Power Operations and Tor Vanishing

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Abstract

We show that there are cofiber sequences relating power operations in Morava E-theory modulo Lubin-Tate parameters $p, \cdots, u_{i-1}$ for various i's and use this to inductively show that certain Tor groups over the algebra of additive operations vanish in nonzero degrees. These Tor groups compute the linearization of the E_2-page of a bar spectral sequence converging to the graded E-cohomology of configuration spaces on $\mathbb{R}^n$.

1 Introduction

Let $E = E(k, \mathbb{G}_0)$ be the Morava E-theory corresponding to a perfect field k of characteristic p and a height h formal group $\mathbb{G}_0$ over k. In [SS25], we consider power operations modulo Lubin-Tate parameters $p, \cdots, u_{i-1}$ for $0 \leq i \leq h$ which act on $\pi_0(-/p, \cdots, u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras. We show that the mod $p, \cdots, u_{i-1}$ analog $\Gamma^{(i)}$ of the additive operations Γ considered in [Rez09, Rez17] is Koszul of length $h - i + 1$ i.e. that its Koszul complex has length $h-i+1$. This supplies us with Koszul (hence finite length) resolutions for computational purposes, which we will use here to gain insight into Morava E-theory of configuration spaces on $\mathbb{R}^n$.

The study of configuration spaces is a classical problem in mathematics. The ordered configuration space of k distinct points on a manifold M is given by

$$\mathrm{Conf}_k(M) = \{(x_1, \cdots, x_k) \in M^k \mid x_i \neq x_j \text{ for all } i \neq j\}$$

and the corresponding unordered configuration space is $B_k(M) = \mathrm{Conf}_k(M)/\Sigma_k$. More generally, the labeled configuration space with labels in a spectrum X is given by

$$B_k(M; X) = \Sigma^\infty_+ \mathrm{Conf}_k(M) \otimes_{h\Sigma_k} X^{\otimes k}.$$

The case when $M = \mathbb{R}^n$ and $X = S^\ell$ is a sphere is of particular significance since the direct sum $B(\mathbb{R}^n, S^\ell) := \bigoplus_{k \geq 0} B_k(\mathbb{R}^n; S^\ell)$ is the free $\mathbb{E}_n$-algebra $\mathrm{free}_{\mathbb{E}_n}(S^\ell)$ on a single generator in degree ℓ, whose homology parameterizes Dyer-Lashof operations on $\mathbb{E}_n$-algebras. If $\ell > 0$, by Snaith's theorem, $\mathrm{free}_{\mathbb{E}_n}(S^\ell)$ is also equivalent to the iterated loop space $\Sigma^\infty_+ \Omega^n S^{n+\ell}$.

The ordinary homology of (unordered) configuration spaces on $\mathbb{R}^n$ is known [CLM76], but not much is known for generalized homology theories. For each prime p and height $h \in \mathbb{N}$, there are two complex oriented cohomology theories central to chromatic homotopy theory: Morava K-theory and Morava E-theory. For height 1, E_1 recovers p-completed complex K-theory, while $K(1)$ is a variant of complex K-theory mod p. The case of height 1 has been solved by Langsetmo [Lan93, Lan96].

The E-theory of the weight p part $B_p(\mathbb{R}^n; S^\ell)$ has been computed by [BHK24]. Their approach utilizes a result of [Knu18] which expresses the stable homotopy type of configuration spaces in terms of certain Lie algebra homologies, and is based on the computation of operations on $K(h)$-local Lie-E-algebras in [Bra17]. In light of the Koszul duality between the $\mathbb{E}_\infty$ and Lie operads, we approach this problem from the $\mathbb{E}_\infty$ side based on an inductive understanding of power operations.

Let σ denote the suspension map $\Gamma^{-q} \to \Gamma^{-q-1}$ relating operations acting on π_q to operations acting on π_{q+1} [Rez09]. Since all the Γ^{-q}'s are isomorphic and σ is an isomorphism if q is odd [Rez09], we will only consider double suspensions σ^2 and regard it as a map $\Gamma \to \Gamma$ where $\Gamma = \Gamma^0$. Our main result is the following.

Theorem A (Tor vanishing, Theorem 6.0.1)**.** *For $p = 2$, $0 \le i < h$, and all $n \ge 0$,*

$$\operatorname{Tor}^*_{\Gamma^{(i)}}({\Gamma^{(i)}}_{\sigma^{2n}}, \overline{E_0/u_{i-1}}) = 0$$

for $* > 0$.

Here $\overline{E_0/u_{i-1}}$ is E_0/u_{i-1} considered as a trivial left $\Gamma^{(i)}$-module, and the subscript in ${\Gamma^{(i)}}_{\sigma^{2n}}$ indicates that $\Gamma^{(i)}$ is considered as a right $\Gamma^{(i)}$-module via σ^{2n}. These Tor groups compute the linearization of the E_2-page of a spectral sequence computing the graded E-cohomology of configuration spaces on $\mathbb{R}^n$ (see Section 2).

We first set up some notations and conventions in Section 3. To prove Tor vanishing, we will show in Section 4 that at the prime 2, there are cofiber sequences relating $\Gamma^{(i)}$ to $\Gamma^{(i+1)}$, which is Koszul of shorter length.

Theorem B (Theorem 4.0.1)**.** *For $p = 2$, $0 \le i < h$, and $n \ge 0$, there is a cofiber sequence*

$${\Gamma^{(i)}}_{\sigma^{2(n+1)}} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}} \to {\Gamma^{(i)}}_{\sigma^{2n}} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}} \to (\Gamma^{(i)} /\!\!/ u_i)_{\sigma^{2n}} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} \overline{E_0/u_i}[1]$$

in the derived category of left $\Gamma^{(i)}$-modules.

Here $[-]$ is the shift functor and $\Gamma^{(i)} /\!\!/ u_i$ is the quotient of $\Gamma^{(i)}$ with respect to the right E_0/u_{i-1}-module structure and is generated from $\Gamma^{(i+1)}$ by an additional operation corresponding to the Frobenius isogeny.

In Section 5, we will describe $\Gamma^{(i)} /\!\!/ u_i$ in terms of isogenies and show that it is a free right $\Gamma^{(i+1)}$-module.

Proposition C (Proposition 5.4.1)**.** *For all primes p, the formal scheme $\operatorname{Sub}_k^{(i)}(\mathbb{G}) /\!\!/ u_i$ corresponding to the dual $\Gamma^{\vee(i)}[k] /\!\!/ u_i$ is the union of closed subschemes*

$$\operatorname{Sub}_k^{(i)}(\mathbb{G}) /\!\!/ u_i = X_k \cup_{X_k/u_{i+1}^{p^{k-1}}} X_{k-1} \cup_{X_{k-1}/u_{i+1}^{p^{k-2}}} \cdots \cup_{X_1/u_{i+1}} X_0$$

where $X_s = \operatorname{Sub}_s^{(i+1)}(\varphi^s \mathbb{G})$ and φ is the Frobenius. This becomes a disjoint union when restricted to a fixed height by inverting u_{i+1}.

We will use this to show that as a right $\Gamma^{(i+1)}$-module, $\Gamma^{(i)} /\!\!/ u_i$ can be expressed as a direct sum of left Frobenius twisted $\Gamma^{(i+1)}$. This allows us to inductively prove Tor vanishing in Section 6 where it is obvious for the base case $\Gamma^{(h)}$.

In Section 7, we use the decomposition in Proposition C to give a congruence criterion analogous to Rezk's, as promised in [SS25]. In Section 8, using a variant of the cofiber sequence in Theorem B, we will show that the Ext groups $\operatorname{Ext}^*_\Gamma(\omega^m, \overline{E}_0)$ considered by [Rez13], where $\omega^m := \pi_{2m}E$ is acted on by Γ^{-2m}, do not vanish. These Ext groups compute (the

E_2-page of a mapping space spectral sequence converging to) $E^*\left(TAQ_{S_{K(n)}}\left(S_{K(n)}^{S_+^{2m-1}}\right)\right)$, the E-cohomology of the topological André-Quillen homology of $S_{K(n)}$-valued cochains on odd spheres. [BR19] shows that this is equivalent to the Morava E-theory $E_*^\wedge(\Phi S^{2m-1})_{K(n)}$ of the Bousfield-Kuhn functor on odd spheres. Rezk showed that for heights $n = 1, 2$ and $a \geq 1$, the groups $\mathrm{Ext}^s_\Gamma(\omega^m, \overline{E}_0)$ vanish except when $s = n$, thus the spectral sequence collapses. We show that this is not true at higher heights.

Proposition D (Proposition 8.0.1)**.** *For $p = 2$ and heights $n \geq 2$,*

- *If $m = 0$, $\mathrm{Ext}^s_\Gamma(\omega^m, null) = 0$ for all s.*
- *If $m = 1$, $\mathrm{Ext}^s_\Gamma(\omega^m, null) \neq 0$ for all $2 \leq s \leq n$.*
- *If $m > 1$, $\mathrm{Ext}^s_\Gamma(\omega^m, null) \neq 0$ for $s = 2$ and n.*

Finally, in Section 9, we consider the Tor groups

$$\mathrm{Tor}^*_{\Gamma^{(i)}}(\overline{E_0/u_{i-1}}, {}_{\sigma^{2n}}\Gamma^{(i)})$$

obtained by swapping the terms of the Tor groups in Theorem A. Interestingly, they do not have nice vanishing properties, even at $p = 2$ and height 2.

Acknowledgements. TODO

2 Motivation

Here we will explain the statement that the Tor groups compute the linearization of the Morava E-theory of configuration spaces on $\mathbb{R}^n$. For a functor $F : \mathcal{A} \to \mathcal{B}$, [Rez17, Section 5] defines its linearization to be a functor $\mathcal{L}_F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{L}_F(X)$ is the coequalizer

$$F(X \oplus X) \underset{F(p_1)+F(p_2)}{\overset{F(p_1+p_2)}{\rightrightarrows}} F(X)$$

where p_1 and p_2 are the projections.

All operads will be nonunital. For $\mathcal{O}$ an operad, the operadic suspension $s\mathcal{O}$ [ACBH25, Def 2.3] is defined such that the free algebra monad for $s\mathcal{O}$ is given by $\mathrm{free}_{s\mathcal{O}} = \Omega \mathrm{free}_{\mathcal{O}} \Sigma$. This is analogous to the suspension σ on power operations. Let L denote the spectral Lie operad which is Koszul dual to the $\mathbb{E}_\infty$ operad; it's operadic suspension sL is the usual Lie operad. [ACBH25] shows that there is an equivalence of symmetric sequences

$$1 \circ_L s^n L \to \mathbb{E}_n$$

where $\circ$ is the composition product. This recovers, for any spectrum X, the equivalence of spectra [Knu18]

$$|\mathrm{Bar}(1, L, \mathrm{free}_{s^n L}(X))| \simeq \mathrm{free}_{\mathbb{E}_n}(X) \simeq \bigoplus_{k \geq 0} B_k(\mathbb{R}^n; X),$$

hence a bar spectral sequence used in [BHK24] whose E^2-page are Lie algebra indecomposables and converges to the free $\mathbb{E}_n$-algebra on X. Considering $|\mathrm{Bar}(1, L, \mathrm{free}_{s^n L}(X))|$ as a graded spectrum, its graded (Spanier-Whitehead) dual is Koszul dual to $\mathrm{free}_{s^n L}(X)$. Here Koszul duality is considered in the graded sense [Heu24, 14.4].

Lemma 2.0.1. *If X is a positively graded spectrum which is levelwise dualizable, then* $\mathrm{free}_{s^nL}(X)$ *is Koszul dual to* $\Sigma^n_{\mathbb{E}_\infty}\mathrm{triv}_{\mathbb{E}_\infty}(\Sigma^{-n}X^\vee)$*, the n-fold suspension in $\mathbb{E}_\infty$-algebras of the trivial $\mathbb{E}_\infty$-algebra (i.e. square zero extension) on $\Sigma^{-n}X^\vee$.*

Proof. The Koszul dual of an operad $\mathcal{O}$ is $K\mathcal{O} := (B\mathcal{O})^\vee$, the levelwise dual of the cooperad $B\mathcal{O} := \mathrm{Bar}(1, \mathcal{O}, 1)$. Koszul duality is the composite

$$\mathrm{Alg}_{\mathcal{O}} \xrightarrow{\mathrm{indec}_{\mathcal{O}}} \mathrm{coAlg}^{\mathrm{dp}}_{B\mathcal{O}} \xrightarrow{(-)^\vee} \mathrm{Alg}_{K\mathcal{O}}{}^{\mathrm{op}},$$

where $\mathrm{coAlg}^{\mathrm{dp}}_{B\mathcal{O}}$ is the category of divided power $B\mathcal{O}$-coalgebras [Heu24] and $\mathrm{indec}_{\mathcal{O}} = \mathrm{Bar}(1, \mathcal{O}, -)$ is the $\mathcal{O}$-indecomposables functor. For graded spectra, $\mathrm{indec}_{\mathcal{O}}$ is an equivalence of categories by [Heu24, Thm 14.14] and the levelwise dual $(-)^\vee$ is an equivalence of categories on levelwise dualizable objects.

We apply this to $\mathcal{O} = L$, where BL is the cocommutative cooperad and $KL = \mathbb{E}_\infty$. We have

$$\mathrm{indec}_L(\mathrm{free}_{s^mL}(X)) = \mathrm{indec}_L(\Omega^n_L \mathrm{free}_L \Sigma^n X) \simeq \Omega^n_{BL}\mathrm{indec}_L(\mathrm{free}_L \Sigma^n X) \simeq \Omega^n_{BL}\mathrm{triv}_{BL}(\Sigma^n X)$$

where triv_{BL} is the trivial BL-coalgebra. To show that $\Omega^n_{BL}\mathrm{triv}_{BL}(\Sigma^n X)$ dualizes to $\Sigma^n_{\mathbb{E}_\infty}\mathrm{triv}_{\mathbb{E}_\infty}(\Sigma^{-n}X^\vee)$, it suffices to show that it is levelwise dualizable.

Since finite products in cocommutative coalgebras are given by the tensor product, Ω_{BL} is computed by the cobar construction $\mathrm{cobar}(-) := \mathrm{cobar}(1, -, 1)$. By [Bur22, A.8,A.9], for a positively graded spectrum Y, $\mathrm{cobar}(Y)$ has a filtration with associated graded given by $(\Sigma^{-1}Y)^{\otimes k}$. Since $(\Sigma^{-1}Y)^{\otimes k}$ is concentrated in degrees $\geq k$, in any fixed degree, $\mathrm{cobar}(Y)$ is computed by a finite limit. Thus, if Y is levelwise dualizable, so is $\Omega_{BL}(Y) \simeq \mathrm{cobar}(Y)$. Applying this repeatedly, we conclude that $\Omega^n_{BL}\mathrm{triv}_{BL}(\Sigma^n X)$ is levelwise dualizable. □

In particular, this holds in the case where $X = S^\ell$ is a sphere. We apply Morava E-theory, where Rezk's monad $\mathbb{T}$ act as an algebraic approximation to the $\mathbb{E}_\infty$ monad. [Rez09] shows that the primitives Γ of the Hopf algebra $\mathbb{T}E_0$ is (noncanonically) isomorphic to the indecomposables Δ, so we can consider Γ as the linearization of $\mathbb{T}E_0$. The Tor groups above are thus the linearization of the bar complex

$$\Sigma^n \mathrm{Bar}\left(s^n\mathbb{E}_\infty, \mathbb{E}_\infty, \mathrm{triv}_{\mathbb{E}_\infty}(\Sigma^{-n-\ell}E)\right) \simeq \mathrm{Bar}\left(\mathbb{E}_\infty, s^{-n}\mathbb{E}_\infty, \mathrm{triv}_{s^{-n}\mathbb{E}_\infty}(\Sigma^{-\ell}E)\right)$$

which computes $\Sigma^n_{\mathbb{E}_\infty}\mathrm{triv}_{\mathbb{E}_\infty}(\Sigma^{-n-\ell}E)$. This is the E_2-page of the cohomological bar spectral sequence computing the graded E-cohomology of $\mathrm{free}_{\mathbb{E}_n}(S^\ell) \simeq B(\mathbb{R}^n; S^\ell)$. The Tor groups in Theorem A are the linearizations of this.

3 Preliminaries

We will freely use the notations in [SS25]. All spectra will be implicitly p-localized. Let $E = E(k, \mathbb{G}_0)$ be a Morava E-theory spectrum associated to a perfect field k of characteristic p and a formal group $\mathbb{G}_0$ of height $0 < h < \infty$ over k. It is an even periodic complex orientable spectrum whose associated formal group $\mathbb{G}$ is the universal deformation of G_0. The coefficient ring of E is given by

$$\pi_* E = \mathbb{W}k[[u_1, \cdots, u_{h-1}]][u^\pm]$$

where $\mathbb{W}k$ is the Witt vectors on k, $u \in \pi_2 E$, and $u_1, \cdots, u_{h-1} \in \pi_0 E$ are the Lubin-Tate parameters pushed down to degree 0. We also set $u_0 := p$ and write $(-)/u_i$ for the quotient $(-)/p, \cdots, u_i$.

We fix a p-typical complex orientation on E and let $t \in E^0(\mathbb{CP}^\infty)$ be the orientation class pushed down to degree 0. The Euler class of the reduced complex standard representation of Σ_p is given by

$$x = \prod_{i=1}^{p-1} [i](t) \equiv \prod_{i=1}^{p-1} \omega^i(t) \equiv (-1)^p t^{p-1} \mod [p](t)$$

where ω is a primitive $(p-1)$st root of unity. Recall the definitions of $\langle p \rangle^{(i)}$ and $f^{(i)}$ in [SS25, Def 4.1.1]

We will use the handedness convention in [SS25, Remark 2.0.2].

3.1 Review of power operations

The algebra Γ of additive power operations acting on π_0 of $K(h)$-local $\mathbb{E}_\infty$-E-algebras is a direct sum of weight p^k operations $\Gamma = \bigoplus_{k\geq 0} \Gamma[k]$. Each $\Gamma[k]$ is both a left and right E_0-module and $E_0 = \Gamma[0]$ is not central in Γ. Its dual $\Gamma^\vee$ has an algebro-geometric interpretation in terms of isogenies of formal groups. Let $\Gamma^\vee[k]$ be the dual of $\Gamma[k]$ with respect to the left E_0-module structure. Then

$$\Gamma^\vee[k] = E^0(B\Sigma_{p^k})/\mathrm{tr} \cong \mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})}$$

classifies degree p^k subgroups of $\mathbb{G}$. Here tr is the ideal generated by the images of the transfer maps associated to the subgroups $\Sigma_i \times \Sigma_{p^k-i} \subset \Sigma_{p^k}$ for $0 < i < p^k$. Let $s, t : E_0 \to \Gamma^\vee[k]$ denote the usual inclusion and the total power operations map P, respectively. On functor of points, s is the source map sending $H \leq \mathbb{G}$ to $\mathbb{G}$ and t is the target map sending $H \leq \mathbb{G}$ to $\mathbb{G}/H$. These give left and right E_0-module structures on $\Gamma^\vee[k]$ dual to those on $\Gamma[k]$. The tensor product $\Gamma^\vee[k_1]{}_t\otimes_{E_0\, s} \cdots {}_t\otimes_{E_0\, s}\Gamma^\vee[k_q]$ classifies the following equivalent data:

- a sequence of subgroups $H_1 < \mathbb{G}_1, H_2 < \mathbb{G}_2, \ldots, H_q < \mathbb{G}_q$ where $\deg H_i = p^{k_i}$, $\mathbb{G}_1 = \mathbb{G}$, and $\mathbb{G}_{i+1} = \mathbb{G}_i/H_i$.
- a chain of subgroups $H_1 \leq \cdots \leq H_q < \mathbb{G}$ where $\deg H_i = p^{k_1+\cdots+k_i}$.
- a sequence $f_1, \ldots, f_{i-1}$ of composable isogenies starting from $\mathbb{G}$ where $\deg f_i = p^{k_i}$.

The multiplicative structure $\Gamma[k_1]_t \otimes_{E\ s} \Gamma[k_2] \to \Gamma[k_1 + k_2]$ on Γ coming from composing power operations is dual to comultiplication $\Gamma^\vee[k_1 + k_2] \to \Gamma^\vee[k_1]_t \otimes_{E\ s} \Gamma^\vee[k_2]$ on $\Gamma^\vee$ coming from composing isogenies.

For each i, there is an algebra of power operations $\Gamma^{(i)} = \bigoplus_{k\geq 0} \Gamma^{(i)}[k]$ which is a subquotient of Γ acting on $\pi_0(-/u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras [SS25]. We have $\Gamma^{(0)} = \Gamma$. Each $\Gamma^{(i)}[k]$ is both a left and right E_0/u_{i-1}-module and $E_0/u_{i-1} = \Gamma^{(i)}[0]$ is not central in $\Gamma^{(i)}$. Its left E_0/u_{i-1}-dual $\Gamma^{\vee(i)}[k]$ of $\Gamma^{(i)}[k]$ is a quotient of $\Gamma^\vee[k]$ with left and right E_0/u_{i-1}-module structures induced by the source and target maps, also denoted s, t. It is finitely generated and free (finite free) as both a left and right E_0/u_{i-1}-module. When restricted to a fixed height by inverting u_i, $u_i^{-1}\Gamma^{\vee(i)}[k]$ classifies degree p^k étale subgroups. We write $\otimes$ for ${}_t\otimes_{E_0/u_{i-1}\ s}$; the ring ($E_0/u_{i-1}$ here) over which the tensor product is taken will be clear depending on the context. The tensor product $\Gamma^{\vee(i)}[k_1] \otimes \cdots \otimes \Gamma^{\vee(i)}[k_q]$ classifies a sequence of subgroups $H_1 < \mathbb{G}_1, \cdots, H_q < \mathbb{G}_q$ where each $H_s < \mathbb{G}_s$ avoids $\ker \varphi^i$ ([SS25, Def 4.4.3]). We do not know if this is the same as a chain of subgroups $H_1 \leq \cdots \leq H_q < \mathbb{G}$ with each H_s avoiding φ^i, but they are equivalent after inverting u_i [SS25, Rmk 4.4.8].

$\Gamma^{(i)}$ can also be described in terms of the Mackey functor $Q^{(i)}$ [SS25, Def 4.2.1] given in terms of the generalized Tate construction. For a poset X with minimal and maximal elements 0 and 1, following [Rez17, Section 6.1], let $\overline{X}$ denote the quotient $X/(\hat{X} \cup \check{X})$ where $\hat{X}$ and $\check{X}$ are the maximal subposets of X not containing 0 and 1, respectively. Explicitly, this is the poset whose nonbasepoint nondegenerate q-simplices are chains $[0 = x_0 < x_1 < \cdots < x_q = 1]$. With this notation, there is an isomorphism of cosimplicial E_0/u_{i-1}-algebras [SS25, Prop 4.2.8]

$$Q_k^{(i)}(\overline{P}_m^\vee) \xrightarrow{\simeq} \overline{\mathcal{B}}(\Gamma^{\vee(i)})[k]$$

where $m = p^k$, P_m is the partition complex and $\overline{\mathcal{B}}(\Gamma^{\vee(i)})[k]$ is the weight k part of the cobar complex for $\Gamma^{\vee(i)}$. This implies that $\Gamma^{(i)}$ is Koszul of length $h+1-i$. In particular, it is generated by operations in $\Gamma^{(i)}[1]$, whose dual $\Gamma^{\vee(i)}[1]$ is the closure of $\mathbb{G}[p] - 0$ in $\mathbb{G}[p]$, the subgroup of p-torsion points. The Koszul terms $C^{(i)}[k] = H_k\overline{\mathcal{B}}(\Gamma^{(i)})[k]$ are finite free left E_0/u_{i-1}-modules and their left E_0/u_{i-1}-duals will be denoted $C^{\vee(i)}[k]$.

There is also an inflated version $\Gamma^{(i)}/\!\!/u_i$ of $\Gamma^{(i+1)}$ ([SS25, Section 6]), where $/\!\!/$ denotes the quotient with respect to the right E_0/u_{i-1}-module structure. It is the largest collection of operations in $\Gamma^{(i)}$ which are defined modulo u_i and contains $\Gamma^{(i+1)}$ as a subalgebra. It is generated by $\Gamma^{(i+1)}$ together with an additional operation.

