MATH 221, PROBLEM SET 10, DUE: MON., NOV. 24.

1.(a) Let A be a Noetherian ring. Let L be a locally free A-module of rank 1. Construct an isomorphism

$$L \underset{A}{\otimes} \mathrm{Hom}_A(L, A) \simeq A.$$

(b) Assume that A is a domain with the field of fractions K. Recall that $\widetilde{\mathrm{Pic}}(A)$ denotes the set of isomorphism classes of (L, α), where L is a locally free A-module of rank 1, and α is an isomorphism of K-vector spaces: $\alpha : K \underset{A}{\otimes} L \simeq K$. Write down the group-law on $\widetilde{\mathrm{Pic}}(A)$ and check the existence of inverses.

2. Let A be a Dedekind domain. Prove that the map

$$\mathrm{Div}(A) \to \widetilde{\mathrm{Pic}}(A)$$

is a group homomorphism.

3. Let A be a Dedekind domain and let K be its fraction field. Let $K^* \to \mathrm{Div}(A)$ be the map $f \mapsto \Sigma\, n_{\mathfrak{p}} \cdot \dot{\mathfrak{p}}$, where $n_{\mathfrak{p}} = v_{\mathfrak{p}}(f)$. Show that the diagram

$$\begin{array}{ccc} K^* & \longrightarrow & \mathrm{Div}(A) \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow \\ K^* & \longrightarrow & \widetilde{\mathrm{Pic}}(A) \end{array}$$

commutes, where the bottom arrow sends $f \in K^*$ to the pair

$$(A, A \underset{K}{\otimes} K \simeq K \overset{f\cdot -}{\simeq} K) \in \widetilde{\mathrm{Pic}}(A).$$

4. Let A be a Dedekind domain. Show that the following conditions are equivalent:

- (i) A is a PID.
- (ii) The map $K^* \to \mathrm{Div}(A)$ is surjective.
- (iii) Every locally free module of rank 1 is isomorphic to A.

5. Let $\phi : M \to N$ be a map of filtered abelian groups, such that $M_i = N_i = 0$ for $i \ll 0$. Assume that $\mathrm{gr}(\phi)$ is injective (resp., surjective). Prove that ϕ itself is injective (resp., surjective). Give an example that the converse is not necessarily true.

6. Let A be a f.g. algebra over a field and let $nilp(A)$ be the ideal of nilpotent elements. Set $A_{red} = A/nilp(A)$. Show that $\dim(A) = \dim(A_{red})$.

Suggested strategy: the inequality $\dim(A) \geq \dim_A(A_{red}) = \dim(A_{red})$ should be evident. To prove the inequality in the opposite direction use the fact that $nilp(A)^n = 0$ for some n, and that each $nilp(A)^i/nilp(A)^{i+1}$ is a A_{red}-module, so $\dim_A(nilp(A)^i/nilp(A)^{i+1}) = \dim_{A_{red}}(nilp(A)^i/nilp(A)^{i+1}) \leq \dim(A_{red})$.

7. Let A be a f.g. algebra over a field.

Date: November 22, 2008.

(a) Let I_1 and I_2 be ideals in A such that $\mathrm{Spec}(A) = V(I_1) \cup V(I_2)$. Show that $\dim(A) = \max(\dim(A/I_1), \dim(A/I_2))$.

(b) Deduce that if $\mathrm{Spec}(A) = \underset{i}{\cup} \mathrm{Spec}(A_i)$ be the decomposition into irreducible components, then $\dim(A) = \max(\dim(A_i)$.

8. Let A and B be finitely generated k-algebras and M and N finitely generated A- and B-modules, respectively. Consider $M \otimes N$ as a module over $A \otimes B$. Show that $\dim_{A\otimes B}(M \otimes N) = \dim_A(M) + \dim_B(N)$.

9.(a) Let A be a f.g. algebra over a field and f a non-nilpotent element. Show that $\dim(A_f) \leq \dim(A)$.

Suggested strategy: Use the fact that $A_f = A[t]/tf - 1$, $\dim(A[t]) = \dim(A) - 1$, and that $tf - 1$ is not a zero divisor in $A[t]$

(b) Assume that f is not a zero divisor in A. Deduce that $\dim(A) = \dim(A_f)$.

10. Let $A \to B$ be an injective homomorphisms of domains f.g. over a field. Assume that $Frac(A) \to Frac(B)$ is a finite field extension. Prove that $\dim(A) = \dim(B)$.

Suggested strategy: Show that one can find an element $f \in A$ such that B_f is f.g. as an A_f module, and use Problem 8.

The following problems will be due on Mon., Dec. 1.

