MATH 221 NOTES

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Date: January 3, 2009.

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1.

LOCALIZATIONS

Definition 1.1. *Let A be a commutative ring. $S \subseteq A$ is a multiplicative subset if it does not contain 0, it is multiplicatively closed [and it does not contain 0]*

[1]

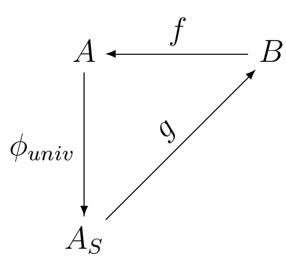
Examples:
1. $f \in A$ non-nilpotent, $S = \{1, f, f^2, \dots\}$
2. A is a domain, $S = A - \{0\}$

Remark 1.1. *$\mathfrak{p} \subseteq A$ is prime if $A - \mathfrak{p}$ is a multiplicative subset.*

Definition 1.2. *Given a commutative ring A and a multiplicative subset $S \subseteq A$, we define the ring A_S by defining an equivalence relation on $A \times S$ by $(a_1, s_1) \sim (a_2, s_2)$ if there exists $s \in S$ such that $a_1 s_2 s = a_2 s_1 s$. We define multiplation component-wise, denote the equivalence class of (a, s) by $\frac{a}{s}$, and define addition by $\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}$. We define a ring homomorphism $\phi_{univ} : A \longrightarrow A_S$ by $\phi_{univ}(a) = \frac{a}{1}$*

Proposition-Construction 1.1. *There exists a bijection between ring homomorphisms $f : A \longrightarrow B$ [B is commutative] such that the image of all elements of S are invertible, and ring homomorphisms $g : A_S \longrightarrow B$ such that $g \circ \phi_{univ} = f$.*

Proof. Given a map f, we define a map g by $g(\frac{a}{s}) = f(a)[f(s)]^{-1}$. It is easily checked that it satisfies ring homomorphism axioms. Given a map g, we define f by $f(a) = g(\frac{a}{1})$. We note that $f(s) = g(\frac{s}{1})$, so that $f(s)$ has inverse $g(\frac{1}{s})$ by homorphism axioms. It is similarly verifiable that this map satisfies axioms and it clearly commutes. ■



Remark 1.2. *This is the universal property of A_S*

Proposition 1.1. *For a non-nilpotent element f and $S = \{1, f, f^2, \dots\}$, $A_f := A_S = A[t]/(1 - tf)$.*

Proof. We define a map $g : A[t]/(1 - tf) \longrightarrow A_f$ by $f(\overline{at^i}) = \frac{a}{f^i}$. This map vanishes on $(1 - tf)A[t]$ since $g(\overline{1 - tf}) = 1 - \frac{f}{f} = 0$. It satisfies ring homomorphisms because

$$g(\overline{at^i} + \overline{bt^j}) = \frac{a}{f^i} + \frac{b}{f^j} = \frac{af^j + bt^i}{f^{i+j}} = g(\overline{(af^j + bf^j)t^{i+j}}) = g(\overline{at^i + bt^j})$$

[1] I have put in brackets that things that are not in my notes but that I think that Dennis left out.

since $\overline{tf} = \overline{(1-tf)+tf} = 1$. The check that it satisfies ring multiplication is trivial. This map is surjective because $\frac{a}{f^i}$ is the image of $\overline{at^i}$. It is injective because for a polynomial $p(t)$ with coefficients a_i,

$$g(p(t)) = g(\overline{\sum_{i=0}^{n} a_i t^i}) = \sum_{i=0}^{n} \frac{a_i}{f^i} = \frac{\sum_{i=0}^{n} a_i f^{n-i}}{f^n} = 0$$

if $\frac{\sum_{i=0}^{n} a_i f^{n-i}}{1} = 0$, since f is non-nilpotent. But as $\overline{tf} = 1$,

$$\frac{\sum_{i=0}^{n} a_i f^{n-i}}{1} = g(\overline{\sum_{i=0}^{n} a_i f^{n-i}(tf)^i}) = g(\overline{\sum_{i=0}^{n} a_i f^n t^i}) = g(\overline{f^n} p(t))$$

so as $\overline{f^n} \neq 0$ for all n, this implies that $p(t) = 0$. ■

Lemma 1.1. *Let A be a finitely generated algebra over k, and $f \in A$ a non-nilpotent element. Then there is a function $\phi : A \to k'$ corresponding to some maximal ideal $\mathfrak{m}$ such that $\phi(f) \neq 0$.*

Proof. We consider the function $A \longrightarrow A_f$, and take any maximal ideal $\mathfrak{p}$ in A_f. Then the composition $A \longrightarrow A_f \longrightarrow A_f/\mathfrak{p} = k'$ has nonzero image of f, since $f \mapsto 0$ implies that the image of f, $\frac{f}{1}$ maps to 0, which means that $\frac{1}{1}$ maps to zero, so that this is a zero map. Thus taking the preimage of this maximal ideal, we have the desired map by this composition. ■

Example: $(A) := A_{A-\{0\}}$, the field of fractions of A.

Proposition 1.2. $(k[x_1, x_2]/x_1x_2)_{x_1} \simeq k[x_1]_{x_1}$

Proof. We define a map $f : k[x_1]_{x_1} \longrightarrow (k[x_1, x_2]/x_1x_2)_{x_1}$ by $\frac{p}{q} \mapsto \frac{p}{q}$. It is clearly injective, and is surjective because $\frac{x_2}{1} = \frac{x_2x_1}{x_1} = 0$, so that all polynomials with x_2 terms are equivalent to polynomials without those terms. ■

Definition 1.3. *Given an A-module M and a multiplicative subset S, we construct the module $M_S := A_S \underset{A}{\otimes} M$. A more explicit description of M_S can be obtained by identifying $\frac{a}{s} \otimes m$ with the fraction $\frac{am}{s}$. Thus we denote M_S by the set of fractions $\frac{m}{s}$ for $m \in M$ and $s \in S$ with addition and action of A_S inherited in this fashion.*

Lemma 1.2. *$M_S = 0$ if and only if $\forall m \in M$, $\exists s \in S$ such that $sm = 0$*

Proof. trivial. ■

Example: $A = k[t]$, $M = k[t]/t$. Then $M_t = 0$

Lemma 1.3. *Let f be a non-nilpotent element. Then $M_f \sim \varinjlim M$, with the maps $M \longrightarrow M$ given by multiplication by f.*

Proof. See 8 on pset 3[2] ■

Corollary 1.1. *$A = k[t]$, $A_t = \varinjlim \frac{1}{t^i} k[t] = k[t, t^{-1}]$*

