MATH 221, PROBLEM SET 11, DUE: MON., DEC. 1.

Let A be a finitely generated domain over a field k. Let

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_n \subset A$$

be a maximal chain of prime ideals. We are going to show that $n = \dim(A)$.

1. Deduce the fact that $n \leq \dim(A)$ from the standard properties of dimension.

2. Show that it is enough to prove the following:

Theorem. Let A be a domain and $\mathfrak{p} \subset A$ a sub-minimal prime ideal (i.e., $\mathfrak{p} \neq 0$, but there are no primes between 0 and $\mathfrak{p}$). Then $\dim(A/\mathfrak{p}) = \dim(A) - 1$.

To prove the Theorem, we'll follow the strategy suggested by Mitka.

3. Let $A \hookrightarrow B$ be an embedding of domains, such that the Theorem holds for A. Show that in this case the Theorem holds also holds for B.

4. Show that the Theorem holds for $A = k[x_1, ..., x_n]$. Hint: use the fact that any sub-minimal prime in $k[x_1, ..., x_n]$ is principal, and construct explicitly an embedding $k[x_1, ..., \hat{x_i}, ..., x_n] \to k[x_1, ..., x_n]/p$ for any polynomial p.

5. Combine Problems 3 and 4 to show that whenever Noether's normalization holds for A, then so does the Theorem.

6. Deduce the general case of the theorem by applying a finite field extension to k, and using the following:

Lemma. Let $A \hookrightarrow B$ be an embedding of domains, such that the Theorem holds for B. Then if the Theorem holds also holds for A.

NB: Even though A is a domain, the tensor product $A \underset{k}{\otimes} k'$ may have zero divisors, so you need to be careful.

7. Prove the lemma. Hint: use the fact that no prime $\mathfrak{p} \in \mathrm{Spec}(B)$ except 0 maps to the 0 prime in $\mathrm{Spec}(A)$.

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