3.2 The suspension map

For each integer $q \in \mathbb{Z}$, we write Γ^{-q} for Rezk's ring of additive degree q power operations which acts naturally on π_q of $K(h)$-local $\mathbb{E}_\infty$-E-algebras [Rez13]. The ring Γ^{-q} is endowed with a canonical weight grading $\Gamma^{-q} = \bigoplus_{k\geq 0} \Gamma^{-q}[k]$ where

$$\Gamma^{-q}[k] \cong \ker\left(E_q^\wedge B\Sigma_m^{q\rho_m} \xrightarrow{\mathrm{tr}} \bigoplus_{0<i<m} E_q^\wedge \left(B\Sigma_i \times \Sigma_{m-i}\right)^{q\rho_{p^k}}\right).$$

Here $m = p^k$, ρ_m is the real standard representation of Σ_m and $E_*^\wedge = \pi_* L_{K(h)}(E \otimes -)$ is the completed homology. There are analogs $\Gamma^{(i)^{-q}}$ of Γ^{-q} acting on $\pi_q(-/u_{i-1})$ [SS25, Rem 4.4.2].

There are suspension homomorphisms $\Gamma^{-q} \to \Gamma^{-(q+1)}$ ([Rez17]) induced by the inclusion $1 \oplus q \cdot \rho_m \to (q+1)\cdot\rho_m$. Explicitly, this is the restriction of the map

$$E_q^\wedge((\mathbb{S}^q)^{\otimes m}_{h\Sigma_m}) \simeq E_{q+1}^\wedge(\mathbb{S}^1 \otimes_{h\Sigma_m} (\mathbb{S}^q)^{\otimes m}) \to E_{q+1}^\wedge((\mathbb{S}^{q+1})^{\otimes m}_{h\Sigma_m}).$$

Since the suspension map is an isomorphism if q is odd, we will only consider double suspensions $\Gamma^{-2q} \to \Gamma^{-2(q+1)}$. We will denote this double suspension map by σ, refer to it as "the suspension map", and regard it as a homomorphism $\Gamma \hookrightarrow \Gamma$ since all the Γ^q's are isomorphic. **Warning:** this is different from the notation used in the introduction and is the one which will be used from now on. For each k, $\sigma : \Gamma^{-2q}[k] \to \Gamma^{-2(q+1)}[k]$ is dual to $\Gamma^\vee[k] \xrightarrow{\cdot c_{p^k}} \Gamma^\vee[k]$, where c_{p^k} is the Euler class of the reduced complex standard representation of Σ_{p^k} [BHK24, Proposition 4.1].

There are mod $p, \cdots, u_{i-1}$ analogs $\sigma : \Gamma^{(i)} \to \Gamma^{(i)}$ given on left E_0/u_{i-1}-duals $\Gamma^{\vee(i)}[k] \xrightarrow{\cdot c_{p^k}} \Gamma^{\vee(i)}[k]$ by multiplication by the Euler class. Recall [SS25, Section 6] that $\Gamma^{(i)}/\!\!/u_i$ is the quotient of $\Gamma^{(i)}$ by u_i with respect to the right E_0/u_{i-1}-module structure. $\Gamma^{(i)}/\!\!/u_i$ is a left

$\Gamma^{(i)}$-module by the projection $\Gamma^{(i)} \to \Gamma^{(i)}/\!\!/u_i$ [SS25, Lemma 6.0.7] and a $\Gamma^{(i+1)}$-bimodule by the inclusion $\Gamma^{(i+1)} \subset \Gamma^{(i)}/\!\!/u_i$ of E_0/u_i-algebras [SS25, Lemma 6.0.6]. These correspond to the fact that we can compose power operations. Since σ is a ring homomorphism whose restriction to $\Gamma^{(i)}[0] = E_0/u_{i-1}$ is the identity, it is a map of E_0/u_{i-1}-bimodules, so it descends to a map σ on $\Gamma^{(i)}/\!\!/u_i$.

4 The cofiber sequence

In this section, we show that for $p = 2$, there are cofiber sequences relating $\Gamma^{(i)}$ to $\Gamma^{(i+1)}$. Let $\overline{E_0/u_{i-1}}$ denote E_0/u_{i-1} with the trivial $\Gamma^{(i)}$-action and let [1] denote the shift functor.

Theorem 4.0.1. *For $p = 2$, $0 \leq i < h$, and $n \geq 0$, there is a cofiber sequence*

$$\Gamma^{(i)}{}_{\sigma^{n+1}} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} (\overline{E_0/u_{i-1}}) \to \Gamma^{(i)}{}_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} (\overline{E_0/u_{i-1}}) \to (\Gamma^{(i)}/\!\!/u_i)_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i})[1]$$

in the derived category of left $\Gamma^{(i)}$-modules.

Here the subscript σ^n indicates that the right $\Gamma^{(i)}$-module structure is given by the n-fold (double) suspension σ^n. The map $\Gamma^{(i)}{}_{\sigma^{n+1}} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}} \to \Gamma^{(i)}{}_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}}$ is given by the suspension map $\sigma : \Gamma^{(i)} \to \Gamma^{(i)}$ on the base ring. In the next section, we will describe this as a map of Koszul complexes. Then, using the description of the Koszul terms as Steinberg summands of certain rings with algebro-geometric interpretations, we will show that $(\Gamma^{(i)}/\!\!/u_i)_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} \overline{E_0/u_i}[1]$ is its cofiber.

4.1 Description of the map

The first map in the cofiber sequence above is given by the suspension map $\sigma : \Gamma^{(i)} \to \Gamma^{(i)}$ on the base ring. We will describe this explicitly in terms of the Koszul complex. Since $\Gamma^{(i)}$ is Koszul and the Koszul terms $C^{(i)}[k]$ are finite free,

$$\cdots \to \Gamma^{(i)} \otimes C^{(i)}[k] \to \Gamma^{(i)} \otimes C^{(i)}[k-1] \to \cdots \to \Gamma^{(i)} \otimes C^{(i)}[0] \to 0 \tag{1}$$

is a Koszul resolution for $\overline{E_0/u_{i-1}}$ of finite free left $\Gamma^{(i)}$-modules. Explicitly, $C^{(i)}[k]$ is the kernel

$$C^{(i)}[k] = \ker\left(\sum_{i=1}^{k-1}(-1)^i d_i : \Gamma^{(i)}[1]^{\otimes k} \to \bigoplus_{i=1}^{k-1} \Gamma^{(i)}[1]^{i-1} \otimes \Gamma^{(i)}[2] \otimes \Gamma^{(i)}[1]^{\otimes k-i-1}\right) \tag{2}$$

where d_i is the face map multiplying the ith and $(i+1)$st terms. The boundary map $\partial : \Gamma^{(i)} \otimes C^{(i)}[k] \to \Gamma^{(i)} \otimes C^{(i)}[k-1]$ is induced by the first face map $\mu \otimes 1^{k-1} : \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k} \to \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k-1}$ of the bar complex $\mathcal{B}(\Gamma^{(i)}, \Gamma^{(i)}, E_0/u_{i-1})$ [BR19, (2.22)].

$\Gamma^{(i)}{}_\sigma \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E/u_{i-1}}$ is computed by the complex

$$\cdots \to \Gamma^{(i)}{}_\sigma \otimes C^{(i)}[k] \to \Gamma^{(i)}{}_\sigma \otimes C^{(i)}[k-1] \to \cdots \to \Gamma^{(i)}{}_\sigma \otimes C^{(i)}[0] \to 0. \tag{3}$$

Since σ is the identity on $E_0/u_{i-1} = \Gamma^{(i)}[0]$, the terms of this complex are equal to those of (1) and its boundary maps are induced by

$$\partial \circ \sigma^n : \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k} \xrightarrow{1 \otimes \sigma^n \otimes 1^{k-1}} \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k} \xrightarrow{\partial} \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k-1}.$$

Since $\sigma : \Gamma \to \Gamma$ is a ring map, $\sigma \otimes \sigma$ restricts to a map on the kernel of the multiplication map

$$\begin{tikzcd}
\ker \arrow[r, dashed, "\sigma\otimes\sigma"] \arrow[d, hook] & \ker \arrow[d, hook] \\
\Gamma^{(i)}[k] \otimes \Gamma^{(i)}[\ell] \arrow[r, "\sigma\otimes\sigma"] \arrow[d, "\mu"'] & \Gamma^{(i)}[k] \otimes \Gamma^{(i)}[\ell] \arrow[d, "\mu"] \\
\Gamma^{(i)}[k+\ell] \arrow[r, "\sigma"] & \Gamma^{(i)}[k+\ell],
\end{tikzcd}$$

thus $\sigma^{\otimes k} : \Gamma^{(i)}[1]^{\otimes k} \to \Gamma^{(i)}[1]^{\otimes k}$ restricts to a map on $C^{(i)}[k]$. The same is true for σ^n.

Definition 4.1.1. *The map $\Gamma^{(i)}{}_{\sigma^{n+1}} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}} \to \Gamma^{(i)}{}_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}}$ is given by the map of chain complexes*

$$1 \otimes \sigma^\bullet : \Gamma^{(i)}{}_{\sigma^{n+1}} \otimes C^{(i)}[\bullet] \to \Gamma^{(i)}{}_{\sigma^n} \otimes C^{(i)}[\bullet]$$

Taking left E_0/u_{i-1}-module duals gives a map of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & \Gamma^{\vee(i)}{}_{\sigma^{n+1}} \otimes C^{\vee(i)}[k-1] \arrow[r] & \Gamma^{\vee(i)}{}_{\sigma^{n+1}} \otimes C^{\vee(i)}[k] \arrow[r] & \cdots \\
0 \arrow[r] & \cdots \arrow[r] & \Gamma^{\vee(i)}{}_{\sigma^{n}} \otimes C^{\vee(i)}[k-1] \arrow[r] \arrow[u, "1\otimes\sigma^{\vee\otimes(k-1)}"] & \Gamma^{\vee(i)}{}_{\sigma^{n}} \otimes C^{\vee(i)}[k] \arrow[r] \arrow[u, "1\otimes\sigma^{\vee\otimes k}"'] & \cdots.
\end{tikzcd}$$

The suspension map $\sigma : \Gamma^{(i)}[1] \to \Gamma^{(i)}[1]$ is injective. Its dual is multiplication by the Euler class of the reduced complex standard representation of Σ_p, which is an injective map $\Gamma^{\vee(i)}[1] \to \Gamma^{\vee(i)}[1]$. Since $\Gamma^{(i)}[1]$ and $\Gamma^{\vee(i)}[1]$ are flat with respect to both the left and right module structures, the map of chain and cochain complexes are componentwise injective, so their cofibers can be computed by taking componentwise cokernels.

For a ring R, we denote by DMod_R the derived category of left R-modules. For $M \in \mathrm{DMod}_R$, let DM denote its derived dual. If M is discrete, let $M^\vee$ denote its underived dual.

Lemma 4.1.2. *Let M be a (discrete) finite free R-module and $f : M \to M$ an injective map whose R-module dual $f^\vee : M^\vee \to M^\vee$ is also injective. Suppose the cokernel M/f is an R/u-module for some nonzero divisor $u \in R$. Then, M/f and $M^\vee/f^\vee$ are dual as R/u-modules.*

Proof. The cofiber sequence $M \xrightarrow{f} M \to M/f$ in DMod_R induces a cofiber sequence

$$D(M/f) \to DM \to DM.$$

Since M is finite free, $DM \simeq M$, so $D(M/f)$ is concentrated in degree -1 with $\pi_{-1} = M^\vee/f^\vee$. Since M/f is an R/u-module, by [SS25, Lemma 6.0.3],

$$\Sigma D(M/f) \simeq D_{R/u}(M/f).$$

On π_0, this tells us that $M^\vee/f^\vee$ is the R/u-dual of M/f. ∎

We will compute the cokernel $K_2[k] := \mathrm{cok}(\sigma^{\vee\otimes k} : C^{\vee(i)}[k] \to C^{\vee(i)}[k])$ and show that it is an E_0/u_i-module. The lemma above lets us recover the cokernel

$$K_1[k] = \mathrm{cok}(\sigma^{\otimes k} : C^{(i)}[k] \to C^{(i)}[k])$$

as the E/u_i-dual of K_2. Thus, we have short exact sequences of chain (resp. cochain) complexes of left $\Gamma^{(i)}$-modules (resp. $\Gamma^{\vee(i)}$-comodules)

$$\Gamma^{(i)}{}_{\sigma^{a+1}} \otimes C^{(i)}[\bullet] \to \Gamma^{(i)}{}_{\sigma^a} \otimes C^{(i)}[\bullet] \to \Gamma^{(i)}{}_{\sigma^a} \otimes K_1[\bullet] \tag{4}$$

$$\Gamma^{\vee(i)}{}_{\sigma^a} \otimes C^{\vee(i)}[\bullet] \to \Gamma^{\vee(i)}{}_{\sigma^{a+1}} \otimes C^{\vee(i)}[\bullet] \to \Gamma^{\vee(i)}{}_{\sigma^a} \otimes K_2[\bullet] \tag{5}$$

with $K_1[k] \cong K_2^\vee[k]$ as E_0/u_i-modules, and all the others are dual as E_0/u_{i-1}-modules.

4.2 The Steinberg idempotent

We recall the Steinberg idempotent, following [San19]. Let $GL_k := GL_k(\mathbb{F}_p)$, and let Σ_k and U_k be the subgroups of permutation matrices and upper triangular matrices, respectively. Associated to these two subgroups are elements $\overline{\Sigma}_k$, $\overline{U}_k$ in the group ring $\mathbb{Z}_{(p)}[GL_k]$

$$\overline{\Sigma}_k = \sum_{\sigma \in \Sigma_k} (-1)^{\mathrm{sgn}(\sigma)} \sigma, \qquad \overline{U}_k = \sum_{u \in U_k} u.$$

For a left GL_k-action, the Steinberg idempotent is

$$e_k = \frac{1}{c_k} \cdot \overline{\Sigma}_k \overline{U}_k \in \mathbb{Z}[GL_k],$$

where

$$c_k = \prod_{i=1}^{k} (p^i - 1).$$

Since c_k is a unit in $\mathbb{Z}_{(p)}$, we will drop it from our notation and write $e_k = \overline{\Sigma}_k \overline{U}_k$. We will also implicitly p-localize everywhere and denote $\mathbb{Z}_{(p)}$ by $\mathbb{Z}$. The left $\mathbb{Z}[GL_k]$-module $\mathrm{St}_k = \mathbb{Z}[GL_k]e_k$ is called the Steinberg module. For a left $\mathbb{Z}[GL_k]$-module M, the $\mathbb{Z}_{(p)}$-module $e_k M = \{e_k m \mid m \in M\} \subseteq M$ is called the Steinberg summand of M. There is a natural isomorphism of $\mathbb{Z}_{(p)}$-modules [San19, Definition 2.2]

$$\begin{aligned} St_k \otimes_{\mathbb{Z}[GL_k]} M &\xrightarrow{\simeq} e_k M, \\ A e_k \otimes m &\mapsto e_k (A^{-1} m). \end{aligned}$$

There are analogous constructions for a right GL_k-action, where the Steinberg idempotent is $\hat{e}_k = \mathrm{unit} \cdot \overline{U}_k \overline{\Sigma}_k$ and $\mathrm{St}_k = e_k \mathbb{Z}[GL_k]$.

If $f : M \to N$ is a map of left $\mathbb{Z}[GL_k]$-modules, then $f(e_k m) = e_k f(n)$. Thus, f restricts to a map on Steinberg summands $f : e_k M \to e_k N$.

4.3 Parsing the terms of the Koszul complex

The terms of the Koszul complex are given by the Steinberg summand

$$C^{(i)}[k] \cong Q_k^{(i)}(\Sigma_m/\Delta_k) \otimes_{\mathbb{Z}[GL_k]} \mathrm{St}_k$$

where $m = p^k$ [Rez17, ADL16, SS25]. In the notation of [ADL16], $\Delta_k = \mathbb{F}_p{}^k$ with automorphism group $GL_k := GL_k(\mathbb{F}_p)$. Choosing an identification of Δ_k with $\mathbf{m} = \{1, \cdots, p^k\}$ and letting Δ_k act on itself by left translation identifies Δ_k as a subgroup of Σ_m with normalizer

$\mathrm{Aff}_k = \Delta_k \rtimes GL_k$. The Weyl group GL_k acts on Σ_m/Δ_k on the right. This gives a right (resp. left) action on the covariant (resp. contravariant) part of the Mackey functor $Q^{(i)}$. In particular, the E_0/u_{i-1}-dual

$$C^{\vee(i)}[k] \cong \mathrm{St}_k \otimes_{\mathbb{Z}[GL_k]} Q_k^{(i)}(\Sigma_m/\Delta_k) = e_k Q_k^{(i)}(\Sigma_m/\Delta_k)$$

for the left GL_k-action.

By definition, $C^{(i)}[k]$ is a subset of $Q_k^{(i)}(\Sigma_m/C_p \wr \cdots \wr C_p)$ (2). Since $Q_k^{(i)}(\Sigma_m/\Sigma_p \wr \cdots \wr \Sigma_p)$ is a direct summand of $Q_k^{(i)}(\Sigma_m/C_p \wr \cdots \wr C_p)$, $C^{(i)}[k]$ can be considered as a direct summand of the latter. We claim that the identification of $C^{(i)}[k]$ with the Steinberg summand is induced by a Σ_p-equivariant map $\Sigma_m/\Delta_k \to \Sigma_m/C_p \wr \cdots \wr C_p$.

Proposition 4.3.1. *An identification $\Delta_k \leftrightarrow \boldsymbol{m}$ gives a map $\Sigma_m/\Delta_k \to \Sigma_m/C_p \wr \cdots \wr C_p$ of Σ_m-sets.*

- *The covariant part of $Q_k^{(i)}$ gives a map $Q_k^{(i)}(\Sigma_m/\Delta_k) \to Q_k^{(i)}(\Sigma_m/C_p \wr \cdots \wr C_p)$ under which $Q_k^{(i)}(\Sigma_m/\Delta_k) \otimes_{\mathbb{Z}[GL_k]} \mathrm{St}_k$ corresponds to $C^{(i)}[k]$.*
- *The contravariant part of $Q_k^{(i)}$ gives a map $Q_k^{(i)}(\Sigma_m/C_p \wr \cdots \wr C_p) \to Q_k^{(i)}(\Sigma_m/\Delta_k)$ under which $C^{\vee(i)}[k]$ corresponds to $\mathrm{St}_k \otimes_{\mathbb{Z}[GL_k]} Q_k^{(i)}(\Sigma_m/\Delta_k)$.*

Proof. We trace through the proof in [ADL16]. Let B be the nerve of the poset of proper nontrivial subgroups of Δ_k, which is the Tits building for GL_k and has a right action by Aff_k. There is an isomorphism of right $\mathbb{Z}[GL_k]$-modules [San19, Proposition 2.7], [Rez17, 6.1]

$$\mathrm{St}_k = e_k\mathbb{Z}[GL_k] \cong H_{k-2}(B; \mathbb{Z}_{(p)}) \cong H_k(\overline{B}; \mathbb{Z}_{(p)}).$$

Let $G = \Sigma_m$, $N = \mathrm{Aff}_k$, $W = GL_k$, and $Q = Q^{(i)}$. Define an N-equivariant map $B \to P := P_m$ by assigning to a subgroup $V \leq \Delta_k$, the partition of $\mathbf{m}$ given by the cosets of V in Δ_k (there is a version for each of the variants in [Rez17, 6.1]). This extends to a G-equivariant map $G \times_N (EW \times B) \to P$ which is an isomorphism in Bredon (co)homology for the Mackey functors $Q^{(i)}$ [ADL16, Thm 1.1].

We will consider spaces as simplicial sets and consider the covariant part of Q (the contravariant part is analogous). For a G-simplicial set X whose terms are finite G-sets, the Bredon chains (resp. cochains) [ADL16, Section 2] is equal to the complex $Q(X)$ coming from the covariant (resp. contravariant) part of Q. Since we are only dealing with finite G-sets, we will identify the two. The G-equivariant map $G \times_N (EW \times \overline{B}) \to \overline{P}$ factors through the projection onto $G \times_N \overline{B}$. This induces a map of chain complexes

$$Q(G \times_N (EW \times \overline{B})) \to Q(G \times_N \overline{B}) \to Q(\overline{P}) \tag{6}$$

hence a map of normalized chain complexes $\mathcal{N}Q(-)$ (obtained by quotienting out by the image of degeneracy maps) [Rez17, 4.1].

The set of q-simplices X_q of a G-simplicial set X is the disjoint union of the G-invariant sets of degenerate and nondegenerate simplices, denoted X_q^0 and X_q^1, respectively.

- $\overline{B}$: Since $\overline{B}$ is a W-simplicial set (hence an N-simplicial set), $\overline{B}_q = \overline{B}_q^0 \sqcup \overline{B}_q^1$. Elements of $\overline{B}_k^1$ are complete flags $0 \subset \mathbb{F}_p \subset \cdots \subset \mathbb{F}_p^k$. W acts transitively on $\overline{B}_k^1$ with stabilizer U_k, so $\overline{B}_k^1 = N/\Delta_k \rtimes U_k$ and $G \times_N \overline{B}_k^1 = G/\Delta_k \rtimes U_k$. Since $\overline{B}_{k+1}^1 = \emptyset$, there is an inclusion $H_k Q(G \times_N \overline{B}) \hookrightarrow \mathcal{N}_k Q(G \times_N \overline{B}) = Q(G/\Delta_k \rtimes U_k)$, where the latter is the kth term of the complex $\mathcal{N}Q(G \times_N \overline{B})$.

- $\mathrm{St_k}$: Since $\mathrm{St_k}$ is the kth homology of $\overline{B}$, there is an inclusion $\mathrm{St}_k \hookrightarrow \mathbb{Z}[\overline{B}^1_k]$.
- $G \times_N (EW \times \overline{B})$: Since $EW \times \overline{B}$ is a W-simplicial set, $(EW \times \overline{B})_q = (EW \times \overline{B})^0_q \sqcup (EW \times \overline{B})^1_q$ as a W-set. Since W acts freely on $EW \times \overline{B}$, there are natural isomorphisms

$$Q(G \times_N (EW \times \overline{B})) \simeq Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[EW \times \overline{B}]$$
$$\mathcal{N}Q(G \times_N (EW \times \overline{B})) \simeq Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathcal{N}\mathbb{Z}[EW \times \overline{B}]$$

of complexes [ADL16, Example 2.3] and the canonical map

$$Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} H_k(\overline{B}) \to H_k\left(Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathcal{N}\mathbb{Z}[EW \times \overline{B}]\right)$$

is an isomorphism [ADL16, Proof of Corollary 1.2]. Since $H_k(\overline{B}) = \mathrm{St}_k$ is a projective $\mathbb{Z}[W]$-module, the inclusion $H_k(\overline{B}) \hookrightarrow \mathbb{Z}[(EW \times \overline{B})^1_k]$ induces an inclusion

$$Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} H_k(\overline{B}) \xrightarrow{\simeq} H_k \hookrightarrow Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[(EW \times \overline{B})^1_k]$$

The G-equivariant map $G \times_N \overline{B} \to \overline{P}$ induces a G-equivariant map $G/\Delta_k \rtimes U_k \to G/\Sigma_p \wr \cdots \wr \Sigma_p$ on nondegenerate k-simplices which restricts to a G-equivariant map $G/\Delta_k \to G/C_p \wr \cdots \wr C_p$. Since we are working with the covariant part of Q, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{N}_k Q(G \times_N (EW \times \overline{B})) \arrow[r] \arrow[d, equal] & \mathcal{N}_k Q(G \times_N \overline{B}) \arrow[r] \arrow[d, equal] & \mathcal{N}_k Q(\overline{P}) \arrow[d, equal] \\
Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[(EW \times \overline{B})^1_k] \arrow[r] \arrow[dr, dashed] & Q(G/\Delta_k \rtimes U_k) \arrow[r] & Q(G/\Sigma_p \wr \cdots \wr \Sigma_p) \\
& Q(G/\Delta_k) \arrow[r] \arrow[u] & Q(G/C_p \wr \cdots \wr C_p). \arrow[u]
\end{tikzcd}$$

We want to show that the dotted map exists. The leftmost horizontal map is induced by the projection $EW \times \overline{B} \to \overline{B}$. Since W acts freely on the normalized complex of $EW \times \overline{B}$, $(EW \times \overline{B})^1_k = (EW_k \times \overline{B}^1_k) \sqcup (EW^1_k \times \overline{B}^0_k)$ is the disjoint union of disjoint unions of W-sets $W/1$. On each component $W/1$, the map $EW_k \times \overline{B}^1_k = \sqcup W/1 \to W/U_k = \overline{B}^1_k$ differs from the canonical projection $W/1 \to W/U_k$ by an automorphism of $W/1$, so it factors as

$$\bigsqcup W/1 \to W/1 \to W/U_k.$$

This induces a factorization

$$Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[EW_k \times \overline{B}^1_k] \cong \bigoplus Q(G/\Delta_k) \to Q(G/\Delta_k) \to Q(G/\Delta_k \rtimes U_k).$$

Since the image of $EW^1_k \times \overline{B}^0_k$ is degenerate in $\overline{B}$, $Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[EW^1_k \times \overline{B}^0_k] \to \mathbb{Z}[\overline{B}^1_k]$ is the zero map. Putting these together gives a factorization

$$Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[(EW \times \overline{B})^1_k] \to Q(G/\Delta_k) \to Q(G/\Delta_k \rtimes U_k)$$

where the first map is induced by tensoring with $Q(G/\Delta_k)$ the $\mathbb{Z}[W]$-module map $\mathbb{Z}[(EW \times \overline{B})^1_k] \cong \bigoplus \mathbb{Z}[W] \to \mathbb{Z}[W]$, thus the dotted map exists.

