1

GROTHENDIECK R-POINTS

Recall that given a map of commutative rings $\phi : A \to B$, we have a map $\Phi : \operatorname{Spec} A \to \operatorname{Spec} B$ given by taking pre-images of prime ideals.

Proposition 1.1 *Let $\phi : A \to B$ be a map of commutative rings such that B is finitely generated as an A-module. Then Φ is a closed map.*

Proof: Let $V(J) \subset \operatorname{Spec} B$ be a closed set. We know from PS 4 that $\overline{\Phi(V(J))} = V(I)$, where $I = \phi^{-1}(J)$. We want to show that $\Phi(V(J))$ is closed, i.e. $\Phi(V(J)) = V(I)$. Equivalently we want the far left map

$$\begin{array}{ccc} \operatorname{Spec}(B/J) & \hookrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \Phi \\ \operatorname{Spec}(A/I) & \hookrightarrow & \operatorname{Spec} A \end{array}$$

to be surjective. Here we are identifying $V(I)$ with $\operatorname{Spec}(A/I)$ and $V(J)$ with $\operatorname{Spec}(B/J)$. Note that by definition, $A/I \hookrightarrow B/J$ is injective. Thus, we are reduced to showing that if $A \hookrightarrow B$ then $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$. Let $\dot{\mathfrak{p}} \in \operatorname{Spec} A$. Then consider the following commutative diagram:

$$\begin{array}{ccccc} \operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}} & \longrightarrow & \operatorname{Spec} B_{\mathfrak{p}} & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} & \longrightarrow & \operatorname{Spec} A_{\mathfrak{p}} & \longrightarrow & \operatorname{Spec} A \end{array}$$

By Nakayama's Lemma, $B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}} \neq 0$, so $\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}}$ is non-empty. Since $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is a field, $\operatorname{Spec} A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ has one point. Therefore, the far left map is surjective. This completes the proof because $\mathfrak{p} \in \operatorname{Spec} A$ has horizontal pre-image $\mathfrak{p}_{\mathfrak{p}}$, which has horizontal pre-image 0. By commutativity of the diagram, we obtain a pre-image in $\operatorname{Spec} B$. ■

In what follows, let k be algebraically closed, and let A be a finitely generated k-algebra. Recall that $\operatorname{Specm} A$ denotes the set of maximal ideals in A. Consider the natural k-algebra structure on $\operatorname{Funct}(\operatorname{Specm} A, k)$. We have a map

$$A \to \operatorname{Funct}(\operatorname{Specm} A, k)$$

which comes from the Weak Nullstellensatz as follows. Maximal ideals $\mathfrak{m} \subset A$ are in bijection with maps $\varphi_{\mathfrak{m}} : A \to k$ where $\ker(\varphi_{\mathfrak{m}}) = \mathfrak{m}$, so we define $a \longmapsto [\mathfrak{m} \longmapsto \varphi_{\mathfrak{m}}(a)]$. If A is reduced, then this map is injective because if $a \in A$ maps to the zero function, then $a \in \cap\, \mathfrak{m} \Rightarrow a$ is nilpotent $\Rightarrow a = 0$.

Definition 1.1 *A function $f \in$ Funct(Specm A, k) is called* **algebraic** *if it is in the image of A under the above map. (Alternate words for this are* **polynomial** *and* **regular***.)*

Let A and B be finitely generated k-algebras and $\phi : A \to B$ a homomorphism. This yields a map Φ :

Specm $B \to$ Specm A given by taking pre-images (see PS4 problem 7).

Definition 1.2 *A map* $\Phi : \mathit{Specm}\, B \to \mathit{Specm}\, A$ *is called* **algebraic** *if it comes from a homomorphism* ϕ *as above.*

To demonstrate how these definitions relate to one another we have the following proposition.