Proposition 1.3. *A_S is flat. (i.e. localization is an exact functor)*

[2] See solution at end

Proof. Suppose that $T : M_1 \hookrightarrow M_2$, and consider the induced map $A_S \otimes M_1 \longrightarrow A_S \otimes M_2$, i.e. the map $L : M_{1_S} \longrightarrow M_{2_S}$. Then $L(\frac{m}{s}) = \frac{T(m)}{s}$ is 0 if $\exists s'$ such that $s' \cdot T(m) = 0$, by definition. But $s' \cdot T(m) = T(s'm) = 0 \implies s'm = 0 \implies \frac{m}{s} = 0$. ■

Proposition 1.4. *A Noetherian implies that A_S is Noetherian.*

Proof. Take an ideal $J \subseteq A_S$. It's preimage under the map $A \longrightarrow A_S$ is an ideal I, so is finitely generated by f_i. We claim that $\frac{f_i}{1}$ generate J. This is true because for any element $b \in J$, $b = \frac{a}{s}$, then by idealness $\frac{s}{1} \cdot \frac{a}{s} = \frac{a}{1} \in J$, and we have $a_i \in A$ such that $a = \sum a_i f_i$. Then $\frac{a}{1} = \sum a_i f_i 1$, so that $b = \frac{a}{s} = \sum \frac{a_i}{s} \cdot \frac{f_i}{1}$, as desired. ■

2.

ZARISKI TOPOLOGY

Definition 2.1. *Let A be a commutative ring. We define* $\mathrm{Spec}(A)$ *to be the set of prime ideals of A, with elements $\dot{\mathfrak{p}}$ corresponding to primes $\mathfrak{p}$.*

We define the Zariski topology by defining closed sets:
$V \subseteq \mathrm{Spec}(A)$ is closed if $\exists I \subseteq A$ such that $V = V(I) := \{\dot{\mathfrak{p}} | \, \mathfrak{p} \supseteq I\}$.

We check that it satisfies topology axioms:
1. $\emptyset = V(A)$ is closed.
2. $\mathrm{Spec}(A) = V(0)$ is closed.
3. For two closed sets $V_1 = V(I_1)$ and $V_2 = V(I_2)$, the set $V_1 \cup V_2 = V(I_1 \cap I_2)$ is closed. (See following lemma)
4. For closed sets $V(I_i)$, $\bigcap_i V(I_i) = V(\sum_i I_i)$ is closed. (See following lemma)

Lemma 2.1. $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$

Proof. $\subseteq$ Let $\dot{\mathfrak{p}} \in (V(I_1) \cup V(I_2))$. Then $\mathfrak{p} \supset I_1$ or I_2. Thus $\mathfrak{p} \supset I_1 \cap I_2$.
$\supseteq$ Suppose $\mathfrak{p} \supset I_1 \cap I_2$, and suppose $\mathfrak{p} \not\supset I_i$ for $i = 1, 2$. Then take $a_1 \in I_1$, $a_2 \in I_2$ such that $a_i \notin I_i$. Then $a_1 a_2 \in (I_1 \cap I_2) \implies a_1 a_2 \in \mathfrak{p}$, which contradicts the primeness of $\mathfrak{p}$. ∎

Lemma 2.2. $\bigcap_i V(I_i) = V(\sum_i I_i)$

Proof. $\dot{\mathfrak{p}} \in \bigcap_i V(I_i) \Leftrightarrow \mathfrak{p} \in V(I_i) \ \forall i \Leftrightarrow \mathfrak{p} \supset I_i \ \forall i \Leftrightarrow \mathfrak{p} \supset \sum I_i$ ∎

Remark 2.1. *If $I_1 \subseteq I_2$, then $V(I_1) \supseteq V(I_2)$*

Proposition 2.1. $V(I) = V(\mathrm{rad}(I))$*;* $\mathrm{rad} I := \{a | \, \exists n \in \mathbb{Z} s.t. a^n \in I\}$

Proof. If $\mathfrak{p}$ contains $\mathrm{rad}(I)$, then as $I \subset \mathrm{rad}(I)$, $\mathfrak{p}$ contains I. If $\mathfrak{p}$ contains I, then by primeness, $\mathfrak{p}$ contains $\mathrm{rad}(I)$. ∎

Corollary 2.1. $V(I_1) = V(I_2)$ *if* $\mathrm{rad}(I_1) = \mathrm{rad}(I_2)$.

Let $f \in A$. We define $U_f := \{\dot{\mathfrak{p}} | \, f \notin \mathfrak{p}\}$. $U_f = \mathrm{Spec}(A) - V(f)$, so is open.

Remark 2.2. *U_f is empty $\Leftrightarrow f \in \mathfrak{p}$ for all $\mathfrak{p} \Leftrightarrow f$ is nilpotent.*

Proposition 2.2. *U_f form a basis of the topology.*

Proof. Take a point $\dot{\mathfrak{p}} \in \mathrm{Spec}(A)$, and suppose that $\dot{\mathfrak{p}} \in U$, an open set. Then consider the set $\mathrm{Spec}(A) - U$. By definition, it is closed, so let $\mathrm{Spec}(A) - U = V(I)$. $\dot{\mathfrak{p}} \notin V(I)$ means that $\mathfrak{p} \not\supseteq I$, so we find $f \in I$ such that $f \notin \mathfrak{p}$. Then $\mathfrak{p} \in U_f$. Further, $U_f \subseteq U$ because $U_f \cap (\mathrm{Spec}(A) - U) = U_f \cap V(I) = \emptyset$ ∎

Lemma 2.3. $U_{f_1} \cap U_{f_2} = U_{f_1 f_2}$

Proof. By primeness of elements on either side. ∎

Lemma 2.4. *Let $f_1, ; f_n$ be elements of A. Then the following are equivalent:*
1. f_i generate the unit ideal, i.e. $\exists a_i$ such that $\sum a_i f_i = 1$.
2. $\bigcup_i U_{f_i} = \mathrm{Spec}(A)$

Proof. $\cup U_{f_i} = \text{Spec}(A) \Leftrightarrow \cap V(f_i) = \emptyset \Leftrightarrow V(\sum I_i) = \emptyset$. In general, if $V(I) = \emptyset$, then $I = A$, because if not, $I \in \mathfrak{m}$ for some maximal $\mathfrak{m}$ by Zorn's lemma, which contradicts. Thus $\sum I_i = A$, as desired. ■

Corollary 2.2. Spec(A) *is compact.*

Proof. Take an open covering, reduce it to basic opens, use the above lemma to express 1 as a finit sum of i_j for $i_j \in I_j$. Then we reduce to the basic opens for which these i_j are non-zero, which is a finite number by definition, and use the above lemma again to obtain that this is our finite subcover. ■

Remark 2.3. $S \subseteq \text{Spec}(A)$. $\bar{S} = V(\underset{\dot{\mathfrak{p}} \in S}{\cap} \mathfrak{p})$

Remark 2.4. $\dot{\mathfrak{p}}$ *is closed if and only if* $\mathfrak{p}$ *is maximal.*

Remark 2.5. *Suppose A is a domain. Then $\eta := \dot{0}$ is the "'generic"' point of* Spec(A).