Since the normalized chain complexes are truncated above, there are inclusions from the top degree homology to the top nonzero terms

$$\begin{tikzcd}
 & & Q(G/C_p \wr \cdots \wr C_p) \arrow[r] & Q(G/\Sigma_p \wr \cdots \wr \Sigma_p) \\
Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[(EW \times \overline{B})^1_k] \arrow[r] & Q(G/\Delta_k) \arrow[r] \arrow[ur] & Q(G/\Delta_k \rtimes U_k) \arrow[ur] & \\
Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} H_k(\overline{B}) \arrow[u, hook] \arrow[rr, hook] & & H_k Q(G \times_N \overline{B}) \arrow[u, hook] \arrow[r, two heads] & H_k Q(P) = C^{(i)}[k]. \arrow[uu, hook]
\end{tikzcd}$$

Since $Q(G/\Sigma_p \wr \cdots \wr \Sigma_p)$ is a direct summand of $Q(G/C_p \wr \cdots \wr C_p)$, $H_k Q(P)$ includes into the latter as a direct summand. The composite (6) induces an isomorphism in homology, so the bottom horizontal composite is an isomorphism and the map $Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} H_k(\overline{B}) \to Q(G/\Delta_k)$ is injective. This is the map taking the Steinberg summand of $Q(G/\Delta_k)$ since it is induced by a $\mathbb{Z}[W]$-module map $H_k(\overline{B}) = \mathrm{St}_k \to \mathbb{Z}[W]$.

Extracting the relevant parts gives the following conclusion. An identification $\Delta_k \leftrightarrow \mathbf{m}$ gives an inclusion $\Delta_k \to \Sigma_m$. Since W acts on G/Δ_k on the right, for the covariant part of Q, there is a right action of W on $Q(G/\Delta_k)$ and the canonical map $Q(G/\Delta_k) \to Q(G/C_p \wr \cdots \wr C_p)$ maps $Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathrm{St}_k$ isomorphically onto $C^{(i)}[k]$.

$$\begin{tikzcd}
Q(G/\Delta_k) \arrow[r] & Q(G/C_p \wr \cdots \wr C_p) \\
Q(G/\Delta_k) \otimes_{\mathbb{Z}[W]} \mathrm{St}_k \arrow[u, hook] \arrow[r, "\simeq"] & C^{(i)}[k] \arrow[u, hook]
\end{tikzcd}$$

Dually, for the contravariant part of Q, there is a left W-action on $Q(G/\Delta_k)$ and the map $Q(G/C_p \wr \cdots \wr C_p) \to Q(G/\Delta_k)$ maps $C^{\vee(i)}[k]$ isomorphically onto $\mathrm{St}_k \otimes_{\mathbb{Z}[W]} Q(G/\Delta_k)$. $\square$

We choose an identification $\Delta_k \leftrightarrow \mathbf{m}$ such that Δ_k embeds diagonally into $C_p \wr \cdots \wr C_p$. This is the inclusion which is defined inductively by

$$\mathbb{F}_p{}^{k-1} \times \mathbb{F}_p \to (C_p \wr \cdots \wr C_p) \times \mathbb{F}_p \xrightarrow{\Delta \times 1} (C_p \wr \cdots \wr C_p)^p \rtimes C_p = C_p \wr \cdots \wr C_p. \tag{7}$$

By [SS25, Cor 3.4.2, Prop 4.2.8], there are isomorphisms

$$\overline{\mathcal{B}}(\Gamma^{(i)})[k] \xrightarrow{\simeq} (Q^{(i)}_k(\overline{P}_m), Q^{(i)}_k(\mathrm{res})) \simeq (Q^{(i)}_k(\overline{P}_m), E^0(\mathrm{tr}))$$
$$(Q^{(i)}_k(\overline{P}^\vee_m), Q^{(i)}_k(\mathrm{tr})) \simeq (Q^{(i)}_k(\overline{P}^\vee_m), E^0(\mathrm{res})) \xrightarrow{\simeq} \overline{\mathcal{B}}(\Gamma^{\vee(i)})[k]$$

of simplicial left E_0/u_{i-1}-modules and cosimplicial E_0/u_{i-1}-algebras, respectively. Unless otherwise stated, we will be working with the contravariant part $(Q^{(i)}_k(\overline{P}^\vee_m), E^0(\mathrm{res}))$ since it has an algebra structure. We have, and will continue to abuse notation by identifying P_m with its dual $P^\vee_m$. We will consider $Q^{(i)}_k(G/H)$ as a quotient of $E^0(BH)$ and say that a map $f : G/H \to G/K$ induces a map $Q^{(i)}_k(G/K) \to Q^{(i)}_k(G/H)$ if it is the quotient of $E^0(f) : E^0(BK) \to E^0(BH)$; this is the map $Q^{(i)}_k(f^\vee)$.

4.3.1 Understanding $Q_k^{(i)}(\Sigma_m/\Delta_k)$

In this section, we write down an explicit formula for $Q_k^{(i)}(\Sigma_m/\Delta_k)$ consider its algebro-geometric description. We first consider the case before modding out by $p, \cdots, u_{i-1}$. Under our choice of identification $\Delta_k \leftrightarrow \mathbf{m}$ (7), the inclusion $\Delta_k = {\mathbb{F}_p}^k \to C_p \wr \cdots \wr C_p$ induces a ring map

$$Q_k(\Sigma_m/C_p \wr \cdots \wr C_p) \to Q_k(\Sigma_m/{\mathbb{F}_p}^k)$$

on the contravariant part of the Mackey functor $Q_k = Q_k^{(0)}$.

Notation 4.3.2. If $f(t) = \sum c_i t^i \in R[[t]]$ is a power series and $\phi : R \to S$ is a ring map, we denote by f^ϕ the power series

$$f^\phi(t) = \sum \phi(c_i) t^i \in S[[t]]$$

obtained by applying ϕ to the coefficients of f.

By the argument in Proposition [SS25, Prop 3.3.4], $Q_k(\Sigma_m/C_p \wr \cdots \wr C_p)$ is the quotient of $E^0(BC_p \wr \cdots \wr C_p)$ by transfers induced by $* \to C_p$ ranging over each component.

$$Q_k(\Sigma_m/C_p \wr \cdots \wr C_p) \cong \left(E^0(BC_p)/\mathrm{tr}\right)^{\otimes k} \cong E_0[[t_1, t_2, \cdots, t_k]]/\langle p\rangle t_1, \langle p\rangle^P t_2, \cdots, \langle p\rangle^{P^{k-1}} t_k,$$

where $\langle p\rangle$ is the reduced p-series and $P^i : E_0 \to E^0(\Sigma_m/C_p \wr \cdots \wr C_p)$ is i iterations of the power operation map P. It is the ring classifying a sequence of points

$$q_s \in \mathbb{G}_s\langle p\rangle, \quad s = 1, \cdots, k$$

where $\mathbb{G}_1 = \mathbb{G}$ and $\mathbb{G}_s = \mathbb{G}_{s-1}/\langle q_{s-1}\rangle$ is the quotient of $\mathbb{G}_{s-1}$ by the degree p subgroup generated by q_{s-1}. Here $\mathbb{G}\langle p\rangle \subset \mathbb{G}[p]$ is the closure $\overline{\mathbb{G}[p] - 0}$ of strict the p-torsion points.

The same argument shows that $Q_k(\Sigma_m/\Delta_k) \cong Q_k(\Sigma_m/{\mathbb{F}_p}^k)$ is the quotient of $E^0(B{\mathbb{F}_p}^k) \cong E^0[[t_1, \cdots, t_k]]/[p]t_1, \cdots [p]t_k \cong \mathcal{O}_{\mathbb{G}[p]^k}$ by transfers from ${\mathbb{F}_p}^{s-1} \times * \times {\mathbb{F}_p}^{k-s}$ for $s = 1, \cdots, k$. Equivalently, it is the image of the map into the Tate construction which is a localization inverting the Euler class c_{p^k}. Since the reduced standard representation of Σ_{p^k} restricts to the reduced regular representation of ${\mathbb{F}_p}^k$ which splits as the tensor product of line bundles, this is the Euler class

$$e\left(\bigoplus_{\substack{(a_1,\cdots,a_k)\in{\mathbb{F}_p}^k \\ (a_1,\cdots,a_k)\neq\vec{0}}} L_1^{a_1} \otimes L_k^{a_k}\right) = \prod_{\substack{(a_1,\cdots,a_k)\in{\mathbb{F}_p}^k \\ (a_1,\cdots,a_k)\neq\vec{0}}} ([a_1]t_1 +_F \cdots +_F [a_k]t_k)$$

where L_s is the tautological line bundle over the sth summand $B\mathbb{F}_p$ of $B{\mathbb{F}_p}^k$.

Definition 4.3.3. *Let*

$$\phi_{\langle i_1,\cdots,i_{s-1}\rangle}(t) = \prod_{\substack{(a_1,\cdots,a_s)\in{\mathbb{F}_p}^s \\ (a_1,\cdots,a_s)\neq\vec{0}}} (t +_F [a_1]t_{i_1} +_F \cdots +_F [a_{s-1}]t_{i_{s-1}}).$$

Then $\tilde{\phi}_{\langle i_1,\cdots,i_{s-1}\rangle}(t) := t\phi_{\langle i_1,\cdots,i_{s-1}\rangle}(t)$ is an isogeny whose kernel is the degree p^{s-1} subgroup generated by $t_{i_1}, \cdots, t_{i_{s-1}}$.

The Tate construction inverts $\tilde{\phi}_{\langle i_1,\cdots,i_{s-1}\rangle}(t_j)$ for all $\{i_1,\cdots,i_{s-1}\}$ and $j \notin \{i_1,\cdots,i_{s-1}\}$. In particular, it inverts each t_s, so there is a surjection $E^0[[t_1,\cdots,t_k]]/\langle p\rangle t_1,\cdots,\langle p\rangle t_k \to Q_k(\Sigma_m/{\mathbb{F}_p}^k)$.

This shows that $Q_k(\Sigma_m/{\mathbb{F}_p}^k)$ is the ring classifying a sequence of points

$$q_s \in \mathbb{G}\langle p\rangle, \quad s = 1,\cdots,k$$

such that for all subsets $\{s_1,\cdots,s_j\} \subseteq \{1,\cdots,k\}$ and $s \notin \{s_1,\cdots,s_j\}$, the image of the point q_s in $\mathbb{G}/\langle q_{s_1},\cdots,q_{s_j}\rangle$ lies in $\left(\mathbb{G}/\langle t_{s_1},\cdots,t_{s_j}\rangle\right)\langle p\rangle$ i.e. q_s is not in the degree p^j subgroup generated by $q_{s_1},\cdots,q_{s_j}$. This is equivalent to the condition that for all s, the image q'_s of q_s in $\mathbb{G}_s = \mathbb{G}/\langle q_1,\cdots,q_{s-1}\rangle$ lies in $\mathbb{G}_s\langle p\rangle$. We can write this in formulas. Over $E_0[[t_1]]/\langle p\rangle t_1$, since the subgroup generated by t_1 is contained in $\mathbb{G}[p]$, the p-series for t_2 factors as

$$[p](t_2) = \tilde{\phi}_{\langle 1\rangle}(t_2)r_1(t_2)$$

for some power series $r_1(t) \in \left(E_0[[t_1]]/\langle p\rangle t_1\right)[[t]]$, so the reduced p-series for t_2 factors as

$$\langle p\rangle(t_2) = \phi_{\langle 1\rangle}(t_2)r_1(t_2).$$

Since the Tate construction inverts $\phi_1(t_2)$, we have $r_1(t_2) = 0$. This is the condition for t_2 to be in $\mathbb{G}\langle p\rangle = \mathbb{G}_1\langle p\rangle$ and to have image in $\mathbb{G}_2\langle p\rangle$ under the quotient map $\mathbb{G}_1 \to \mathbb{G}_2$. So,

$$Q_2(\Sigma_{p^2}/{\mathbb{F}_p}^2) \cong E_0[[t_1,t_2]]/\langle p\rangle t_1, r_1(t_2).$$

In general, for $k < n$, over $Q_k(\Sigma_m/{\mathbb{F}_p}^k)$, the p-series is divisible by $\tilde{\phi}_{\langle 1,\cdots,k\rangle}(t)$ since the subgroup generated by $t_1,\cdots,t_k$ is contained in $\mathbb{G}[p]$.

Definition 4.3.4. *For $0 \le k < n$, define r_k to be the power series over $Q_k(\Sigma_m/{\mathbb{F}_p}^k)$ such that*

$$[p](t) = \tilde{\phi}_{\langle 1,\cdots,k\rangle}(t)r_k(t).$$

In particular, $\langle p\rangle(t) = \phi_{\langle 1,\cdots,k\rangle}(t)r_k(t)$. r_k is well-defined since $\tilde{\phi}_{\langle 1,\cdots,k\rangle}(t)$ is not a zero divisor in $Q_k(\Sigma_m/{\mathbb{F}_p}^k)[[t]]$. Indeed, the constant term in $\phi_{\langle 1,\cdots,k\rangle}(t)$ is the Euler class c_{p^k} which is inverted by the Tate construction, so c_{p^k} is not a zero divisor in $Q_k(\Sigma_m/{\mathbb{F}_p}^k)$.

Since the Tate construction inverts $\phi_{\langle 1,\cdots,k\rangle}(t_{k+1})$, $r_k(t_{k+1}) = 0$ in $Q_{k+1}(\Sigma_{p^{k+1}}/{\mathbb{F}_p}^{k+1})$. This is the condition for t_{k+1} to be in $\mathbb{G}\langle p\rangle$ and to have image in $(\mathbb{G}/\langle t_1,\cdots,t_k\rangle)\langle p\rangle$. Thus,

$$Q_{k+1}(\Sigma_{p^{k+1}}/{\mathbb{F}_p}^{k+1}) \cong E_0[[t_1,\cdots,t_{k+1}]]/\langle p\rangle t_1, r_1(t_2),\cdots,r_k(t_{k+1}).$$

Note that $Q_{k+1}(\Sigma_{p^{k+1}}/{\mathbb{F}_p}^{k+1})$ is symmetric in $t_1,\cdots,t_{k+1}$ and we could have replaced $\{1,\cdots,k+1\}$ by any of its permutations. In particular,

$$Q_{k+1}(\Sigma_{p^{k+1}}/{\mathbb{F}_p}^{k+1}) \cong E_0[[t_{k+1},\cdots,t_1]]/\langle p\rangle(t_{k+1}), r_1(t_k),\cdots,r_k(t_1).$$

If $k > h$, the reduced p-series for t_{h+1} is both 0 and invertible in the Tate construction since it is equal to $\phi_{\langle 1,\cdots,h\rangle}(t_{h+1})$, so $Q_k(\Sigma_m/{\mathbb{F}_p}^k) = 0$.

Since the isomorphism $Q_k(\Sigma_m/\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}}) \cong \Gamma^\vee[k_1]_t \otimes_s \Gamma^\vee[k_2]$ reverses the order of the k_is, $E^0(BC_p)/\mathrm{tr} \otimes E^0(BC_p)/\mathrm{tr}$ corresponds to $t_1,\cdots,t_k$ while $C_p \wr \cdots \wr C_p$ corresponds to $t_k,\cdots,t_1$.

Thus, on functor of points, the map $Q_k(\Sigma_m/C_p \wr \cdots \wr C_p) \to Q_k(\Sigma_m/{\mathbb{F}_p}^k)$ induced by the inclusion ${\mathbb{F}_p}^k \to C_p \wr \cdots \wr C_p$ is given by sending $q_1, \cdots, q_k$ to $q'_k, \cdots, q'_1$. This is the map

$$E_0[[t_1, t_2, \cdots, t_k]]/\langle p\rangle t_1, \langle p\rangle^P t_2, \cdots, \langle p\rangle^{P^{k-1}} t_k \to E_0[[t_1, \cdots, t_k]]/\langle p\rangle t_1, r_1(t_2), \cdots, r_{k-1}(t_k)$$

sending t_s to $\tilde{\phi}_{\langle k,\cdots,k-s+2\rangle}(t_{k-s+1})$.

We modify the above argument for $Q^{(i)}$. We will construct the analogs $r_s^{(i)}$ of r_s in the proof of the following lemma.

Lemma 4.3.5. *For $1 \leq k \leq h-i$,*

$$Q_k^{(i)}(\Sigma_m/\mathbb{F}_p^k) \cong E_0/u_{i-1}[[t_1, \cdots, t_k]]/\langle p\rangle^{(i)}(t_1), r_1^{(i)}(t_2), \cdots, r_k^{(i)}(t_k)$$

is a regular local Noetherian ring with a regular system of parameters $(t_1, \cdots, t_k, u_{i+k}, \cdots, u_{n-1})$. If $k > h-i$, $Q_k^{(i)}(\Sigma_m/\mathbb{F}_p^k) = 0$.

Proof. Let $S[k] = Q_k^{(i)}(\Sigma_m/\mathbb{F}_p^k)$. It is clear that $S[k]$ is a local Noetherian ring being a quotient of the local Noetherian ring $E_0/u_{i-1}[[t_1, \cdots, t_k]]$. It remains to show that it is regular. We will denote the maximal ideal in a local ring by $\mathfrak{m}$ and use the following facts about regular local rings.

(i) A power series ring over a regular local ring is a regular local ring.

(ii) [May, Theorem 2.1] The quotient of a regular local ring by a subset of a regular system of parameters is a regular local ring.

$S[k]$ is the ring classifying a sequence of points

$$q_s \in \mathbb{G}^{(i)}\langle p\rangle, \quad s = 1, \cdots, k$$

such that for all $\{s_1, \cdots, s_j\} \subseteq \{1, \cdots, k\}$ and $s \notin \{s_1, \cdots, s_j\}$, the image of the point q_s in $\mathbb{G}/\langle q_{s_1}, \cdots, q_{s_j}\rangle$ lies in $\left(\mathbb{G}/\langle t_{s_1}, \cdots, t_{s_j}\rangle\right)^{(i)}\langle p\rangle$, where $\mathbb{G}^{(i)}\langle p\rangle$ is the closure $\overline{\mathbb{G}[p]-0}$ of strict the p-torsion points over $\mathrm{Spf}(E_0/u_{i-1})$. $S[k] = 0$ if $k > h-i$.

For $k = 1$, $E_0/u_{i-1}[[t_1]]$ is a regular local ring with $(t_1, u_i, \cdots, u_{h-1})$ a regular system of parameters. Since $\langle p\rangle^{(i)}(t_1) = u_i + t_1(\cdots)$, it is part of a basis of $\mathfrak{m}/\mathfrak{m}^2$, so

$$S[1] \cong E_0/u_{i-1}[[t_1]]/\langle p\rangle^{(i)}(t_1)$$

is regular with $(t_1, u_{i+1}, \cdots, u_{h-1})$ a regular system of parameters.

Over E_0/u_{i-1}, and hence $S[1]$, $[p](t_2) = g(t_2^{p^i}) = t_2^{p^i} g'(t_2^{p^i})$ and $\langle p\rangle^{(i)}(t_2) = g'(t_2^{p^i})$ for some power series g and g' (the isogeny $\mathbb{G} \to \mathbb{G}/\mathbb{G}[p] \simeq \mathbb{G}$ factors through φ^i as $\mathbb{G} \to \varphi^i\mathbb{G} \to \mathbb{G}$). Over $S[1]$, g' has roots $t_1^{p^i}, ([2]t_1)^{p^i}, \cdots, ([p-1]t_1)^{p^i}$. Since $S[1]$ is regular, hence a UFD,

$$g'(y) = \left(\prod_{j=1}^{p-1}\left(y - ([j]t_1)^{p^i}\right)\right) r(y)$$

for some power series r, so $\prod_{j=1}^{p-1}\left(t_2^{p^i} - ([j]t_1)^{p^i}\right)$, which is a unit multiple of $\phi_{\langle 1\rangle}(t)^{p^i}$, divides $\langle p\rangle^{(i)}(t)$. So,

$$\langle p\rangle^{(i)}(t_2) = \phi_{\langle 1\rangle}(t)^{p^i} r_1^{(i)}(t_2)$$

where $r_1^{(i)}(t_2)$ is of the form $h(t_2^{p^i})$.

Under the map $S[1][[t_2]] \to E_0/u_i[[t_2]]$ which quotients out t_1, $\langle p\rangle^{(i)}(t_2)$ maps to $t^{p^i(p-1)}\langle p\rangle^{(i+1)}(t_2)$ and $\prod_{j=1}^{p-1}\left(t_2^{p^i} - ([j]t_1)^{p^i}\right)$ maps to $t^{p^i(p-1)}$, so $r_1^{(i)}(t_2)$ maps to $\langle p\rangle^{(i+1)}(t_2) = u_{i+1} + t_2(\cdots)$ since the target is a UFD. This implies that

$$r_1^{(i)}(t_2) = u_{i+1} + t_2(\cdots) + t_1(\cdots)$$

in $S[1][[t_2]]$. Since $S[1][[t_2]]$ is regular with $(t_1, t_2, u_{i+1}, \cdots, u_{h-1})$ a regular system of parameters, $r_1(t_2)$ is part of a basis for $\mathfrak{m}/\mathfrak{m}^2$, so the quotient

$$S[2] \cong S[1][[t_2]]/r_1^{(i)}(t_2) \cong E_0/u_{i-1}[[t_1, t_2]]/\langle p\rangle^{(i)}(t_1), r_1^{(i)}(t_2)$$

is regular with a regular system of parameters $(t_1, t_2, u_{i+2}, \cdots, u_{h-1})$.

Repeating this process, we obtain power series $r_s^{(i)}$ such that

$$S[k] \cong E_0/u_{i-1}[[t_1, \cdots, t_k]]/\langle p\rangle^{(i)}(t_1), r_1^{(i)}(t_2), \cdots, r_{k-1}^{(i)}(t_k)$$

is regular with $(t_1, \cdots, t_k, u_{i+k}, \cdots, u_{h-1})$ a regular system of parameters. Over $S[k]$, g has roots $([j]t_s)^{p^i}$ for $1 \leq s \leq k$ and $0 \leq j \leq p-1$. Since g is an isogeny on $\varphi^i\mathbb{G}$, all linear combinations $([a_1]t_1)^{p^i} +_{F^i} \cdots +_{F^i} ([a_k]t_k)^{p^i} = ([a_1]t_1 +_F \cdots +_F [a_k]t_k)^{p^i}$ are roots of g, where F^i denotes the formal group law on $\varphi^i\mathbb{G}$. The same argument as before shows that over $S[k]$, $\phi_{\langle 1,\cdots,k\rangle}(t)^{p^i}$ divides $\langle p\rangle^{(i)}(t)$.

Definition 4.3.6. *For $0 \leq k < h-i$, define $r_k^{(i)}$ to be the power series over $Q_k^{(i)}(\Sigma_m/{\mathbb{F}_p}^k)$ such that*

$$[p]^{(i)}(t) = \tilde{\phi}_{\langle 1,\cdots,k\rangle}(t)^{p^i} r_k^{(i)}(t).$$

$r_k^{(i)}(t)$ is of the form $h(t^{p^i})$. Under the map $S[k][[t_{k+1}]] \to E_0/u_{i+k-1}[[t_{k+1}]]$ which quotients out $t_1, \cdots, t_k$, $\langle p\rangle^{(i)}(t_{k+1})$ maps to $t_{k+1}^{p^i(p^k-1)}\langle p\rangle^{(i+k)}(t_{k+1})$ and $\tilde{\phi}_{\langle 1,\cdots,k\rangle}(t_{k+1})^{p^i}$ maps to $t_{k+1}^{p^i(p^k-1)}$, so $r_k^{(i)}(t_{k+1})$ maps to $\langle p\rangle^{(i+k)}(t_{k+1}) = u_{i+k} + t_{k+1}(\cdots)$. This implies that in $S[k][[t_{k+1}]]$,

$$r_k^{(i)}(t_{k+1}) = u_{i+k} + t_{k+1}(\cdots) + \text{ terms involving at least one of } t_1, \cdots, t_k.$$

Thus,

$$S[k+1] \cong S[k][[t_{k+1}]]/r_k^{(i)}(t_{k+1}) \cong E_0/u_{i-1}[[t_1, \cdots, t_{k+1}]]/\langle p\rangle^{(i)}(t_1), r_1^{(i)}(t_2), \cdots, r_k^{(i)}(t_{k+1})$$

is regular with a regular system of parameters $(t_1, \cdots, t_{k+1}, u_{i+k+1}, \cdots, u_{h-1})$. □

To summarize, we have the following.

