomorphically to

$$(4.13) \quad \mathrm{Hom}_{\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})}(\pi^!(\mathcal{F}), \pi^! \circ p_{\mathrm{Bun}_G}^!(V)) \simeq \mathrm{Hom}_{\mathfrak{D}(\mathrm{Gr}_{\mathrm{Ran} X})}(\pi^!(\mathcal{F}), p_{\mathrm{Gr}_{\mathrm{Ran} X}}^!(V)).$$

However, the right-hand side is isomorphic to

$$\mathrm{Hom}_{\mathrm{Vect}} \left(\Gamma_{\mathrm{dR},c} \left(\mathrm{Gr}_{\mathrm{Ran} X}, \pi^!(\mathcal{F}) \right), V \right),$$

by definition. □

Remark 4.3.3. Note that Corollary 4.3.2 is specific to Bun_G , i.e., it would not work for $\mathrm{Bun}_G^{\mathrm{level} D}$, because in the latter case the indschemes $\mathrm{Gr}_{X^I}^{\mathrm{level} D}$ are no longer ind-proper.

4.3.4. Being a left adjoint to a continuous functor, the functor $\Gamma_{\mathrm{dR},c}(\mathrm{Bun}_G, -)$ sends compact objects to compact ones, i.e., to bounded complexes of vector space with finite-dimensional cohomologies.

Let us recall now that the main theorem of [DrGa1] asserts that $\mathfrak{D}(\mathrm{Bun}_G)$ is compactly generated. In fact in *loc.cit.* a stronger assertion is established:

It is shown that $\mathfrak{D}(\mathrm{Bun}_G)$ can be presented as a union of quasi-compact open substacks \mathcal{U}_α that are *co-truncative* (see [DrGa1, Theorem 4.1.12]). By definition (see [DrGa1, Sect. 4.1]), this means that for the open embedding

$$\mathcal{U}_\alpha \xrightarrow{j_\alpha} \mathrm{Bun}_G,$$

the *a priori partially defined functor* $(j_\alpha)_!$, left adjoint to $j_\alpha^!$, is defined on all of $\mathfrak{D}(\mathcal{U}_\alpha)$.

A generating set of compact objects in $\mathfrak{D}(\mathrm{Bun}_G)$ is provided by

$$(j_\alpha)_!(\mathcal{F}_\alpha)$$

for $\mathcal{F}_\alpha \in \mathfrak{D}(\mathcal{U}_\alpha)^c$. (The categories $\mathfrak{D}(\mathcal{U}_\alpha)$ themselves are compactly generated because \mathcal{U}_α 's are quasi-compact.)

4.3.5. From Corollary 4.3.2 we obtain:

Corollary 4.3.6. *For each open substack \mathcal{U}_α as above, the functor*

$$\Gamma_{\mathrm{dR},c}(\mathcal{U}_\alpha, -) : \mathfrak{D}(\mathcal{U}_\alpha) \rightarrow \mathrm{Vect},$$

left adjoint to $p_{\mathcal{U}_\alpha}^!$, is defined on all of $\mathfrak{D}(\mathcal{U}_\alpha)$.

Thus, each of the quasi-compact substacks \mathcal{U}_α also exhibits features of a proper scheme.

4.3.7. Applying (4.11) and (4.10) to $p_{\mathcal{U}_\alpha} : \mathcal{U}_\alpha \rightarrow \mathrm{pt}$, we obtain:

Corollary 4.3.8. *For \mathcal{U}_α as above, the functor*

$$\Gamma_{\mathrm{dR}}(\mathcal{U}_\alpha, -) : \mathfrak{D}(\mathcal{U}_\alpha) \rightarrow \mathrm{Vect}$$

sends compact objects to finite-dimensional vector spaces.

Applying (4.12) to $p_{\mathcal{U}_\alpha} : \mathcal{U}_\alpha \rightarrow \mathrm{pt}$, we obtain:

Corollary 4.3.9. *The functor $\Gamma : \mathrm{QCoh}(\mathcal{U}_\alpha) \rightarrow \mathrm{Vect}$ sends $\mathrm{Coh}(\mathcal{U}_\alpha)$ to Vect^c , i.e., to bounded complexes of vector spaces with finite-dimensional cohomologies.*

4.4. **The Hecke-equivariant category.**

4.4.1. We are going to consider another functor $(\mathbf{fSet})^{op} \rightarrow \mathbf{PreStk}$, denoted $\mathcal{H}_{X^{\mathbf{fSet}}}$, and referred to as “the Hecke stack”:

For a finite set I and a test scheme S we set $\mathbf{Maps}(S, \mathcal{H}_{X^I})$ to be the groupoid of triples $(x^I, \mathcal{P}', \mathcal{P}'', \alpha)$, where $x^I : S \rightarrow X^I$, $\mathcal{P}', \mathcal{P}''$ are G -bundles on $S \times X$, and α is an isomorphism

$$\mathcal{P}'_{S \times X - \{x^I\}} \simeq \mathcal{P}''_{S \times X - \{x^I\}}.$$

Let

$$\mathcal{H}_{\mathbf{Ran} X} := \mathop{\mathrm{colim}}_{(\mathbf{fSet})^{op}} \mathcal{H}_{X^{\mathbf{fSet}}} \in \mathbf{PreStk}.$$

We have the natural projections

$$\begin{array}{ccc} & \mathcal{H}_{\mathbf{Ran} X} & \\ \swarrow \bar{h} & \downarrow f_{\mathcal{H}} & \searrow \bar{h} \\ \mathbf{Bun}_G & & \mathbf{Bun}_G \\ & \downarrow & \\ & \mathbf{Ran} X & \end{array}$$

where \bar{h} and \bar{h} remember the data of \mathcal{P}' and \mathcal{P}'' , respectively.

An important property of the projections \bar{h} and \bar{h} is that for each $I \in \mathbf{fSet}$, the corresponding maps

$$\bar{h}|_{\mathcal{H}_{X^I}}, \bar{h}|_{\mathcal{H}_{X^I}} : \mathcal{H}_{X^I} \rightarrow \mathbf{Bun}_G$$

are ind-schematic and ind-proper. Hence, the maps \bar{h} and \bar{h} are pseudo ind-proper.

In particular, the functors $\bar{h}^!, \bar{h}^! : \mathcal{D}(\mathbf{Bun}_G) \rightarrow \mathcal{D}(\mathcal{H}_{\mathbf{Ran} X})$ admit left adjoints, denoted $\bar{h}_!$ and $\bar{h}_!$, respectively.

4.4.2. Note now that the projections (\bar{h}, \bar{h}) make $\mathcal{H}_{\mathbf{Ran} X}$ into a groupoid over \mathbf{Bun}_G . The composition map is defined as follows:

We send an S -point $(x^{I_1}, \mathcal{P}'_1, \mathcal{P}''_1, \alpha_1)$ of $\mathcal{H}_{X^{I_1}}$ as above and an S -point $(x^{I_2}, \mathcal{P}'_2, \mathcal{P}''_2, \alpha_2)$ of $\mathcal{H}_{X^{I_2}}$ with $\mathcal{P}''_1 \simeq \mathcal{P}'_2$, to the S -point of $\mathcal{H}_{X^{I_1 \sqcup I_2}}$ with $(x^{I_1 \sqcup I_2}, \mathcal{P}', \mathcal{P}'', \alpha)$ given by

- $x^{I_1 \sqcup I_2} := (x^{I_1}, x^{I_2})$,
- $\mathcal{P}' := \mathcal{P}'_1, \mathcal{P}'' := \mathcal{P}''_2$, and
- α equal to the composition

$$\mathcal{P}'_1|_{S \times X - \{x^{I_1}, x^{I_2}\}} \xrightarrow{\alpha_1} \mathcal{P}''_1|_{S \times X - \{x^{I_1}, x^{I_2}\}} \simeq \mathcal{P}'_2|_{S \times X - \{x^{I_1}, x^{I_2}\}} \xrightarrow{\alpha_2} \mathcal{P}''_2|_{S \times X - \{x^{I_1}, x^{I_2}\}}.$$