Remark 2.6. Spec(A) *is not necessarily T1 (i.e. for $y \neq x$, $\exists$ a neighborhood of x that does not contain y). For example, let A be a domain and $y = \eta$*

Example: $A = k[t]$ with k algebraically closed. Then Spec(A) is isomorphic to $\eta \cup k$, and if $S \subseteq k$ is an infinite subset, then $\bar{S} = k$ since $k[t]$ is a PID.

Key results from HW

1. Given a map $\phi : A \to B$ of rings, we have an associated map $\Phi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ given by $\mathfrak{p} \longrightarrow \phi^{-1}(\mathfrak{p})$. This map is continuous, and $V(\phi^{-1}(J)) = \overline{\Phi(V(J))}$.
2. If A is Noetherian, we have an irreducible decomposition $\text{Spec}(A) = \cup_{i=1}^{n} V(\mathfrak{p}_i)$ where $\mathfrak{p}_i$ are all the minimal primes.

3.

LOCAL PROPERTIES

Definition 3.1. *We say that a property Q is local if given an A-module M, $[\exists f \in A$ s.t. $\dot{\mathfrak{p}} \in U_f$ and Q holds for M_f as an A_f-module $\forall \mathfrak{p}] \implies [Q$ holds for $M]$. Equivalently, a property is local if for some f_i that generate the unit ideal, the property holding for M_{f_i} implies that the property holds for M.*

Lemma 3.1. *The property of being 0 is local.*

Proof. Suppose M is a module s.t. $\forall \mathfrak{p}$, $\exists f$ s.t. $p \in U_f$ and $M_f = 0$. Then for all $\mathfrak{p}$, we take the set $U_{f_\mathfrak{p}}$ that satisfies this property. This is an open cover of Spec(A), so as Spec(A) is quasi-compact, we have a finite subcover U_{f_i}, $i = 1, \ldots, n$ of Spec(A) such that $M_{f_i} = 0$. However, $M_{f_i} = 0 \implies \exists n_i$ s.t. $f_i^{n_i} m = 0$ for all $m \in M$. By Lemma 2.4, these f_i generate A, so that there is some combination $\sum a_i f_i = 1$. But now we have $m = 1 \cdot m = 1^{n \max\{n_i\}} = (\sum a_i f_i)^{n \max\{n_i\}} m = 0$. ■

Lemma 3.2. *The property of being surjective/injective/bijective is local.*

Proof. Suppose we have a map $M \longrightarrow N$.

Surjective. Let L be the cokernel, so that we have the exact sequence $M \longrightarrow N \longrightarrow L \longrightarrow 0$. Applying the previous lemma, if for all $\mathfrak{p}$, there is some f such that $\dot{\mathfrak{p}} \in U_f$ and $M_f \longrightarrow N_f$ is surjective, then L_f is zero for these f, so L is 0.

Injective. Let L be the kernel, so that we have the exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N$. Then again, if the $M_f \longrightarrow N_f$ are injective, then L_f is zero, so L is zero. ■

Lemma 3.3. *The property of being finitely generated is local.*

Proof. Suppose that for some f_i that generate the unit ideal, M_{f_i} is finitely generated over A_{f_i}. Let these generators be m_i^j. We can assume that these generators lie in the image of the map $M \longrightarrow M_{f_i}$ by clearing denominators (replace $\frac{m}{s}$ with $\frac{ms}{1}$). In other words, for each i, we have a surjection $\bigoplus_j A_{f_i} \longrightarrow M_{f_i}$. We claim that the preimages of all of these m_i^j generate M over A. This is equivalent to the assertion that the map $\bigoplus_i \bigoplus_j A \longrightarrow M$ is surjective, which is true because it is locally surjective. ■

Definition 3.2. *Given a prime $\mathfrak{p}$, we denote $A_\mathfrak{p} := A_{A-\mathfrak{p}}$ and $M_\mathfrak{p} := M_{A_\mathfrak{p}}$.*

Definition 3.3. *A ring A is called local if it has only one maximal ideal.*

Lemma 3.4. $\mathfrak{m} \subseteq A$. *Then A is local with unique maximal ideal $\mathfrak{m}$ if and only if every element in $A - \mathfrak{m}$ is invertible.*

Proof. $\Rightarrow$ Suppose that A is local with unique maximal ideal $\mathfrak{m}$. Suppose that $f \in A - \mathfrak{m}$ is not invertible. Then (f) is an ideal not equal to A, so by Zorn's lemma, is in a maximal ideal $\mathfrak{m}'$. But since this $\mathfrak{m}'$ has f, $\mathfrak{m} \neq \mathfrak{m}'$, which contradicts localness.

$\Leftarrow$ Let $\mathfrak{m}'$ be a maximal ideal, and $\mathfrak{m}' \neq \mathfrak{m}$. Then $\mathfrak{m}' \not\subset \mathfrak{m}$ by maximality, so $\exists a \in \mathfrak{m}'$ such that $a \notin \mathfrak{m}$. Then $(a) \subset \mathfrak{m}'$, but as a is invertible, $(a) = A$ - contradiction. ■

Example: Let $A = \mathbb{C}[x]$, and $\mathfrak{p} = (x)$. Then $A_\mathfrak{p} = \mathbb{C}[x]_{\mathbb{C}[x]-(x)}$ is not finitely generated (do infinite prime trick on denominators)

Lemma 3.5. *$A_\mathfrak{p}$ is local.*

Proof. Consider the prime $\mathfrak{p}_\mathfrak{p} \subseteq A_\mathfrak{p}$. Every element in $A_\mathfrak{p} - \mathfrak{p}_\mathfrak{p}$ is invertible by construction. ∎

Result from homework 5.1: The property of being flat is local

Definition 3.4. *Given an A-module M, we define* $\text{supp}(M) = \{\dot{\mathfrak{p}} | \ M_\mathfrak{p} \neq 0\} \subseteq \text{Spec}(A)$ *to be the support of M*

Proposition 3.1. *Let M be a finitely generated A-module. Then* $\text{supp}(M) = V(I)$, *where $I = Ann(M) := \{a \in A | \ am = 0 \ \forall m \in m\}$.*

Proof. We prove the following lemma: $M_S = 0$ if and only if $S \cap \text{Ann}(M) \neq \emptyset$.