Lemma 4.3.7. *For the contravariant part $Q_k^{(i)}$,*

- *The ring*

$$Q_k^{(i)}(\Sigma_{p^k}/C_p \wr \cdots \wr C_p) \cong E_0/u_{i-1}[[t_1, \cdots, t_k]]/\langle p\rangle^{(i)}t_1, \cdots, {\langle p\rangle^{(i)}}^{P^{k-1}}t_k$$

classifies a sequence of points $\{q_s \in \mathbb{G}_s^{(i)}\langle p\rangle\}_{s=1}^k$ where $\mathbb{G}_1 = \mathbb{G}$ and $\mathbb{G}_s = \mathbb{G}_{s-1}/\langle q_{s-1}\rangle$ is the quotient of $\mathbb{G}_{s-1}$ by the degree p subgroup generated by q_{s-1}.

- *The ring*

$$Q_k^{(i)}(\Sigma_{p^k}/\mathbb{F}_p{}^k) \cong E_0/u_{i-1}[[t_1, \cdots, t_k]]/\langle p\rangle^{(i)}(t_1), r_1^{(i)}(t_2), \cdots, r_{k-1}^{(i)}(t_k)$$

  classifies a sequence of points $\{q_s \in \mathbb{G}^{(i)}\langle p\rangle\}_{s=1}^k$ such that for all s, the image q_s' of q_s in $\mathbb{G}_s$ lies in $\mathbb{G}_s^{(i)}\langle p\rangle$. It is 0 if $k > h - i$.

- *Under our choice of inclusion $\mathbb{F}_p{}^k \to C_p \wr \cdots \wr C_p$ (7), the map $Q_k^{(i)}(\Sigma_{p^k}/C_p \wr \cdots \wr C_p) \to Q_k^{(i)}(\Sigma_{p^k}/\mathbb{F}_p{}^k)$ sends t_s to $\tilde{\phi}_{\langle k,\cdots,k-s+2\rangle}(t_{k-s+1})$. It is the map which sends $q_1, \cdots, q_k$ to $q_k', \cdots, q_1'$.*

4.3.2 The GL_k-action

Since the standard basis of $\mathbb{F}_p{}^k$ corresponds to $q_1, \cdots, q_k$, a matrix $g \in GL_k$ acts on the right on the scheme $\mathrm{Spf}\left(Q_k^{(i)}(\Sigma_m/\mathbb{F}_p{}^k)\right)$ by sending a sequence of points $\begin{bmatrix} q_1 & \cdots & q_k\end{bmatrix}$ to $\begin{bmatrix} q_1 & \cdots & q_k\end{bmatrix} \cdot g$. On its ring of functions $Q_k^{(i)}(\Sigma_m/\mathbb{F}_p{}^k)$, g acts on the left by sending t_s to $\vec{g_s} \cdot \vec{t}$, where $\vec{g_s}$ is the sth column of g and $\vec{t} = \begin{bmatrix} t_1 & \cdots & t_k\end{bmatrix}$.

Example 4.3.8. For $k = 2$, $g = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$ acts on the scheme $\mathrm{Spf}\left(Q_2^{(i)}(\Sigma_{p^2}/\mathbb{F}_p{}^2)\right)$ by

$$\begin{bmatrix} q_1 & q_2\end{bmatrix} \cdot g = \begin{bmatrix} q_1 & q_2\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} aq_1 + cq_2 & bq_1 + dq_2\end{bmatrix}$$

and on the ring $Q_2^{(i)}(\Sigma_{p^2}/\mathbb{F}_p{}^2)$ by

$$\begin{aligned} t_1 &\mapsto at_1 + ct_2 \\ t_2 &\mapsto bt_1 + dt_2, \end{aligned}$$

where all the operations are formal group law operations ($+ := +_F$, $a := [a]_F$).

4.4 Computing the cokernel

The goal of this section is to show the following.

Proposition 4.4.1. *For $p = 2$, the cokernel of $\sigma^{\vee\otimes k} : C^{\vee(i)}[k] \to C^{\vee(i)}[k]$ is isomorphic to $C^{\vee(i+1)}[k-1]$.*

Let

$$R[k] = Q_k^{(i)}(\Sigma_{p^k}/C_p \wr \cdots \wr C_p) \tag{8}$$

$$S[k] = Q_k^{(i)}(\Sigma_{p^k}/\mathbb{F}_p{}^k) \tag{9}$$

The map $\sigma^{\vee\otimes k} : C^{\vee(i)}[k] \to C^{\vee(i)}[k]$ is induced by the map on $\Gamma^{\vee(i)}[1]^{\otimes k}$ which is multiplication by the Euler class $x_1 \cdots x_k$. On $R[k]$ and $S[k]$, it corresponds to

$$\prod_{s=1}^k \left(\prod_{i=1}^{p-1}[i]t_s\right) = (-1)^{pk}(t_1 \cdots t_k)^{p-1} \quad \text{and} \quad \eta_k := \prod_{\substack{(a_1,\cdots,a_k)\in\mathbb{F}_p{}^k \\ (a_1,\cdots,a_k)\neq\vec{0}}} ([a_1]t_1 +_F \cdots +_F [a_k]t_k),$$

respectively. By Proposition 4.3.1, under our choice of map $R[k] \to S[k]$, $C^{\vee(i)}[k]$ corresponds to the Steinberg summand of $S[k]$. Since the Steinberg module is a projective $\mathbb{Z}[GL_k]$-module and the Euler class in $S[k]$ is GL_k-equivariant, the cokernel of $\sigma^{\vee\otimes k} : C^{\vee(i)}[k] \to C^{\vee(i)}[k]$ is the Steinberg summand of the cokernel $A[k]$ of multiplication by η_k on $S[k]$.

Let

$$A[k] = S[k]/\eta_k$$

$$B[k] = S[k]/\prod_{i\neq 0}[i]t_1 = S/t_1^{p-1} = E_0/u_i[[t_1, t_2, \cdots, t_k]]/t_1^{p-1}, r_1^{(i)}(t_2), \cdots, r_{k-1}^{(i)}(t_k)$$

$$B'[k-1] = S[k]/t_1 = E_0/u_i[[t_2, \cdots, t_k]]/\langle p\rangle^{(i+1)}t_2, \cdots, r_{k-2}^{(i+1)}(t_k) \cong Q_{k-1}^{(i+1)}(\Sigma_{p^{k-1}}/\mathbb{F}_p^{k-1}).$$

Note that $B[k]$ and $B'[k-1]$ are E_0/u_i-modules and there are canonical projection maps $A[k] \to B[k] \to B'[k-1]$.

Example 4.4.2. For $k = 0$, $A[0] = 0$. For $k = 1$, $S[1] \cong E^0[[t_1]]/\langle p\rangle^{(i)}t_1$, $e_1S[1] \cong \Gamma^{\vee(i)}[1] \cong E_0/u_{i-1}[[x_1]]/f^{(i)}(x_1)$, and the Euler class is $x_1 = \prod_{i=1}^{p-1}[i]t_1 = (-1)^p t^{p-1}$. Since e_1 is taking C_p fixed-points,

$$e_1A[1] = E_0/u_i = C^{\vee(i+1)}[0].$$

We will use results about the Steinberg idempotent from [HP14, Section 1.3].

Notation 4.4.3.

- A left G-action on a set S is equivalent to a right G^{op}-action on S by $g \cdot x := xg^{-1}$. For a subgroup $H \leq G$, the left G-set G/H can be identified with the right G-set of left cosets $H\backslash G$ by $gH \leftrightarrow Hg^{-1}$. Under this identification, the corresponding right G^{op}-action on $H\backslash G$ is equivalent to the left G-action on G/H.

- Let $\mathrm{Inj}(k-1,k)$ be the set of all $\mathbb{F}_p$-linear injections ${\mathbb{F}_p}^{k-1} \to {\mathbb{F}_p}^{k}$, which can be regarded as a subset of matrices $\mathrm{Mat}_{k\times(k-1)}(\mathbb{F}_p)$. Then $\mathrm{Inj} := \mathbb{F}_p[\mathrm{Inj}(k-1,k)]$ is naturally a GL_k-GL_{k-1}-bimodule.

- Let $\mathrm{Pr}(k,k-1)$ be the set of all $\mathbb{F}_p$-linear surjections ${\mathbb{F}_p}^{k} \to {\mathbb{F}_p}^{k-1}$, which can be regarded as a subset of matrices $\mathrm{Mat}_{(k-1)\times k}(\mathbb{F}_p)$. Then $\mathrm{Pr} := \mathbb{F}_p[\mathrm{Pr}(k,k-1)]$ is naturally a GL_{k-1}-GL_k-bimodule. Equivalently, it as a GL_k^{op}-GL_{k-1}^{op}-bimodule where $g \cdot W \cdot h := h^{-1}Wg^{-1}$.

- Let $U_1 \leq GL_k$ be the subgroup of matrices $\left[\begin{array}{c|c} & * \\ I & \vdots \\ & * \\ \hline 0 & * \end{array}\right]$. Then $U_1 = \mathrm{Stab}_{GL_k}\left(\begin{bmatrix} I \\ 0 \end{bmatrix}\right)$ for the left GL_k-action. Its normalizer $B_{k1} = N_{GL_k}(U_1) \leq GL_k$ is the subgroup of matrices $\left[\begin{array}{c|c} & * \\ * & \vdots \\ & * \\ \hline 0 & * \end{array}\right]$ and its Weyl group $W_{GL_k}(U_1) = GL_{k-1} \times 1$. $\mathrm{Inj}(k-1,k) = GL_k/U_1$ has a left action of GL_k and a right action of $W_{GL_k}(U_1)$.

- Let $U_2 \leq GL_k$ is the subgroup of matrices $\left[\begin{array}{c|c} * & * \ \cdots \ * \\ \hline 0 & I \end{array}\right]$. Then $U_2 = \mathrm{Stab}_{GL_k}\left(\begin{bmatrix} 0 & I \end{bmatrix}\right)$ for the right GL_k-action, equivalently, the left GL_k^{op}-action. Its normalizer $B_{k2} =$

$N_{GL_k}(U_2) \leq GL_k$ is the subgroup of matrices $\left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & & * & \end{array}\right]$ and its Weyl group $W_{GL_k}(U_2) = 1 \times GL_{k-1}$. $\Pr(k, k-1) = U_2 \backslash GL_k$ as a right GL_k-set and a left $W_{GL_k}(U_2)$-set. Equivalently, $\Pr(k, k-1) = GL_k / U_2$ as a left GL_k-set and a right $W_{GL_k}(U_2)$-set.

Proof strategy. To prove Proposition 4.4.1, we compute the cokernel

$$\operatorname{cok}\left(\sigma^{\vee \otimes k} : C^{\vee(i)}[k] \to C^{\vee(i)}[k]\right) = e_k A[k]$$

as follows. In Proposition 4.4.9, we will show that there exists a GL_k-equivariant map

$$A[k] \to \Pr \otimes_{GL_{k-1}} B'[k-1].$$

This induces a map on Steinberg summands

$$e_k A[k] \to e_k \Pr \otimes_{GL_{k-1}} B'[k-1] \cong e_{k-1} B'[k-1]$$

where the isomorphism comes from a key fact in [HP14, Prop 1.3.8]. We then show (Proposition 4.4.13) that for $p = 2$, the map $e_k A[k] \to e_{k-1} B'[k-1]$ is injective with image isomorphic to $e_{k-1} B'[k-1]$, which is $C^{\vee(i+1)}[k-1]$. This is the only part where we use the fact that $p = 2$.

4.4.1 The GL_k-equivariant map

We will construct a GL_k-equivariant map $A[k] \to \Pr \otimes_{GL_{k-1}} B'[k-1]$ which induces a map $e_k A[k] \to e_{k-1} B'[k-1]$ on Steinberg summands.

Lemma 4.4.4. *For A a left GL_k-module and B a left GL_{k-1}-module, there are natural bijections*

$$\operatorname{Hom}_{GL_k}(A, \operatorname{Inj} \otimes_{GL_{k-1}} B) \leftrightarrow \operatorname{Hom}_{B_{k1}}(A, B)$$

where B_{k1} acts on B by the projection $B_{k1} \twoheadrightarrow GL_{k-1} \times 1$ onto the top left and

$$\operatorname{Hom}_{GL_k}(A, \Pr \otimes_{GL_{k-1}} B) \leftrightarrow \operatorname{Hom}_{B_{k2}}(A, B)$$

where B_{k-2} acts on B by the projection $B_{k2} \twoheadrightarrow 1 \times GL_{k-1}$ onto the bottom right.

Proof. For a subgroup $N \leq G$, the restriction from $\mathbb{Z}[G]$-modules to $\mathbb{Z}[N]$-modules has left and right adjoints $\mathbb{Z}[G] \otimes_{\mathbb{Z}[N]} -$ and $\operatorname{Hom}_{\mathbb{Z}[N]}(\mathbb{Z}[G], -)$, which are naturally isomorphic.

$$\begin{tikzcd} \operatorname{Mod}_{\mathbb{Z}[N]} \arrow[r, bend left, "\mathbb{Z}[G]\otimes_{\mathbb{Z}[N]}-"] \arrow[r, bend right, "\operatorname{Hom}_{\mathbb{Z}[N]}(\mathbb{Z}[G],-)"'] & \operatorname{Mod}_{\mathbb{Z}[G]} \arrow[l, "\mathrm{res}"'] \end{tikzcd}$$

Under the restriction and coinduction adjunction, an N-equivariant map $f : A \to B$ corresponds to a G-equivariant map $A \to \prod_i g_i N \otimes_N B$. Since we will need this later, we give an explicit description of this correspondence.

Let $g_1N, \cdots, g_rN$ be a complete set of left cosets in G/N. Then $Ng_1^{-1}, \cdots, Ng_r^{-1}$ is a complete set of right cosets in $N\backslash G$. These give bases for $\mathbb{Z}[G]$ as a free right and left $\mathbb{Z}[N]$-module, respectively. Thus, there are isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}[N]}(\mathbb{Z}[G], -) \cong \prod_i \mathrm{Hom}_N(Ng_i^{-1}, -) \cong \prod_i g_iN \otimes_N - \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[N]} -.$$

For $f : A \to B$ an N-equivariant map, the G-equivariant map $A \to \prod_i g_iN \otimes_N B$ is given by

$$x \mapsto \prod_i g_iN \otimes_N f(g_i^{-1}x).$$

This notation means that the image of x on the component indexed by i is $f(g_i^{-1}x)$. The map f can be recovered by postcomposing this with the projection onto the component indexed by N.

Let $G = GL_k$, $H = U_i$, $N = B_{ki}$, and $W = N/H = GL_{k-1} \times 1$ or $1 \times GL_{k-1}$. Let A be a left GL_k-module and B a left GL_{k-1}-module considered as a B_{ki}-module by the projection of $B_{ki} \twoheadrightarrow W$. By the restriction and coinduction adjunction, an N equivariant map $A \to B$ corresponds to a G-equivariant map $A \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[N]} B \cong \mathbb{Z}[G/H] \otimes_{\mathbb{Z}[W]} B$ since H acts on B trivially. The result follows since $\mathrm{Inj} = \mathbb{Z}[GL_k/U_1]$ and $\mathrm{Pr} = \mathbb{Z}[GL_k/U_2]$. $\square$

We apply this to $A = A[k]$, $B = B[k]$, and $B' = B'[k-1]$.

Lemma 4.4.5. *The projection $A \twoheadrightarrow B'$ is a B_{k2}-equivariant map. Thus, it induces a GL_k-equivariant map $A \to \mathrm{Pr} \otimes_{GL_{k-1}} B'$.*

Proof. The map $A \twoheadrightarrow B'$ sends t_1 to 0 and $t_{i>1}$ to t_i. For $h = \left[\begin{array}{c|ccc} \lambda & * & \cdots & * \\ \hline & & & \\ 0 & & h' & \\ & & & \end{array}\right] \in B_{k2}$, $h \cdot t_1 = \lambda t_1$ and $h \cdot t_{i>1} = *t_1 + h' \cdot t_i$. These map to 0 and $h' \cdot t_i$ in B', respectively. $\square$

Lemma 4.4.6. *There is an isomorphism of $\mathbb{F}_p[GL_k]$-modules*

$$\mathrm{Pr} \otimes_{GL_{k-1}} B' \cong \prod A/(\vec{v} \cdot \vec{t}),$$

where $\vec{v}$ ranges over all 1-dimensional subspaces of ${\mathbb{F}_p}^k$ and $\vec{v} \cdot \vec{t} = [v_1]t_1 +_F \cdots +_F [v_k]t_k$ for any representative $\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$ of v.

Proof. Let $g = \begin{bmatrix} | & & | \\ \vec{g_1} & \cdots & \vec{g_k} \\ | & & | \end{bmatrix} \in GL_k$. The action of g on $\prod A/\vec{v} \cdot \vec{t}$ is induced by its action on A which sends t_1 to $\vec{g_1} \cdot \vec{t_1}$

$$\begin{array}{ccc} A & \xrightarrow{g} & A \\ \downarrow & & \downarrow \\ A/t_1 & \xrightarrow[g]{} & A/\vec{g_1} \cdot \vec{t_1} \end{array}$$

where the vertical maps are the canonical projections.

The B_{k2}-equivariant map $A \to B'$ corresponds to a GL_k-equivariant map $A \to \mathrm{Pr} \otimes_{GL_{k-1}} B'$. As a free right $\mathbb{Z}[GL_{k-1}]$-module, Pr has basis $\{W\}$ consisting of equivalence classes of projections $\mathbb{F}_p^k \to \mathbb{F}_p^{k-1}$, which can be represented by $(k-1) \times k$ matrices. Thus, $\mathrm{Pr} \otimes_{GL_{k-1}} B' \cong \prod_W W \otimes B'$ and the isomorphism of the lemma is given by identifying $W \otimes B'$ with $A/\vec{v} \cdot \vec{t}$ for any nonzero $\vec{v}$ in the orthogonal complement of W. In particular, $\begin{bmatrix} 0 & I \end{bmatrix} \otimes B'$ corresponds to A/t_1 under the projection $A \twoheadrightarrow A/t_1 = B'$. □

The following key fact from [HP14] allows us to relate e_k to e_{k-1}.

Lemma 4.4.7. *[HP14, Prop 1.3.8] For a left GL_{k-1}-module M, there is an isomorphism*

$$e_{k-1}M \xrightarrow{\simeq} e_k \,\mathrm{Pr} \otimes_{GL_{k-1}} M.$$

Proof. We reproduce the proof in [HP14, Proposition 1.3.8], keeping track of the corresponding actions. Consider Pr as a subset of matrices $\mathrm{Mat}_{(k-1)\times k}(\mathbb{F}_p)$ with its left GL_k^{op} and right GL_{k-1}^{op} actions. Let $T_k \leq \Sigma_k$ be the cyclic subgroup generated by $(1, \cdots, k)$ and let $\overline{T}_k = \sum_{\sigma \in T_k} (-1)^\sigma \sigma$. The map $f : e_{k-1}M \to e_k \,\mathrm{Pr} \otimes_{GL_{k-1}} M$ is given by

$$e_{k-1}m \mapsto I_{k-1,k}\overline{T}_k \otimes e_{k-1}m,$$

where $I_{k-1,k} = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathrm{Pr}$. For $A \in \mathrm{Pr}$, $e_k \cdot A = A\overline{U}_k\overline{\Sigma}_k = Ae_k$. Note that the first e_k denotes the Steinberg idempotent for the left GL_k^{op}-action and the latter for the right GL_k-action.

Since Pr is a free right GL_{k-1}^{op} and $I_{k-1,k}\overline{T}_k$ is a sum of distinct basis elements, this map is a monomorphism. To show that it is an epimorphism, let $A = \begin{bmatrix} A' & v \end{bmatrix} \in \mathrm{Pr}$ and $m \in M$. $e_k \cdot A \otimes m = \begin{bmatrix} A' & v \end{bmatrix} e_k \otimes m = 0$ if $\operatorname{rank} A' \leq k-2$ [HP14, Lemma 1.3.4]. If $\operatorname{rank} A' = k-1$, there exists $C \in GL_{k-1}$ and $B \in U_k$ such that $CAB = I_{k-1,k}$. Then

$$\begin{aligned}
e_k \cdot A \otimes m &= C^{-1} I_{k-1,k} B^{-1} e_k \otimes m \\
&= C^{-1} I_{k-1,k} e_k \otimes m \\
&= I_{k-1,k} e_k \otimes Cm \\
&= (e_{k-1} I_{k-1,k} \overline{T}_k) \otimes Cm, \text{ by [HP14, Lemma 1.3.7]} \\
&= I_{k-1,k}\overline{T}_k \cdot e_{k-1} \otimes Cm \\
&= I_{k-1,k}\overline{T}_k \otimes e_{k-1}Cm = f(e_{k-1}Cm).
\end{aligned}$$

□

Remark 4.4.8. Note that for any $\sigma \in T_k$, $I_{k-1,k}\overline{T}_k = (-1)^\sigma (I_{k-1,k}\sigma)\overline{T}_k$. Thus, the isomorphism $f : e_{k-1}M \to e_k \,\mathrm{Pr} \otimes_{GL_{k-1}} M$ above could have been defined by

$$f(e_{k-1}m) = (I_{k-1,k}\sigma)\overline{T}_k \otimes e_{k-1}m.$$

With this definition, there is a commutative diagram

$$\begin{tikzcd}
e_k \,\mathrm{Pr} \otimes_{GL_{k-1}} M \arrow[r, leftrightarrow, "\simeq"] \arrow[d, hook] & e_{k-1}M \arrow[d, hook] \\
\mathrm{Pr} \otimes_{GL_{k-1}} M \arrow[r, twoheadrightarrow] & M
\end{tikzcd}$$

where the bottom map is projection onto the factor indexed by $I_{k-1,k}\sigma$.

Putting things together, with $\sigma = (1 \cdots k)$ in Remark 4.4.8, we obtain the following.

Proposition 4.4.9. *The map*

$$A[k] \to \prod A[k]/\vec{v} \cdot \vec{t} \cong \mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1]$$

given by canonical projections onto each component is GL_k-equivariant, where the product ranges over 1-dimensional subspaces of ${\mathbb{F}_p}^k$. Taking Steinberg summands induces a map

$$e_k A[k] \to e_{k-1} B'[k-1]. \tag{10}$$

4.4.2 The cokernel on Steinberg summands

To identify $e_k A[k]$, we first compute the image of the map $e_k A[k] \to e_{k-1} B'[k-1]$, then show that this map is injective for $p = 2$.

Proposition 4.4.10. *The map $e_k A[k] \to e_{k-1} B'[k-1]$ (10) has image $\eta_{k-1} \cdot e_{k-1} B'[k-1] \cong e_{k-1} B'[k-1]$.*

We consider Pr as a GL_k^{op}-GL_{k-1}^{op}-bimodule (Notation 4.4.3).

Lemma 4.4.11. *The image of the map (10) contains $\eta_{k-1} \cdot e_{k-1} B'[k-1]$.*

Proof. Let $\{W\} \subset \mathrm{Pr} = \mathrm{Mat}_{(k-1)\times k}(\mathbb{F}_p)$ be a basis for $\mathbb{F}_p[\mathrm{Pr}]$ as a free right $\mathbb{F}_p[GL_{k-1}]$-module such that each matrix W is in row reduced echelon form. For each W, let $g_W \in GL_k$ be such that $W = g_W \cdot \begin{bmatrix} 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} g_W^{-1}$. The canonical projection $f : A[k] \to B'[k-1]$ corresponds to the GL_k-equivariant map $A[k] \to \mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1] \cong \prod_W W \otimes B'[k-1] \cong \sum_W W \otimes B'[k-1]$ (Lemma 4.4.4) given by

$$y \mapsto \prod_W W \otimes f(g_W^{-1} y) = \sum_W W \otimes f(g_W^{-1} y).$$

This notation means that it is $f(g_W^{-1} y)$ on the component indexed by W.