4.4.3. In general, whenever

$$\begin{array}{ccc} & \mathfrak{G} & \\ \swarrow \bar{h} & & \searrow \bar{h} \\ \mathfrak{Y} & & \mathfrak{Y} \end{array}$$

is a groupoid in PreStk , we define the \mathcal{G} -equivariant category $\mathfrak{D}(\mathcal{Y})^{\mathcal{G}}$ as the totalization of the corresponding cosimplicial category $\mathfrak{D}(\mathcal{G}^\bullet)$, where

$$\mathcal{G}^n := \underbrace{\mathcal{G} \times \dots \times \mathcal{G}}_{n+1}$$

In other words,

$$\mathfrak{D}(\mathcal{Y})^{\mathcal{G}} \simeq \mathfrak{D}(|\mathcal{G}^\bullet|),$$

where $|\mathcal{G}^\bullet|$ is the geometric realization of \mathcal{G}^\bullet in PreStk .

4.4.4. We define the Hecke equivariant category

$$\mathfrak{D}(\text{Bun}_G)^{\text{Hecke}} := \mathfrak{D}(\text{Bun}_G)^{\mathcal{H}_{\text{Ran } X}}.$$

By definition, we have a conservative continuous functor

$$\mathbf{oblv}^{\text{Hecke}} : \mathfrak{D}(\text{Bun}_G)^{\text{Hecke}} \rightarrow \mathfrak{D}(\text{Bun}_G).$$

We will prove:

Proposition 4.4.5. *The functor $\mathbf{oblv}^{\text{Hecke}}$ admits a left adjoint (denoted $\mathbf{ind}^{\text{Hecke}}$). The resulting monad $\mathbf{oblv}^{\text{Hecke}} \circ \mathbf{ind}^{\text{Hecke}}$, viewed as an endo-functor of $\mathfrak{D}(\text{Bun}_G)$ is canonically isomorphic to $\overleftarrow{h}_! \circ \overrightarrow{h}^!$.*

We will prove the proposition in the general framework of Sect. 4.4.3, when the morphisms \overleftarrow{h} and \overrightarrow{h} are pseudo ind-proper (see Sect. 1.2.5, where this notion is introduced). We will show that in this case the forgetful functor

$$\mathbf{oblv}^{\mathcal{G}} : \mathfrak{D}(\mathcal{Y})^{\mathcal{G}} \rightarrow \mathfrak{D}(\mathcal{Y})$$

admits a left adjoint, denoted $\mathbf{ind}^{\mathcal{G}}$, and that the resulting endo-functor $\mathbf{oblv}^{\mathcal{G}} \circ \mathbf{ind}^{\mathcal{G}}$ of $\mathfrak{D}(\mathcal{Y})$ is isomorphic to $\overleftarrow{h}_! \circ \overrightarrow{h}^!$.¹³

4.4.6. *Digression: the Beck-Chevalley condition.* Let us note the following feature of pseudo ind-proper maps in PreStk . Let $g : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ be a pseudo ind-proper map. In this case, the functor

$$g^! : \mathfrak{D}(\mathcal{Y}_2) \rightarrow \mathfrak{D}(\mathcal{Y}_1),$$

admits a left adjoint, which we denote by $g_!$. Moreover, for a Cartesian square

$$(4.14) \quad \begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g'} & \mathcal{Y}'_2 \\ t_1 \downarrow & & \downarrow t_2 \\ \mathcal{Y}_1 & \xrightarrow{g} & \mathcal{Y}_2. \end{array}$$

the map

$$(4.15) \quad g'_! \circ t_1^! \rightarrow t_2^! \circ g_!$$

that arises by adjunction from the isomorphism

$$t_1^! \circ g^! \simeq g'^! \circ t_2^!,$$

is an isomorphism, i.e., the above adjunction satisfies what is sometimes referred to as “the Beck-Chevalley condition”.

¹³We are sure that the assertion of the proposition is known in this general context. We are supplying a proof for completeness.

The isomorphism (4.15) follows from fact that the Beck-Chevalley condition is satisfied for schemes:

For a Cartesian diagram of schemes

$$\begin{array}{ccc} S'_1 & \xrightarrow{g'} & S'_2 \\ t_1 \downarrow & & \downarrow t_2 \\ S_1 & \xrightarrow{g} & S_2. \end{array}$$

with g proper, the resulting map $g'_1 \circ t_1^! \rightarrow t_2^! \circ g_1$ equals the base change isomorphism

$$g'_{\mathrm{dR},*} \circ t_1^! \rightarrow t_2^! \circ g_{\mathrm{dR},*}.$$

4.4.7. *Proof of Proposition 4.4.5.* Consider the shifted simplicial object $\mathcal{G}^{1+\bullet}$ of PreStk . We have a canonical map of simplicial objects

$$g^\bullet : \mathcal{G}^{1+\bullet} \rightarrow \mathcal{G}^\bullet.$$

Pullback defines a map of cosimplicial categories

$$(g^\bullet)^! : \mathfrak{D}(\mathcal{G}^\bullet) \rightarrow \mathfrak{D}(\mathcal{G}^{1+\bullet}).$$

However, the fact that the Beck-Chevalley condition holds implies that the term-wise left adjoint $(g^\bullet)_!$ of $(g^\bullet)^!$ is also a map of cosimplicial categories. In particular, we obtain adjoint functors between the totalizations:

$$(4.16) \quad \mathrm{Tot}((g^\bullet)_!) : \mathrm{Tot}(\mathfrak{D}(\mathcal{G}^{1+\bullet})) \rightleftarrows \mathrm{Tot}(\mathfrak{D}(\mathcal{G}^\bullet)) : \mathrm{Tot}((g^\bullet)^!).$$

Note now that the simplicial object $\mathcal{G}^{1+\bullet}$ is augmented by \mathcal{Y} and split. Hence, the category $\mathrm{Tot}(\mathfrak{D}(\mathcal{G}^{1+\bullet}))$ identifies with $\mathfrak{D}(\mathcal{Y})$. In particular, the above functor $\mathrm{Tot}((g^\bullet)_!)$ provides a left adjoint to $\mathbf{oblv}^{\mathcal{G}} : \mathfrak{D}(\mathcal{Y})^{\mathcal{G}} \rightarrow \mathfrak{D}(\mathcal{Y})$.

Moreover, it is easy to see that the composition

$$\mathfrak{D}(\mathcal{Y}) \xrightarrow{\simeq} \mathrm{Tot}(\mathfrak{D}(\mathcal{G}^{1+\bullet})) \xrightarrow{\mathrm{Tot}((g^\bullet)_!)} \mathrm{Tot}(\mathfrak{D}(\mathcal{G}^\bullet)) \xrightarrow{\mathrm{Tot}((g^\bullet)^!)} \mathrm{Tot}(\mathfrak{D}(\mathcal{G}^{1+\bullet})) \xrightarrow{\simeq} \mathfrak{D}(\mathcal{Y})$$

is isomorphic to $\overleftarrow{h}_! \circ \overrightarrow{h}^!$, as required. \square

4.4.8. We are now ready to formulate the main theorem of this subsection. Let recall that $\mathbf{1}$ denotes the unit point of Bun_G , and $\iota_{\mathbf{1}}$ the corresponding map $\mathrm{pt} \rightarrow \mathrm{Bun}_G$.

We have:

Theorem 4.4.9. *The composed functor*

$$\mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}} \xrightarrow{\mathbf{oblv}^{\mathrm{Hecke}}} \mathfrak{D}(\mathrm{Bun}_G) \xrightarrow{\iota_{\mathbf{1}}^!} \mathrm{Vect}$$

is an equivalence.