Proposition 1.2 *A map* $\Phi : \mathit{Specm}\, B \to \mathit{Specm}\, A$ *is algebraic if and only if for any algebraic function* $f \in \mathit{Funct}(\mathit{Specm}\, A, k)$*, the pullback* $f \circ \Phi \in \mathit{Funct}(\mathit{Specm}\, B, k)$ *is algebraic.*

Proof: [⇒] Suppose that Φ is algebraic. It suffices to check that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Funct}(\mathrm{Specm}\, A, k) & \xrightarrow{-\circ\Phi} & \mathrm{Funct}(\mathrm{Specm}\, B, k) \\ \uparrow & & \uparrow \\ A & \xrightarrow[\phi]{} & B \end{array}$$

where $\phi : A \to B$ is the map that gives rise to Φ.

[⇐] Suppose that for all algebraic functions $f \in \mathrm{Funct}(\mathrm{Specm}\, A, k)$, the pull-back $f \circ \Phi$ is algebraic. Then we have an induced map, obtained by chasing the diagram counter-clockwise:

$$\begin{array}{ccc} \mathrm{Funct}(\mathrm{Specm}\, A, k) & \xrightarrow{-\circ\Phi} & \mathrm{Funct}(\mathrm{Specm}\, B, k) \\ \uparrow & & \uparrow \\ A & \dashrightarrow_{\phi} & B \end{array}$$

From ϕ, we can construct the map $\Phi' : \mathrm{Specm}\, B \to \mathrm{Specm}\, A$ given by $\Phi'(\mathfrak{m}) = \phi^{-1}(\mathfrak{m})$. I claim that $\Phi = \Phi'$. If not, then for some $\mathfrak{m} \in \mathrm{Specm}\, B$ we have $\Phi(\mathfrak{m}) \neq \Phi'(\mathfrak{m})$. By definition, for all algebraic functions $f \in \mathrm{Funct}(\mathrm{Specm}\, A, k)$, $f \circ \Phi = f \circ \Phi'$ so to arrive at a contradiction we show the following lemma:
Given any two distinct points in $\mathrm{Specm}\, A = V(I) \subset k^n$, there exists some algebraic f that separates them. This is trivial when we realize that any polynomial function is algebraic, and such polynomials separate points. ■

Definition 1.3 *A* **space** *(or* **functor***)* X *is an assignment of every ring* R *to a set* $X(R)$ *such that for any homomorphism* $\alpha : R \to R'$*, there exists a map of sets* $X(\alpha) : X(R) \to X(R')$*. Furthermore,*
(i) If $\alpha = id$*, then* $X(\alpha) = id$*.*
(ii) If $\alpha : R \to R'$ *and* $\beta : R' \to R''$ *then* $X(\beta \circ \alpha) = X(\beta) \circ X(\alpha)$*.*

Example: Any ring A gives rise to a space Spec A defined as follows:

$$(\mathrm{Spec}\, A)(R) := \mathrm{Hom}_{k-alg}(A, R)$$

Definition 1.4 *Let* X *and* Y *be spaces. A map of spaces (or* **natural transformation***)* $\Phi : X \to Y$ *is an* *assignment for any R, $\Phi_R : X(R) \to Y(R)$ such for any homomorphism $\alpha : R \to R'$ the following diagram commutes:*

$$\begin{array}{ccc} X(R) & \xrightarrow{\Phi_R} & Y(R) \\ {\scriptstyle X(\alpha)}\downarrow & & \downarrow{\scriptstyle Y(\alpha)} \\ X(R') & \xrightarrow[\Phi_{R'}]{} & Y(R') \end{array}$$

Example: Let $\varphi : A \to B$ be a ring homomorphism. This yields a map of spaces from $\operatorname{Spec} B \to \operatorname{Spec} A$ by pre-composition. It satisfies the axions since the following diagram commutes.

$$\begin{array}{ccc} \operatorname{Hom}(B,R) & \xrightarrow{-\circ\varphi} & \operatorname{Hom}(A,R) \\ {\scriptstyle \alpha\circ-}\downarrow & & \downarrow{\scriptstyle \alpha\circ-} \\ \operatorname{Hom}(B,R') & \xrightarrow[-\circ\varphi]{} & \operatorname{Hom}(A,R') \end{array}$$

It turns out that such maps of spaces are the *only* ones from $\operatorname{Spec} B \to \operatorname{Spec} A$. More precisely,

Proposition 1.3 *(Yoneda's Lemma) For two k-algebras A and B, there is a natural bijection between maps of k-algebras from $A \to B$ and maps of spaces $\operatorname{Spec} B \to \operatorname{Spec} A$, given by pre-composition.*

