MATH 221, PROBLEM SET 1, DUE: SEPT. 22.

1. Let R be a ring. In the exercise we will not yet assume that we know the existence of tensor products. Write complete proofs of the following statements:
(a) $R \underset{R}{\otimes} N \simeq N$ with the pairing $\phi_{univ}(r,n) = r \cdot n$.
(b) Let M_i be right R-modules for the index i running over a set I. Set $M = \underset{i\in I}{\oplus} M_i$. Assume that $M_i \underset{R}{\otimes} N$ exist for $i \in I$. Then $M \underset{R}{\otimes} N$ exists and the natural map

$$\underset{i\in I}{\oplus} (M_i \underset{R}{\otimes} N) \to M \underset{R}{\otimes} N$$

is an isomorphism.
(c) Let M_1 and M_2 be two right R-modules such that $M_i \underset{R}{\otimes} N$ exist. Let $\alpha : M_1 \to M_2$ be an R-module map and denote $M_3 := \text{coker}(\alpha)$. Then $M_3 \underset{R}{\otimes} N$ also exists and the sequence

$$M_1 \underset{R}{\otimes} N \to M_2 \underset{R}{\otimes} N \to M_3 \underset{R}{\otimes} N \to 0$$

is exact.
(d) Let $(M \underset{R}{\otimes} N)', \phi'_{univ}$ be another pair satisfying the same universal property as $M \underset{R}{\otimes} N, \phi_{univ}$. Show that there exists a unique isomorphism of abelian groups $\psi : M \underset{R}{\otimes} N \to (M \underset{R}{\otimes} N)'$ with $\psi \circ \phi_{univ} = \phi'_{univ}$.

2.(a) Let R be a ring and I a right ideal. Consider the right R-module $I\backslash R$. Show that for any left R-module N,

$$I\backslash R \underset{R}{\otimes} N \simeq I \cdot N \backslash N.$$

(b) Assume that R is commutative, and let $N = R/J$ for another ideal J. Assume that I and J are coprime, i.e., $I + J = R$. Deduce that $R/I \underset{R}{\otimes} R/J = 0$. In particular, $\mathbb{Z}/n\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/m\mathbb{Z} = 0$ if and only if $(m,n) = 1$.
(c) Consider the polynomial ring in one variable $\mathbb{Z}[t]$. Show that $\mathbb{Z}[t] \underset{\mathbb{Z}}{\otimes} R \simeq R[t]$ (as rings=$\mathbb{Z}$-algebras). For an R-module M describe explicitly the $R[t]$-module $R[t] \underset{R}{\otimes} M$.

3. Let R_1 and R_2 be two rings, let L be a right R_1-module, N a left R_2-module and M an (R_1, R_2)-bi-module. We've show that $L \underset{R_1}{\otimes} M$ is naturally a right R_2-module, and $M \underset{R_2}{\otimes} N$ is naturally a left R_1-module. Construct a natural isomorphism

$$(L \underset{R_1}{\otimes} M) \underset{R_2}{\otimes} N \simeq L \underset{R_1}{\otimes} (M \underset{R_2}{\otimes} N).$$

4. Do Problems 2.14, 2.15, 2.18, 2.19, 2.20 from Atiyah-MacDonald.

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