$\Rightarrow$ Take m_i generators of A over M. Then for all i, $\exists s_i \in S$ such that $s_i m_i = 0$. Then $\prod s_i$ kills all $m \in M$, so $\prod s_i \in \text{Ann}(M)$, but as S is a multiplicative subset, it is closed under multiplication, so $\prod s_i \in S$.

$\Leftarrow$ Take $s \in S \cap \text{Ann}(M) \neq \emptyset$. Then $\forall \frac{m}{s'} \in M_S$, $\frac{m}{s'} = \frac{sm}{ss'} = 0$.

In light of this lemma, we see that $M_\mathfrak{p} = 0$ if and only if $(A - \mathfrak{p}) \cap \text{Ann}(M) \neq \emptyset$, which occurs if and only if $\mathfrak{p} \not\supset \text{Ann}(M)$, i.e. $\mathfrak{p} \notin V(\text{Ann}(M))$. ∎

Remark 3.1. *The above local property stuff is for the definition above, not for the general definition of a $\mathfrak{p}$-local property, which is that if a property holds for all localizations at primes $\mathfrak{p}$, then it holds in general. These defintions are not equivalent, as we shall see.*

For an example of a module that is finitely generated at localizations $M_\mathfrak{p}$ over all primes $\mathfrak{p}$, but is not itself finitely generated, see problem 5.4(c)

Extra Stuff not in notes:

Definition 3.5. *A property Q is said to be $\mathfrak{p}$-local if [Q holds for M] if and only if [Q holds for $M_\mathfrak{p}$ for all primes $\mathfrak{p}$].*

From pg 40-41 in Atiyah, we have that the property of being 0 is $\mathfrak{p}$-local. In fact, it is $\mathfrak{m}$-local (i.e. for maximals ideals), as is the property of being injective/surjective/bijective and flatness.

Lemma 3.6. *(Serre) Let A be a ring, $f_1, \ldots, f_n$ elements that generate the unit ideal. For an A-module M, we define a map $M \xrightarrow{\phi} \bigoplus\limits_i M_{f_i}$ by compiling the maps $M \longrightarrow M_{f_i}$. We claim that*

1. *The map ϕ is injective.*
2. *The image of M inside $\bigoplus\limits_i M_{f_i}$ are elements (m_i) such that when mapped further onto $\bigoplus\limits_{i,j} M_{f_i f_j}$, the images of m_i and m_j coincide in $M_{f_i f_j}$.*

$$\begin{array}{ccc} M & \longrightarrow & M_{f_i} \\ \downarrow & & \downarrow \\ M_{f_j} & \longrightarrow & M_{f_i f_j} \end{array}$$

Proof.

1. We localize the function at each f_i, and obtain that since $(M_{f_i})_{f_i} = M_{f_i}$, then the function $\phi_i : M_{f_i} \longrightarrow \oplus (M_{f_j})_{f_i}$ is injective (look at the $i - th$ coordinate, where we have a bijection), so that since injectiveness is a local property, ϕ is injective.

2. We denote the set of these kinds of elements M'. It is clear that $Im(M) \subset M'$ by the commutative diagram above.

Case 1: Suppose that f_1 is invertible. Then $A_{f_1} = A$, so $M_{f_1} = M$. For an element $(m_1, \ldots, m_n)$ in M', we see that m_1 is an element of M. By the definition of M', the image of m_1 in $M_{f_1 f_j} = (M_{f_1})_{f_j} = M_{f_j}$ is the same as the image of m_j in M_{f_j}. Thus the element (m_i) is the image of $m_1 \in M$. Thus $M' \subset \mathrm{Im}(M)$.

Case 2: We see that for an element g, $M'_g =$ the set of elements $(m_i) \in \oplus M_{f_i g}$ such that the image in $\bigoplus_{i,j} M_{f_i f_j g}$ coincide. Thus we take $g = f_i$ for each i, apply case 1, and now obtain that $M'_{f_i} \subset (\mathrm{Im}(M))_{f_i}$ for all i. Thus as injectiveness is a local property, we have $M' \subset \mathrm{Im}(M)$, as desired. ■

4.

ARTINIAN RINGS

Definition 4.1. *A commutative ring A is Artinian if every descending chain of ideals $I_1 \supset I_2 \supset \dots$ stabilizes.*

Examples

1. $\mathbb{Z}$ is not Artinian $Z \supset 2Z \supset 4Z \supset \dots$ does not stabilize.
2. $\mathbb{C}[t]$ is not Artinian $(t) \supset (t^2) \supset \dots$ does not stabilize.
3. $\mathbb{C}[t]/t^{100001}$ is Artinian as it is a finite dimensional vector space over a field
4. For V a finite dimensional vector space, the subalgebra of $End(V)$ consisting of elements T_i such that $T_iT_j = T_jT_i$ with multiplication consisting of composition is Artinian because it is a subspace of $End(V)$, which is finite dimensional over a field.

Proposition 4.1. *Let A be Artinian. Then every prime ideal is maximal.*

Proof. Consider $A/\mathfrak{p}$, for a prime ideal $\mathfrak{p}$. We note that $A/\mathfrak{p}$ is also Artinian, since chains of ideals can be lifted to chains of ideals in A. Further, since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is a domain. We take an element $a \in A/\mathfrak{p}$, and consider the descending chain $(a) \supset (a^2) \supset \dots$. It stabilizes, so that we have $(a^n) = (a^{n+1})$ for some n. In particular, we have that $\exists b$ such that $a^n = a^{n+1}b$, or $a^n(1 - ab) = 0$. Since $A/\mathfrak{p}$ is a domain, this implies that either a^n or $1 - ab$ is zero. But domainness again implies that $a^n = 0 \implies a = 0$, so we can ignore this case. Thus b is an inverse for a, so that $A/\mathfrak{p}$ is a field, i.e. $\mathfrak{p}$ is maximal. ■

Corollary 4.1. *If A is also Noetherian, then as all points are closed, 6 on problem set 4 implies that from our decomposition into irreducibles, there are a finite number of maximal ideals. Thus A has a finite number of prime ideals.*