On Steinberg summands, $e_k y$ maps to $\sum_W \left(e_k \cdot W \otimes f(g_W^{-1} y)\right) = \sum_W \left(W e_k \otimes f(g_W^{-1} y)\right)$, where $W e_k$ denotes the Steinberg summand for a right GL_k-action (4.2). By analogs of Lemmas 1.3.3 and 1.3.4 in [HP14] for general primes, $W e_k = 0$ if $W = \begin{bmatrix} W' & \vec{v} \end{bmatrix}$ with $\operatorname{rank} W' \leq k-2$. If $\operatorname{rank} W' = k-1$, then $W = \begin{bmatrix} I & \vec{v} \end{bmatrix}$, so $e_k y$ maps to

$$\sum_{\vec{v} \in \mathbb{F}_p^{k-1}} e_k \cdot \begin{bmatrix} I & \vec{v} \end{bmatrix} \otimes f \left(\left[\begin{array}{c|c} 0 & 1 \\ \hline I & \vec{v} \end{array} \right] y \right)$$

Since

$$e_k \cdot \begin{bmatrix} I & \vec{v} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \left[\begin{array}{c|c} I & \vec{v} \\ \hline 0 & 1 \end{array} \right] \overline{U}_k \overline{\Sigma}_k = I_{k-1,k} \overline{U}_k \overline{\Sigma}_k = I_{k-1,k} e_k = e_{k-1} I_{k-1,k} \overline{T}_k = (I_{k-1,k} \overline{T}_k) \cdot e_{k-1},$$

this is equal to

$$\sum_{\vec{v}\in\mathbb{F}_p^{k-1}} I_{k-1,k}\overline{T}_k \otimes e_{k-1} f(g_W^{-1}y) = I_{k-1,k}\overline{T}_k \otimes \left(\sum_{\vec{v}\in\mathbb{F}_p^{k-1}} e_{k-1} f(g_W^{-1}y) \right).$$

Let ψ denote the map (10). By Lemma 4.4.7, $\psi(e_k y) = \sum_{\vec{v}\in\mathbb{F}_p^{k-1}} e_{k-1} f(g_W^{-1}y)$.

To simplify notation, we shift the indices in $B'[k-1] = A[k]/t_1$ down by 1 such that $t_2, \cdots, t_k$ becomes $t_1, \cdots, t_{k-1}$, $B'[k-1] = E_0/u_i[[t_1, \cdots, t_{k-1}]]/\langle p\rangle^{(i+1)} t_1, \cdots, r_{k-2}^{(i+1)}(t_{k-1})$, and the projection map f is given by $f(t_1) = 0$ and $f(t_{i<k}) = t_{i-1}$. For $W = \begin{bmatrix} I & v \end{bmatrix}$, $g_W^{-1} = \left[\begin{array}{c|c} 0 & 1 \\ \hline I & \vec{v} \end{array}\right]$ and

$$\begin{aligned} f(g_W^{-1} t_{i<k}) &= t_i \\ f(g_W^{-1} t_k) &= \vec{v}\cdot\vec{t} \end{aligned}$$

where $\vec{t} = t_1, \cdots, t_{k-1}$.

Elements in $B'[k-1]$ can be represented by power series $h(t_1, \cdots, t_{k-1})$. If $y = h(t_1, \cdots, t_{k-1}) \prod_{\vec{w}\in\mathbb{F}_p^{k-1}-\vec{0}} (t_k +_F \vec{w}\cdot\vec{t})$,

$$\psi(e_k y) = \sum_{\vec{v}\in\mathbb{F}_p^{k-1}} e_{k-1} \left(h(t_1, \cdots, t_{k-1}) \prod_{\vec{w}\in\mathbb{F}_p^{k-1}-\vec{0}} (\vec{v}\cdot\vec{t} +_F \vec{w}\cdot\vec{t}) \right).$$

If $\vec{v} \neq \vec{0}$, then $-\vec{v} \neq \vec{0}$ so $\prod_{\vec{w}\in\mathbb{F}_p^{k-1}-\vec{0}} (\vec{v}\cdot\vec{t} +_F \vec{w}\cdot\vec{t}) = 0$. Thus,

$$\psi(e_k y) = e_{k-1} \left(h(t_1, \cdots, t_{k-1}) \prod_{\vec{w}\neq\vec{0}} \vec{w}\cdot\vec{t} \right).$$

Each $g \in GL_{k-1}$ permutes the set $\mathbb{F}_p^{k-1} - \vec{0}$, so it fixes $\prod_{\vec{w}\neq\vec{0}} \vec{w}\cdot\vec{t} = \eta_{k-1}$, which is the Euler class in $B'[k-1]$.

$$\begin{aligned} \psi(e_k y) &= \sum_{\sigma\in\Sigma_{k-1}, b\in U_{k-1}} (-1)^\sigma \left(\sigma b \cdot h(t_1, \cdots, t_{k-1})\right) (\sigma b \cdot \eta_{k-1}) \\ &= \eta_{k-1} e_{k-1} h(t_1, \cdots, t_{k-1}). \end{aligned}$$

Thus, $\eta_{k-1} \cdot e_{k-1} B'[k-1]$ is contained in the image of (10). $\square$

Proof of Proposition 4.4.10. $A[k]$ is generated by elements $h(t_1, \cdots, t_{k-1}) t_k^i$ ranging over all power series h and $i \geq 0$. If $y = h(t_1, \cdots, t_{k-1})$, then $e_k y = 0$ since the stabilizer of y in U_k has order divisible by p [HP14, Lemma 1.3.3]. If $y = h(t_1, \cdots, t_{k-1}) t_k^i$ with $i > 0$,

$$\psi(e_k y) = \sum_{\vec{v}\in\mathbb{F}_p^{k-1}} e_{k-1} \left(h(t_1, \cdots, t_{k-1}) (\vec{v}\cdot\vec{t})^i \right) = e_{k-1} \left(h(t_1, \cdots, t_{k-1}) \left(\sum_{\vec{v}\neq\vec{0}} (\vec{v}\cdot\vec{t})^i \right) \right).$$

Each $g \in GL_{k-1}$ permutes the set $\mathbb{F}_p^{k-1} - \vec{0}$, so $g \cdot \sum_{\vec{v} \neq \vec{0}} (\vec{v} \cdot \vec{t})^i = \sum_{\vec{v} \neq \vec{0}} (\vec{v} \cdot \vec{t})^i$.

$$\begin{aligned}
\psi(e_k y) &= \sum_{\sigma \in \Sigma_{k-1} b \in U_{k-1}} (-1)^\sigma \left(\sigma b \cdot h(t_1, \cdots, t_{k-1})\right) \left(\sigma b \cdot \sum_{\vec{v} \neq \vec{0}} (\vec{v} \cdot \vec{t})^i \right) \\
&= \left(\sum_{\vec{v} \neq \vec{0}} (\vec{v} \cdot \vec{t})^i \right) e_{k-1} h(t_1, \cdots, t_{k-1}).
\end{aligned}$$

Since elements in $B'[k-1]$ are power series $h(t_1, \cdots, t_{k-1})$, the image of (10) is the sum $+_{i \geq 1} p_i \cdot e_{k-1} B'[k-1]$ where $p_i = p_i \left(\vec{v} \cdot \vec{t} \mid \vec{v} \in \mathbb{F}_p^{k-1} \right) = \sum_{\vec{v} \neq \vec{0}} (\vec{v} \cdot \vec{t})^i$ is the ith power sum in the elements $\vec{v} \cdot \vec{t}$. Since the image of (10) contains $\eta_{k-1} \cdot e_{k-1} B'[k-1]$, to prove the proposition, it suffices to show that $\eta_{k-1} | p_i$ in $B'[k-1]$ for all $i \geq 1$.

Consider the isogeny $\mathbb{G} \to \mathbb{G}/H$ over $B'[k-1]$ which quotients out the subgroup H generated by $t_1, \cdots, t_{k-1}$. On coordinates, it is given by $t \mapsto f(t)$ for $f(t) = \prod_{\vec{v} \in \mathbb{F}_p^{k-1}} (t +_F \vec{v} \cdot \vec{t}) =: \sum_{i \geq 1} c_i t^i$. Since the Weierstrass polynomial

$$w(t) = \prod_{\vec{v} \in \mathbb{F}_p^{k-1}} (t + \vec{v} \cdot \vec{t}), \tag{11}$$

for f is an isogeny with kernel H, $\eta_{k-1} | c_1$. In $B'[k-1]/\eta_{k-1}$, $t^2 | f(t)$, so $f(t) \equiv g(t^p) \mod \eta_{k-1}$ for some power series g [Rav23, Lemma A2.2.7]. We say that a polynomial or power series $h(t) = \sum c_i t^i$ over $B'[k-1]$ satisfies $\bigstar$ if $\eta_{k-1} | c_i$ if $p \nmid i$. Thus, f satisfies $\bigstar$. $\bigstar$ is equivalent to the condition that the image of h in $(B'[k-1]/\eta_{k-1})[[t]]$ lands in $(B'[k-1]/\eta_{k-1})[[t^p]]$. The condition $\bigstar$ is closed under multiplication and inverses (if exists).

In a math overflow answer, Lubin gives an algorithm to compute the Weierstrass polynomial [hL]. Let $d = p^{k-1}$ and let S denote the degree-d shift operation: $Sf = \sum_{i \geq 0} c_{d+i} t^i$. Step 0: $f_1 = f/Sf$, so $Sf_1 \equiv 1 \mod \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal in $B'[k-1]$. Step i: repeat. At each step we get a power series f_i such that $Sf_i \equiv 1 \mod \mathfrak{m}^i$. The Weierstrass polynomial is the limit of the f_is. Since f satisfies $\bigstar$, so does Sf, $1/Sf$, and hence f_1. Repeating, we get that for all i, f_i satisfies $\bigstar$, thus so does the limit.

By uniqueness of the Weierstrass polynomial, this limit is equal to (11). The coefficient of t^{d-i} in w is the elementary symetric function $e_i := e_i \left(\vec{v} \cdot \vec{t} \mid \vec{v} \in \mathbb{F}_p{}^{k-1} \right)$. Since w satisfies $\bigstar$, $\eta_{k-1} | e_i$ if $p \nmid d - i$ i.e. $p \nmid i$. The power sums are related to the elementary symmetric functions by the following identities.

$$p_s = \begin{cases} (-1)^{s-1} s e_s + \sum_{i=1}^{s-1} (-1)^{s-1+i} e_{s-i} p_i & \text{if } 1 \leq s \leq d \\ \sum_{i=s-d}^{s-1} (-1)^{s-1+i} e_{s-i} p_i & \text{if } 1 \leq d < s. \end{cases}$$

Since $\eta_{k-1} \mid p$, by induction, $\eta_{k-1} | p_i$ for all $i \geq 1$ (if $p - 1 \nmid i$, we could have used the fact that $p_i = 0$). Thus, the image of (10) is equal to $\eta_{k-1} \cdot e_{k-1} B'[k-1]$.

For the last part, since $B'[k-1] \cong S[k-1]$ in an integral domain (Lemma 4.3.5), multiplication by η_{k-1} on $B'[k-1]$ and hence $e_{k-1} B'[k-1]$ is injective, so $\eta_{k-1} \cdot e_{k-1} B'[k-1] \cong e_{k-1} B'[k-1]$. $\square$

Lemma 4.4.12. *The GL_k-equivariant map*

$$A[k] \to \prod_{\vec{v}} \left(A[k] / \prod_{i=1}^{p-1} [i](\vec{v} \cdot \vec{t}) \right) \cong \prod_{\vec{v}} B[k]$$

is injective, where $\vec{v}$ ranges over all 1-dimensional subspaces of ${\mathbb{F}_p}^k$ and $\vec{v} \cdot \vec{t} = [v_1]t_1 +_F \cdots +_F [v_k]t_k$ for any representative $\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$ of v.

In particular, for $p = 2$, the map

$$A[k] \to \mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1] \cong \prod_{\vec{v}} B'[k-1]$$

is injective since $B[k] = B'[k-1]$. Thus, the map $e_k A[k] \to e_{k-1} B'[k-1]$ in (10) obtained by taking Steinberg summands is injective.

Proof. The kernel of the map into the product is the intersection of the kernels

$$\bigcap_{\vec{v}} \ker \left(A[k] \to \left(A[k] / \prod_{i=1}^{p-1} [i](\vec{v} \cdot \vec{t}) \right) \right) = \bigcap_{\vec{v}} \left(\prod_{i=1}^{p-1} [i](\vec{v} \cdot \vec{t}) \right) =: \bigcap_{\vec{v}} I_{\vec{v}}.$$

We claim that the intersection is equal to the product of ideals

$$\bigcap_{\vec{v}} I_{\vec{v}} = \prod_{\vec{v}} I_{\vec{v}} = \eta_k = 0$$

in $A[k]$, which will imply that the map from $A[k]$ into the product is injective.

Suppose by induction that for any $n-1$ distinct 1-dimensional subspaces $\vec{v}_1, \cdots, \vec{v}_{n-1}$, the intersection $I_{\vec{v}_1} \cap \cdots \cap I_{\vec{v}_{n-1}}$ is equal to the product $I_{\vec{v}_1} \cdots I_{\vec{v}_{n-1}}$. We need to show that for any n distinct subspaces $\vec{v}_1, \cdots, \vec{v}_n$,

$$I_{\vec{v}_1} \cap \cdots \cap I_{\vec{v}_n} = (\eta_{1,\cdots,n-1}) \cap I_{\vec{v}_n} = I_{\vec{v}_1} \cdots I_{\vec{v}_n} = (\eta_{1,\cdots,n-1}\pi)$$

in $A[k]$, where $\eta_{1,\cdots,n-1} = \prod_{i_1,\cdots,i_{n-1}=1}^{p-1} (i_1\vec{v}_1 \cdot \vec{t}) \cdots (i_{n-1}\vec{v}_{n-1} \cdot \vec{t})$ and $\pi = \prod_{i=1}^{p-1} (i\vec{v}_n \cdot \vec{t})$. An element in the intersection of ideals can be represented by $g(t_1, \cdots, t_k)\eta_{1,\cdots,n-1} = h(t_1, \cdots, t_k)\pi$ in $A[k] = S[k]/\eta_k$. This means that η_k divides the difference $g(\vec{t})\eta_{1,\cdots,n-1} - h(\vec{t})\pi$ in $S[k]$. In the UFD $S[k]$, $\eta_{1,\cdots,n-1}, \pi \mid \eta_k$, so $\eta_{1,\cdots,n-1} \mid h(\vec{t})\pi$ and $\pi \mid g(\vec{t})\eta_{1,\cdots,n-1}$. For a nonzero vector $\vec{w}$, $\vec{w} \cdot \vec{t}$ is irreducible in $S[k]$ since there is a change of coordinates taking it to t_1, so $S[k]/\vec{w} \cdot \vec{t} \cong S[k]/t_1 \cong S[k-1]$ is an integral domain. Since the irreducible factors in $\eta_{1,\cdots,n-1}$ and π are not unit multiples of each other, we must have $\eta_{1,\cdots,n-1} \mid h(\vec{t})$ and $\pi \mid g(\vec{t})$. Thus, $g(\vec{t})\eta_{1,\cdots,n-1} = h(\vec{t})\pi \in I_{\vec{v}_1} \cdots I_{\vec{v}_n}$ and $I_{\vec{v}_1} \cap \cdots \cap I_{\vec{v}_n} = I_{\vec{v}_1} \cdots I_{\vec{v}_n}$. □

Combining Proposition 4.4.10 and Lemma 4.4.12 gives the following.

Proposition 4.4.13. *For $p = 2$, the map $e_k A[k] \to e_{k-1} B'[k-1]$ of (10) is injective with image $\eta_{k-1} \cdot e_{k-1} B'[k-1] \cong e_{k-1} \cdot B'[k-1]$.*

Conjecture 4.4.14. *For all primes p, the GL_k-equivariant map $A[k] \to \mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1]$ induces an injective map on Steinberg summands*

$$e_k A[k] \to e_k \mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1] \xrightarrow{\cong} e_{k-1} B'[k-1].$$

Thus, the map $e_k A[k] \to e_{k-1} B'[k-1]$ of (10) is injective with image $\eta_{k-1} \cdot e_{k-1} B'[k-1] \cong e_{k-1} \cdot B'[k-1]$.

4.4.3 Compatibility with differentials

Having identified the levelwise cokernels in (4) and (5), we now identify the boundary maps to conclude the proof of Theorem 4.0.1. For simplicity, we do the case $n = 0$. The general case follows by postcomposing with the n-fold suspension σ^n. Recall that $R[k] \cong R[1]^{\otimes k}$ and the dual to the map $\Gamma^{(i)}_\sigma \otimes^{\mathbb{L}}_{\Gamma^{(i)}} E_0/u_{i-1} \to E_0/u_{i-1}$ in Definition 4.1.1 is induced by commutative squares

$$\begin{tikzcd} \Gamma^{\vee(i)} \otimes R[k] \arrow[r, "1\otimes\sigma^{\vee\otimes k}"] & \Gamma^{\vee(i)} \otimes R[k] \\ \Gamma^{\vee(i)} \otimes R[k-1] \arrow[u, "\partial^\vee"] \arrow[r, "1\otimes\sigma^{\vee\otimes(k-1)}"'] & \Gamma^{\vee(i)} \otimes R[k-1]. \arrow[u, "(\partial\circ\sigma)^\vee"'] \end{tikzcd}$$

Lemma 4.4.15. *There are commutative squares*

$$\begin{tikzcd} \Gamma^{\vee(i)} \otimes S[k] \arrow[r, "1\otimes\eta_k"] & \Gamma^{\vee(i)} \otimes S[k] \\ \Gamma^{\vee(i)} \otimes S[k-1] \arrow[u, "\partial^\vee"] \arrow[r, "1\otimes\eta_{k-1}"'] & \Gamma^{\vee(i)} \otimes S[k-1] \arrow[u, "(\partial\circ\sigma)^\vee"'] \end{tikzcd}$$

compatible with the maps $R[k] \to S[k]$.

Proof. Recall that the boundary maps in the Koszul complex $\Gamma^{(i)} \otimes C^{(i)}[\bullet]$ are induced by the first face map $\partial = \mu\otimes 1^{k-1} : \Gamma^{(i)}\otimes\Gamma^{(i)}[1]^{\otimes k} \to \Gamma^{(i)}\otimes\Gamma^{(i)}[1]^{\otimes k-1}$ of the bar complex. $\mu\otimes 1^{k-1}$ is induced by the C_p-version $\mu\otimes 1^{k-1} : \Gamma^{(i)}\otimes Q^{(i)}_1(\Sigma_p/C_p)^{\otimes k} \to \Gamma^{(i)}\otimes\Gamma^{(i)}[1]\otimes Q^{(i)}_1(\Sigma_p/C_p)^{\otimes k-1} \to \Gamma^{(i)} \otimes Q^{(i)}_1(\Sigma_p/C_p)^{\otimes k-1}$. Since $\Gamma^{\vee(i)} = \oplus\Gamma^{\vee(i)}[\ell]$, the dual $\partial^\vee$ is the direct sum of maps

$$\Gamma^{\vee(i)}[\ell] \otimes R[1]^{\otimes k} \xleftarrow{\quad\partial^\vee\quad} \Gamma^{\vee(i)}[\ell+1] \otimes R[1]^{\otimes k-1}$$

$$(H < \mathbb{G}^{(i)}, \{q_s \in \mathbb{G}^{(i)}_s\langle p\rangle\}_{s=1}^k) \longmapsto (H + \langle q_1\rangle, \{q_s\}_{s=2}^k)$$

where $\mathbb{G}_1 = \mathbb{G}/H$, $\mathbb{G}_s = \mathbb{G}_{s-1}/\langle q_{s-1}\rangle$, and $H + \langle q_1\rangle < \mathbb{G}$ is the subgroup generated by H and the inverse image of $\langle q_1\rangle$. Since $R[k] \to S[k]$ maps t_i to $\tilde{\phi}_{\langle k,\cdots,k-i+2\rangle}(t_{k-i+1})$, these correspond to maps

$$\Gamma^{\vee(i)}[\ell] \otimes Q^{(i)}_k(\Sigma_{p^k}/{\mathbb{F}_p}^k) \xleftarrow{\quad\partial^\vee\quad} \Gamma^{\vee(i)}[\ell+1] \otimes Q^{(i)}_{k-1}(\Sigma_{p^{k-1}}/{\mathbb{F}_p}^{k-1})$$

$$(H < \mathbb{G}^{(i)}, \{q_s \in \mathbb{G}^{(i)}/H\}_{s=1}^k) \longmapsto (H + \langle q_k\rangle, \{q'_s \in \mathbb{G}^{(i)}/H + \langle q_k\rangle\}_{s=2}^k),$$

where q'_s is the image of q_s in $H + \langle q_k\rangle$. Since all maps involved are ring maps, by checking on functor of points, we obtain a commutative diagram

$$\begin{tikzcd} \Gamma^{\vee(i)} \otimes S[k] \arrow[r] & \Gamma^{\vee(i)} \otimes R[k] \\ \Gamma^{\vee(i)} \otimes S[k-1] \arrow[u, "\partial^\vee"] \arrow[r] & \Gamma^{\vee(i)} \otimes R[k-1]. \arrow[u, "\partial^\vee"'] \end{tikzcd}$$

Similarly, the twisted differentials are induced by

$$\partial \circ \sigma : \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k} \xrightarrow{1\otimes\sigma\otimes 1^{k-1}} \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes k} \xrightarrow{\partial} \Gamma^{(i)} \otimes \Gamma^{(i)}[1]^{\otimes (k-1)},$$

so $(\partial \circ \sigma)^\vee$ is induced by

$$(\partial \circ \sigma)^\vee : \Gamma^{\vee(i)} \otimes R[1]^{\otimes (k-1)} \xrightarrow{\partial^\vee} \Gamma^{\vee(i)} \otimes R[1]^{\otimes k} \xrightarrow{\cdot t_1^{p-1}} \Gamma^{\vee(i)} \otimes R[1]^{\otimes k}.$$

This corresponds to the composite

$$(\partial \circ \sigma)^\vee : \Gamma^{\vee(i)} \otimes S[k-1] \xrightarrow{\partial^\vee} \Gamma^{\vee(i)} \otimes S[k] \xrightarrow{\cdot t_k^{p-1}} \Gamma^{\vee(i)} \otimes S[k].$$

Since the maps $R[k] \to S[k]$ are compatible with both maps, they are compatible with $(\partial\circ\sigma)^\vee$.

$\sigma^{\vee k}$ is multiplication by the Euler class $(t_1 \cdots t_k)^{p-1}$ and corresponds to multiplication by the Euler class η_k on $S[k]$. We want to show that for all ℓ, the square

$$\begin{tikzcd}
\Gamma^{\vee(i)}[\ell] \otimes S[k] \arrow[r, "1\otimes\eta_k"] & \Gamma^{\vee(i)}[\ell] \otimes S[k] \\
\Gamma^{\vee(i)}[\ell+1] \otimes S[k-1] \arrow[u, "\partial^\vee"] \arrow[r, "1\otimes\eta_{k-1}"'] & \Gamma^{\vee(i)}[\ell+1] \otimes S[k-1] \arrow[u, "(\partial\circ\sigma)^\vee"']
\end{tikzcd}$$

commutes. Since $\partial^\vee$ factors as

$$\partial^\vee : \Gamma^{\vee(i)}[\ell+1] \otimes S[k-1] \to \Gamma^{\vee(i)}[\ell] \otimes \Gamma^{\vee(i)}[1] \otimes S[k-1] \xrightarrow{1\otimes\partial^\vee} \Gamma^{\vee(i)}[\ell] \otimes S[k],$$

it is enough to show that the square commutes for $\ell = 0$.