Remark 4.4.10. Let us observe that, like Theorem 4.1.6, there should not be much surprise in Theorem 4.4.9:

Note that the Cartesian square (4.1) can be extended to a diagram in which all squares are Cartesian:

$$(4.17) \quad \begin{array}{ccccc} \mathbf{Maps}(X, G)_{\mathrm{Ran} X}^{\mathrm{rat}} & \xrightarrow{i'_1} & \mathrm{Gr}_{\mathrm{Ran} X} & \xrightarrow{p_{\mathrm{Gr}_{\mathrm{Ran} X}}} & \mathrm{pt} \\ & & \downarrow i''_1 & & \downarrow i_1 \\ & & \mathcal{H}_{\mathrm{Ran} X} & \xrightarrow{\vec{h}} & \mathrm{Bun}_G \\ & \downarrow & \leftarrow \vec{h} \downarrow & & \\ & \mathrm{pt} & \xrightarrow{i_1} & \mathrm{Bun}_G, & \end{array}$$

where the composed arrow $\overleftarrow{h} \circ i''_1 : \mathrm{Gr}_{X^{\mathrm{fSet}}} \rightarrow \mathrm{Bun}_G$ is π .

Thus, if we imagine $\mathcal{H}_{\mathrm{Ran} X}$ as a groupoid acting transitively on Bun_G , we obtain that the stabilizer of the point $\mathbf{1}$ is $\mathbf{Maps}(X, G)_{\mathrm{Ran} X}^{\mathrm{rat}}$ which is contractible. In such situation it is natural to expect that the category of groupoid-equivariant objects be equivalent to Vect .

4.4.11. Let us note that the object

$$\omega_{\mathrm{Bun}_G} \in \mathfrak{D}(\mathrm{Bun}_G)$$

naturally upgrades to an object of $\mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}}$. Namely, it corresponds to the object

$$\omega_{|\mathcal{H}_{\mathrm{Ran} X} \bullet|} \in \mathfrak{D}(|\mathcal{H}_{\mathrm{Ran} X} \bullet|) =: \mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}}.$$

Since $i_1^!(\omega_{\mathrm{Bun}_G}) \simeq k$, Theorem 4.4.9 implies that under the equivalence of the theorem, ω_{Bun_G} corresponds to $k \in \mathrm{Vect}$.

Therefore, the inverse to the functor of the theorem is given by $V \mapsto V \otimes \omega_{\mathrm{Bun}_G}$.

4.4.12. We claim that under the identification $\mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}} \simeq \mathrm{Vect}$ of Theorem 4.4.9, the functor

$$\mathbf{ind}^{\mathrm{Hecke}} : \mathfrak{D}(\mathrm{Bun}_G) \rightarrow \mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}}$$

corresponds to the functor

$$\Gamma_{\mathrm{dR}, c}(\mathrm{Bun}_G, -) : \mathfrak{D}(\mathrm{Bun}_G) \rightarrow \mathrm{Vect}.$$

Indeed, by Sect. 4.4.11 above, the right adjoint functor

$$\mathrm{Vect} \simeq \mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}} \xrightarrow{\mathbf{oblv}^{\mathrm{Hecke}}} \mathfrak{D}(\mathrm{Bun}_G)$$

identifies with $p_{\mathrm{Bun}_G}^!$, and our assertion follows from the $(\Gamma_{\mathrm{dR}, c}(\mathrm{Bun}_G, -), p_{\mathrm{Bun}_G}^!)$ -adjunction.

4.4.13. *Variant with level structure.* We can study a variant of $\mathfrak{D}(\mathrm{Bun}_G)^{\mathrm{Hecke}}$ for the stack $\mathrm{Bun}_G^{\mathrm{level} D}$ (see Sect. 4.1.11). The corresponding groupoid is given by identifying the G -bundles generically, ignoring the level structure.

Note that in this case the functor

$$\mathbf{oblv}^{\mathrm{Hecke}} : \mathfrak{D}(\mathrm{Bun}_G^{\mathrm{level} D})^{\mathrm{Hecke}} \rightarrow \mathfrak{D}(\mathrm{Bun}_G^{\mathrm{level} D})$$

does not admit a left adjoint because the corresponding map

$$\overleftarrow{h} : \mathcal{H}_{\mathrm{Ran} X}^{\mathrm{level} D} \rightarrow \mathrm{Bun}_G^{\mathrm{level} D}$$

is no longer pseudo ind-proper.

However, we claim that $\mathfrak{D}(\mathrm{Bun}_G^{\mathrm{level} D})^{\mathrm{Hecke}}$ is still equivalent to Vect .

Indeed, the groupoid $\mathcal{H}_{\text{Ran } X}^{\text{level } D}$ is by definition the pullback of the groupoid $\mathcal{H}_{\text{Ran } X}$ under the forgetful map $\text{Bun}_G^{\text{level } D} \rightarrow \text{Bun}_G$, i.e.,

$$\mathcal{H}_{\text{Ran } X}^{\text{level } D} \simeq (\text{Bun}_G^{\text{level } D} \times_{\text{Bun}_G \times \text{Bun}_G} \text{Bun}_G^{\text{level } D}) \times_{\text{Bun}_G \times \text{Bun}_G} \mathcal{H}_{\text{Ran } X}.$$

Now, since the map $\text{Bun}_G^{\text{level } D} \rightarrow \text{Bun}_G$ is faithfully flat, and in particular, satisfies descent of D-modules, we have:

$$\mathfrak{D}(\text{Bun}_G^{\text{level } D})^{\mathcal{H}_{\text{Ran } X}^{\text{level } D}} \simeq \mathfrak{D}(\text{Bun}_G)^{\mathcal{H}_{\text{Ran } X}}.$$

4.4.14. *Proof of Theorem 4.4.9.* Let us denote the functor in the theorem by \mathbf{G} . Consider its left adjoint, which is given by

$$\mathbf{F} : \text{Vect} \xrightarrow{(\iota_1)_!} \mathfrak{D}(\text{Bun}_G) \xrightarrow{\text{ind}^{\text{Hecke}}} \mathfrak{D}(\text{Bun}_G)^{\text{Hecke}}.$$

By construction, both functors \mathbf{F} and \mathbf{G} are continuous (i.e., commute with colimits). We have:

Lemma 4.4.15. *The functor \mathbf{G} is conservative.*

The proof will be given in Sect. 4.4.18. Let us continue with the proof of the theorem. We obtain that the pair of functors

$$\mathbf{F} : \text{Vect} \rightleftarrows \mathfrak{D}(\text{Bun}_G)^{\text{Hecke}} : \mathbf{G}$$

satisfies the conditions of the Barr-Beck-Lurie theorem, see e.g., [GL:DG], Proposition 3.1.1.

Hence, to prove the theorem, it suffices to show that the adjunction map

$$k \rightarrow \mathbf{G} \circ \mathbf{F}(k)$$

is an isomorphism.