11. Let A be a Noetherian domain, and let M and N be f.g. locally free modules, and $f : M \hookrightarrow N$ an injective map. Show that the following conditions are equivalent:

- (i) The quotient N/M is flat (equivalently, locally free).
- (ii) For every prime ideal $\mathfrak{p}$ the map $M_\mathfrak{p} \underset{A_\mathfrak{p}}{\otimes} k_\mathfrak{p} \to N_\mathfrak{p} \underset{A_\mathfrak{p}}{\otimes} k_\mathfrak{p}$ is injective, where $k_\mathfrak{p}$ is the residue field at $\mathfrak{p}$, i.e., the quotient of $A_\mathfrak{p}$ by its maximal ideal.

Hint: recall the criterion for flatness in terms of constancy of dimensions of $- \underset{A_\mathfrak{p}}{\otimes} k_\mathfrak{p}$.

12. Let A be a Dedekind domain. Let L be a locally free A-module of rank 1 and M another locally free A-module. Let $f : L \to M$ be a non-zero (equivalently, injective–show it!) map. Show that there exists a unique effective divisor D, such that the map f extends to map

$$f' : L \underset{A}{\otimes} I_{-D} \to M,$$

such that the quotient $M/(L \underset{A}{\otimes} I_{-D})$ is flat.

NB: Above we view A as a submodule of I_{-D}, and hence L as a submodule of $L \underset{A}{\otimes} I_{-D}$.

Suggested strategy: show that the question is local, and hence one can assume that $L = A_\mathfrak{p}$ and $M = A_\mathfrak{p}^n$, so that the map f is given by an n-tuple $(a_1, .., a_n)$ of elements of $A_\mathfrak{p}$. Use Problem 11 to show that the contribution of $\mathfrak{p}$ to D is the integer $n_\mathfrak{p}$ equal to $min(v_\mathfrak{p}(a_i))$.

13. Let A be a Dedekind domain. Show that any finitely generated torsion-free module is (non-canonically) isomorphic to a direct of locally free modules of rank one. Hint: use Problem 12.

The next problems are fun and not terribly difficult, but optional.

14.** Let A be a commutative ring, and M an A-module. Let V_1, V_2 be two non-intersecting closed subsets of $\mathrm{Spec}(A)$. Consider the following two sets:

- (i) Submodules $M' \subset M$, such that $M'_{\mathfrak{p}} = M_{\mathfrak{p}}$ for $\mathfrak{p} \notin V_1 \cup V_2$.
- (ii) Pairs of submodules $M^1, M^2 \subset M$, such that $M^1_{\mathfrak{p}} = M_{\mathfrak{p}}$ for $\mathfrak{p} \notin V_1$, and $M^2_{\mathfrak{p}} = M_{\mathfrak{p}}$ for $\mathfrak{p} \notin V_2$.

Consider the map (ii) $\to$ (i) given by $M^1, M^2 \mapsto M^1 \cap M^2$. Show that it is well-defined and is an isomorphism.

Suggested strategy: to construct the inverse map use the idea from Problem 6 on PS 5 to show that M/M' splits canonically as a direct sum $N_1 \oplus N_2$, where $supp(N_i) \subset V_i$.

15.** Let A be a domain and M a torsion-free A-module. Let K denote the field of fractions of A, and let M_K denote the K-vector space $M \underset{A}{\otimes} K$.

Definition: A modification of M is a torsion-free A-module M' endowed with an isomorphism of K-vector spaces

$$M' \underset{A}{\otimes} K \simeq M_K.$$

Let V_1, V_2 be two non-intersecting closed subsets of $\mathrm{Spec}(A)$. Consider the following two sets:

- (i) Modifications M' of M, such that $M'_{\mathfrak{p}} = M_{\mathfrak{p}}$ as subsets of M_K for $\mathfrak{p} \notin V_1 \cup V_2$.
- (ii) Pairs of modifications (M'_1, M'_2) of M such that $M'_{1\mathfrak{p}} = M_{\mathfrak{p}}$ as subsets of M_K for $\mathfrak{p} \notin V_1$, and $M'_{2\mathfrak{p}} = M_{\mathfrak{p}}$ as subsets of M_K for $\mathfrak{p} \notin V_2$.

Construct a bijection between the sets (i) and (ii) with the following properties: for $M' \in$ (i) and the corresponding $M'_1, M'_2 \in$ (ii),

$$M'_{1\mathfrak{p}} = M'_{\mathfrak{p}} \text{ for } \mathfrak{p} \in V_1 \text{ and } M'_{2\mathfrak{p}} = M'_{\mathfrak{p}} \text{ for } \mathfrak{p} \in V_2.$$