MATH 221, PROBLEM SET 7, DUE: MON., NOV. 3.

1.(a) Let $\phi : M^\bullet \to N^\bullet$ be a map of complexes such that ϕ_i is isomorphism for $i = 1, -1$, surjective for $i = -2$ and injective for $i = 2$. Show that ϕ_0 is an isomorphism.

(b) Let $0 \to M_1^\bullet \to M_2^\bullet \to M_3^\bullet \to 0$ be a short exact sequence of complexes. Construct the connecting map $H^i(M_3^\bullet) \to H^{i+1}(M_1^\bullet)$ and show that

$$... \to H^{i-1}(M_3^\bullet) \to H^i(M_1^\bullet) \to H^i(M_2^\bullet) \to H^i(M_3^\bullet) \to H^{i+1}(M_1^\bullet) \to ...$$

is an acyclic complex.

2. Construct a map between complexes of abelian groups, which induces a zero map on cohomology, but which is not homotopic to zero. Hint: consider $M^\bullet$ with $M^0 = \mathbb{Z}$, $M^1 = \mathbb{Z}$ and zero everywhere else, with $d^0 : \mathbb{Z} \to \mathbb{Z}$ being multiplication by 2, and $N^\bullet$ with $N^0 = \mathbb{Z}$ and zero everywhere else.

3. A bi-complex (of R-modules) is a bi-graded R-module $M^{\bullet,\bullet}$ with maps $d_v^{i,j} : M^{i,j} \to M^{i+1,j}$ and $d_h^{i,j} : M^{i,j} \to M^{i,j+1}$ such that $d_v^{i+1,j} \circ d^{i,j} = 0$, $d_h^{i,j+1} \circ d_h^{i,j} = 0$ and $d_v^{i,j+1} \circ d_h^{i,j} = d_h^{i+1,j} \circ d_v^{i,j}$.

(a) To a bi-complex we associate a complex $\mathrm{Tot}(M^{\bullet,\bullet})$ whose n-th term is $\underset{i+j=n}{\oplus} M^{i,j}$, and the differential is given by $d^n(m^{i,j}) = d_v^{i,j}(m^{i,j}) + (-1)^j \cdot d_h^{i,j}(m^{i,j})$. Show that this is indeed a complex.

(b) Assume now that for each j, the complex $(M^{\bullet,j}, d_v^{\bullet,j})$ is acyclic. Assume also that $M^{i,j} = 0$ if $i < 0$ *and* $j > 0$. Show that $\mathrm{Tot}(M^{\bullet,\bullet})$ is acyclic.

(c) Let $f : M^\bullet \to N^\bullet$ be a map of complexes. Realize $\mathrm{Cone}(f)$ a particular case of the above construction. Prove the assertion mentioned in class about the connecting homomorphism for the short exact sequence $0 \to N^\bullet \to \mathrm{Cone}(f) \to M^\bullet[1] \to 0$.

4. Let M and N be R-modules. By an extension of M by N we shall mean a short exact sequence of R-modules

$$0 \to N \to E \to M \to 0.$$

By a map of extensions we shall mean a map of R-modules $\alpha : E \to E'$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & {\scriptstyle \mathrm{id}_N}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \mathrm{id}_M}\downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes. (Any such α is automatically an isomorphism.)

(a) Construct a bijection between the set of isomorphism classes of extensions as above and the set $\mathrm{Ext}^1(M, N)$.

Suggested strategy: pick a projective module P that surjects onto M and consider the corresponding long exact cohomology sequence of Exts into N.

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(b) Note that $\text{Ext}^1(M, N)$ is naturally an abelian group. Describe explicitly the group structure on the set of isomorphism classes of extensions.

5. Calculate:

(a) $\text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all i.

(b) $\text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ for all i.

(c) $\text{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all i.

6. Let k be a field. Calculate:

(a) $\text{Ext}^i_{k[t]}(k[t]/t^n, k[t]/t^m)$ for all i.

(b) $\text{Ext}^i_{k[t]}(k[t]/t^n, k[t])$ for all i.

(c) $\text{Tor}^{k[t]}_i(k[t]/t^n, k[t]/t^m)$ for all i.