We are going to calculate the object $\mathbf{G} \circ \mathbf{F}(k)$ explicitly. We have:

$$(4.18) \quad \mathbf{G} \circ \mathbf{F}(k) \simeq \iota_1^! \circ \text{oblv}^{\text{Hecke}} \circ \text{ind}^{\text{Hecke}} \circ (\iota_1)_!(k).$$

By Proposition 4.4.5, we have an isomorphism of endo-functors on $\mathfrak{D}(\text{Bun}_G)$:

$$\text{oblv}^{\text{Hecke}} \circ \text{ind}^{\text{Hecke}} \simeq \overleftarrow{h}_! \circ \overrightarrow{h}^!,$$

so we have:

$$(4.19) \quad \mathbf{G} \circ \mathbf{F}(k) \simeq \iota_1^! \circ \overleftarrow{h}_! \circ \overrightarrow{h}^! \circ (\iota_1)_!(k).$$

From the right Cartesian square in (4.17), we obtain a map

$$(4.20) \quad (\iota_1'')_! \circ (p_{\text{GrRan } X})^!(k) \rightarrow \overrightarrow{h}^! \circ (\iota_1)_!(k) \in \mathfrak{D}(\mathcal{H}_{\text{Ran } X}).$$

Lemma 4.4.16. *The map (4.20) is an isomorphism.*

The proof of the lemma is given below, see Sect. 4.4.17. Assuming the lemma, we obtain that

$$\mathbf{G} \circ \mathbf{F}(k) \simeq \iota_1^! \circ \overleftarrow{h}_! \circ (\iota_1'')_! \circ p_{\text{GrRan } X}^!(k) \simeq \iota_1^! \circ \pi_!(\omega_{\text{GrRan } X}).$$

However, by Corollary 4.1.10, $\pi_!(\omega_{\text{GrRan } X}) \simeq \omega_{\text{Bun}_G}$. Hence, we obtain that there exists an isomorphism

$$(4.21) \quad k \simeq \mathbf{G} \circ \mathbf{F}(k).$$

By tracing through the sequence of isomorphisms that led to (4.21), one could show directly that it equals the adjunction map $k \rightarrow \mathbf{G} \circ \mathbf{F}(k)$. However, instead of doing so, we will use a shortcut:

From (4.21), we obtain that the unit map of the adjunction

$$(4.22) \quad \mathrm{Id}_{\mathrm{Vect}} \rightarrow \mathbf{G} \circ \mathbf{F}$$

is either zero or an isomorphism.

However, we claim that (4.22) is non-zero. Indeed, whenever we have a pair of adjoint functors between stable categories

$$\mathbf{F} : \mathbf{C}_1 \rightleftarrows \mathbf{C}_2 : \mathbf{G},$$

if the unit of the adjunction is zero, this implies that for any $\mathbf{c}_1 \in \mathbf{C}_1$ and $\mathbf{c}_2 \in \mathbf{C}_2$,

$$\mathrm{Hom}(\mathbf{c}_1, \mathbf{G}(\mathbf{c}_2)) = 0.$$

However, this is clearly not the case in our situation: take $\mathbf{c}_1 = k$ and $\mathbf{c}_2 = \omega_{\mathrm{Bun}_G}$.

Thus, we conclude that the unit map (4.22) is an isomorphism. Applying the Barr-Beck-Lurie theorem mentioned above, we conclude that \mathbf{G} is an equivalence. \square

4.4.17. *Proof of Lemma 4.4.16.* We will prove the following generalization:

Instead of $\nu_1 : \mathrm{pt} \rightarrow \mathrm{Bun}_G$ we will take an arbitrary schematic morphism $g : \mathcal{Y} \rightarrow \mathrm{Bun}_G$, and instead of $k \in \mathfrak{D}(\mathrm{pt}) = \mathrm{Vect}$ an arbitrary $\mathcal{F} \in \mathfrak{D}(\mathcal{Y})$, for which $g_!(\mathcal{F}) \in \mathfrak{D}(\mathrm{Bun}_G)$ is defined.

We will show that in the notation of the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} \times_{\mathrm{Bun}_G} \mathcal{H}_{\mathrm{Ran} X} & \xrightarrow{g'} & \mathcal{H}_{\mathrm{Ran} X} \\ \pi_{\mathcal{Y}} \downarrow & & \downarrow \pi \\ \mathcal{Y} & \xrightarrow{g} & \mathrm{Bun}_G \end{array}$$

the map

$$g'_! \circ \pi_{\mathcal{Y}}^!(\mathcal{F}) \rightarrow \pi^! \circ g_!(\mathcal{F}) \in \mathfrak{D}(\mathcal{H}_{\mathrm{Ran} X})$$

is an isomorphism (in particular, the left-hand side is defined).

In fact, we will show that the isomorphism takes place for every individual finite set I , i.e., for the diagram

$$\begin{array}{ccc} \mathcal{Y} \times_{\mathrm{Bun}_G} \mathcal{H}_{X^I} & \xrightarrow{g'} & \mathcal{H}_{X^I} \\ \pi(I)_{\mathcal{Y}} \downarrow & & \downarrow \pi(I) \\ \mathcal{Y} & \xrightarrow{g} & \mathrm{Bun}_G \end{array}$$

and the corresponding functors.

For each I -tuple λ^I of dominant coweights of G , let $\mathcal{H}_{X^I}^{\lambda^I}$ be the corresponding closed substack of \mathcal{H}_{X^I} . We have

$$\mathcal{H}_{X^I} \simeq \mathrm{colim}_{\lambda^I \in (\Lambda^+)^I} \mathcal{H}_{X^I}^{\lambda^I},$$

where the set $(\Lambda^+)^I$ is given the standard ordering.

We will show that the indicated isomorphism takes place for every individual λ^I , for the functors in the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}_{\text{Bun}_G} \times \mathcal{H}_{X^I}^{\lambda^I} & \xrightarrow{g'} & \mathcal{H}_{X^I}^{\lambda^I} \\ \pi(I)_{\mathcal{Y}}^{\lambda^I} \downarrow & & \downarrow \pi(I)^{\lambda^I} \\ \mathcal{Y} & \xrightarrow{g} & \text{Bun}_G. \end{array}$$

The latter follows from the fact that, locally in the smooth topology on $X^I \times \text{Bun}_G$, the stack $\mathcal{H}_{X^I}^{\lambda^I}$ is isomorphic to the product $\text{Gr}_{X^I}^{\lambda^I} \times \text{Bun}_G$. (In fact $\mathcal{H}_{X^I}^{\lambda^I}$ is a fiber bundle over $X^I \times \text{Bun}_G$ obtained by twisting $\text{Gr}_{X^I}^{\lambda^I}$ by a torsor with respect to a certain group-scheme acting on it.) \square

4.4.18. *Proof of Lemma 4.4.15.* The idea of the proof is the following: if $\mathcal{F} \in \mathfrak{D}(\text{Bun}_G)^{\text{Hecke}}$ is such that $\iota_1^!(\mathcal{F}) = 0$, then $\iota_{\mathbf{g}}^!(\mathcal{F}) = 0$ for any other point $\mathbf{g} \in \text{Bun}_G$, because any two points of Bun_G can be connected by a Hecke correspondence.

We articulate this idea as follows. Let $\mathcal{F} \in \mathfrak{D}(\text{Bun}_G)^{\text{Hecke}}$ be an object such that $\iota_1^!(\mathbf{oblv}^{\text{Hecke}}(\mathcal{F})) = 0$. We wish to show that $\mathcal{F} = 0$. Since the functor $\mathbf{oblv}^{\text{Hecke}}$ is conservative by definition, it suffices to show that $\mathcal{F}' := \mathbf{oblv}^{\text{Hecke}}(\mathcal{F}) \in \mathfrak{D}(\text{Bun}_G)$ vanishes.

Let S be a scheme equipped with a map $g : S \rightarrow \text{Bun}_G$. We need to show that $g^!(\mathcal{F}') = 0$.

As in the proof of Theorem 4.1.6, possibly after passing to an étale cover of S , there exists a finite set I and a map $x^I : S \rightarrow X^I$, such that the pullback of the universal bundle to $S \times X - \{x^I\}$ admits a trivialization.