Proof: This was problem 4 on PS6, so we omit the proof here. ■

Proposition 1.4 *Let X be a space. Then we have $\operatorname{Hom}_{spaces}(\operatorname{Spec} R, X) = X(R)$.*

Proof: Let Φ be a map of spaces, so we have an assignment $\Phi_R : (\operatorname{Spec} R)(R) \to X(R)$. Since $(\operatorname{Spec} R)(R) = \operatorname{Hom}(R,R)$ we can take $\Phi_R(\mathrm{id}) \in X(R)$. Conversely, suppose we are given an element $x \in X(R)$. We want for each R' a map from $\operatorname{Hom}(R,R') \to X(R')$. We define such a map as follows. If $\varphi : R \to R'$ then

$$\varphi \longmapsto X(\varphi)(x) \in X(R')$$

It is trivial to check that this is indeed a map of spaces, and that the two constructions are inverses of each other. ■

Proposition 1.5 *(Cayley-Hamilton Theorem)*

Proof: ?

2

HOMOLOGICAL ALGEBRA

Let R be a commutative ring.

Definition 2.1 *A* **complex** $M^\bullet$ *is a sequence of* R*-modules* $\{M^i\}$ *with maps* $d^i : M^i \to M^{i+1}$

$$\dots \xrightarrow{d^{-3}} M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} \dots$$

such that $d^i \circ d^{i-1} = 0$*, i.e.* $\operatorname{Im} d^{i-1} \subset \ker d^i$*.*

Definition 2.2 *The* i**-th cohomology** *is the quotient module*

$$H^i(M^\bullet) := \ker d_i / \operatorname{Im} d_{i-1}$$

A complex is called **acyclic** *if it is exact at each index, i.e.* $H^i(M^\bullet) = 0$ *for all* i*.*

Definition 2.3 *We define* $\operatorname{Hom}_R(M^\bullet, N^\bullet)$ *to be the set of maps of complexes from* $M^\bullet \to N^\bullet$*. Such a map is an element* $\{\varphi^i\} \in \prod_i \operatorname{Hom}_R(M^i, N^i)$ *such that for all* i*, the following diagram is commutative.*

$$\begin{array}{ccc} M^i & \xrightarrow{d_M^i} & M^{i+1} \\ \varphi^i \downarrow & & \downarrow \varphi^{i+1} \\ N^i & \xrightarrow{d_N^i} & N^{i+1} \end{array}$$

Proposition-Construction 2.1 *A map of complexes* $\varphi : M^\bullet \to N^\bullet$ *induces a map of cohomologies* $H^i(M^\bullet) \to H^i(N^\bullet)$ *for all* i*.*

Proof: We define the map by restricting φ^i to $\ker d_M^i$. Since each square is commutative, φ^i maps $\ker d_M^i \to \ker d_N^i$ and $\operatorname{Im} d_M^{i-1} \to \operatorname{Im} d_N^{i-1}$. Thus, the induced map is well-defined on $H^i(M^\bullet)$. ■

Definition 2.4 *A map of complexes is a* **quasi-isomorphism** *if it induces an isomorphism of cohomologies.*

Definition 2.5 *Let* φ *and* ψ *be maps of complexes from* $M^\bullet \to N^\bullet$*. A homotopy from* φ *to* ψ *is an element* $\{h^i\} \in \prod \operatorname{Hom}_R(M^i, Ni-1)$ *such that*

$$\varphi^i - \psi^i = h^{i+1} \circ d_M^i + d_N^{i-1} \circ h^i$$

Lemma 2.1 *If* φ *and* ψ *are homotopic, then their induced maps of cohomologies coincide.*

Proof: Let $m \in \ker(d_M^i)$. Then

$$\varphi^i(m) - \psi^i(m) = h^{i+1} \circ d_M^i(m) + d_N^{i-1} \circ h^i(m) = d_N^{i-1} \circ h^i(m) \in \operatorname{Im}(d_N^{i-1})$$

which is zero in the cohomology $H^i(N^\bullet)$. ■

Proposition 2.1 *If we have a short exact sequence of complexes $0 \to M_1^\bullet \to M_2^\bullet \to M_3^\bullet \to 0$, this induces a long exact sequence of cohomologies:*

$$\cdots \to H^{i-1}(M_3^\bullet) \to H^i(M_1^\bullet) \to H^i(M_2^\bullet) \to H^i(M_3^\bullet) \to H^{i+1}(M_1) \to \dots$$