For $a \in \Gamma^{\vee(i)}[1]$ and $b \in S[k-1]$.

$$\begin{aligned}
(\partial\circ\sigma)^\vee \circ (1\otimes\eta_{k-1})(a\otimes b) &= t_k^{p-1}\partial^\vee(a \otimes \eta_{k-1} b) = t_k^{p-1}\partial^\vee(\eta_{k-1})\partial^\vee(a)\partial^\vee(b) \\
(1\otimes\eta_k)\circ\partial^\vee(a\otimes b) &= \eta_k \partial^\vee(a)\partial^\vee(b).
\end{aligned}$$

We claim that $t_k^{p-1}\partial^\vee(\eta_{k-1}) = \eta_k$ in S[k]. To see this, consider the commutative square

$$\begin{tikzcd}
R[1]\otimes R[k-1] \cong R[k] \arrow[r] & S[k] \\
\Gamma^{\vee(i)}[1] \otimes R[k-1] \arrow[u, "\partial^\vee"] \arrow[r] & \Gamma^{\vee(i)}[1] \otimes S[k-1]. \arrow[u, "\partial^\vee"']
\end{tikzcd}$$

Since the right vertical map sends $q_1, \cdots, q_k$ to $\langle q_k\rangle, q_1', \cdots, q_{k-1}'$, where q_s' is the image of q_s in $\mathbb{G}^{(i)}/\langle q_k\rangle$, $\partial^\vee(x_1\otimes 1) = t_k^{p-1}$. The images of $x_1 \otimes (t_2\cdots t_k)^{p-1} \in \Gamma^{\vee(i)}[1]\otimes R[k-1]$ under the two ways of going around the square are the same, so $\eta_k = \partial^\vee(x_1\otimes\eta_{k-1}) = t_k^{p-1}\partial^\vee(\eta_{k-1})$. Thus, the square commutes for $\ell = 0$, hence for all ℓ. □

This induces a map on the cokernels

$$(\partial\circ\sigma)^\vee : \Gamma^{\vee(i)} \otimes A[k-1] \to \Gamma^{\vee(i)} \otimes A[k].$$

The map $A[k] \to B'[k-1] = A[k]/t_1$ sends $q_2, \cdots, q_k$ to $0, q_2, \cdots, q_k$. By checking on functor of points, there is a commutative diagram

$$\begin{tikzcd}
\Gamma^{\vee(i)}[\ell] \otimes S[k] \arrow[r, two heads] & \Gamma^{\vee(i)}[\ell] \otimes B'[k-1] \\
\Gamma^{\vee(i)}[\ell+1] \otimes S[k-1] \arrow[r, two heads] \arrow[u, "\partial^\vee"] & \Gamma^{\vee(i)}[\ell+1] \otimes B'[k-2] \arrow[u, "\partial^\vee"']
\end{tikzcd}$$

for $\ell = 0$, hence for all ℓ, as in the proof of the above lemma (here the rightmost vertical map is obtained by tensoring the $\Gamma^{\vee(i+1)}$-module map $\partial^\vee$ with $\Gamma^{\vee(i)}/u_i$ on the left). Postcomposing with multiplication by t_k^{p-1} gives a corresponding diagram for $(\partial \circ \sigma)^\vee$. Since the horizontal maps factor through $A[k]$ and $A[k-1]$ and the projection maps $S[k] \to A[k] \to B'[k-1]$ are surjective, there are commutative diagrams

$$\begin{tikzcd}
\Gamma^{\vee(i)}[\ell] \otimes S[k] \arrow[r, two heads] & \Gamma^{\vee(i)}[\ell] \otimes A[k] \arrow[r, two heads] & \Gamma^{\vee(i)}[\ell] \otimes B'[k-1] \\
\Gamma^{\vee(i)}[\ell+1] \otimes S[k-1] \arrow[r, two heads] \arrow[u, "(\partial\circ\sigma)^\vee"] & \Gamma^{\vee(i)}[\ell+1] \otimes A[k-1] \arrow[r, two heads] \arrow[u, "(\partial\circ\sigma)^\vee"'] & \Gamma^{\vee(i)}[\ell+1] \otimes B'[k-2]. \arrow[u, "(\partial\circ\sigma)^\vee"']
\end{tikzcd}$$

Since the leftmost vertical map restricts to a map $\Gamma^{\vee(i)}[\ell+1] \otimes e_{k-1}S[k-1] \to \Gamma^{\vee(i)}[\ell] \otimes e_k S[k]$ and the projection $S[k] \to A[k]$ is GL_k-equivariant, the middle vertical map restricts to $\Gamma^{\vee(i)}[\ell+1] \otimes e_{k-1}A[k-1] \to \Gamma^{\vee(i)}[\ell] \otimes e_k A[k]$. By Remark 4.4.8, there is a commutative diagram

$$\begin{tikzcd}
e_k A[k] \arrow[r] \arrow[d, hook] & e_k \operatorname{Pr} \otimes_{GL_{k-1}} B'[k-1] \arrow[r, leftrightarrow, "\simeq"] \arrow[d, hook] & e_{k-1}B'[k-1] \arrow[d, hook] \\
A[k] \arrow[r] & \operatorname{Pr} \otimes_{GL_{k-1}} B'[k-1] \arrow[r] & B'[k-1]
\end{tikzcd}$$

where the bottom row is the GL_k-equivariant map $A[k] \to \operatorname{Pr} \otimes_{GL_{k-1}} B'[k-1]$ followed by projection onto the component indexed by $\begin{bmatrix} 0 & I \end{bmatrix}$. This composite is the projection map $A[k] \to B'[k-1]$, which is compatible with $(\partial \circ \sigma)^\vee$. Since the vertical maps are inclusions and $(\partial \circ \sigma)^\vee$ on $A[k]$ and $B'[k-1]$ restrict to the Steinberg summands, $e_k A[k] \to e_{k-1}B'[k-1]$ is compatible with $(\partial \circ \sigma)^\vee$.

Combining this with Proposition 4.4.10 gives a commutative diagram

$$\begin{tikzcd}
\Gamma^{\vee(i)} \otimes e_k A[k] \arrow[r] \arrow[rr, bend left, "\simeq"] & \Gamma^{\vee(i)} \otimes e_{k-1}B'[k-1] & \Gamma^{\vee(i)} \otimes e_{k-1}B'[k-1] \arrow[l, hook', "1\otimes\eta_{k-1}"'] \\
\Gamma^{\vee(i)} \otimes e_{k-1}A[k-1] \arrow[r] \arrow[u, "(\partial\circ\sigma)^\vee"] \arrow[rr, bend right, "\simeq"'] & \Gamma^{\vee(i)} \otimes e_{k-2}B'[k-2] \arrow[u, "(\partial\circ\sigma)^\vee"'] & \Gamma^{\vee(i)} \otimes e_{k-2}B'[k-2] \arrow[l, hook, "1\otimes\eta_{k-2}"] \arrow[u, dashed]
\end{tikzcd} \tag{12}$$

where the top and bottom composites are isomorphisms if $p = 2$. Since multiplication by η_{k-1} and η_{k-2} are injective, there is exactly one possibility for the dotted map. By Lemma 4.4.15, it must be $\partial^\vee$.

For each k, $B'[k]$ is an E_0/u_i-module, so $\Gamma^{\vee(i)} \otimes_{E_0/u_{i-1}} e_{k-1}B'[k-1] \cong (\Gamma^{\vee(i)} /\!\!/ u_i) \otimes_{E_0/u_i} C^{\vee(i+1)}[k-1]$. Thus, we obtain the following.

Lemma 4.4.16. *For $p = 2$, there is a cofiber sequence*

$$(\Gamma^{\vee(i)} \otimes_{E_0/u_{i-1}} C^{\vee(i)}[\bullet], \partial^\vee) \to (\Gamma^{\vee(i)} \otimes_{E_0/u_{i-1}} C^{\vee(i)}[\bullet], (\partial \circ \sigma)^\vee) \to (\Gamma^{\vee(i)} /\!\!/ u_i \otimes_{E_0/u_i} C^{\vee(i+1)}[\bullet - 1], \partial^\vee).$$

We now dualize this to obtain the cofiber sequence in Theorem 4.0.1. Recal the cofibers sequences (4) and (5). Since $K_1[k] \cong K_2^\vee[k]$ is a left E_0/u_i-module, (4) is equivalent to the cofiber sequence

$$\Gamma^{(i)}{}_\sigma \otimes C^{(i)}[\bullet] \to \Gamma^{(i)} \otimes C^{(i)}[\bullet] \to \Gamma^{(i)} \otimes K_2^\vee[\bullet] \simeq (\Gamma^{(i)} /\!\!/ u_i) \otimes K_2^\vee[\bullet]$$

of left $\Gamma^{(i)}$-modules.

Lemma 4.4.17. *The boundary map ∂ of $(\Gamma^{(i)} /\!\!/ u_i) \otimes K_2^\vee[\bullet]$ is*

$$\partial = (\Gamma^{(i)} /\!\!/ u_i) \otimes_{\Gamma^{(i+1)}} \partial^{(i+1)},$$

where $\partial^{(i+1)}$ is the boundary map of the Koszul complex for E_0/u_i.

Proof. By [SS25, Cor 6.0.5], applying $\Sigma D \simeq D_{E_0/u_i}$ to the cofiber $\Gamma^{\vee(i)} \otimes K_2[\bullet] \simeq (\Gamma^{\vee(i)} /\!\!/ u_i) \otimes K_2[\bullet] \in \mathrm{DMod}_{E_0/u_i}$ shows that it is E_0/u_i-dual as a complex to $(\Gamma^{(i)} /\!\!/ u_i) \otimes K_2^\vee[\bullet]$. By (12) and checking on functor of points, the boundary map $\partial^\vee$ of $(\Gamma^{\vee(i)} /\!\!/ u_i) \otimes K_2[\bullet]$ sits in the commutative square

$$\begin{tikzcd}
(\Gamma^{\vee(i)} /\!\!/ u_i) \otimes K_2[k] \arrow[r, two heads] & \Gamma^{\vee(i+1)} \otimes K_2[k] \\
(\Gamma^{\vee(i)} /\!\!/ u_i) \otimes K_2[k-1] \arrow[r, two heads] \arrow[u, "\partial^\vee"] & \Gamma^{\vee(i+1)} \otimes K_2[k-1]. \arrow[u, "\partial^{\vee(i+1)}"']
\end{tikzcd}$$

Since this is a diagram in DMod_{E_0/u_i}, applying ΣD shows that ∂ fits in the diagram

$$\begin{tikzcd}
(\Gamma^{(i)} /\!\!/ u_i) \otimes K_2^\vee[k] & \Gamma^{(i+1)} \otimes K_2^\vee[k] \arrow[l, hook'] \arrow[d, "\partial^{(i+1)}"] \\
(\Gamma^{(i)} /\!\!/ u_i) \otimes K_2^\vee[k-1] \arrow[from=u, "\partial"'] & \Gamma^{(i+1)} \otimes K_2^\vee[k-1]. \arrow[l, hook']
\end{tikzcd}$$

$(\Gamma^{(i)} /\!\!/ u_i) \otimes K_2^\vee[\bullet]$ is a derived left $\Gamma^{(i)}$-module with the usual action of $\Gamma^{(i)}$. In particular, the boundary map ∂ is a map of left $\Gamma^{(i)}$-modules and is uniquely determined by its values on $1 \otimes y$ for all $y \in K_2[k]$. Since the map $\Gamma^{(i+1)} \to \Gamma^{(i)} /\!\!/ u_i$ is an inclusion of algebras, its image contains $1 \otimes y$, so $\partial(1 \otimes y) = \partial^{(i+1)}(1 \otimes y)$. But $(\Gamma^{(i)} /\!\!/ u_i) \otimes_{\Gamma^{(i+1)}}$ is a $\Gamma^{(i)}$-linear map which agrees with $\partial^{(i+1)}$ on all $1 \otimes y$. By uniqueness, we must have $\partial = (\Gamma^{(i)} /\!\!/ u_i) \otimes_{\Gamma^{(i+1)}} \partial^{(i+1)}$. ☐

Thus, for $p = 2$, the cofiber $\Gamma^{(i)} \otimes K_1[\bullet]$ is equivalent to $(\Gamma^{(i)} /\!\!/ u_i) \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i})$. Post-composing with σ^n completes the proof of Theorem 4.0.1.

5 Inflated $\Gamma^{(i)}$

In order to use the cofiber sequence in Theorem 4.0.1 to prove Tor vanishing, we need to understand the cofiber $(\Gamma^{(i-1)} /\!\!/ u_{i-1})_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} \overline{E_0/u_{i-1}}$, where we have shifted the index i for notational convenience. In particular, we would like to understand $\Gamma^{(i-1)} /\!\!/ u_{i-1}$ as a right $\Gamma^{(i)}$-module. The main goal of this section is to prove the following.

Proposition 5.0.1. *$\Gamma^{(i-1)}/\!\!/u_{i-1}$ is a free right $\Gamma^{(i)}$-module.*

Example 5.0.2. $\Gamma^{(h)} = k$ is the base field and $\Gamma^{(h-1)}/\!\!/u_{n-1} = k[Q]$ is a polynomial algebra on the Frobenius Q.

$\Gamma^{(i-1)}/\!\!/u_{i-1}$ can be thought of as an inflated version of $\Gamma^{(i)}$ generated by an additional operation corresponding to the Frobenius isogeny φ [SS25, Section 6]. We will see that an isogeny classified by $\Gamma^{\vee(i-1)}/\!\!/u_{i-1}$ is given by a sequence of Frobenii followed by a subgroup avoiding $\ker\varphi^i$ ([SS25, Def 4.4.3]). Heuristically, this allows us to decompose $\Gamma^{(i-1)}/\!\!/u_{i-1}$ into pieces isomorphic to $\Gamma^{(i)}$ corresponding to the number of iterates of Frobenii.

Notation 5.0.3.

- If R is a ring, $u \in R$, and X is a scheme over R, we will write X/u for the base change $\mathrm{Spf}(R/u) \times_{\mathrm{Spf}(R)} X$.
- If R is an E_0-algebra, we write $\mathbb{G}_R$ for the base change of the universal formal group $\mathbb{G}$ to R.
- Let $(H_1, \cdots, H_q)$ denote a chain of subgroups $H_s < \mathbb{G}_s = \mathbb{G}_{s-1}/H_{s-1}$ and $\overline{(H_1, \cdots, H_q)}$ denote the subgroup of $\mathbb{G}$ which is the kernel of the composition of isogenies given by $H_1, \cdots, H_q$.
- Let $\mathcal{S}_{k_1,\cdots,k_q} = \Gamma^\vee[k_1]_t \otimes \cdots \otimes_s \Gamma^\vee[k_q]$ and $\mathrm{Sub}_{k_1,\cdots,k_q}(\mathbb{G}) = \mathrm{Spf}(\mathcal{S}_{k_1,\cdots,k_q})$. We will often drop the group $\mathbb{G}$ when the formal group is clear.
- Let $\mathcal{S}^{(i)}_{k_1,\cdots,k_q} = \Gamma^{\vee(i)}[k_1]_t \otimes \cdots \otimes_s \Gamma^{\vee(i)}[k_q]$ and $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G}) = \mathrm{Spf}(\mathcal{S}^{(i)}_{k_1,\cdots,k_q})$. Then $\mathrm{Sub}^{(i)}_k(\mathbb{G}) = \mathrm{Spf}(\mathcal{S}_k)$ is a closed subscheme of $\mathrm{Sub}_k(\mathbb{G})$ and $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G})$ is the closed subscheme of $\mathrm{Sub}_{k_1,\cdots,k_q}(\mathbb{G})$ such that each $H_i < \mathbb{G}_i$ is in $\mathrm{Sub}^{(i)}_{k_i}(\mathbb{G}_i)$.

Note that for an E_0/u_{i-1}-algebra R, $\mathrm{Sub}_k(\mathbb{G}_R) = \mathrm{Sub}_k(\mathbb{G})_R := \mathrm{Spf}(R) \times_{\mathrm{Spf}(E_0/u_{i-1})} \mathrm{Sub}_k(\mathbb{G})$ ([Str97]). For $i > 0$, we do not know whether a sequence of subgroups $\{H_s < \mathbb{G}_s \mid 1 \leq s \leq q, |H_s| = p^{k_s}\}$ with $H_s \in \mathrm{Sub}^{(i)}_{k_s}(\mathbb{G}_s)$ is the same as a filtration of subgroups $\{H_1 < \cdots < H_q < \mathbb{G}_R \mid |H_s/H_{s-1}| = p^{k_s}\}$ with $H_s \in \mathrm{Sub}^{(i)}_{k_s}(\mathbb{G}_s)$. However, they are the same after inverting u_i [SS25, Remark 4.4.8].

5.1 Preliminaries

Definition 5.1.1. *Let $X = \mathrm{Spf}(R)$ be a scheme and $W = \mathrm{Spf}(R/I)$, $Z = \mathrm{Spf}(R/J)$ closed subschemes. The intersection and union of W and Z in X are defined to be*

$$W \cap Z = \mathrm{Spf}(R/I + J) \quad \text{and} \quad W \cup Z = \mathrm{Spf}(R/I \cap J),$$

respectively. These fit into pullback and pushout diagrams

$$\begin{tikzcd} W \cap Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ W \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} W \cap Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ W \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & W \cup Z. \end{tikzcd}$$

Lemma 5.1.2. *Let $X = W \cup Z$ be a union of closed subschemes. Let $s, t : X \to Y$ be flat maps of schemes. Then the fibered product $X_t{\times_Y}_s X$ is the union of closed subschemes*

$$X \times X = (W \times W) \cup (W \times Z) \cup (Z \times W) \cup (Z \times Z),$$

where $\times := {}_t{\times_Y}_s$.

Proof. It suffices to show that $X \times Z = (W \times Z) \cup (Z \times Z)$. Since s, t are flat, applying $- \times Z$ to the pullback and pushout square

$$\begin{tikzcd} W \cap Z \arrow[r] \arrow[d] & Z \arrow[d] \\ W \arrow[r] & X = W \cup Z \end{tikzcd}$$

gives a pullback and pushout square

$$\begin{tikzcd} (W \cap Z) \times Z \arrow[r] \arrow[d] & Z \times Z \arrow[d] \\ W \times Z \arrow[r] & X \times Z \end{tikzcd}$$

Since intersections commute with pullbacks,

$$(W \times Z) \cap (Z \times Z) = (W \cap Z) \times Z.$$

Thus, $X \times Z = (W \times Z) \cup (Z \times Z)$. □

Definition 5.1.3. *Let $R \to S$ be a map of rings. The image of a closed subscheme* $\mathrm{Spf}(S/I)$ *is* $\mathrm{Spf}\,(R/\ker(R \to S \to S/I))$.

Lemma 5.1.4. *If $R \hookrightarrow S$ is an injective map of rings and W and Z are closed subschemes, then the union of the images of W and Z in* $\mathrm{Spf}(R)$ *is the image of the union*

$$\mathrm{im}(W \cup Z) = \mathrm{im}(W) \cup \mathrm{im}(Z)$$

and the intersection of the images contains the image of the intersection

$$\mathrm{im}(W) \cap \mathrm{im}(Z) \supseteq \mathrm{im}(W \cap Z).$$

Proof. We may assume that $R \hookrightarrow S$ is an inclusion. Let $W = \mathrm{Spf}(S/I)$ and $Z = \mathrm{Spf}(S/J)$. Since $W \cup Z = \mathrm{Spf}(S/I \cap J)$,

$$\mathrm{im}(W \cup Z) = \mathrm{Spf}(R/R \cap I \cap J) = \mathrm{Spf}(R/(R \cap I) \cap (R \cap J)) = \mathrm{im}(W) \cup \mathrm{im}(Z).$$

The statement about intersections follows from the surjection

$$\mathcal{O}_{\mathrm{im}(W) \cap \mathrm{im}(Z)} = R/(R \cap I + R \cap J) \twoheadrightarrow R/R \cap (I + J) = \mathcal{O}_{\mathrm{im}(W \cap Z)}.$$

□

5.2 Union of subschemes

Our goal here is to prove Proposition 5.2.6, which expresses $\mathrm{Sub}_k^{(i-1)}(\mathbb{G})/\!\!/u_{i-1} := \mathrm{Spf}(\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1})$ as a union of closed subschemes. This is saying that a subgroup classified by $\mathrm{Sub}_k^{(i-1)}(\mathbb{G})/\!\!/u_{i-1}$ is given by a sequence of $k - s$ Frobenii followed by a subgroup avoiding $\ker \varphi^i$, for some $0 \le s \le k$.

Let $i \ge 1$, $R = \Gamma^{\vee(i-1)}[1]$, and $f = \tilde{f}^{(i)}(x)$ [SS25, Lemma 4.3.2]. Then $R/x = E_0/u_{i-1}$, $R/f = \Gamma^{\vee(i)}[1]$, $R/xf = \Gamma^{\vee(i-1)}[1]/\!\!/u_{i-1}$, and

$$(R/f) \otimes_R (R/x) = E_0/u_i.$$

Lemma 5.2.1. *There is a pullback of rings*

$$\begin{tikzcd} R/xf \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R/x \arrow[d] \\ R/f \arrow[r] & E_0/u_i \end{tikzcd}$$

Proof. $E_0/u_i = R/(x, f)$ and the maps $R/f \to E_0/u_i$ and $R/x \to E_0/u_i$ are the quotient maps. The pullback is given by

$$\begin{aligned} R/f \times_{E_0/u_i} R/x &= \{(r_1 \mod f, r_2 \mod x) \in R/f \times R/x \mid r_1 \equiv r_2 \mod (x, f)\} \\ &= \{(r + xs_1 + fq_1 \mod f, r + xs_2 + fq_2 \mod x) \mid s_1, s_2, q_1, q_2 \in R\} \\ &= \{(r + xs_1 \mod f, r + fq_2 \mod x) \mid s_1, q_2 \in R\}. \end{aligned}$$

The map $R/xf \to R/f \times R/x$ given by the respective quotients on each components has kernel $(f) \cap (x)$. R is a UFD since it is a regular local ring. Since $R/x = E_0/u_{i-1}$ and $R/f = \Gamma^{\vee(i)}[1]$ are integral domains, x and f are irreducible, hence $(f) \cap (x) = (xf)$ and the map $R/xf \to R/f \times R/x$ is injective. It is surjective since for any $s_1, q_2 \in R$, $r + xs_1 + fq_2 \in R/xf$ is an element which maps to $(r + xs_1 \mod f, r + fq_2 \mod x) \in R/f \times_{E_0/u_i} R/x$. □

Let $X = \mathrm{Spf}(R/xf) = \mathrm{Sub}_1^{(i-1)} /\!\!/ u_{i-1}$, $W = \mathrm{Spf}(R/f) = \mathrm{Sub}_1^{(i)}(\mathbb{G})$, and $Z = \mathrm{Spf}(R/x) = \mathrm{Spf}(E_0/u_{i-1})$. Z classifies the degree p subgroup "$\ker \varphi$" where φ is the Frobenius isogeny, which is the relative Frobenius $t \mapsto t^p$ on $\mathbb{G}$ [SS25, Section 2.1]. The lemma above tells us that $X = W \cup Z$ and $W \cap Z = \mathrm{Spf}(E_0/u_i)$ i.e. that a degree p subgroup in $\mathrm{Sub}_1^{(i-1)}(\mathbb{G}) /\!\!/ u_{i-1}$ is either $\ker \varphi$ or avoids $\ker \varphi^i$.

By [SS25, Lemma 6.0.2],

$$\mathcal{S}^{(i-1)}_{\underbrace{1,\ldots,1}_{k}} /\!\!/ u_{i-1} = \left(\Gamma^{\vee(i-1)}[1]^{\otimes k}\right) /\!\!/ u_{i-1} \cong \left(\Gamma^{\vee(i-1)}[1] /\!\!/ u_{i-1}\right)^{\otimes k} = (R/xf)^{\otimes k}.$$

Since both the source and target maps factor as $E_0/u_{i-2} \to \Gamma^{\vee(i-1)}[k] \hookrightarrow \Gamma^{\vee(i-1)}[k_1] \otimes \Gamma^{\vee(i-1)}[k_2]$ for all $k = k_1 + k_2$,

$$\mathcal{S}^{(i-1)}_{k_1,\cdots,k_q} /\!\!/ u_{i-1} \cong (\Gamma^{\vee(i-1)}[k_1] /\!\!/ u_{i-1}) \otimes \cdots \otimes (\Gamma^{\vee(i-1)}[k_q] /\!\!/ u_{i-1}).$$

Writing $\times$ for the pullback ${}_t\times_{\mathrm{Spf}\, E_0/u_{i-1}}\, {}_s$, by Lemma 5.1.2,

$$\mathrm{Sub}^{(i-1)}_{1,\ldots,1}(\mathbb{G}) /\!\!/ u_{i-1} \cong X \times \cdots \times X = \bigcup_{Y_s \in \{W, Z\}} Y_1 \times \cdots \times Y_k$$

classifies a sequence of subgroups, where each one is either $\ker \varphi$ or avoids $\ker \varphi^i$ (or both). This gives the following.

Proposition 5.2.2. $\mathrm{Sub}^{(i-1)}_{\underbrace{1,\ldots,1}_{k}}(\mathbb{G}) /\!\!/ u_{i-1}$ *is the union of closed subschemes*

$$\mathrm{Sub}^{(i-1)}_{\underbrace{1,\ldots,1}_{k}} /\!\!/ u_{i-1} = \bigcup_{\mu \in \{0,1\}^k} \mathrm{Spf}(R_\mu),$$

where $R_0 = R/x$, $R_1 = R/f$, *and* $R_\mu = R_{\mu_1} \otimes \cdots \otimes R_{\mu_k}$ *if* $\mu = (\mu_1, \cdots, \mu_k)$.

Note that the source and target maps $s, t : E_0/u_{i-1} \to R/x = E_0/u_{i-1}$ are the identity and Frobenius maps, respectively. These are not the same as the source and target maps for $\Gamma^{\vee(i)}[0] = E_0/u_{i-1}$, which are both the identity map. Thus, $\mathrm{Spf}\left((R/x)^{\otimes(k-s)} \otimes (R/f)^{\otimes s}\right) = \mathrm{Sub}^{(i)}_{\underbrace{1,\ldots,1}_{s}}(\varphi^{k-s}\mathbb{G})$ is a closed subscheme of $\mathrm{Sub}^{(i-1)}_{\underbrace{1,\ldots,1}_{k}}(\mathbb{G})/u_{i-1}$, where $\varphi\mathbb{G}$ is the pullback of $\mathbb{G}$ under the Frobenius φ [SS25, Def 2.1.3].