A choice of such trivialization is the same as a lift of the map $(g, x^I) : S \rightarrow \text{Bun}_G \times X^I$ to a map $g' : S \rightarrow \mathcal{H}_{X^I}$, such that

$$\overleftarrow{h} \circ g' = g \text{ and } \overrightarrow{h} \circ g' = \iota_1 \circ p_S.$$

Hence,

$$g^!(\mathcal{F}') \simeq g'^!(\overleftarrow{h}^!(\mathcal{F}')) \simeq g'^!(\overrightarrow{h}^!(\mathcal{F}')) \simeq p_S^!(\iota_1^!(\mathcal{F}')) = 0,$$

where the isomorphism

$$\overleftarrow{h}^!(\mathcal{F}') \simeq \overrightarrow{h}^!(\mathcal{F}')$$

takes place because \mathcal{F}' came from a Hecke-equivariant object \mathcal{F} . \square

4.5. **Another version of the Hecke category.** Let us note that one could define the category $\mathfrak{D}(\text{Bun}_G)^{\text{Hecke}}$ slightly differently.

4.5.1. Namely, instead of taking the simplicial object of PreStk to be $(\mathcal{H}_{\text{Ran } X})^\bullet$ defined above, we can take

$$(\mathcal{H}^\bullet)_{\text{Ran } X},$$

where we take iterated Cartesian products of $\mathcal{H}_{\text{Ran } X}$ “over” $\text{Ran } X$, as well as over Bun_G .¹⁴

¹⁴Quotation marks for “over” are due to the fact that we are not actually taking Cartesian products over $\text{Ran } X$ in PreStk , but rather at the level of functors $(\text{fSet})^{\text{op}} \rightarrow \text{PreStk}$, and then pass to the colimit. The former is an ill-behaved operation because the category $(\text{fSet})^{\text{op}}$ is not filtered.

I.e., each term $(\mathcal{H}^n)_{\text{Ran } X}$ is by definition

$$\text{colim}_{I \in (\text{fSet})^{op}} (\mathcal{H}^n)_{X^I},$$

where

$$\begin{aligned} \text{Maps}(S, (\mathcal{H}^n)_{X^I}) &= \{x^I \in \text{Maps}(S, X^I), \mathcal{P}^0, \dots, \mathcal{P}^n \in \text{Maps}(S, \text{Bun}_G), \\ &\mathcal{P}^0|_{S \times X - \{x^I\}} \xrightarrow{\alpha_1} \mathcal{P}^1|_{S \times X - \{x^I\}}, \dots, \mathcal{P}^{n-1}|_{S \times X - \{x^I\}} \xrightarrow{\alpha_n} \mathcal{P}^n|_{S \times X - \{x^I\}}\}. \end{aligned}$$

Let us denote the version of the Hecke-equivariant category defined using $(\mathcal{H}^\bullet)_{\text{Ran } X}$ by $'\mathcal{D}(\text{Bun}_G)^{\text{Hecke}}$. We claim that is in fact equivalent to $\mathcal{D}(\text{Bun}_G)^{\text{Hecke}}$.

4.5.2. Note that there are maps of simplicial objects of PreStk :

$$(4.23) \quad (\mathcal{H}^\bullet)_{\text{Ran } X} \rightleftarrows (\mathcal{H}_{\text{Ran } X})^\bullet$$

By definition, the right-hand side is

$$\text{colim}_{(I_1, \dots, I_n) \in (\text{fSet}^{\times n})^{op}} \mathcal{H}_{X^{I_1}} \times_{\text{Bun}_G} \dots \times_{\text{Bun}_G} \mathcal{H}_{X^{I_n}}.$$

It receives a map from the left-hand side via the identification

$$(\mathcal{H}^n)_{X^I} \simeq X^I \times_{(X^I)^{\times n}} (\mathcal{H}_{\text{Ran } X})^n,$$

where

$$X^I \rightarrow (X^I)^{\times n} = \underbrace{X^I \times \dots \times X^I}_n$$

is the diagonal map.

The map \leftarrow in (4.23) is constructed via the procedure of Sect. 1.5.4: we send

$$(\text{fSet}^{\times n})^{op} \rightarrow (\text{fSet})^{op}$$

by the disjoint union functor

$$(I_1, \dots, I_n) \mapsto I_1 \sqcup \dots \sqcup I_n.$$

The corresponding map of indschemes sends the datum of

$$\begin{aligned} \{(x^{I_1}, \mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^0|_{S \times X - \{x^{I_1}\}} \xrightarrow{\alpha_1} \mathcal{P}^1|_{S \times X - \{x^{I_1}\}}), \dots \\ \dots, (x^{I_n}, \mathcal{P}^{n-1}, \mathcal{P}^n, \mathcal{P}^{n-1}|_{S \times X - \{x^{I_n}\}} \xrightarrow{\alpha_n} \mathcal{P}^n|_{S \times X - \{x^{I_n}\}})\} \end{aligned}$$

to the tautologically defined point of $(\mathcal{H}^n)_{\text{Ran } X}$, where we regard each α_k , $k = 1, \dots, n$ as defined over $S \times X - \{x^{I_1}, \dots, x^{I_n}\}$.

Pullback along the maps in (4.23) defines functors

$$(4.24) \quad '\mathcal{D}(\text{Bun}_G)^{\text{Hecke}} \rightleftarrows \mathcal{D}(\text{Bun}_G)^{\text{Hecke}},$$

and we claim that these functors are mutually inverse equivalences.

4.5.3. The objects $(\mathcal{H}_{\text{Ran } X})^\bullet$ and $(\mathcal{H}^\bullet)_{\text{Ran } X}$, viewed as simplicial objects of PreStk equipped with pseudo ind-proper maps to Bun_G , define simplicial monads

$$(\overleftarrow{h}^\bullet)_! \circ (\overrightarrow{h}^\bullet)! \text{ and } ({}'\overleftarrow{h}^\bullet)_! \circ ({}'\overrightarrow{h}^\bullet)!,$$

acting on $\mathfrak{D}(\text{Bun}_G)$, where $(\overleftarrow{h}^\bullet, \overrightarrow{h}^\bullet)$ and $({}'\overleftarrow{h}^\bullet, {}'\overrightarrow{h}^\bullet)$ are the maps

$$(\mathcal{H}_{\text{Ran } X})^\bullet \rightrightarrows \text{Bun}_G \text{ and } (\mathcal{H}^\bullet)_{\text{Ran } X} \rightrightarrows \text{Bun}_G,$$

respectively.

The monads

$$\mathbf{oblv}^{\text{Hecke}} \circ \mathbf{ind}^{\text{Hecke}} \text{ and } {}'\mathbf{oblv}^{\text{Hecke}} \circ {}'\mathbf{ind}^{\text{Hecke}},$$

acting on $\mathfrak{D}(\text{Bun}_G)$ and responsible for $\mathfrak{D}(\text{Bun}_G)^{\text{Hecke}}$ and ${}'\mathfrak{D}(\text{Bun}_G)^{\text{Hecke}}$, are given by

$$|(\overleftarrow{h}^\bullet)_! \circ (\overrightarrow{h}^\bullet)!| \text{ and } |({}'\overleftarrow{h}^\bullet)_! \circ ({}'\overrightarrow{h}^\bullet)!|.$$

respectively.

However, we claim that the maps (4.23) define simplicial isomorphisms of monads

$$(\overleftarrow{h}^\bullet)_! \circ (\overrightarrow{h}^\bullet)! \text{ and } ({}'\overleftarrow{h}^\bullet)_! \circ ({}'\overrightarrow{h}^\bullet)!.$$

This follows from a relative and iterated version of Corollary 2.6.6.

Alternatively, for the reader who skipped Sect. 2, one can prove this directly, by repeating the argument of [BD1], Theorem 4.3.6.

