MATH 221, PROBLEM SET 7, DUE: MON., NOV. 10.

Problems marked with (*) are more difficult but mandatory.

Problems marked with (**) are optional.

1. Prove the essential uniqueness (up to homotopy) of injective resolutions.

2.(a) Let R be a ring and M a module. Show that M is injective if and only if $\mathrm{Ext}^1(R/I, M) = 0$ for any ideal $I \subset R$.

(b) Show that if R is a PID, a module M is injective if and only if it is divisible (i.e., $\forall m \in M, r \in R\, \exists m' \in M \,|\, r \cdot m' = m\}$).

(c) Show that for a PID, $\mathrm{Ext}^i(M, N)$ and $\mathrm{Tor}_i(M, N)$ are zero for $i \geq 2$.

(d) Show that a direct product of injective modules in injective.

(e) Prove that every $\mathbb{Z}$-module can be embedded into an injective one.

(f) Show that any R-module can be embedded into an injective one.

(g**) Show that if R is Noetherian, then a direct sum of injective modules is injective.

3. Composition of Exts.** Let M_1, M_2, M_3 be R-modules. Let $P_1^\bullet, P_2^\bullet, P_3^\bullet$ be their respective projective resolutions.

(a) Show that $\underline{\mathrm{Hom}}(P_i^\bullet, P_j^\bullet)$ is canonically quasi-isomorphic to $\underline{\mathrm{Hom}}(P_i^\bullet, M_j)$. Use this fact to define the composition maps

$$\mathrm{Ext}^n(M_1, M_2) \otimes \mathrm{Ext}^m(M_2, M_3) \to \mathrm{Ext}^{n+m}(M_1, M_3).$$

(b) Let a class in $\mathrm{Ext}^1(M_1, M_2)$ be represented by an extension

$$0 \to M_2 \to E \to M_1 \to 0,$$

and a class in $\mathrm{Ext}^1(M_2, M_3)$ be represented by an extension

$$0 \to M_3 \to F \to M_2 \to 0.$$

Show that their composition, i.e., the resulting class in $\mathrm{Ext}^2(M_1, M_3)$ is zero if and only if there exists a module G such that

$$0 \to F \to G \xrightarrow{\alpha} M_1 \to 0;\;\; 0 \to M_3 \xrightarrow{\beta} G \to E \to 0,$$

such that the composition $M_3 \hookrightarrow F \hookrightarrow G$ equals β, and $G \twoheadrightarrow E \twoheadrightarrow M_1$ equals α.

4*. Let V be a finite-dimensional vector space over a field k. Consider the symmetric algebra $A = \mathrm{Sym}(V)$. (If we choose a basis $x_1, ..., x_n$ of V, we can identify $\mathrm{Sym}(V) \simeq k[x_1, ..., x_n]$.) Consider k as an A-module by letting V act trivially on it.

(a) Show that $\mathrm{Tor}_i(k, k) \simeq \Lambda^i(V)$.

(b) Show that $\mathrm{Ext}^i(k, k) \simeq \Lambda^i(V^*)$, where V^* is the dual vector space.

(c) Show that $\mathrm{Ext}^i(k, A) = 0$ for $i \neq \dim(V)$, and $\mathrm{Ext}^{\dim(V)}(k, A) \simeq \Lambda^{\dim(V)}(V)$.

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Hint: Prove the existence of the following explicit resolution of k as an A-module (the Koszul complex):

$$... \to A \otimes \Lambda^2(V) \to A \otimes V \to A.$$

For the latter, consider using the following trick: let $K_1^\bullet$ and $K_2^\bullet$ be complexes of vector spaces. Consider the bi-complex $K_1^\bullet \underset{k}{\otimes} K_2^\bullet$. Assume that both $K_1^\bullet$ and $K_2^\bullet$ live in non-positive cohomological degrees and are acyclic off degree 0. Show that $\mathrm{Tot}(K_1^\bullet \underset{k}{\otimes} K_2^\bullet)$ is also acyclic off degree 0 and its H^0 is canonically isomorphic to $H^0(K_1^\bullet) \otimes H^0(K_2^\bullet)$.

5. Let R be a ring. A module M over R is said to be of finite length if there exists a finite sequence of submodules (a.k.a. filtration)

$$0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_n = M,$$

such that the successive quotients M_i/M_{i-1} are *irreducible*, i.e., contain no proper submodules.

(a) Let M be of finite length. Show that the length of any filtration on M is bounded by n.

(b*) Let N be an irreducible R-module. Let $M : N$ be the integer equal to the number of indices i such that $M_i/M_{i-1} \simeq N$. Show that this integer is independent of the choice of a filtration with irreducible quotients.

(c) Show that any irreducible module over R is of the form $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal left ideal.

From now we shall assume that R is commutative.

(d) Show that R is of finite length as a module over itself if and only if it is Artinian. Show that any f.g. module over an Artinian ring if of finite length.

(e) Let M be finitely generated over R. Show that M is of finite length $\Leftrightarrow$ $R/\operatorname{Ann}(M)$ is Artinian $\Leftrightarrow$ *supp*(M) is a finite collection of closed points.