Definition 5.2.3. *Let* $\mathcal{S}^{(i)}_{\varphi^{k_0},k_1,\cdots,k_q} = (R/x)^{\otimes k_0} \otimes \Gamma^{\vee(i)}[k_1] \otimes \cdots \otimes \Gamma^{\vee(i)}[k_q]$. *Then* $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\varphi^{k_0}\mathbb{G}) = \mathrm{Spf}(\mathcal{S}^{(i)}_{\varphi^{k_0},k_1,\cdots,k_q})$. *We will denote* $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\varphi^{k_0}\mathbb{G})$ *by* $\mathrm{Sub}^{(i)}_{\varphi^{k_0},k_1,\cdots,k_q}(\mathbb{G})$.

Lemma 5.2.4. *For* $0 \le s \le k$, $\mathrm{Sub}^{(i)}_{\varphi^{k-s},s}(\mathbb{G})$ *is a closed subscheme of* $\mathrm{Sub}^{(i-1)}_{k}(\mathbb{G})/\!\!/u_{i-1}$. *It is the image of* $\mathrm{Sub}^{(i)}_{\varphi^{k-s},\underbrace{1,\ldots,1}_{s}}(\mathbb{G})$ *in* $\mathrm{Sub}^{(i-1)}_{k}(\mathbb{G})/\!\!/u_{i-1}$ *under the restriction map* $\mathcal{S}^{(i-1)}_{k}/\!\!/u_{i-1} \hookrightarrow \mathcal{S}^{(i-1)}_{1,\cdots,1}/\!\!/u_{i-1} = (R/xf)^{\otimes k}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \arrow[r, hook] \arrow[d, hook] & (R/xf)^{\otimes k} \arrow[d, "\simeq"] \\
(R/xf)^{\otimes(k-s)} \otimes \Gamma^{\vee(i-1)}[s]/u_{i-1} \arrow[r, hook] \arrow[d, two heads] & (R/xf)^{\otimes(k-s)} \otimes (R/xf)^{\otimes s} \arrow[d, two heads] \\
(R/x)^{\otimes(k-s)} \otimes \Gamma^{\vee(i)}[s] \arrow[r, hook] & (R/x)^{\otimes(k-s)} \otimes (R/f)^{\otimes s}.
\end{tikzcd}$$

The map $\mathrm{Sub}^{(i)}_{\varphi^{k-s},s}(\mathbb{G}) \to \mathrm{Sub}^{(i-1)}_{k}(\mathbb{G})/\!\!/u_{i-1}$ corresponds to the left vertical composite. We want to show that it is a surjective map of rings.

Since the source and target are finite free over the base E_0/u_{i-1}, it is a finite map. It is injective on functor of points: if $(\ker\varphi^{k-s}, H)$ and $(\ker\varphi^{k-s}, H')$ compose to the same isogeny, then $H = H'$ since φ^{k-s} is an isogeny, and isogenies are epimorphism. So, it is a monomorphism of schemes, thus an epimorphism of rings, and finite epimorphisms of rings are surjective. $\square$

For a binary sequence μ, let $|\mu| = \mu_1 + \cdots + \mu_k$ denote the sum of its digits.

Lemma 5.2.5. *If* $|\mu| = s$, *the image of* $\mathrm{Spf}(R_\mu)$ *in* $\mathrm{Sub}^{(i-1)}_{k}(\mathbb{G})/\!\!/u_{i-1}$ *is contained in the image of* $\mathrm{Spf}\left((R/x)^{\otimes(k-s)} \otimes (R/f)^{\otimes s}\right)$. *Equivalently, the map* $\mathcal{S}^{(i-1)}_{p^k}/\!\!/u_{i-1} \hookrightarrow (R/xf)^{\otimes k} \twoheadrightarrow R_\mu$ *factors through the quotient* $(R/xf)^{\otimes k} \twoheadrightarrow (R/x)^{\otimes(k-s)} \otimes (R/f)^{\otimes s}$.

Since the map $\mathrm{Sub}^{(i-1)}_{1,\cdots,1}(\mathbb{G})/\!\!/u_{i-1} \to \mathrm{Sub}^{(i-1)}_{k}(\mathbb{G})/\!\!/u_{i-1}$ is given by composition of isogenies, this is equivalent to the fact that every isogeny $g(t)$ can be written as $h(t^{p^j})$ for some $j \ge 0$ with $h'(0) \neq 0$ ([Rav23, Lemma A2.2.7]), so we can always move the Frobenius to the left.

Proof. It suffices to prove the lemma for $k = 2$ and $\mu = (1, 0)$. The map $\mathrm{Spf}(R_\mu) = \mathrm{Sub}^{(i)}_{1}(\mathbb{G}) \to \mathrm{Sub}^{(i-1)}_{2}(\mathbb{G})/\!\!/u_{i-1}$ sends $H < \mathbb{G}$ to the subgroup $\overline{(H, \varphi)} = \overline{(\varphi, \varphi H)}$ [SS25, Def 2.1.3] of $\mathbb{G}$. By

[SS25, Lemma 5.0.3], if H avoids $\ker\varphi^i$, then so does φH. So, $\mathrm{Sub}_1^{(i)}(\mathbb{G}) \to \mathrm{Sub}_2^{(i-1)}(\mathbb{G})/\!\!/u_{i-1}$ factors as

$$\mathrm{Sub}_1^{(i)}(\mathbb{G}) \to \mathrm{Sub}_{\varphi,1}^{(i)}(\mathbb{G}) \to \mathrm{Sub}_2^{(i-1)}(\mathbb{G})/\!\!/u_{i-1},$$

where the first map is the relative Frobenius sending H to φH and the second sends $H' < \varphi\mathbb{G}$ to $\overline{(\varphi, H')}$. Thus, the image of $\mathrm{Sub}_1^{(i)}(\mathbb{G}) = \mathrm{Spf}(R/f)$ is contained in the image of $\mathrm{Sub}_{\varphi,1}^{(i)}(\mathbb{G}) = \mathrm{Spf}(R/x \otimes R/f)$. □

Proposition 5.2.6. $\mathrm{Sub}_k^{(i-1)}(\mathbb{G})/\!\!/u_{i-1}$ *is the union of closed subschemes*

$$\mathrm{Sub}_k^{(i-1)}(\mathbb{G})/\!\!/u_{i-1} = \bigcup_{s=0}^{k} \mathrm{Sub}_s^{(i)}(\varphi^{k-s}\mathbb{G}).$$

Proof. Since the restriction map $\mathcal{S}_k^{(i-1)}/\!\!/u_{i-1} \hookrightarrow \mathcal{S}_{1,\cdots,1}^{(i-1)}/\!\!/u_{i-1} = (R/xf)^{\otimes k}$ is an injective map of rings, by Lemma 5.1.4 and Proposition 5.2.2, $\mathrm{Sub}_k^{(i-1)}(\mathbb{G})/\!\!/u_{i-1}$ is the union of the images of $\mathrm{Spf}(R_\mu)$ for all length k binary sequences μ. Thus, the proposition follows from Lemmas 5.2.4 and 5.2.5. □

5.3 Intersection of subschemes

Since we have a cover of $\mathrm{Sub}_k^{(i-1)}(\mathbb{G})/\!\!/u_{i-1}$ by closed subschemes, we would like to understand their intersections.

Lemma 5.3.1. *For* $0 \le j < s \le k$,

$$\mathrm{Sub}^{(i)}_{\varphi^{k-j},\underbrace{1,\cdots,1}_{j}}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},\underbrace{1,\cdots,1}_{s}}(\mathbb{G}) = \mathrm{Spf}\left((R/x)^{\otimes(k-s)} \otimes E_0/u_i \otimes (R/x)^{\otimes(s-j-1)} \otimes (R/f)^{\otimes j}\right).$$

Proof. Since R/x and R/f are flat as left and right E_0/u_{i-1}-modules, it suffices to prove the lemma for $j = 0$ and $s = k$. $(R/x)^{\otimes k} = R/I$ and $(R/f)^{\otimes k} = R/J$ where

$$I = (x_1, \cdots, x_k)$$
$$J = (f(x_1), f^P(x_2), \cdots, f^{P^{k-1}}(x_k)).$$

Since $f(x_1) = u_i + x_1(\cdots)$,

$$I + J = (x_1, \cdots, x_k, u_i, P(u_i), \cdots, P^{k-1}(u_i)).$$

By [Rez09, Proposition 3.25], the composite of P followed by projection $E_0 \xrightarrow{P} E^0(B\Sigma_p) \to E_0$ is the Frobenius, so $P(u_i) = u_i^p \mod x$ in R and

$$I + J = (x_1, \cdots, x_k, u_i, u_i^p, \cdots, u_i^{p^{k-1}}) = (x_1, \cdots, x_k, u_i).$$

Thus,

$$R/(I+J) = R/(x, u_i) \otimes (R/x)^{\otimes(k-1)} = E_0/u_i \otimes (R/x)^{\otimes(k-1)}.$$

□

Lemma 5.3.2. *For $0 \le j < s \le k$, the image of* $\mathrm{Sub}^{(i)}_{\varphi^{k-j},\underbrace{1,\cdots,1}_{j}}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},\underbrace{1,\cdots,1}_{s}}(\mathbb{G})$ *in* $\mathrm{Sub}^{(i-1)}_k(\mathbb{G})/\!\!/u_{i-1}$ *is*

$$\mathrm{Spf}\left((R/x)^{\otimes(k-s)} \otimes E_0/u_i \otimes (R/x)^{\otimes(s-j-1)} \otimes \Gamma^{\vee(i)}[j]\right).$$

Proof. There is a commutative diagram

$$\begin{tikzcd}
\Gamma^{\vee(i-1)}[k]/u_{i-1} \arrow[r, hook] \arrow[d, two heads] & \Gamma^{\vee(i-1)}[1]^{\otimes k}/u_{i-1} \arrow[d, two heads] \\
(R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j] \arrow[r, hook] \arrow[d, two heads] & (R/x)^{\otimes(k-j)} \otimes (R/f)^{\otimes j} \arrow[d, two heads] \\
(R/x)^{\otimes(k-s)} \otimes (E_0/u_i) \otimes (R/x)^{\otimes(s-j-1)} \otimes \Gamma^{\vee(i)}[j] \arrow[r] & (R/x)^{\otimes(k-s)} \otimes (E_0/u_i) \otimes (R/x)^{\otimes(s-j-1)} \otimes (R/f)^{\otimes j}
\end{tikzcd}$$

where the top part is from Lemma 5.2.4. Since $\Gamma^{\vee(i)}[j]$ is a direct summand of $\Gamma^{\vee(i)}[1]^{\otimes j} = (R/f)^{\otimes j}$, the bottom horizontal map is an inclusion of a direct summand, hence injective. □

Lemma 5.3.3. *For $0 \le j < s \le k$,*

$$\begin{aligned}
\mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},s}(\mathbb{G}) &= \mathrm{Spf}\left((R/x)^{\otimes(k-s)} \otimes E_0/u_i \otimes (R/x)^{\otimes(s-j-1)} \otimes \Gamma^{\vee(i)}[j]\right) \\
&= \mathrm{im}\left(\mathrm{Sub}^{(i)}_{\varphi^{k-j},\underbrace{1,\cdots,1}_{j}}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},\underbrace{1,\cdots,1}_{s}}(\mathbb{G})\right).
\end{aligned}$$

Proof. By Lemmas 5.1.4 and 5.3.1, the right hand side is contained in the left hand side. In terms of rings, both sides are given by $(R/x)^{\otimes(k-s)} \otimes$ stuff. Since $(R/x)^{\otimes(k-s)}$ is flat as both a left and right E_0/u_{i-1}-module, it suffices to do the case $s = k$.

$\mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},s}(\mathbb{G})$ corresponds to the ring

$$A = \left((R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j]\right) \otimes_{\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1}} \Gamma^{\vee(i)}[k],$$

which is both a quotient of $B = (R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j]$ and $C = \Gamma^{\vee(i)}[k]$. A surjects onto the right hand side $(E_0/u_i) \otimes (R/x)^{\otimes(k-j-1)} \otimes \Gamma^{\vee(i)}[j]$, which is the quotient of B by u_i. To show that they are the same, it suffices to show that $u_i = 0$ in A.

Over $\Gamma^{\vee(i)}[k]$, the universal degree p^k subgroup has equation [Str98, Proof of Theorem 9.2]

$$f_K(y) = c_{p^k} y + y^2(\cdots),$$

while the degree p^k subgroup coming from $(R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j]$ has equation

$$f_H(y) = g(y^{p^{k-j}}) = y^{p^{k-j}} + \cdots$$

for some power series g. In order for f_K and f_H to define the same subgroup over A, they must differ by a unit in $A[[y]]$. Since y is an indeterminate, this means that $c_{p^k} = 0$ in A, so the quotient map $C \twoheadrightarrow A$ factors through the projection $C \twoheadrightarrow C/c_{p^k}$. By [SS25, Lemma 4.4.7], c_{p^k} divides u_i in C. Thus, $C \twoheadrightarrow A$ factors through the projection $C \twoheadrightarrow C/u_i$, so $u_i = 0$ in A. □

Corollary 5.3.4. *If* $0 \le j < s < s' \le k$,

$$\mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},s}(\mathbb{G}) \supset \mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s'},s'}(\mathbb{G}).$$

Thus,

$$\mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G}) \cap \bigcup_{j+1\le s\le k} \mathrm{Sub}^{(i)}_{\varphi^{k-s},s}(\mathbb{G}) = \mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-j-1},j+1}(\mathbb{G}) = \mathrm{Sub}^{(i)}_{\varphi^{k-j},j}(\mathbb{G})/P^{k-j-1}(u_i).$$

Here P^s is the target map on $(R/x)^{\otimes s}$, which is the s-fold Frobenius on E_0/u_{i-1}, so $P^{k-j-1}(u_i) = u_i^{p^{k-j-1}}$ in $(R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j]$ and $P^0(u_i) = u_i$. Killing $P^{s-1}(u_i)$ in $(R/x)^{\otimes s}$ is equivalent to killing u_i on the sth R/x factor in $(R/x)^{\otimes s}$ i.e. $(R/x)^{\otimes s}/P^s(u_i) = (R/x)^{\otimes s}/\!\!/u_i$ and $\left((R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j]\right)/P^{k-j-1}(u_i) = (R/x)^{\otimes(k-j-1)} \otimes E_0/u_i \otimes (R/x) \otimes \Gamma^{\vee(i)}[j]$.

For X a scheme over $\mathrm{Spf}(E_0/u_{i-1})$, let $u_i^{-1}X$ denote its base change to $\mathrm{Spf}(u_i^{-1}E_0/u_{i-1})$. The following tells us that any two closed subschemes in the cover intersect trivially away from u_i, so the union becomes a disjoint union.

Corollary 5.3.5. *For* $0 \le j < s \le k$,

$$u_i^{-1}\mathrm{Sub}^{(i)}_j(\varphi^{k-j}\mathbb{G}) \cap u_i^{-1}\mathrm{Sub}^{(i)}_s(\varphi^{k-s}\mathbb{G}) = \emptyset.$$

Thus, $u_i^{-1}\mathrm{Sub}^{(i-1)}_k(\mathbb{G})/\!\!/u_{i-1}$ *is the disjoint union of closed subschemes*

$$u_i^{-1}\mathrm{Sub}^{(i-1)}_k(\mathbb{G})/\!\!/u_{i-1} = \bigsqcup_{s=0}^{k} u_i^{-1}\mathrm{Sub}^{(i)}_s(\varphi^{k-s}\mathbb{G}).$$

Proof. The localization $E_0/u_{i-1} \to u_i^{-1}E_0/u_{i-1}$ commutes with unions and intersections of schemes. Since the target map on R/x is the ring Frobenius, the ring corresponding to the intersection $\mathrm{Sub}^{(i)}_{\varphi^{k-j},p^j}(\mathbb{G}) \cap \mathrm{Sub}^{(i)}_{\varphi^{k-s},p^s}(\mathbb{G})$ is u_i-power torsion by Lemma 5.3.3, so it is zero after inverting u_i. □

5.4 A direct sum decomposition

Proposition 5.2.6 and Corollary 5.3.4 allow us to express $\mathrm{Sub}^{(i-1)}_k(\mathbb{G})/\!\!/u_{i-1}$ as an iterated pushout of schemes, equivalently, $\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1}$ as an iterated pullback of rings.

Proposition 5.4.1. *There is an isomorphism of rings*

$$\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \cong \mathcal{S}^{(i)}_{\varphi^k} \times_{\mathcal{S}^{(i)}_{\varphi^k}/P^{k-1}(u_i)} \mathcal{S}^{(i)}_{\varphi^{k-1},p} \times_{\mathcal{S}^{(i)}_{\varphi^{k-1},p}/P^{k-2}(u_i)} \cdots \times_{\mathcal{S}^{(i)}_{\varphi,p^{k-1}}/P^0(u_i)} \mathcal{S}^{(i)}_{p^k}.$$

The following will allow us to write this pullback as a direct sum decomposition.

Lemma 5.4.2. *Let R be a ring and M, N left R-modules. Suppose M is an R-algebra with $u \in M$ a non-zerodivisor. Given a map $h : N \to M$ of R-modules, we can form the pullback $M \times_{M/u} N$ by postcomposing with the the projection map $M \to M/u$. There is an isomorphism of (left) R-modules*

$$M \times_{M/u} N \cong M \oplus N$$

natural in (M, N, h, u). Explicitly, given (M, N, h, u), (M', N', h', u'), a map of R-algebras $M \to M', u \mapsto u'$ and a map of R-modules and $N \to N'$ compatible with h and h', the following diagram commutes.

$$\begin{array}{ccc} M \times_{M/u} N & \longrightarrow & M' \times_{M'/u'} N' \\ \wr\| & & \wr\| \\ M \oplus N & \longrightarrow & M' \oplus N' \end{array}$$

Proof. The isomorphism is given by $M \oplus N \to M \times_{M/u} N$ by $(m, n) \mapsto (um + h(n), n)$. Its inverse $(m, n) \mapsto \left(\frac{m-h(n)}{u}, n\right)$ is well-defined since u is not a zero divisor. Naturality is straightforward. $\square$

Proposition 5.4.3. *There is an isomorphism of left E_0/u_{i-1}-modules*

$$\Gamma^{\vee(i-1)}[k] /\!\!/ u_{i-1} \cong \bigoplus_{j=0}^{k} \mathcal{S}^{(i)}_{\varphi^{k-j}, p^j}.$$

Proof. We will apply the above lemma to the pullback decomposition in Proposition 5.4.1. For $0 \le j \le k$, let

$$X_j = \bigcup_{j \le s \le k} \mathrm{Sub}_{\varphi^{k-s}, p^s}.$$

By Corollary 5.3.4, $\mathrm{Sub}_{\varphi^{k-j}, p^j} \cap X_{j+1} = \mathrm{Sub}_{\varphi^{k-j}, p^j} / P^{k-j-1}(u_i)$, so the map $\mathcal{O}_{X_{j+1}} \twoheadrightarrow \mathcal{S}^{(i)}_{\varphi^{k-j}, p^j} / P^{k-j-1}(u_i)$ factors through the projection $\mathcal{O}_{X_{j+1}} \twoheadrightarrow \mathcal{S}^{(i)}_{\varphi^{k-j-1}, p^{j+1}}$. To apply Lemma 5.4.2, we need to provide E_0/u_{i-1}-module lifts

$$\mathcal{S}^{(i)}_{\varphi^{k-j-1}, p^{j+1}} \dashrightarrow \mathcal{S}^{(i)}_{\varphi^{k-j}, p^j} \to \mathcal{S}^{(i)}_{\varphi^{k-j}, p^j} / P^j(u_i).$$

Let $h_1 : R/f \to R/x$ be the E_0/u_{i-1}-bimodule map sending $Q_0 + Q_1 x + \cdots$ to Q_0. The desired lifts are given by tensoring

$$h_{j+1} : \Gamma^{\vee(i)}[j+1] \hookrightarrow (R/f) \otimes \Gamma^{\vee(i)}[j] \xrightarrow{h_1 \otimes 1} (R/x) \otimes \Gamma^{\vee(i)}[j]$$

with $(R/x)^{k-j-1}$. Thus, we obtain the direct sum decomposition by repeated application of Lemma 5.4.2.

$\square$

Let $E_0/{u_{i-1}}_{\varphi^k}$ denote E_0/u_{i-1} with the usual left E_0/u_{i-1}-module structure and with a right E_0/u_{i-1}-module structure given by k iterations of the Frobenius. Since the target map t on R/x is the Frobenius, $E_0/{u_{i-1}}_{\varphi^k} = (R/x)^{\otimes k}$.

Corollary 5.4.4. *There are isomorphisms of left E_0/u_{i-1}-modules*

$$\Gamma^{\vee(i-1)} /\!\!/ u_{i-1} \cong \bigoplus_{k \ge 0} (R/x)^{\otimes k} \otimes \Gamma^{\vee(i)} = \bigoplus_{k \ge 0} E_0/{u_{i-1}}_{\varphi^k} \otimes \Gamma^{\vee(i)}$$

$$\Gamma^{(i-1)} /\!\!/ u_{i-1} \cong \bigoplus_{k \ge 0} E_0/{u_{i-1}}_{\varphi^k} \otimes \Gamma^{(i)}.$$

Proof.

$$\Gamma^{\vee(i-1)}/\!\!/u_{i-1} = \bigoplus_{k\geq 0} \Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \cong \bigoplus_{\substack{k\geq 0\\ 0\leq j\leq k}} (R/x)^{\otimes(k-j)} \otimes \Gamma^{\vee(i)}[j] \cong \bigoplus_{k\geq 0} (R/x)^{\otimes k} \otimes \Gamma^{\vee(i)}$$

as left E_0/u_{i-1}-modules. The dual statement follows by self-duality of $E_0/u_{i-1}{}_{\varphi^k}$. □

Indeed, for a local ring S, if P, Q are S-bimodules which are finitely generated projective left S-modules, then $\operatorname{rank}(P \otimes_S Q) = \operatorname{rank} P \cdot \operatorname{rank} Q$. Since $R/x = E_0/u_{i-1}$ is a left E_0/u_{i-1}-module of rank 1,

$$\Gamma^{\vee(i-1)}/\!\!/u_{i-1} \cong \bigoplus_{k\geq 0} (R/x)^{\otimes k} \otimes \Gamma^{\vee(i)} \cong \bigoplus_{k\geq 0} \Gamma^{\vee(i)}$$

as left E_0/u_{i-1}-modules.

Proposition 5.4.5. *There is an isomorphism of right $\Gamma^{\vee(i)}$-comodules*

$$\Gamma^{\vee(i-1)}/\!\!/u_{i-1} \cong \bigoplus_{k\geq 0} (R/x)^{\otimes k} \otimes \Gamma^{\vee(i)} \cong \bigoplus_{k\geq 0} E_0/u_{i-1}{}_{\varphi^k} \otimes \Gamma^{\vee(i)},$$

dually, an isomorphism of right $\Gamma^{(i)}$-modules

$$\Gamma^{(i-1)}/\!\!/u_{i-1} \cong \bigoplus_{k\geq 0} E_0/u_{i-1}{}_{\varphi^k} \otimes \Gamma^{(i)}.$$

Proof. We will show the statement on the dual. Let $\Gamma_s^{\vee(i)}[j] := (R/x)^s \otimes \Gamma^{\vee(i)}[j]$. The right $\Gamma^{\vee(i)}$-comodule structure is given by maps

$$\Gamma^{\vee(i-1)}[k+r]/\!\!/u_{i-1} \hookrightarrow \Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \otimes \Gamma^{\vee(i-1)}[r]/\!\!/u_{i-1} \twoheadrightarrow \Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \otimes \Gamma^{\vee(i)}[r]$$

for $k, r \geq 0$ coming from comultiplication on $\Gamma^{\vee(i-1)}/\!\!/u_{i-1}$. For $r \leq j \leq k+r$, there are commutative diagrams

$$\begin{tikzcd}
\Gamma^{\vee(i-1)}[k+r]/\!\!/u_{i-1} \arrow[r, hook] \arrow[d, two heads] & \Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \otimes \Gamma^{\vee(i-1)}[r]/\!\!/u_{i-1} \arrow[r, two heads] \arrow[d, two heads] & \Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \otimes \Gamma^{\vee(i)}[r] \arrow[dl, two heads] \\
\Gamma^{\vee(i)}_{k-j}[j+r] \arrow[r, hook] & \Gamma^{\vee(i)}_{k-j}[j] \otimes \Gamma^{\vee(i)}[r]. &
\end{tikzcd}$$

These induce a map on pullbacks

$$\Gamma_k^{\vee(i)}[r] \times_{\Gamma_k^{\vee(i)}[r]/P^{k-1}(u_i)} \cdots \times \Gamma_0^{\vee(i)}[k+r] \to (\Gamma_k^{\vee(i)}[0] \otimes \Gamma^{\vee(i)}[r]) \times_{(\Gamma_k^{\vee(i)}[0]/P^{k-1}(u_i)) \otimes \Gamma^{\vee(i)}[r]} \cdots \times (\Gamma_0^{\vee(i)}[k] \otimes \Gamma^{\vee(i)}[r]), \tag{13}$$

where the target is isomorphic to $\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \otimes \Gamma^{\vee(i)}[r]$ since $\Gamma^{\vee(i)}[r]$ is a finite free left E_0/u_{i-1}-module hence distributes over the pullback decomposition of $\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1}$.