5. A REDERIVATION OF THE ATIYAH-BOTT FORMULA

5.1. **A local expression for the homology of Bun_G .** The goal of this section is to explain how the isomorphism of Corollary 4.3.2 reproduces the Atiyah-Bott formula for

$$\Gamma_{\text{dR}}^\bullet(\text{Bun}_G, k) =: H^\bullet(\text{Bun}_G).$$

5.1.1. Note that Corollary 4.3.2 implies the following isomorphism:

Corollary 5.1.2. $\Gamma_{\text{dR},c}(\text{Bun}_G, \omega_{\text{Bun}_G}) \simeq \Gamma_{\text{dR},c}(\text{Gr}_{\text{Ran } X}, \omega_{\text{Gr}_{\text{Ran } X}}).$

5.1.3. First, we observe that the left-hand side in Corollary 5.1.2 is really the homology of Bun_G in the sense that

$$\text{Hom}_{\mathbf{Vect}}^\bullet(\Gamma_{\text{dR},c}(\text{Bun}_G, \omega_{\text{Bun}_G}), k) \simeq H^\bullet(\text{Bun}_G).$$

Indeed, for an Artin stack \mathcal{Y} , let $\underline{k}_{\mathcal{Y}} \in \mathfrak{D}(\mathcal{Y})$ denote the ‘‘constant sheaf’’ object, i.e., one for which

$$\text{Hom}^\bullet(\underline{k}_{\mathcal{Y}}, -) \simeq \Gamma_{\text{dR}}(\mathcal{Y}, -).$$

Then by definition

$$H^\bullet(\mathcal{Y}) := \Gamma_{\text{dR}}(\mathcal{Y}, \underline{k}_{\mathcal{Y}}) \simeq \text{Hom}_{\mathfrak{D}(\mathcal{Y})}^\bullet(\underline{k}_{\mathcal{Y}}, \underline{k}_{\mathcal{Y}}).$$

Now, if the left adjoint $(p_{\mathcal{Y}})_! = \Gamma_{\text{dR},c}(\mathcal{Y}, -)$ to $p_{\mathcal{Y}}^!$ is defined on $\omega_{\mathcal{Y}}$, we have by definition

$$\text{Hom}^\bullet(\Gamma_{\text{dR},c}(\mathcal{Y}, \omega_{\mathcal{Y}}), k) \simeq \text{Hom}_{\mathfrak{D}(\mathcal{Y})}^\bullet(\omega_{\mathcal{Y}}, \omega_{\mathcal{Y}}).$$

However, it is easy to see that

$$\text{Hom}_{\mathfrak{D}(\mathcal{Y})}^\bullet(\omega_{\mathcal{Y}}, \omega_{\mathcal{Y}}) \simeq \text{Hom}_{\mathfrak{D}(\mathcal{Y})}^\bullet(\underline{k}_{\mathcal{Y}}, \underline{k}_{\mathcal{Y}}).$$

Indeed, by pulling back to schemes Z mapping smoothly to \mathcal{Y} , the latter isomorphism follows from the fact that $\mathbb{D}_{\mathfrak{D}(Z)}(\underline{k}_Z) \simeq \omega_Z$.

5.1.4. Let us now observe that the right-hand side in Corollary 5.1.2 can be rewritten as follows:

$$(5.1) \quad \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, f_!(\omega_{\mathrm{Gr}_{\mathrm{Ran} X}})).$$

Recall now that the basic feature of the functor $\mathrm{Gr}_{X^{\mathrm{fSet}}} : (\mathrm{fSet})^{\mathrm{op}} \rightarrow \mathrm{IndSch}$ is *factorization* with respect to X^{fSet} , see [BD2], Sect. 5.3.12.

This implies that the object

$$\mathcal{B} := f_!(\omega_{\mathrm{Gr}_{\mathrm{Ran} X}}) \in \mathfrak{D}(\mathrm{Ran} X)$$

has a structure of *factorization algebra* in the terminology of [BD1], Sect. 3.4.4 (or *factorization \mathcal{D} -module* in the terminology of [FrGa], Sect. 2.4).

Remark 5.1.5. Let us observe that unlike the situation of Remark 1.7.9, the object $f_!(\omega_{\mathrm{Gr}_{\mathrm{Ran} X}}) \in \mathfrak{D}(\mathrm{Ran} X)$ can be calculated term-wise, i.e.,

$$(\Delta^I)!(f_!(\omega_{\mathrm{Gr}_{\mathrm{Ran} X}})) \simeq f(I)!(\omega_{\mathrm{Gr}_{X^I}}).$$

This is due to the fact that the properness of Gr_{X^I} over X^I and the proper base change formula.

5.1.6. Recall now the notion of chiral algebra on X , see [BD1], Sect. 3.3 (see also [FrGa], Sect. 2.4, where the definition is spelled out in the setting of higher categories; note that in *loc.cit.*, chiral algebras on X are referred to as “chiral Lie algebras on X ”).

We shall denote by $\mathrm{Lie}\text{-alg}^{\mathrm{ch}}(X)$ the category of chiral algebras on X .

Thus, by Theorem 1.2.4 of [FrGa], \mathcal{B} corresponds to a *chiral algebra* B on X .

5.1.7. Recall that for a chiral algebra B , its chiral homology $\int_X B$ is defined as $\Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \mathcal{B})$, where \mathcal{B} is the corresponding factorization algebra, viewed as an object of $\mathfrak{D}(\mathrm{Ran} X)$.

Thus, we obtain that

$$(5.2) \quad H_\bullet(\mathrm{Bun}_G) \simeq \int_X B,$$

for the above chiral algebra B on X .

5.1.8. One can regard the operation of taking chiral homology as a local-to-global principle on X . In this sense, (5.2) gives a “local on X ” expression for the homology of Bun_G .

5.2. The Atiyah-Bott formula.

5.2.1. Recall that for any space \mathcal{Y} , the category $\mathfrak{D}(\mathcal{Y})$ has a natural symmetric monoidal structure: the monoidal operation corresponds to the composed functor

$$\mathfrak{D}(\mathcal{Y}) \otimes \mathfrak{D}(\mathcal{Y}) \rightarrow \mathfrak{D}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta_{\mathcal{Y}}^\dagger} \mathfrak{D}(\mathcal{Y}).$$

Recall also ([BD1], Sect. 3.3.1) that commutative algebras in $\mathfrak{D}(X)$ in the above symmetric monoidal structure give rise to chiral algebras.

If A is a commutative chiral algebra, the vector space $\int_X A$ has a natural structure of commutative algebra, which satisfies the following universal property (see [BD1], Sect. 4.6.1):

Note that the functor $p_X^\dagger : \mathrm{Vect} \rightarrow \mathfrak{D}(X)$ has a natural symmetric monoidal structure (this is true for the functor g^\dagger for any morphism of prestacks $g : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$). Thus, p_X^\dagger gives rise to the (eponymous) functor

$$p_X^\dagger : \mathrm{Com}\text{-alg}(\mathrm{Vect}) \rightarrow \mathrm{Com}\text{-alg}(\mathfrak{D}(X)).$$

For $A \in \text{Com-alg}(\mathfrak{D}(X))$, viewed as a chiral algebra, and $A' \in \text{Com-alg}(\text{Vect})$ we have:

$$(5.3) \quad \text{Hom}_{\text{Com-alg}(\text{Vect})} \left(\int_X A, A' \right) \simeq \text{Hom}_{\text{Com-alg}(\mathfrak{D}(X))} (A, p_X^!(A')).$$

For $A \in \text{Com-alg}(\text{Vect})$ we will use a short-hand notation $\int_X A$ for $\int_X p_X^!(A)$.

5.2.2. We apply the above discussion to $A := H^\bullet(BG)$ and $A' = H^\bullet(\text{Bun}_G)$, where BG is the stack pt/G .

Pullback along the universal map $\text{Bun}_G \times X \rightarrow BG$ gives rise to a map in $\text{Com-alg}(\mathfrak{D}(X))$

$$(5.4) \quad p_X^!(H^\bullet(BG)) \rightarrow p_X^!(H^\bullet(\text{Bun}_G)).$$

Thus, from (5.3) we obtain a map

$$(5.5) \quad \int_X H^\bullet(BG) \rightarrow H^\bullet(\text{Bun}_G).$$

The Atiyah-Bott formula says that when G is semi-simple and simple connected, the map (5.5) is an isomorphism.

