MATH 221, PROBLEM SET 6, DUE: MON., OCT. 27.

1. Let M be a f.g. module over a commutative ring A and $\mathfrak{p} \subset A$ a prime ideal such that $M_{\mathfrak{p}} \neq 0$. Show that $M \underset{A}{\otimes} Frac(A/\mathfrak{p}) \neq 0$.

2. Let $A \to B$ be an injective map of k-algebras over a field with B being f.g. as an A-module.

(a) Show that $\text{Specm}(B) \to \text{Specm}(A)$ is surjective.

(b) Show that for an algebraically closed field k' containing k, the map

$$\text{Hom}_{k-alg}(B, k') \to \text{Hom}_{k-alg}(A, k')$$

is surjective. Hint: tensor with k' over k.

(c) Give an example that the conclusion of (b) would be false if k' wasn't algebraically closed.

3. Let A and B be two algebras over an algebraically closed field k with B reduced (no nilpotents). Show that the set of $\phi : \text{Hom}_{k-alg}(A, B)$ is in bijection with the set of $\Phi : \text{Specm}(B) \to \text{Specm}(A)$ such that for every $f \in \text{Funct}(\text{Specm}(A), k)$, which is algebraic, i.e., lies in the image of $A \to \text{Funct}(\text{Specm}(A), k)$, the pull-back $f \circ \Phi \in \text{Funct}(\text{Specm}(B), k)$ is also algebraic. Give a counter-example to this statement when B is not reduced.

4. Prove Yoneda's lemma: for two k-algebras A and B, there is a natural bijection between the set of $\phi \in \text{Hom}_{k-alg}(A, B)$ and the set of maps of spaces $\text{Spec}(A) \to \text{Spec}(B)$, i.e., systems of maps $\phi(R) : \text{Hom}_{k-alg}(B, R) \to \text{Hom}_{k-alg}(A, R)$ defined for every for each k-algebra R, such that for any homomorphism $\psi : R_1 \to R_2$, the diagram

$$\begin{array}{ccc} \text{Hom}_{k-alg}(B, R_1) & \xrightarrow{\phi(R_1)} & \text{Hom}_{k-alg}(A, R_1) \\ {\scriptstyle \psi\circ -}\downarrow & & {\scriptstyle \psi\circ -}\downarrow \\ \text{Hom}_{k-alg}(B, R_2) & \xrightarrow{\phi(R_2)} & \text{Hom}_{k-alg}(A, R_2). \end{array}$$

5. Let A be a k-algebra.

(a) Show that polynomial maps $\text{Spec}(A) \to \mathbb{A}^1 := \text{Spec}(k[x])$ are in bijection with elements of A.

(b) Show that polynomial maps $\text{Spec}(A) \to \mathbb{G}_m := \text{Spec}(k[x, x^{-1}])$ are in bijection with invertible elements of A.

(c) Show that polynomial maps $\text{Spec}(A) \to \text{Spec}(k[x, y, z]/x^2 + y^2 + z^2)$ are in bijection with the set $\{a_1, a_2, a_3 \in A \mid a_1^2 + a_2^2 + a_3^2 = 0\}$.

6. Give 3 (different) examples of a non-representable spaces.

7. Show that if functors F_1 and F_2 are spaces representable by k-algebras A_1 and A_2, respectively, then the space $F_1 \times F_2$ defined as

$$R \mapsto F_1(R) \times F_2(R),$$

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is represented by $A_1 \otimes A_2$. Show that $\mathbb{A}^{n_1} \times \mathbb{A}^{n_2} \simeq \mathbb{A}^{n_1+n_2}$ as spaces.

8. Consider the space Pol^n of monic polynomials of degree n. I.e., for a k-algebra R, $\mathrm{Pol}^n(R)$ is the set of all monic polynomials $p[t]$ with coefficients in R of degree n. Let Sol^n be the space that assigns to a k-algebra R the set of pairs $\{(p[t] \in \mathrm{Pol}^n(R), r \in R) \,|\, p(r) = 0\}$. Let σ denote the natural map of spaces $\mathrm{Sol}^n \to \mathrm{Pol}^n$ that sends $(p[t], r) \mapsto p[t]$.

(a) Show that Pol^n is representable; in fact $\mathrm{Pol}^n = \mathrm{Spec}(A)$, where $A = k[a_1, ..., a_n]$.

(b) Show that Sol^n is representable; in fact $\mathrm{Sol}^n = \mathrm{Spec}(B)$, where

$$B = k[a_0, ..., a_n, z]/z^n + a_1 \cdot z^{n-1} + ... + a_{n-1} \cdot z + a_n.$$

Write down explicitly the algebra homomorphism $A \to B$ that corresponds to σ.

(c) Prove that B is f.g. as an A-module.

9. Consider the map of spaces $\pi : \mathbb{A}^n \to \mathrm{Pol}^n$ that assigns to every k-algebra R the map

$$(r_1, ..., r_n) \in \mathbb{A}^n(R) \mapsto \prod_i (t - r_i) \in \mathrm{Pol}^n(R).$$

(a) Think of $\mathbb{A}^n$ as $\mathrm{Spec}(k[x_1, ..., x_n])$ and $\mathrm{Pol}^n = \mathrm{Spec}(k[a_1, ..., a_n])$. Write explicitly the homomorphism $k[a_1, ..., a_n] \to k[x_1, ..., x_n]$ that corresponds to π. Do you know the name of the resulting subalgebra of $k[x_1, ..., x_n]$?

(b) For every $i = 1, ..., n$ show that there exists a map of spaces $\xi_i : \mathbb{A}^n \to \mathrm{Sol}^n$ that sends $(r_1, ..., r_n) \mapsto (\prod_i (t - r_i), r_i)$, and that satisfies $\pi = \sigma \circ \xi_i$. Write the algebra homomorphism corresponding to ξ_i.

(c) Prove that $k[x_1, ..., x_n]$ is f.g. as a $k[a_1, ..., a_n]$-module.

(d) Deduce that there exists an ideal $I \subset k[a_1, ..., a_n]$ with the following property: for an algebraically closed field k' containing k, a point $p[t] \in \mathrm{Pol}^n(k')$ belongs to the subset

$$V(I)(k') \subset \mathrm{Spec}(k[a_1, ..., a_n])(k') = \mathrm{Pol}^n(k')$$

if and only if $p[t]$ *does not* have pairwise distinct roots.

10. Consider the product functor $\mathrm{Pol}^{n_1} \times \mathrm{Pol}^{n_2}$, which by Problem 7 is representable by $k[a^1_1, ..., a^1_{n_1}] \otimes k[a^2_1, ..., a^2_{n_2}] \simeq k[a^1_1, ..., a^1_{n_1}, a^2_1, ..., a^2_{n_2}]$. Show that there exists an ideal $J \subset k[a^1_1, ..., a^1_{n_1}, a^2_1, ..., a^2_{n_2}]$ such that for an algebraically closed field k' containing k a point $(p^1[t], p^2[t]) \in \mathrm{Pol}^{n_1}(k') \times \mathrm{Pol}^{n_2}(k')$ belongs to

$$V(J)(k') \subset \mathrm{Spec}(k[a^1_1, ..., a^1_{n_1}, a^2_1, ..., a^2_{n_2}])(k') \simeq \mathrm{Pol}^{n_1}(k') \times \mathrm{Pol}^{n_2}(k')$$

if and only if the sets of roots of $p^1[t]$ and $p^2[t]$ *have* a non-empty intersection.