MATH 221, PROBLEM SET 1, DUE: SEPT. 29.

Problems marked with (*) are more difficult, but still mandatory.

1*. Let M be an R-module. Show that the following conditions are equivalent:
(a) Any ascending sequence $M_1 \subset M_2 \subset ...$ of submodules such that $\underset{i}{\cup} M_i = M$, stabilizes.
(b) For any set of R-modules N_i, $i \in I$, for any map $M \to \underset{i \in I}{\oplus} N_i$ there exists a finite subset $I^0 \subset I$ such that the above map factors through

$$\underset{i \in I^0}{\oplus} N_i \subset \underset{i \in I}{\oplus} N_i,$$

i.e., the map

$$\underset{i \in I}{\oplus} \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \underset{i \in I}{\oplus} N_i)$$

is an isomorphism. Note that the above map always injective.

Hint: let $j \mapsto (M_j \subset M)$ be a chain of submodules of M, such that $\underset{j}{\cup} M_j = M$. Can you map M to $\underset{j}{\oplus} M/M_j$? When will such a map (if it exists) be a finite sum of maps to some particular M/M_j's?

2. Let $\phi : A \to B$ be a homomorphism of rings with A left-Noetherian.
(a) Assume that ϕ is surjective. Show that B is left-Noetherian.
(b) Assume that ϕ makes B a f.g. left A-module. Show that B is left-Noetherian.

3. Let k be a field and let $R = k[x_1, x_2, ...,]$ be the polynomial algebra on $\mathbb{N}$-many generators. By definition, it is isomorphic to $\underset{\substack{\longrightarrow \\ n}}{lim}\, k[x_1, ..., x_n]$. Consider the homomorphism of k-algebras $\phi : R \to k$ that sends all x_i to 0. Show that $\ker(\phi)$ is not finitely generated.

Hint: pass from $R = k[x_1, x_2, ...,]$ to a quotient algebra by killing x_i^2 for all i.

4*. Let A be a ring, and consider the ring of formal power series $A[[x]]$. Modify the proof of Hilbert's basis theorem to show that if A is Noetherian, then so is $A[[x]]$.

In the rest of the PS, all rings are commutative.

5. Let $A \to B \to C$ be homomorphisms of rings. Assume that B is finite over A and C if finite over B. Show that C is finite over A.

6. Let $A \to B$ be a homomorphism, with A Noetherian.
(a) Let $b \in B$ be integral over A. Show that for the corresponding homomorphism $\phi : A[x] \to B$, any element in $\mathrm{Im}(\phi)$ is integral over A.
(b) Let $b_1, ..., b_n \in B$ be integral over A. Show that for the corresponding homomorphism $\phi_n : A[x_1, ..., x_n] \to B$, any element in $\mathrm{Im}(\phi)$ is integral over A.
(c) Show that the set of elements in B that are integral over A forms a subring.

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7. Let $A \to B$ be a homomorphism. We say B is integral over A if every element $b \in B$ is integral over A.

Assume that A is Noetherian. Show that B is finite over A if and only if it is integral and finitely generated as an A-algebra.

8. Let $A \to B$ be injective, with A Noetherian and B integral over A. Assume that neither A nor B have zero divisors.

(a) Show that A is a field then so is B.

(b) Deduce that a field k is algebraically closed (i.e., every polynomial has a root) if and only for every finite field extension $k \subset k'$ (i.e., k' is f.d. as a k-vector space) we have $k = k'$.

(c) Show that if B is a field, then so is A.

9. Let k be an arbitrary field (not necessarily algebraically closed). Recall that the Weak Nullstellensatz says that every field extension $k \subset k'$, such that k' is f.g. as a k-algebra, is finite. Deduce from it the following statements:

(a) Every maximal ideal in $k[x_1, ..., x_n]$ is the kernel of a surjective k-algebra homomorphism $\phi : k[x_1, ..., x_n] \to k'$, where k' is a finite field extension of k. Show that for any two choices (k'_1, ϕ_1) and (k'_2, ϕ_2) there exists a unique k-algebra homomorphism $\psi : k'_1 \to k'_2$ such that $\psi \circ \phi_1 = \phi_2$.

(b) For every maximal ideal in $\mathfrak{m} \subset k[x_1, ..., x_n]$ there exists a field extension k' and a point $(c'_1, ..., c'_n) \in (k')^n$ such that $\mathfrak{m} = \mathfrak{m}_{(c'_1,...,c'_n)} \cap k[x_1, ..., x_n] \subset k'[x_1, ..., x_n]$. Give an example, how for the same field extension k' two different choices of $(c'_1, ..., c'_n)$ give rise to the same ideal $\mathfrak{m}$.