Proof: This was problem 1(b) on PS7, so we omit the proof here. ■

Definition 2.6 *A map is* **null-homotopic** *if it is homotopic to the zero map.*

Definition 2.7 *A map $\varphi : M^\bullet \to N^\bullet$ is a homotopy equivalence if there exists some $\psi : N^\bullet \to M^\bullet$ such that*

$$id_{N^\bullet} \simeq \varphi \circ \psi$$
$$id_{M^\bullet} \simeq \psi \circ \varphi$$

where $\simeq$ denotes homotopy.

Lemma 2.2 *A homotopy equivalence is a quasi-isomorphism.*

Proof: This follows directly from the definition.

Example: Not every quasi-isomorphism is a homotopy equivalence. Consider the complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \to 0 \to \dots$$

so $H^0 = \mathbb{Z}/2\mathbb{Z}$ and all cohomologies are 0. We have a quasi-isomorphism from the above complex to the complex

$$\cdots \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to 0 \to \dots$$

but no inverse can be defined (no map from $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$).

Definition 2.8 *If $M^\bullet$ is a complex then for any integer k, we define a new complex $M^\bullet[k]$ by shifting indices, i.e. $(M^\bullet[k])^i := M^{i+k}$.*

Definition 2.9 *If $f : M^\bullet \to N^\bullet$ is a map of complexes, we define a complex $Cone(f) := \{N^i \oplus M^{i+1}\}$ with differential*

$$d(n^i, m^{i+1}) := (d_N^i(n_i) + (-1)^i \cdot f(m^{i+1}, d_M^{i+1}(m^{i+1}))$$

Remark: This is a special case of the total complex construction to be seen later.

Proposition 2.2 *A map $f : M^\bullet \to N^\bullet$ is a quasi-isomorphism if and only if $Cone(f)$ is acyclic.*

Proof: Note that by definition we have a short exact sequence of complexes

$$0 \to N^\bullet \to \mathrm{Cone}(f) \to M^\bullet[1] \to 0$$

so by Proposition 2.1, we have a long exact sequence

$$\cdots \to H^{i-1}(\mathrm{Cone}(f)) \to H^i(M) \to H^i(N) \to H^i(\mathrm{Cone}(f)) \to \dots$$

so by exactness, we see that $H^i(M) \simeq H^i(N)$ if and only if $H^{i-1}(\mathrm{Cone}(f)) = 0$ and $H^i(\mathrm{Cone}(f)) = 0$. Since this is the case for all i, the claim follows. ■

Definition 2.10 *Let $M^\bullet$ and $N^\bullet$ be complexes. We define the* **inner Hom** *complex $(\underline{\mathrm{Hom}}(M^\bullet, N^\bullet))^\bullet$ as:*

$$(\underline{\mathrm{Hom}}(M^\bullet, N^\bullet))^i := \prod_n \mathrm{Hom}(M^n, N^{n+i})$$

with differential $(d\varphi)(m^n) := d_N^{n+i} \circ \varphi^n(m^n) + (-1)^{i+1} \cdot \varphi^{n+1} \circ d_M^n(m^n)$.

Remark: From the definition of the inner Hom complex, we have that $\ker(d^0) = \mathrm{Hom}(M^\bullet, N^\bullet)$, the usual maps of complexes. Similarly, $\mathrm{Im}(d^{-1})$ are those maps that are null-homotopic. Thus, the cohomology $H^0((\underline{\mathrm{Hom}}(M^\bullet, N^\bullet))^\bullet)$ can be thought of us as maps of complexes, up to homotopy. This is denoted

$$hHom(M^\bullet, N^\bullet) := H^0((\underline{\mathrm{Hom}}(M^\bullet, N^\bullet))^\bullet)$$