Lemma 4.1. *Let $\operatorname{Spec}(A) = V_1 \cup V_2$. Then there is a unique decomposition $A = A_1 \oplus A_2$ such that $V_i \simeq \operatorname{Spec}(A_i)$.*

Proof. See 6 on pset 5. ■

Lemma 4.2. *Let A be a local Artinian ring. Then $\exists n$ such that $\mathfrak{m}^n = 0$, for $\mathfrak{m}$ its maximal ideal.*

Proof. Since the intersection of maximal/prime ideals is the set of nilpotent elements, we have that every element of $\mathfrak{m}$ is nilpotent. By quasi-compactness of $\operatorname{Spec}(A)$, we take a finite number of elements f_i that generate the unit ideal. Then by nilpotence, $\exists n_i$ such that $f_i^{n_i} = 0$. Thus $\mathfrak{m}^{\max\{n_i\}} = 0$. ■

Remark 4.1. *From Homework 5.7, every Artinian ring is Noetherian*

Random Lemma

Lemma 4.3. *Suppose that B is rational (i.e. not nilpotent). Then for $\phi_1, \phi_2 : A \longrightarrow B$, we look at the associated maps $\Phi_i : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$. If $\Phi_1 = \Phi_2$, then $\phi_1 = \phi_2$.*

Proof. We want to show that $\phi_1(f) = \phi_2(f)$ for all $f \in A$. For every maximal ideal $\dot{\mathfrak{m}} \in \operatorname{Spec}m(B)$, consider the map $\psi_{\mathfrak{m}} : B \longrightarrow B/\mathfrak{m}$. From the assumption, we have $\psi_{\mathfrak{m}}(\phi_1(f)) = \psi_{\mathfrak{m}}(\phi_2(f))$. Thus by the Nullstellensatz, $\phi_1(f) - \phi_2(f)$ is nilpotent. Thus by assumption of rationality, it is 0. ■

5.

NAKAYAMA, LOCALLY FREE

Lemma 5.1. *(Nakayama) Let A be a local ring with maximal ideal $\mathfrak{m}$, M a finitely generated A-module. Consider $M \underset{A}{\otimes} A/\mathfrak{m} = M/\mathfrak{m}M$. Then $M/\mathfrak{m}M = 0$ implies that $M = 0$.*

Proof. let $m_1, \dots, m_n$ generate M over A By assumption, we have $M/\mathfrak{m}M = 0$, i.e. $\mathfrak{m}M = M$. Therefore, we can represent each b_i as an element of the form am, with $m \in M$ and $a \in \mathfrak{m}$. Representing m with the generators, we obtain $b_i = \sum_{j=1}^n a a_j b_j$. As $a \in \mathfrak{m}$, we have $a a_j \in \mathfrak{m}$. Thus we have $b_i = \sum_{j=1}^n a_{ij} b_j$ for $a_{ij} \in \mathfrak{m}$. We consider the commutative diagram

$$\begin{array}{ccc} A^n & \twoheadrightarrow & M \\ {\scriptstyle T}\downarrow & & \downarrow \\ A^n & \longrightarrow & M \end{array}$$

Where the map T is given by $T(e_i) = e_i - \sum_{j=1}^n a_{ij} e_j$, and the maps across are given by $e_i \mapsto b_i$. The matrix for T is given by

$$\begin{pmatrix} 1 - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a21 & & & \\ \dots & & & \\ -an1 & & & 1 - a_{nn} \end{pmatrix}$$

Taking the determinant of this matrix, we obtain an element of the form $1 + m$, where $m \in \mathfrak{m}$. Thus this element is invertible, so is nonzero, so T is invertible. Therefore T is an isomorphism. However, going down and across in the diagram, we obtain $e_i \mapsto b_i - \sum_{j=1}^n a_{ij} b_j = b_i - b_i = 0$, so for this diagram to commute, we put the zero map on the right hand side. However, T being an isomorphism means that this map is an isomorphism, so thus $M = 0$, as desired. ∎

Corollary 5.1. *Let A be a local ring, M_1 and M_2 finitely generated modules. If $\phi : M_1 \longrightarrow M_2$ is a function such that $M_1/\mathfrak{m}M_1 \twoheadrightarrow M_2/\mathfrak{m}M_2$, then ϕ is surjective.*

Proof. Let $\operatorname{coker}(\phi) = K$. Then $K/\mathfrak{m}K$ is the cokernel of the map $M_1/\mathfrak{m}M_1 \twoheadrightarrow M_2/\mathfrak{m}M_2$. Thus it is zero, so by Nakayama, K is zero. ∎

Example: $A = \mathbb{Z}_p$, $M = p\mathbb{Z}_p$. Then $\mathbb{Q}_p$ is a module for which Nakayama's does not hold ($\mathbb{Z}_p$ is not local), since $\mathbb{Q}_p/p\mathbb{Q}_p = 0$.

Lemma 5.2. *M free $\Rightarrow$ M projective $\Rightarrow$ M flat.*

Proof. If M is free, then $M = \oplus A$. Since A is projective and direct sums of projective modules are projective, then M is projective.

Now suppose that M is projective, i.e. is a direct summand of a free module, $\oplus A = M \oplus N$. Then as $\oplus A$ is flat, we have that for an injection $M_1 \hookrightarrow M_2$, $M_1 \otimes \oplus A \hookrightarrow M_2 \otimes \oplus A$ so $(M_1 \otimes M) \oplus (M_1 \otimes N) \hookrightarrow (M_2 \otimes M) \oplus (M_2 \otimes N)$, so in particular, as each summand maps to each summand, $M_1 \otimes M \hookrightarrow M_2 \otimes M$. ∎

Theorem 5.1. *Let A be a local Noetherian ring and M a finitely generated A-module. Then TFAE*

1. *M is free*
2. *M is projective*
3. *M is flat*

Proof.

$1 \Rightarrow 2 \Rightarrow 3$ by lemma above.

