MATH 221, PROBLEM SET 10, DUE: MON., NOV. 17.

1. Let A be a commutative ring, and M an A-module.

(a) Let $M^1, M^2 \subset M$ be two submodules such that $M^1_{\mathfrak{p}} \subset M^2_{\mathfrak{p}}$ as submodules of $M_{\mathfrak{p}}$ for any prime $\mathfrak{p}$. Show that $M_1 \subset M_2$.

(b) Let A be an integral domain and K its field of fractions. Show that $f \in K$ belongs to A if and only if $f \in A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$.

2. Let A be a commutative ring.

(a) Let $S \subset A$ be a multiplicative subset, and let N be an A_S-module. Show that $supp_A(N) = \overline{supp_{A_S}(N)}$ as subsets of $\mathrm{Spec}(A)$.

(b) Let $\mathfrak{m} \in \mathrm{Specm}(A)$ be such that $\dot{\mathfrak{m}} \in \mathrm{Spec}(A_S)$, and let N be as in point (a). Deduce that $supp_A(N) = \{\dot{\mathfrak{m}}\} \Leftrightarrow supp_{A_S}(N) = \{\dot{\mathfrak{m}}_S\}$.

(c) Let L be an A-module and $\mathfrak{m}$ a maximal ideal. Show that $supp(L) = \{\dot{\mathfrak{m}}\} \Leftrightarrow$ $\forall\, l \in L\, \forall\, a \in \mathfrak{m}\, \exists k \,|\, a^k \cdot l = 0$.

(d) Show that for L with $supp(L) = \{\dot{\mathfrak{m}}\}$, the map $N \to N_{\mathfrak{m}}$ is an isomorphism.

3. Let A be a Noetherian domain of Krull dimension 1.

(a) Show that for any non-zero ideal $I \subset A$, the subset $V(I) = \mathrm{Spec}(A/I) = supp(A/I) =\subset \mathrm{Spec}(A)$ is a union of a finite collection of closed points.

(b) Show that for any non-zero $a \in A$, for all but finitely many primes $a \notin \mathfrak{p}$.

4. Let A be a Noetherian domain of Krull dimension 1, and let K denote its field of fractions.

(a) Assume that A is local with the unique maximal ideal $\mathfrak{p}$. Show that for any non-zero ideal $I \subset A$, $supp(A/I) = \{\dot{\mathfrak{p}}\}$ and $supp(K/A) = \{\dot{\mathfrak{p}}\}$.

(b) Let $\mathfrak{p}$ and $\mathfrak{q}$ be two distinct maximal ideals of A; let $I_{\mathfrak{p}}$ be a non-zero ideal in $A_{\mathfrak{p}}$. Show that $(A_{\mathfrak{p}}/I_{\mathfrak{p}})_{\mathfrak{q}} = 0$ and $(K_{\mathfrak{p}}/A_{\mathfrak{p}})_{\mathfrak{q}} = 0$. Hint: use Problem 2.

(c) Prove that the natural map $K \mapsto \prod_{\mathfrak{p}} K/A_{\mathfrak{p}}$ defines an isomorphism

$$K/A \simeq \bigoplus_{\mathfrak{p}} K/A_{\mathfrak{p}}.$$

5.(a) Let A be as in Problem 4. Prove that the assignments

$$I \mapsto \{\mathfrak{p} \mapsto I_{\mathfrak{p}}\}$$

and

$$\{\mathfrak{p} \mapsto I_{\mathfrak{p}}\} \mapsto \ker(K \to \bigoplus_{\mathfrak{p}} K/I_{\mathfrak{p}})$$

establish mutually inverse bijections between the set of f.g. A-submodules $I \subset K$ and the set of collections of f.g. $A_{\mathfrak{p}}$-submodules $I_{\mathfrak{p}} \subset K$ such that $I_{\mathfrak{p}} = A_{\mathfrak{p}}$ for almost all $\mathfrak{p}$.

Date: November 17, 2008.

(b) Prove that for I as above we have an isomorphism

$$K/I \simeq \bigoplus_{\mathfrak{p}} K/I_{\mathfrak{p}}.$$

6. Let A be a commutative ring, and let $S \subset A - 0$ a multiplicative set.

(a) Show that $Tor_i^A(M, N)$ has a natural structure of A-module, and

$$(Tor_i^A(M, N))_S \simeq Tor_i^A(M, N_S) \simeq Tor_i^A(M_S, N) \simeq Tor_i^{A_S}(M_S, N_S).$$

(b) Assume that A is Noetherian and that M is finitely generated. Show that $Ext^i_A(M, N)$ has a natural structure of A-module and

$$(Ext_i^A(M, N))_S \simeq Ext^i_A(M, N_S) \simeq Ext^i_{A_S}(M_S, N_S).$$

7. Let A be a Dedekind domain. Show that $Ext^i_A(M, N)$ and $Tor_i^A(M, N)$ vanish for $i \geq 2$

8.(a) Let $\phi : A \to B$ be a homomorphism of rings and $\dot{\mathfrak{p}} \in \mathrm{Spec}(A)$ a point. Construct an isomorphism of sets $\Phi^{-1}(\dot{\mathfrak{p}}) \simeq \mathrm{Spec}(B \underset{A}{\otimes} k_{\mathfrak{p}})$, where $k_{\mathfrak{p}}$ is the residue field at $\mathfrak{p}$, i.e., the residue field of the local rings $A_{\mathfrak{p}}$.

(b) Assume that $\mathfrak{p}$ above is a maximal ideal $\mathfrak{m}$. Show that $\Phi^{-1}(\dot{\mathfrak{m}}) \cap \mathrm{Specm}(B) = \mathrm{Specm}(B \underset{A}{\otimes} k_{\mathfrak{m}})$ as subsets of $\Phi^{-1}(\dot{\mathfrak{m}}) \simeq \mathrm{Spec}(B \underset{A}{\otimes} k_{\mathfrak{m}})$.