Lemma 2.3 *Let $M^\bullet$ be an acyclic complex. Let $P^\bullet$ be a complex of projective modules that is bounded from above, i.e. $P^i = 0$ for $i > 0$. Then the complex $\underline{\mathrm{Hom}}(P^\bullet, M^\bullet)$ is acyclic.*

Proof: This can be shown by a simple diagram chase. ■

Corollary 2.1 *Let $M_1^\bullet \to M_2^\bullet$ be a quasi-isomorphism, and let $P^\bullet$ be as in the lemma above. Then $\underline{\mathrm{Hom}}(P^\bullet, M_1^\bullet) \to \underline{\mathrm{Hom}}(P^\bullet, M_2^\bullet)$ is a quasi-isomorphism (let us call this map ϕ).*

Proof: Consider the acyclic complex $\mathrm{Cone}(f)$. By the lemma, $\underline{\mathrm{Hom}}(P^\bullet, \mathrm{Cone}(f))$ is acyclic. We want to show that $\mathrm{Cone}(\phi)$ is acyclic. I claim that the two complexes are isomorphic:

$$\underline{\mathrm{Hom}}(P^\bullet, M_2^\bullet)^i \oplus \underline{\mathrm{Hom}}(P^\bullet, M_1^\bullet)^{i+1} \simeq \prod_n \mathrm{Hom}(P^n, M_2^{n+i} \oplus M_1^{n+i+1})$$

which is true by the universal property of the direct sum. It can be checked that the differentials are the same. ■

Proposition 2.3 *Let M be an R-module.*
(i) There exists a complex of projective modules called the **projective resolution** *of M:*

$$\begin{array}{ccccccc} \cdots \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow 0 \\ & & & & & \downarrow & \\ & & & & & M & \end{array}$$

such that $H^0(P^\bullet) = M$ and $H^i(P^\bullet) = 0$ for $i \neq 0$.

(ii) If we have two such resolutions $P_1^\bullet$ and $P_2^\bullet$, then there exist unique (up to homotopy) maps of complexes α and β such that $\alpha \circ \beta = id$, $\beta \circ \alpha = id$, and the triangle below commutes (up to homotopy):

$$\begin{array}{ccc} P_1^\bullet & \overset{\alpha}{\underset{\beta}{\rightleftarrows}} & P_2^\bullet \\ & \searrow \qquad \swarrow & \\ & M & \end{array}$$

Proof: (i) Since free R-modules are projective, we can just take a free resolution, i.e. let P^0 be a free module surjecting onto M with kernel K^0, P^1 a free module surjecting onto K^0 and so on.
(ii) For this, we consider M as a complex:

$$\cdots \to 0 \to M \to 0 \to \dots$$

Since Cone(ϕ) is acyclic, we have that ϕ is a quasi-isomorphism. In particular,

$$H^0(\underline{\mathrm{Hom}}(P_1^\bullet, P_2^\bullet)) \simeq H^0(\underline{\mathrm{Hom}}(P_1^\bullet, M))$$

The resolution gives us a map of complexes $P_1^\bullet \to M^\bullet$, i.e. an element of the right-hand side. The corresponding element of the left-hand side is α. An analagous construction yields β, and they are inverses by uniqueness of the construction. ■

Definition 2.11 *Let M and N be R-modules. Let $P^\bullet$ be a projective resolution for M. We define*

$$Tor_i^R(M, N) := H^{-i}(P^\bullet \underset{A}{\otimes} N)$$

Remark: $\mathrm{Tor}_0^R(M, N) = \mathrm{coker}(P^{-1} \otimes N \to P^0 \otimes N) \simeq \mathrm{coker}(P^{-1} \to P^0) \otimes N \simeq M \otimes N$, so Tor can be seen as a generalization of the tensor product. Also, as a direct consequence of this definition, we see that M is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N.

Proposition 2.4 *Let M and N be R-modules. Then*
(i) $Tor_i^R(M, N)$ is independent of the choice of projective resolution.
(ii) $Tor_i^R(M, N) \simeq Tor_i^R(N, M)$, despite the asymmetry in the definition.