$3 \Rightarrow 1$ Take n such that $A^n \twoheadrightarrow M$. Tensoring with $A/\mathfrak{m}$, we obtain a map $(A/\mathfrak{m})^n \twoheadrightarrow M/\mathfrak{m}M$. Since these are both vector spaces, we can adjust n so that this map is an isomorphism. Then by the corrolary to Nakayama above, for this n, we still have $A^n \twoheadrightarrow M$. We thus consider the exact sequence

$$0 \longrightarrow M' \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

By Noetherianess, M', as a submodule of a finitely generated module, is finitely generated. Therefore, showing that $M'/\mathfrak{m}M' = 0$ shows that $M' = 0$. However, from number 4 on the tensor problem homework, we obtain that M is flat gives up the exact sequence

$$0 \longrightarrow M' \otimes A/\mathfrak{m} \longrightarrow A^n \otimes A/\mathfrak{m} \longrightarrow M \otimes A/\mathfrak{m} \longrightarrow 0$$

Thus since n was chosen so that the latter arrow was an isomorphism, we have $M' \otimes A/\mathfrak{m} = M'/\mathfrak{m}M' = 0$. Thus the map $A^n \to M$ is an isomorphism. ■

Definition 5.1. *Let A be a commutative ring, M an A-module. Then M is locally free if $\forall \dot{\mathfrak{p}} \in$ Spec(A), $\exists f$ s.t. $\dot{\mathfrak{p}} \in U_f$ s.t. M_f is locally free over A_f. Equivalently, for f_i that generate the unit ideal, M_{f_i} are free over A_{f_i}.*

Lemma 5.3. *Suppose that M is finitely generated and locally free. Then for the f_i in the definition, $M_{f_i} \simeq A^n_{f_i} \implies M_{f_j} \simeq A^n_{f_j}$, i.e. M can be said to be locally free of rank n (all the localizations at f_i are isomorphic to $A^n_{f_i}$).*

Proof. Take $i \neq j$, and consider $M_{f_i} = A^{n_i}_{f_i}$ and $M_{f_j} = A^{n_j}_{f_j}$. Then as

$$(M_{f_i})_{f_j} = M \underset{A}{\otimes} A_{f_i} \underset{A}{\otimes} A_{f_j} = M \underset{A}{\otimes} A_{f_j} \underset{A}{\otimes} A_{f_i} = (M_{f_j})_{f_i}$$

Thus we have $A^{n_i}_{f_i f_j} = (A^{n_i}_{f_i})_{f_j} = (A^{n_j}_{f_j})_{f_i} = A^{n_j}_{f_i f_j}$, so that $n_i = n_j$, as desired. ■

Theorem 5.2. *Let A be Noetherian, M finitely generated. Then the following are equivalent*

1. *M is locally free*
2. *M is projective*
3. *M is flat*

Proof. From 2(d) on problem set 5, the property of being finitely generated and projective is local. Localizing, therefore, we obtain that M is locally projective, and further, as M is finitely generated, then $M_f = M \otimes A_f$ is finitely generated over A_f (take the same generators). Thus M is projective. From the lemma above, we have $2 \Rightarrow 3$. Thus it remains to show that $3 \Rightarrow 1$.

$3 \Rightarrow 1$ M flat over A implies, as localizations are flat, that $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$. Then the previous lemma, as $A_\mathfrak{p}$ is local, implies that $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$, so suppose that $M_\mathfrak{p} = A^n_\mathfrak{p}$. We prove the following lemma

Let N be a finitely generated module over A such that $N_\mathfrak{p} = 0$ for some $\mathfrak{p}$. Then there is some $f \notin \mathfrak{p}$ such that $N_f = 0$.

Proof. $N_\mathfrak{p} = 0 \implies \mathfrak{p} \notin$supp$(N)$. By proposition 3.1, supp(N) is closed, so it's complement is open, so there is some open set containing $\mathfrak{p}$ such that this set does not intersect supp(N). Taking this set to be a basic open, we have $\mathfrak{p} \in U_f$ such that $U_f \cap$supp$(N) = U_f \cap V(\text{Ann}(N)) = \emptyset$, which means that $f \in$Ann(M). Thus $N_f = 0$.

In light of this lemma, we look at the map $A^n_\mathfrak{p} \xrightarrow{\sim} M_\mathfrak{p}$. We can take the basis elements e_i of $A^n_\mathfrak{p}$ to map to elements of the form $\frac{m}{1}$ by ring action of $A_\mathfrak{p}$, so that we can consider the corresponding map $A^n \longrightarrow M$ and exact sequence

$$0 \longrightarrow K \longrightarrow A^n \longrightarrow M \longrightarrow C \longrightarrow 0$$

Then since $A_{\mathfrak{p}}^n \simeq M_{\mathfrak{p}}$, we have $K_{\mathfrak{p}} = C_{\mathfrak{p}} = 0$. Thus by our lemma, K_f and C_f are zero, so that the map $A_f^n \longrightarrow M_f$ is an isomorphism, as desired. ■

Remark 5.1. *In light of this theorem, we have M locally free if and only if M is flat, and as flatness is a $\mathfrak{p}$-local property and localizations are flat, M is flat if and only if $M_{\mathfrak{p}}$ is flat for all $\mathfrak{p}$, which by the theorem before is equivalent to $M_{\mathfrak{p}}$ free over $A_{\mathfrak{p}}$. Thus M is locally free if and only if $M_{\mathfrak{p}}$ is free over $A_{\mathfrak{p}}$ for all $\mathfrak{p}$.*

Lemma 5.4. *Let B be a domain. $N \subseteq B^n$ such that $N \otimes Frac(B) = 0$ implies that $N = 0$*

Proof. Take $n \in N$. Suppose that this is nonzero, i.e. as a submodule of B^n, $\exists 0 \neq f \in B$ such that $fn \neq 0$. Then $N \otimes Frac(B) = 0$ implies that $\exists 0 \neq b \in B$ such that $bfn = 0 \implies bf = 0$, as B is a domain. But this is impossible because B is again a domain. ■

Theorem 5.3. *Let A be a local Noetherian domain and M a finitely generated module. TFAE:*

1. *M is free.*
2. *$dim_{A/\mathfrak{m}}(M/\mathfrak{m}M) = dim_{Frac(A)}(M \otimes Frac(A))$.*

Proof.

1 ⇒ 2 *Suppose that M is free. Then $M = A^n$, so $M/\mathfrak{m}M = M \otimes A/\mathfrak{m} = A^n \otimes A/\mathfrak{m} = (A/\mathfrak{m})^n$, so $dim_{A/\mathfrak{m}}(M/\mathfrak{m}M) = n$. Also, $M \otimes Frac(A) = A^n \otimes Frac(A) = (Frac(A))^n$, so that the RHS is also n.*

2 ⇒ 1 *Suppose that $dim_{A/\mathfrak{m}}(M/\mathfrak{m}M) = n = dim_{Frac(A)}(M \otimes Frac(A))$. Then we have the surjection $(A/\mathfrak{m})^n \xrightarrow{\sim} M/\mathfrak{m}M$. From the corollary to Nakayama, we have the surjection $A^n \longrightarrow M$. Thus we consider the short exact sequence*

$$0 \longrightarrow M' \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

Tensoring this with $Frac(A)$, we obtain

$$0 \longrightarrow M' \otimes Frac(A) \longrightarrow A^n \otimes Frac(A) = Frac(A)^n \longrightarrow M \otimes Frac(A) = Frac(A)^n \longrightarrow 0$$

by assumption, so we $M' \otimes Frac(A) = 0$. As A is a domain, and $M' \subset A^n$, we can apply the previous lemma to obtain that this implies that $M' = 0$, i.e. $A^n \sim M$, as desired. ■

Theorem 5.4. *Let A be a Noetherian domain and M a finitely generated A-module. Then TFAE:*

1. *M is locally free of rank n.*
2. *There exists n such that $dim(M_{\mathfrak{p}} \underset{A_{\mathfrak{p}}}{\otimes} A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}}) = n$ as a vector space over $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}}$ for all $\mathfrak{p}$.*

Proof.