Remark 5.2.3. We emphasize that although the map (5.5) is an isomorphism only when G is semi-simple and simply connected, formula (5.2) is valid for any reductive G .

5.2.4. Let us bring the above version of the Atiyah-Bott formula to a more familiar form. Let us recall that the commutative algebra $H^\bullet(BG)$ is free, i.e., it is isomorphic to $\text{Sym}(\mathfrak{a}_G)$ for some particular object $\mathfrak{a}_G \in \text{Vect}$.

In fact

$$\mathfrak{a}_G \simeq \bigoplus_e k[-2 \cdot e],$$

where e runs through the set of exponents of G .

By [BD1], Proposition 4.6.2, chiral homology of a free commutative chiral algebra $\text{Sym}(V)$ is computed by

$$\int_X \text{Sym}(V) \simeq \text{Sym}(V \otimes H_\bullet(X)).$$

Taking $V = \mathfrak{a}_G$, we obtain that (5.5) gives rise to an isomorphism

$$(5.6) \quad \text{Sym}(\mathfrak{a}_G \otimes H_\bullet(X)) \simeq H^\bullet(\text{Bun}_G),$$

which is the more usual form of the Atiyah-Bott formula.

5.3. The rederivation of the formula. Let us now explain the equivalence of the isomorphisms (5.5) and (5.2). We will only give a sketch; a detailed proof will appear in the forthcoming joint paper of the author and J. Lurie, [GaLu].

5.3.1. Let \mathfrak{b}_G be the Lie algebra in \mathbf{Vect} that governs the homotopy type of BG tensored with k . I.e., by definition, \mathfrak{b}_G is the Lie algebra such that

$$C^\bullet(\mathfrak{b}_G) \simeq H^\bullet(BG),$$

where C^\bullet denotes the cohomological Chevalley complex of a Lie algebra.

In the case of BG , the Lie algebra \mathfrak{b}_G is abelian, which corresponds to the fact that $H^\bullet(BG)$ is free as a commutative algebra:

We have

$$\mathfrak{a}_G \simeq \mathfrak{b}_G^\vee[-1],$$

where $\mathfrak{a}_G \in \mathbf{Vect}$ is the vector space from Sect. 5.2.4.

5.3.2. Recall the notion of Lie- \star algebra on X , see [BD1], Sect. 2.5 (this is what is called a \star -Lie algebra on X in [FrGa], Sect. 2.4). Let $\mathbf{Lie}\text{-alg}^\star(X)$ denote the category of Lie- \star algebras on X .

Recall also that if L is a Lie algebra in \mathbf{Vect} , the object $p_X^{\mathrm{dR},\star}(L) \in \mathfrak{D}(X)$ naturally upgrades to one in $\mathbf{Lie}\text{-alg}^\star(X)$. Here

$$p_X^{\mathrm{dR},\star} : \mathbf{Vect} \rightarrow \mathfrak{D}(X)$$

is the functor of \star -pullback, left adjoint to the direct image functor $(p_X)_! \simeq (p_X)_{\mathrm{dR},\star}$.

Finally, recall that the forgetful functor

$$\mathbf{Lie}\text{-alg}^{\mathrm{ch}}(X) \rightarrow \mathbf{Lie}\text{-alg}^\star(X)$$

admits a left adjoint, called the functor of *chiral universal envelope*, and denoted by U^{ch} .

We have:

Proposition 5.3.3. *For G semi-simple and simply connected, the chiral algebra B of (5.2) is canonically isomorphic to $U^{\mathrm{ch}}(p_X^{\mathrm{dR},\star}(\mathfrak{b}_G))$.*

Remark 5.3.4. The assertion of the above proposition is well-known in topology. At the level of $!$ -stalks at points of X it says that

$$H_\bullet(\mathrm{Gr}_x) \simeq C_\bullet(\mathfrak{b}_G[-2]),$$

where x is any point of X , and Gr_x is the fiber of Gr_X over $x \in X$, and $\mathfrak{b}_G[-2]$ is the Lie algebra obtained by looping \mathfrak{b}_G twice. This results from the fact that Gr_x is homotopy-equivalent to the double-loop space of BG .

Proof. We shall only sketch the proof; details will be supplied in [GaLu].

Let Z be any functor $(\mathbf{fSet})^{\mathrm{op}} \rightarrow \mathbf{IndSch}$ as in Sect. 2.4.1, which is a unital *factorization monoid*, see [BD1], Sect. 3.10.16. In this case $\mathcal{B} := f_!(\omega_{Z_{\mathrm{Ran} X}})$ is a unital and augmented factorization algebra in $\mathfrak{D}(\mathrm{Ran} X)$. Let B be the corresponding chiral algebra on X . We let \overline{B} denote the augmentation ideal of B , and $\overline{\mathcal{B}}$ the corresponding factorization algebra on $\mathrm{Ran} X$.

The diagonal map $Z \rightarrow Z \times Z$ defines on B a structure of unital augmented commutative co-algebra object in the category of chiral algebras.

Now, the functor U^{ch} canonically factors through a functor

$$(5.7) \quad \mathbf{Lie}\text{-alg}^\star(X) \rightarrow \mathbf{Com}\text{-coalg}(\mathbf{Lie}\text{-alg}_{\mathrm{unital},\mathrm{augm}}^{\mathrm{ch}}(X)),$$

followed by the forgetful functor

$$\mathbf{Com}\text{-coalg}(\mathbf{Lie}\text{-alg}_{\mathrm{unital},\mathrm{augm}}^{\mathrm{ch}}(X)) \rightarrow \mathbf{Lie}\text{-alg}^{\mathrm{ch}}(X).$$

Moreover, the functor of (5.7) induces an equivalence on the subcategories of objects satisfying an appropriate connectivity hypothesis. In particular, Sect. 1.6.3 implies that this hypothesis is satisfied by B if the fibers of $Z(\{1\})$ over X are connected.

Let us return now to the situation when $Z = \mathrm{Gr}_{X^{\mathrm{fSet}}}$. If G is semi-simple and simply connected, the fibers of Gr_X over X are connected. Hence, we obtain that in this case

$$B \simeq U^{\mathrm{ch}}(L)$$

for a canonically defined Lie- $*$ algebra L on X .

Let \mathcal{A} denote the factorization algebra corresponding to the commutative chiral algebra $A := p_X^!(H^\bullet(BG))$. Let $\overline{\mathcal{A}}$ denote the augmentation ideal of \mathcal{A} , corresponding to the unit point of BG . Let $\overline{\mathcal{B}}$ denote the corresponding factorization algebra.

A local version of the map (5.4) gives rise to a map

$$(5.8) \quad \mathrm{union}_1(\overline{\mathcal{B}} \boxtimes \overline{\mathcal{A}}) \rightarrow \omega_{\mathrm{Ran} X},$$

compatible with the commutative co-algebra structure on $\overline{\mathcal{B}}$ and the commutative algebra structure on $\overline{\mathcal{A}}$.