Proof: (i) If we take two different projective resolutions $P_1^\bullet$ and $P_2^\bullet$, then by proposition 2.3(ii), we have α and β which induce isomorphisms on the cohomologies:

$$P_1^\bullet \otimes N \overset{\alpha}{\underset{\beta}{\rightleftarrows}} P_2^\bullet \otimes N$$

(ii) Let $P^\bullet$ be a projective resolution for M and $Q^\bullet$ a projective resolution for N. Consider the bi-complex $P^\bullet \otimes Q^\bullet$ and define $\mathrm{Tot}(P^\bullet \otimes Q^\bullet)$ complex with n-th term

$$\bigoplus_{i+j=n} P^i \otimes Q^j$$

and differential $d^n(m^{i,j}) := d_v^{i,j}(m^{i,j}) + (-1)^i \cdot d_h^{i,j}(m^{i,j})$. From problem 3 of PS7, we see that this is indeed a complex, and there is a canonical quasi-isomorphism from $\mathrm{Tot}(P^\bullet \otimes Q^\bullet)$ to $P^\bullet \otimes N$ and to $M \otimes Q^\bullet$. ■

Corollary 2.2 *If $0 \to M_1 \to M_2 \to M_3$ is a short exact sequence of R-modules, then for any R-module N, there exists a long exact sequence:*

$$\cdots \to Tor_i^R(M_1, N) \to Tor_i^R(M_2, N) \to Tor_i^R(M_3, N) \to Tor_{i-1}^R(M_1, N) \to \dots$$

Proof: Take a projective resolution $Q^\bullet$ for N. Since projective implies flat, we have a short exact sequence of complexes:

$$0 \to M_1 \otimes Q^\bullet \to M_2 \otimes Q^\bullet \to M_3 \otimes Q^\bullet \to 0$$

the result follows from applying the long exact cohomology sequence construction. ■

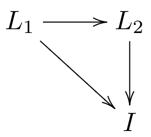
Definition 2.12 *Let M and N be R-modules. Let $P^\bullet$ be a projective resolution for M. Consider the complex:*

$$0 \to \mathrm{Hom}(P^0, N) \to \mathrm{Hom}(P^{-1}, N) \to \mathrm{Hom}(P^{-2}, N) \to \dots$$

We define $Ext_R^i(M, N)$ to be the i-th cohomology of this complex.

Remark: From the definition, $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}(M, N)$ and M is projective if and only if $\mathrm{Ext}_R^1(M, N) = 0$ for all R-modules N.

Definition 2.13 *A module I is* **injective** *if given an injection $L_1 \hookrightarrow L_2$ and a map from $L_1 \to I$, there exists a map from $L_2 \to I$ such that the following triangle commutes:*



Proposition 2.5 *Any module can be imbedded into an injective module.*

Proof: This was problem 2 on PS7, so we omit the proof here. ■

Remark: This allows us to take injective resolutions $0 \to M \to I^0 \to I^1 \to \dots$ that are unique up to homotopy (also shown on PS7).

Proposition 2.6 *Let M and N be R-modules. Let $I^\bullet$ be a projective resolution for M. Consider the complex:*

$$0 \to \mathrm{Hom}(M, I^0) \to \mathrm{Hom}(M, I^1) \to \mathrm{Hom}(M, I^2) \to \dots$$

We can define $Ext_R^i(M, N)$ as the i-th cohomology of this complex as well.

Proof: Use the same argument as for Tor symmetry (with the Tot complex).

Proposition 2.7 *(i) If $0 \to N_1 \to N_2 \to N_3$ is a short exact sequence of R-modules, then for any R-module M, there exists a long exact sequence:*

$$\dots \to Ext^i_R(M, N_1) \to Ext^i_R(M, N_2) \to Ext^i_R(M, N_3) \to Ext^{i+1}_R(M, N_1) \to \dots$$

(ii) If $0 \to M_1 \to M_2 \to M_3$ is a short exact sequence of R-modules, then for any R-module N, there exists a long exact sequence:

$$\dots \to Ext^i_R(M_3, N) \to Ext^i_R(M_2, N) \to Ext^i_R(M_1, N) \to Ext^{i+1}_R(M_3, N) \to \dots$$

Proof: Use the same argument as for the Tor long exact sequence.