1 ⇒ 2 From the lemma in the theorem above, we have $A_f^n \simeq M_f^n$ implies that $A_{\mathfrak{p}}^n \simeq M_{\mathfrak{p}}$. Thus tensoring both sides with $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$, we get the desired result.

2 ⇒ 1 By assumption, we have $(A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}})^n \simeq M_p/\mathfrak{p}_{\mathfrak{p}}M_{\mathfrak{p}}$. By Nakayama, this implies that the function $A_{\mathfrak{p}}^n \longrightarrow M_{\mathfrak{p}}$ is an isomorphism (the cokernel and kernel are 0 by Nakayama). Thus $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$, and as flatness is a $\mathfrak{p}$-local property, then M is flat, so by the previous lemma, M is locally free, as desired. ■

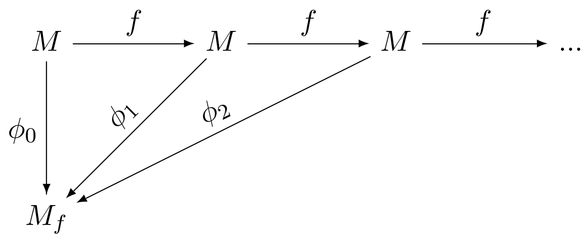
Definition 5.2. *$M_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}M_{\mathfrak{p}} = M \underset{A}{\otimes} A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}}$ is called the fiber of M at $\mathfrak{p}$*

Definition 5.3. *$M_{\mathfrak{p}}$ is called the stalk of M at $\mathfrak{p}$.*

6.

SOLUTIONS

3.8 To define a map $\underset{\rightarrow}{l\,im}\,(M_0 \xrightarrow{f} M_1 \xrightarrow{f} M_2 ...) \mapsto M_f$ (the $M_i = M$, just indexed conveniently by i), we define maps $\phi_i : M_i \mapsto M_f$ by $\phi_0(m) = \frac{m}{1}$, $\phi_i(m) = \frac{m}{f^i}$, and see that the diagram:



commutes, and that the map is a map of modules. Then by the universal property of the direct limit, there is a unique map $\phi_{univ} : \underset{\rightarrow}{l\,im}\,(M_0 \xrightarrow{f} M_1 \xrightarrow{f} M_2 ...) \mapsto M_f$ s.t. $\phi_i = \phi_{univ} \circ f\ \forall i$.

To map $M_f \mapsto \underset{\rightarrow}{l\,im}\,(M_0 \xrightarrow{f} M_1 \xrightarrow{f} M_2 ...)$, we define a map ψ by $\psi(\frac{m}{f^i}) = \overline{(0, .., m, ..)}$ which is nonzero in the k position. It is clearly a map of modules, and under this map, images of elements of the set $\{1, f^i\}$ are invertible, as $f^i(\psi(\frac{m}{f^i})) = \overline{(0, .., f^i m, ..)} \sim \overline{(1, 0, 0..)}$. Finally, by problem set 1 we have that all elements of the direct limit can be expressed as $\overline{(0, ..m, ..)}$, so

$$\psi(\phi(\overline{(0, ..m, ..)})) = \psi(\frac{m}{f^i}) = \overline{(0, ..m, ..)}$$

$$\phi(\psi(\frac{m}{f^i})) = \phi(\overline{(0, ..m, ..)}) = \frac{m}{f^i}$$

and we have the desired isomorphism.

5.1 Suppose that for an A-module M, for every $\mathfrak{p} \in A$, $\exists f \notin \mathfrak{p}$ s.t. $\mathfrak{p} \in U_f$ and M_f is flat. Then we claim that M is flat.

Consider the covering of $Spec(A)$ by these U_f for each prime. Then by quasi-compactness $\exists f_1 ... f_n$ s.t. $U_{f_1} ... U_{f_n}$ cover $Spec(A)$, so by homework before, $f_1 ... f_n$ generate A. Now for any injection between A-modules N_1 and N_2, we consider the exact sequences:

$$Ker \hookrightarrow N_1 \otimes M \longrightarrow N_2 \otimes M$$

$$Ker_{f_i} \hookrightarrow N_1 \otimes M_{f_i} \longrightarrow N_2 \otimes M_{f_i}$$

Where Ker is the kernel of this map. Then by flatness of M_{f_i}, Ker_{f_i} is 0, and as the property of being 0 is local, we have that Ker is 0, as desired.

5.4(c) Let M be $\bigoplus_{a \in \mathbb{C}} (\mathbb{C}[x]/(x-a))$. It is not finitely generated over $\mathbb{C}[x]$, as it is an infinite direct sum. Primes of $\mathbb{C}[x]$ are ideals generated by $x - b$, where $b \in \mathbb{C}$. Now consider $M_{(x-b)} = \bigoplus_{a \in \mathbb{C}} (\mathbb{C}[x]/(x-a))_{x-b}$, so consider $(\mathbb{C}[x]/(x-a))_{x-b}$. If $b \neq a$, $x - a \in \mathbb{C}[x] - ((x-b))$, so $(\mathbb{C}[x]/(x-a))_{x-b} = 0$. If $b = a$, $(x-a) \cap (\mathbb{C}[x] - ((x-b)) = \emptyset$, so by the last pset, $(\mathbb{C}[x]/(x-a))_{x-a} \neq 0$. Further, we claim that $(\mathbb{C}[x]/(x-a))_{x-a} \simeq \mathbb{C}[x]/(x-a)$.