Since $H^\bullet(BG) \simeq \mathrm{Sym}(\mathfrak{a}_G)$, the above properties of the map (5.8) define a map of Lie- $*$ algebras

$$(5.9) \quad L \rightarrow \mathbb{D}_{\mathfrak{D}(X)}(p_X^!(\mathfrak{a}_G))[-1] \simeq p_X^{\mathrm{dR},*}(\mathfrak{b}_G).$$

Now, a non-degeneracy property of (5.8) implies that the map (5.9) is an isomorphism. \square

5.3.5. Finally, let us see how (5.2), combined with Proposition 5.3.3, reproduces the isomorphism (5.5).

By [BD1], Theorem 4.8.1 (see also [FrGa], Corollary 6.3.4), for a Lie- $*$ algebra L on X we have:

$$\int_X U^{\mathrm{ch}}(L) \simeq C_\bullet(\Gamma_{\mathrm{dR},c}(X, L)),$$

where $\Gamma_{\mathrm{dR},c}(X, L) \in \mathrm{Vect}$ acquires a natural structure of Lie algebra by [FrGa], Sect. 6.2.1. We apply this to $L = p_X^{\mathrm{dR},*}(\mathfrak{b}_G)$, and we obtain that

$$(5.10) \quad H_\bullet(\mathrm{Bun}_G) \simeq \int_X B \simeq C_\bullet\left(\Gamma_{\mathrm{dR},c}(X, p_X^{\mathrm{dR},*}(\mathfrak{b}_G))\right).$$

Hence,

$$(5.11) \quad \begin{aligned} H^\bullet(\mathrm{Bun}_G) &\simeq (H_\bullet(\mathrm{Bun}_G))^\vee \simeq \\ &\simeq \left(C_\bullet\left(\Gamma_{\mathrm{dR},c}(X, p_X^{\mathrm{dR},*}(\mathfrak{b}_G))\right)\right)^\vee \simeq C^\bullet\left(\Gamma_{\mathrm{dR},c}(X, p_X^{\mathrm{dR},*}(\mathfrak{b}_G))\right). \end{aligned}$$

5.3.6. On the other hand, we have $H^\bullet(BG) \simeq C^\bullet(\mathfrak{b}_G)$, and therefore we can identify the commutative chiral algebra $p_X^!(H^\bullet(BG))$ with

$$C_{\mathfrak{D}(X)}^\bullet(p_X^{\mathrm{dR},*}(\mathfrak{b}_G)),$$

where for a $L \in \mathrm{Lie}\text{-alg}^*(X)$ we denote by

$$C_{\mathfrak{D}(X)}^\bullet(L) \in \mathrm{Com}\text{-alg}(\mathfrak{D}(X))$$

the corresponding Chevalley algebra, see [BD1], Sect. 1.4.14.

By [BD1], Proposition 4.7.1, we have:

$$\int_X C_{\mathfrak{D}(X)}^\bullet(L) \simeq C^\bullet(\Gamma_{\mathrm{dR},c}(X, L)).$$

Thus, we obtain that

$$(5.12) \quad \int_X H^\bullet(BG) \simeq C^\bullet(\Gamma_{\mathrm{dR},c}(X, p_X^{\mathrm{dR},*}(\mathfrak{b}_G))).$$

Comparing (5.11) with (5.12), we deduce the desired isomorphism

$$(5.13) \quad H^\bullet(\mathrm{Bun}_G) \simeq \int_X H^\bullet(BG).$$

Remark 5.3.7. The construction of the isomorphism $\mathcal{B} \simeq U^{\mathrm{ch}}(p_X^{\mathrm{dR},*}(\mathfrak{b}_G))$ sketched in Proposition 5.3.3 implies that the isomorphism (5.13) coincides with that of (5.5):

The map (5.8), whose construction we have omitted is designed so that the diagram

$$\begin{array}{ccc} \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \mathrm{union}_!(\overline{\mathcal{B}} \boxtimes \overline{\mathcal{A}})) & \xrightarrow{\sim} & \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \overline{\mathcal{B}}) \otimes \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \overline{\mathcal{A}}) \\ \downarrow & & \downarrow \\ \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \omega_{\mathrm{Ran} X}) & & \overline{H}_\bullet(\mathrm{Bun}_G) \otimes \overline{H}^\bullet(\mathrm{Bun}_G) \\ \downarrow & & \downarrow \\ k & \xrightarrow{\mathrm{id}} & k \end{array}$$

commutes, where $\overline{H}_\bullet(\mathrm{Bun}_G)$ (resp., $\overline{H}^\bullet(\mathrm{Bun}_G)$) denotes the reduced homology (resp., cohomology) of Bun_G with respect to the base point $\mathbf{1} \in \mathrm{Bun}_G$. This property is true for any G , not necessarily semi-simple and simply connected.

Now, for a Lie-* algebra L , if we denote by \mathcal{B} the factorization algebra corresponding to the chiral algebra $U^{\mathrm{ch}}(L)$, and by \mathcal{A} the factorization algebra corresponding to the chiral algebra $C_{\mathfrak{D}(X)}^\bullet(L)$, and by $\overline{\mathcal{B}}$ and $\overline{\mathcal{A}}$ their reduced counterparts (i.e., the factorization algebras corresponding to the augmentation ideals in the corresponding chiral algebras), the canonical map

$$\mathrm{union}_!(\overline{\mathcal{B}} \boxtimes \overline{\mathcal{A}}) \rightarrow \omega_{\mathrm{Ran} X}$$

makes the following diagram commute:

$$\begin{array}{ccc}
 \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \mathrm{union}_!(\overline{\mathcal{B}} \boxtimes \overline{\mathcal{A}})) & \xrightarrow{\sim} & \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \overline{\mathcal{B}}) \otimes \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \overline{\mathcal{A}}) \\
 \downarrow & & \downarrow \sim \\
 \Gamma_{\mathrm{dR},c}(\mathrm{Ran} X, \omega_{\mathrm{Ran} X}) & & \overline{\mathcal{C}}_{\bullet}(\Gamma_{\mathrm{dR},c}(X, L)) \otimes \overline{\mathcal{C}}^{\bullet}(\Gamma_{\mathrm{dR},c}(X, L)) \\
 \downarrow & \xrightarrow{\mathrm{id}} & \downarrow \\
 k & & k.
 \end{array}$$

6. APPENDIX: CONTRACTIBILITY OF THE RAN SPACE

Here is the promised proof of Theorem 1.6.5. By Sect. 1.6.3, $H_{\bullet}(\mathrm{Ran} X)$ is connective, and $H_0(\mathrm{Ran} X)$ maps isomorphically to k .

Assume by contradiction that for some $n > 0$ we have $H_n(\mathrm{Ran} X) \neq 0$. With no restriction of generality, we can assume that n is minimal such integer. By the Künneth formula, we obtain that the maps $\mathrm{id} \times p_{\mathrm{Ran} X}$ and $p_{\mathrm{Ran} X} \times \mathrm{id}$

$$\mathrm{Ran} X \times \mathrm{Ran} X \rightrightarrows \mathrm{Ran} X$$

define an isomorphism

$$H_n(\mathrm{Ran} X \times \mathrm{Ran} X) \rightarrow H_n(\mathrm{Ran} X) \oplus H_n(\mathrm{Ran} X).$$

Denote $M := H_n(\mathrm{Ran} X)$.

The map

$$\mathrm{union} : \mathrm{Ran} X \times \mathrm{Ran} X \rightarrow \mathrm{Ran} X$$

defines, therefore, a map

$$M \oplus M \rightarrow M,$$

which by symmetry must be of the form $u \oplus u$ for some map $u : M \rightarrow M$. By associativity, for any $k \geq 2$, the corresponding map

$$(\mathrm{Ran} X)^{\times k} \xrightarrow{\mathrm{union}^k} \mathrm{Ran} X$$

acts on H_n as $u^{\oplus k}$.

For an integer k , consider now the diagonal map $\mathrm{Ran} X \rightarrow (\mathrm{Ran} X)^{\times k}$. It induces on H_n the diagonal map $M \rightarrow M^{\oplus k}$.

By Sect. 2.2.10, the composition

$$\mathrm{Ran} X \rightarrow (\mathrm{Ran} X)^{\times k} \xrightarrow{\mathrm{union}^k} \mathrm{Ran} X$$

is the identity map. Hence, we obtain that $k \cdot u = \mathrm{id}$ for any $k \geq 2$. Taking k to be, e.g., 2 and 3, we obtain that $2 \cdot u = 3 \cdot u$, i.e., $u = 0$. Hence, $\mathrm{id} : M \rightarrow M$ is the zero map, which is a contradiction. \square

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