We define a map from the right side to the left by sending $\overline{p(x)}$ to $\frac{\overline{p(x)}}{1}$. This map is injective because $\frac{\overline{p(x)}}{1} = \frac{\overline{g(x)}}{1}$ implies that there exists an element $s(x)$ of the multiplicative set $\mathbb{C}[x] - ((x-a))$ such that $s(x)\overline{g(x)} = s(x)\overline{p(x)}$. But since $\overline{s(x)p(x)} = \overline{p(x)}$ this implies that $\overline{g(x)} = \overline{p(x)}$, so the map is injective. To prove surjectivity, we first note that $\mathbb{C}[x]/x - a \simeq \mathbb{C}$. Now take an element

$\frac{\overline{m}}{s(x)}$, and let $\overline{s(x)} = \overline{n}$. Then as $\frac{\overline{m}}{s(x)} = \frac{\frac{\overline{m}}{n}}{1}$, so it is mapped to by the element $\frac{\overline{m}}{n}$, so the map is also surjective.

Also, we claim that $\mathbb{C}[x]/(x-a) \simeq \mathbb{C}$ (this is true as it is a 1-dimensional field extension of $\mathbb{C}$, but we'll show it more explicitly). We do this by mapping $\overline{p(x)}$ to its "'remainder"' when divided by $x - a$. This map is well defined because if $p(x)$ $q(x)$, $\exists m(x) \in (x-a)$ s.t. $p(x) = q(x) + m(x)$, so their images are the same because the remainder of $q(x) + m(x)$ is the remainder of $q(x)$ plus the remainder of $m(x)$, and the remainder of $m(x)$ is zero. Further, this map is injective because the images of $\overline{p(x)}$ and $\overline{q(x)}$ are the same if and only if $\exists m(x) \in (x-a)$ that is their difference. It is also surjective because for any $b \in \mathbb{C}$, b is the image of $\overline{b}$.

Therefore, for any prime $\mathfrak{p}$, $M_\mathfrak{p} \simeq \mathbb{C}$ is finitely generated.($\mathbb{C}$ is finitely generated by 1, with the action of $\mathbb{C}[x]$ given by multiplication of constants), but as mentioned before, M is not finitely generated.

5.6 Suppose $V_i = V(I_i)$. Then $Spec(A) = V(I_1) \cup V(I_2) = V(I_1 I_2)$ implies that elements of $I_1 I_2$ are nilpotent. Further, $V(I_1) \cap V(I_2) = V(I_1 + I_2) = \emptyset$ implies that $I_1 + I_2 = A$. Thus as $1 \in A$, then $\exists i_1, i_2$ s.t. $i_1 + i_2 = 1$. As both are nilpotent, let raising each to the n kill both. Now consider the element of A, $(i_1 + i_2)^{2n} = i_1^{2n} + a_1 i_1^{2n-1} i_2 ... a_n i_1^n i_2^n + a_{n+1} i_1^{n-1} i_2^{n+1} + ... i_2^{2n}$, where the a_i are constants. Then the term $a_n i_1^n i_2^n$ is zero. Let the terms before equal a, and the terms after equal b. Then $a \in I_1$, $b \in I_2$, $a \cdot b = 0$, $a + b = 1$. Now let $A_1 = A/(a)$, $A_2 = A/(b)$. Then we claim $A_1 \oplus A_2 = A$.

We define a map from $A \to A_1 \oplus A_2$ by sending an element to its projections. Then if an element c is sent to $(0, 0)$, then $c \in (a)$ and $c \in (b)$, which implies that $bc = ac = 0 \implies (a+b)c = c = 0$. Thus this map is injective. Finally, given an element of the image, $(\overline{m}, \overline{n})$, we have that $(\overline{m}, \overline{n})$ is the image of $an + bm$, so the map is surjective.

Now, we claim that $V(I_1) = V(0 \oplus A_2)$, where we view $0 \oplus A_2$ as its preimage in A, (a), i.e. we claim that $V(I_1) = V(a)$.

$\Rightarrow$ If $\mathfrak{p} \supset I_1$, then as $a \in I_1$, $\mathfrak{p} \supset (a)$

$\Leftarrow$ Suppose $\mathfrak{p} \supset (a)$. Then as $Spec(A) = V(I_1) \cup V(I_2)$, where this is a disjoin union, $\mathfrak{p} \supset$ either I_1 or I_2. However, if $\mathfrak{p} \supset I_2$, then as $b \in I_2$, $b \in \mathfrak{p}$, but then this would imply that $a, b \in \mathfrak{p}$, and as $(a) + (b) = A$, this would imply that $p = A$, which is a contradiction. Thus $\mathfrak{p} \supset I_1$.

Now, we claim that $V((a)) \simeq Spec(A_1)$. This is true because we have the surjection $\phi : A \to A_1 = A/(a)$, so by 2. on the last problem set, $Spec(A_1)$ is homeomorphic to $V(ker(\phi)) = V((a))$, as desired. We do similarly for A_2.

TP.4 In class, we showed that tensoring is a right-exact functor. Take an R module N_3, and surject onto from a free module N_2 module, and let N_1 be the kernel of this projection, and inject it into N_2. Then $0 \mapsto N_1 \mapsto N_2 \mapsto N_3 \mapsto 0$ is exact. Then we have the following exact commutative diagram:

$$\begin{array}{ccccc}
N_1 \otimes M_1 & \xrightarrow{1} & N_1 \otimes M_2 & \xrightarrow{2} & N_1 \otimes M_3 \\
\downarrow 3 & & \downarrow 4 & & \downarrow 5 \\
N_2 \otimes M_1 & \xrightarrow{6} & N_2 \otimes M_2 & \xrightarrow{7} & N_2 \otimes M_3 \\
\downarrow 8 & & \downarrow 9 & & \downarrow 10 \\
N_3 \otimes M_1 & \xrightarrow{11} & N_3 \otimes M_2 & \xrightarrow{12} & N_3 \otimes M_3
\end{array}$$

We want to show that 11 is injective. Suppose that $a \in N_3 \otimes M_1$ maps to zero in 11, $a \neq 0$. As 8 is surjective, it has a nonzero preimage $b \in N_2 \otimes M_1$. By injectivity of 6, $6(b) = c \in N_2 \otimes M_2$ is nonzero, and by commutativity, $9(c) = 0 \Rightarrow c \in ker(9) \Rightarrow c \in Im(4) \Rightarrow \exists$ nonzero $d \in N_1 \otimes M_2$ s.t. $4(d) = c$. But as $c \in Im(6)$, $c \in ker(7)$, so $7(c) = 0$. Then as 5 is injective, this implies by commutivity that $2(d) = 0 \Rightarrow d \in ker(2) \Rightarrow d \in Im(1)$. But $b \notin ker(8) \Rightarrow b \notin Im(3)$, so we have a contradiction.