To apply naturality of Lemma 5.4.2, we need to check that the lifts provided in the proof of Proposition 5.4.3 are compatible with the right $\Gamma^{\vee(i)}$-comodule structure. Ignoring the $(R/x)^{k-j}$ factor, this is the condition that the left square in

$$\begin{tikzcd}
\Gamma^{\vee(i)}[j+r] \arrow[r, hook] \arrow[d, "h_{j+r}"] & \Gamma^{\vee(i)}[j] \otimes \Gamma^{\vee(i)}[r] \arrow[r, hook] \arrow[d, "h_j \otimes 1"] & (R/f) \otimes (R/f)^{j+r-1} \arrow[d, "h_1 \otimes 1"] \\
(R/x) \otimes \Gamma^{\vee(i)}[j-1+r] \arrow[r, hook] & (R/x) \otimes \Gamma^{\vee(i)}[j-1] \otimes \Gamma^{\vee(i)}[r] \arrow[r, hook] & (R/x) \otimes (R/f)^{j+r}
\end{tikzcd}$$

commutes. This is true since the right and outer square commute and the horizontal maps are injective. Hence, the map (13) commutes with the direct sum decomposition on both sides.

By mapping the terms $\Gamma^{\vee(i)}_{k+r-j}[j]$ to 0 if $j < r$, we get the following commutative diagram

$$\begin{tikzcd}
\Gamma^{\vee(i-1)}[k+r]/\!\!/u_{i-1} \arrow[r] \arrow[d] & \Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} \otimes \Gamma^{\vee(i)}[r] \arrow[d] \\
\Gamma^{\vee(i)}_{k+r}[0] \times_{\Gamma^{\vee(i)}_{k+r}[0]/P^{k+r-1}(u_i)} \cdots \times \Gamma^{\vee(i)}_0[k+r] \arrow[r] \arrow[d] & (\Gamma^{\vee(i)}_k[0] \otimes \Gamma^{\vee(i)}[r]) \times_{\dots} \cdots \times (\Gamma^{\vee(i)}_0[k] \otimes \Gamma^{\vee(i)}[r]) \arrow[d] \\
\Gamma^{\vee(i)}_{k+r}[0] \oplus \cdots \oplus \Gamma^{\vee(i)}_0[k+r] \arrow[r] & (\Gamma^{\vee(i)}_k[0] \otimes \Gamma^{\vee(i)}[r]) \oplus \cdots \oplus (\Gamma^{\vee(i)}_0[k] \otimes \Gamma^{\vee(i)}[r]).
\end{tikzcd}$$

Thus, the direct sum decomposition is compatible with the right $\Gamma^{\vee(i)}$-comodule structure on $\Gamma^{\vee(i-1)}/\!\!/u_{i-1}$, equivalently, the right $\Gamma^{(i)}$-module structure on $\Gamma^{(i-1)}/\!\!/u_{i-1}$.

□

Since $E_0/u_{i-1} = \mathbb{F}_p[[u_i, \cdots, u_{n-1}]]$, the Frobenius $\varphi : E_0/u_{i-1} \to E_0/u_{i-1}$ is a finite free map, so $E_0/u_{i-1}{}_{\varphi^k}$ is a finite free right E_0/u_{i-1}-module. Therefore, $\Gamma^{(i-1)}/\!\!/u_{i-1}$ is a free right $\Gamma^{(i)}$-module, concluding the proof of Proposition 5.0.1.

Remark 5.4.6. Proposition 5.4.5 is not an isomorphism of left $\Gamma^{(i)}$-modules. The lifts provided in the proof of Proposition 5.4.3 are not compatible with the left $\Gamma^{\vee(i)}$-comodule structure since the left $\Gamma^{\vee(i)}$-comodule structure involves commuting past a series of Frobenii.

However, things are much simpler after inverting u_i. Corollary 5.3.5 shows that all intersections are empty away from u_i, so the pullback decomposition in Proposition 5.4.1 is just a finite product.

Corollary 5.4.7. *There is an isomorphism of left and right $u_i^{-1}\Gamma^{\vee(i)}$-comodules*

$$u_i^{-1}\Gamma^{\vee(i-1)}/\!\!/u_{i-1} \cong \bigoplus_{k\geq 0} u_i^{-1}E_0/u_{i-1}{}_{\varphi^k} \otimes u_i^{-1}\Gamma^{\vee(i)},$$

dually, an isomorphism of $u_i^{-1}\Gamma^{(i)}$-bimodules

$$u_i^{-1}\Gamma^{(i-1)}/\!\!/u_{i-1} \cong \bigoplus_{k\geq 0} u_i^{-1}E_0/u_{i-1}{}_{\varphi^k} \otimes u_i^{-1}\Gamma^{(i)}.$$

6 Tor vanishing

Putting things together, we obtain the following.

Theorem 6.0.1 (Tor vanishing)**.** *For $p = 2$, $0 \leq i < h$, and all $n \geq 0$,*

$$\mathrm{Tor}^*_{\Gamma^{(i)}}(\Gamma^{(i)}{}_{\sigma^n}, \overline{E_0/u_{i-1}}) = 0$$

for $* > 0$.

Here $\overline{E_0/u_{i-1}}$ is the trivial left $\Gamma^{(i)}$-module and the subscript in $\Gamma^{(i)}{}_{\sigma^n}$ indicates that $\Gamma^{(i)}$ is considered as a right $\Gamma^{(i)}$-module by the n-fold (double) suspension σ^n.

Proof. We proceed by descending induction i. For $i = h$, $\Gamma^{(h)}$ is the base field so the statement is true. Suppose by induction that the statement is true for $i+1$. We will prove the statement for i by induction on n, using the cofiber sequence

$$\Gamma^{(i)}{}_{\sigma^{n+1}} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} (\overline{E_0/u_{i-1}}) \to \Gamma^{(i)}{}_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} (\overline{E_0/u_{i-1}}) \to (\Gamma^{(i)} /\!\!/ u_i)_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i})[1]$$

in Theorem 4.0.1.

For $n = 0$, Tor vanishing is obvious. Suppose by induction that Tor vanishing is true for n, so the middle term of the cofiber sequence is discrete. Since $\sigma : \Gamma^{(i+1)} \to \Gamma^{(i)} /\!\!/ u_i$ factors as $\Gamma^{(i+1)} \xrightarrow{\sigma} \Gamma^{(i+1)} \hookrightarrow \Gamma^{(i)} /\!\!/ u_i$,

$$(\Gamma^{(i)} /\!\!/ u_i)_{\sigma^{a+1}} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i}) \simeq (\Gamma^{(i)} /\!\!/ u_i) \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} \Gamma^{(i+1)}{}_{\sigma^{n+1}} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i}).$$

By Proposition 5.0.1, $\Gamma^{(i)} /\!\!/ u_i \cong \bigoplus \Gamma^{(i+1)}$ is a free right $\Gamma^{(i+1)}$-module, so the last term

$$\begin{aligned}(\Gamma^{(i)} /\!\!/ u_i)_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i}) &\simeq \bigoplus \Gamma^{(i+1)} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} \Gamma^{(i+1)}{}_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i}) \\ &\simeq \bigoplus \Gamma^{(i+1)}{}_{\sigma^n} \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} (\overline{E_0/u_i})\end{aligned}$$

in the cofiber sequence is discrete. Thus, the first term is discrete by the long exact sequence in homology, completing the proof. $\square$

7 Congruence criterion

In [Rez09, Thm A], Rezk shows that a p-torsion free Γ-algebra B lifts (necessarily uniquely) to a $\mathbb{T}$-algebra if and only if it satisfies the congruence criterion

$$\sigma(b) \equiv b^p \mod p$$

for all $b \in B$, where $\sigma \in \Gamma$ is a lift of $\overline{\sigma} \in \Gamma/p$ classifying the Frobenius isogeny. Here we give a congruence criterion analogous to Rezk's, as promised in [SS25]. This will be an application of Corollary 5.3.5.

In [SS25], we saw that there is a monad $\mathbb{T}^{(i)}$ analogous to $\mathbb{T}$. Let $i > 0$ and $\widetilde{\Gamma}^{(i)} := \Gamma^{(i-1)} /\!\!/ u_{i-1}$ and let $\mathbb{T}^{(i)}$ be the monad in [SS25]. Recall that $\mathrm{Alg}_{\mathbb{T}^{(i)}} \subset \mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$ is the full subcategory consisting of objects on which Q_0 acts by the ring Frobenius, where $Q_0 \in \widetilde{\Gamma}^{(i)}[1]$ is the operation corresponding to the ring map $\widetilde{\Gamma}^{(i)}[1] \to E_0/u_{i-1}, x \mapsto 0$. The mod $p, \cdots, u_{i-1}$ analog of the congruence criterion will give a sufficient condition for a $\Gamma^{(i)}$-algebra whose underlying E_0/u_{i-1}-module is u_i-torsion free, to lift to a $\mathbb{T}^{(i)}$-algebra. This is much easier than the original case in Rezk since unlike $\mathbb{T}(E_0)$, $\mathbb{T}^{(i)}(E_0/u_{i-1})$ is an algebra of power operations.

Let $Q_0, \cdots, Q_N \in \widetilde{\Gamma}^{\vee(i)}[1]$ and $Q'_0, \cdots, Q'_{N-1}$ be dual to the bases $1, x, \cdots, x^N$ and $1, x, \cdots, x^{N-1}$ for $\widetilde{\Gamma}^{\vee(i)}[1]$ and $\Gamma^{\vee(i)}[1]$, respectively, where $N = \operatorname{rank} \Gamma^{\vee(i)}[1]$.

Lemma 7.0.1. *There exists at most one lift of a u_i-torsion free $\Gamma^{(i)}$-algebra to a $\mathbb{T}^{(i)}$-algebra.*

Proof. Write $f^{(i)}(x) = c_0 + c_1 x + \cdots + c_{N-1}x^{N-1} + x^N$ where $c_0 = u \cdot u_i$ for some unit u. Then $Q'_j = Q_j - c_j Q_N$ for $0 \le j \le N-1$. Since Q_0 acts on a $\mathbb{T}^{(i)}$-algebra by the ring Frobenius and u_i is not a zero divisor, there is at most one way to extend a $\Gamma^{(i)}$-algebra structureto a $\mathbb{T}^{(i)}$-algebra structure. $\square$

[Rez09, Prop 8.3] shows every rational Γ-algebra extends uniquely to a $\mathbb{T}$-algebras. Let $\mathrm{Alg}_{\mathbb{T}^{(i)},u_i} \subset \mathrm{Alg}_{\mathbb{T}^{(i)}}$ and $\mathrm{Alg}_{\Gamma^{(i)},u_i} \subset \mathrm{Alg}_{\Gamma^{(i)}}$ denote the full subcategories on objects in which u_i is invertible.

Lemma 7.0.2. *The forgetful functor* $\mathrm{Alg}_{\mathbb{T}^{(i)}} \to \mathrm{Alg}_{\Gamma^{(i)}}$ *induces an equivalence of subcategories* $\mathrm{Alg}_{\mathbb{T}^{(i)},u_i} \to \mathrm{Alg}_{\Gamma^{(i)},u_i}$.

Proof. Given $A \in \mathrm{Alg}_{\Gamma^{(i)},u_i}$, by Lemma 7.0.1, it suffices to show that there exists a $\mathbb{T}^{(i)}$-algebra structure on A compatible with the $\Gamma^{(i)}$-algebra structure. By Corollary 5.3.5, $u_i^{-1}\widetilde{\Gamma}^{\vee(i)}[k]$ splits as a product. Since $A \otimes_s \widetilde{\Gamma}^{\vee(i)}[k] \cong A \otimes_s u_i^{-1}\widetilde{\Gamma}^{\vee(i)}[k]$, to construct a map $P : A \to A \otimes \widetilde{\Gamma}^{\vee(i)}[k]$, it suffices to construct maps $P : A \to (u_i^{-1}R/x)^{k-s} \otimes u_i^{-1}\Gamma^{\vee(i)}[s]$ for each $0 \le s \le k$, where $R/x = \Gamma^{\vee(i-1)}[1]/x$ as in Section 4.

Define $P : A \to A\otimes u_i^{-1}R/x \cong A$ to be the ring Frobenius. This is the map $u_i^{-1}\mathrm{Sub}_{\varphi}^{(i)}(\mathbb{G}_A) \to \mathrm{Spf}(A)$ classifying $\ker\varphi$. All the other maps are obtained by composing operations. Explicitly,

$$A \xrightarrow{P} A \otimes u_i^{-1}R/x \xrightarrow{P\otimes 1} A \otimes (u_i^{-1}R/x)^{k-1} \otimes (u_i^{-1}R/x)$$

is the k-fold Frobenius and $A \to A \otimes (u_i^{-1}R/x)^{k-s} \otimes u_i^{-1}\Gamma^{\vee(i)}[s]$ is the composite

$$A \xrightarrow{P} A \otimes u_i^{-1}\Gamma^{\vee(i)}[s] \xrightarrow{P\otimes 1} A \otimes (u_i^{-1}R/x)^{k-s} \otimes u_i^{-1}\Gamma^{\vee(i)}[s]$$

where the first map comes from the $\Gamma^{(i)}$-algebra structure of A.

This makes A into a $\widetilde{\Gamma}^{(i)}$-algebra compatible with the $\Gamma^{(i)}$-algebra structure. Since $u_i^{-1}\widetilde{\Gamma}^{\vee(i)}[1] \cong u_i^{-1}R/x \times u_i^{-1}\Gamma^{\vee(i)}[1]$ where the projection onto the first factor is given by $x \mapsto 0$, R/x corresponds to Q_0, so A is indeed a $\mathbb{T}^{(i)}$-algebra. $\square$

Lemma 7.0.3. *If* $A \in \mathrm{Alg}_{\Gamma^{(i)}}$*, then* $u_i^{-1}A \in \mathrm{Alg}_{\Gamma^{(i)}}$ *and the localization map* $A \to u_i^{-1}A$ *is a map of* $\Gamma^{(i)}$*-algebras.*

Proof. The power operations map $P : E_0/u_{i-1} \to \Gamma^{\vee(i)}[k]$ extends to a map $u_i^{-1}E_0/u_{i-1} \to P(u_i)^{-1}\Gamma^{\vee(i)}[k]$. Since $P(u_i)$ divides u_i in $\Gamma^{\vee(i)}[k]$ ([SS25, (10)]), inverting u_i inverts $P(u_i)$, so we can postcompose this with the localization map to $u_i^{-1}\Gamma^{\vee(i)}[k] \cong u_i^{-1}E_0/u_{i-1} \otimes \Gamma^{\vee(i)}[k]$. Thus, $u_i^{-1}E_0/u_{i-1}$ is (uniquely) a $\mathbb{T}^{(i)}$-algebra. Since $u_i^{-1}A \cong u_i^{-1}E_0/u_{i-1} \otimes A$ and $\mathrm{Alg}_{\Gamma^{(i)}}$ is symmetric monoidal, the lemma follows. $\square$

The following says that the only "critical weight" ([Rez09]) is p.

Lemma 7.0.4. *Let* $M \in \mathrm{Alg}_{\mathbb{T}^{(i)}}$ *and* $N \subseteq M$ *a subobject in* $\mathrm{Alg}_{\Gamma^{(i)}}$*. If the dotted map* P_k *in*

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, dashed, "P_k"'] & M \arrow[d] \\ N \otimes \widetilde{\Gamma}^{\vee(i)}[k] \arrow[r, hook] & M \otimes \widetilde{\Gamma}^{\vee(i)}[k], \end{tikzcd} \tag{14}$$

exists for $k = 1$*, then it exists for all* $k \ge 0$*. In particular, there is a unique* $\mathbb{T}^{(i)}$*-algebra structure on* N *such that the inclusion* $N \to M$ *is a map of* $\mathbb{T}^{(i)}$*-algebras.*

Proof. P_0 clearly exists and P_1 exists by assumption. Suppose by induction that we have defined P_{k-1}. For $k > 1$, consider the diagram

$$\begin{tikzcd}
N \arrow[rrr, hook] \arrow[d, "P_1"'] \arrow[ddrr, dashed, "P_k"] & & & M \arrow[d] \arrow[ddr] & \\
N \otimes \widetilde{\Gamma}^{\vee(i)}[1] \arrow[rrr, hook] \arrow[dd, "P_{k-1}\otimes 1"'] & & & M \otimes \widetilde{\Gamma}^{\vee(i)}[1] \arrow[dd] & \\
& & N \otimes \widetilde{\Gamma}^{\vee(i)}[k] \arrow[rrr, hook] \arrow[dll, hook] & & & M \otimes \widetilde{\Gamma}^{\vee(i)}[k] \arrow[dll, hook] \\
N \otimes \widetilde{\Gamma}^{\vee(i)}[k-1] \otimes \widetilde{\Gamma}^{\vee(i)}[1] \arrow[rrr, hook] & & & M \otimes \widetilde{\Gamma}^{\vee(i)}[k-1] \otimes \widetilde{\Gamma}^{\vee(i)}[1] & &
\end{tikzcd}$$

where P_k is the dotted arrow. Then $\operatorname{im} N$, the image of N in $M \otimes \widetilde{\Gamma}^{\vee(i)}[k-1] \otimes \widetilde{\Gamma}^{\vee(i)}[1]$, is contained in the intersection of $N \otimes \widetilde{\Gamma}^{\vee(i)}[k-1] \otimes \widetilde{\Gamma}^{\vee(i)}[1]$ and $M \otimes \widetilde{\Gamma}^{\vee(i)}[k]$, which is equal to $N \otimes \widetilde{\Gamma}^{\vee(i)}[k]$ since $\widetilde{\Gamma}^{\vee(i)}[k]$ is a direct summand of $\widetilde{\Gamma}^{\vee(i)}[k-1] \otimes \widetilde{\Gamma}^{\vee(i)}[1]$. Thus, P_k exists. $\square$

Definition 7.0.5. *We say that an object $A \in \mathrm{Alg}_{\Gamma^{(i)}}$ satisfies the congruence condition if $Q_0'a \equiv a^p \mod u_i$ for all $a \in A$.*

Note that Q_0' is a lift of a class in $\Gamma^{(i)}[1]/u_i$ which classifies the Frobenius isogeny.

Proposition 7.0.6 (Congruence criterion)**.** *An object $A \in \mathrm{Alg}_{\Gamma^{(i)}}$ which is u_i-torsion free admits the structure of a $\mathbb{T}^{(i)}$-algebra (necessarily uniquely), if and only if it satisfies the congruence condition.*

Proof. By Lemma 7.0.4 applied to the localization map $A \to u_i^{-1}A$, it suffices to show that the map P_1 in (14) exists. By the proof of Lemma 7.0.1, $Q_0' = Q_0 + u_i Q_N$. Since Q_0 acts on a $\mathbb{T}^{(i)}$-algebra by the ring Frobenius, the action of Q_N is uniquely determined if u_i is not a zero divisor. This applies to both A and $u_i^{-1}A$. Since $A \to u_i^{-1}A$ is a map of $\Gamma^{(i)}$-algebras, the action of Q_N on A and $u_i^{-1}A$ agree. Thus, there exists a map $P_1 : A \to A \otimes \widetilde{\Gamma}^{\vee(i)}[1]$ making the diagram (14) commute for $k = 1$. $\square$

8 Rezk's Stuff

In [Rez13], Rezk considers the Ext groups $\mathrm{Ext}^*_\Gamma(\omega^m, \mathrm{null})$, which compute (the E_2-page of a mapping space spectral sequence converging to) $E^*\left(TAQ_{S_{K(h)}}\left(S_{K(h)}^{S_+^{2m-1}}\right)\right)$, the E-cohomology of $K(h)$-local TAQ of $S_{K(h)}$-valued cochains on odd spheres; [BR19] identifies these groups with the Morava E-theory $E_*^\wedge(\Phi S^{2m-1})_{K(h)}$ of the Bousfield-Kuhn functor on odd spheres.

Recall that the Γ-action for a Γ-module M is denoted by $P_{M,k} : M \to M \otimes_s \Gamma^\vee[k]$ for all $k \geq 0$. We will be dealing with 3 different Γ-modules below, all whose underlying E_0-module is E_0.

- E_0 has the usual Γ-action $P_k : E_0 \to \Gamma^\vee[k]$.
- null, also denoted $\overline{E_0}$, has the trivial Γ-action i.e. $P_{\mathrm{null},0} = \mathrm{id}$ and $P_{\mathrm{null},k} = 0$.

- $\omega^m = \pi_{2m}E$ is acted on by Γ^{-2m}. It is obtained by postcomposing the Γ-action on E_0 with $\sigma^m : \Gamma \to \Gamma$, so
$$P_{\omega^m,k} : E_0 \xrightarrow{P_k} \Gamma^\vee[k] \xrightarrow{\cdot c_{p^k}^m} \Gamma^\vee[k]$$
where c_{p^k} is the Euler class of the reduced complex standard representation of Σ_{p^k}.

Rezk showed that for heights $n = 1$ and 2, the groups $\mathrm{Ext}^s_\Gamma(\omega^m, \mathrm{null})$ vanish except when $s = n$, thus the spectral sequence collapses. Unfortunately, this is not true at higher heights.

Proposition 8.0.1. *For $p = 2$ and heights $n \geq 2$,*

- *If $m = 0$, $\mathrm{Ext}^s_\Gamma(\omega^m, null) = 0$ for all s.*
- *If $m = 1$, $\mathrm{Ext}^s_\Gamma(\omega^m, null) \neq 0$ for all $2 \leq s \leq n$.*
- *If $m > 1$, $\mathrm{Ext}^s_\Gamma(\omega^m, null) \neq 0$ for $s = 2$ and n.*

Since Γ is Koszul of length $n+1$, these Ext groups vanish when $s > n$. We will prove the proposition using some variant of the cofiber sequence in Theorem 4.0.1. There are obvious mod u_i versions of E_0, null, and ω^m, which we also denote by E_0/u_i, null, and ω^m.

Let
$$\begin{aligned}
A_k(\mathbb{G}, E_0/u_{i-1}) &= \mathcal{O}_{\mathrm{Sub}^{(i)}_{\underbrace{1\cdots 1}_{k}}(\mathbb{G})} = \Gamma^{\vee(i)}[1]^k \\
B_k(\mathbb{G}, E_0/u_{i-1}) &= \bigoplus_{j=1}^{k-1} \mathcal{O}_{\mathrm{Sub}^{(i)}_{1\cdots 2\cdots 1}(\mathbb{G})} = \bigoplus_{j=1}^{k-1} \Gamma^{\vee(i)}[1]^{j-1} \otimes \Gamma^{\vee(i)}[2] \otimes \Gamma^{\vee(i)}[1]^{k-j-1} \\
K_k(\mathbb{G}, E_0/u_{i-1}) &= \mathrm{cok}(B_k(\mathbb{G}, E_0/u_{i-1}) \to A_k(\mathbb{G}, E_0/u_{i-1})) \cong C^{\vee(i)}[k],
\end{aligned}$$
where $\mathbb{G}$ the universal formal group over E_0/u_{i-1}. Let R be a $u_i^{-1}E_0/u_{i-1}$-algebra. Let $A_k(G,R)$, $B_k(G,R)$, and $K_k(G,R)$ denote their base changes to R. By [SS25, Lemma 5.0.5], since u_i is invertible in R, $R \otimes_{E_0/u_{i-1}} \mathcal{O}_{\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G})}$ classifies filtrations $(H_1 < \cdots < H_q < G_R)$ consisting of étale subgroups. We will often drop the G and R from the notation when it is clear. Let $\mathcal{D}(\mathbb{G}, E_0/u_{i-1})$ be the diagram

$$\mathcal{D}(\mathbb{G}, E_0/u_{i-1}) = \begin{tikzcd} \vdots & \vdots \\ B_k \arrow[r] \arrow[u] & A_k \arrow[u] \\ B_{k-1} \arrow[r] \arrow[u, "1^{k-1}\otimes s"] & A_{k-1} \arrow[u, "1^{k-1}\otimes s"'] \\ \vdots \arrow[u] & \vdots \arrow[u] \end{tikzcd}$$

where the vertical maps $1^{k-1} \otimes s : A_{k-1} = A_1^{k-1} \otimes R \to A_1^{k-1} \otimes A_1 = A_k$ are given on functor of points by forgetting the last subgroup
$$(H_1 < \mathbb{G}_1, \cdots, H_k < \mathbb{G}_k) \mapsto (H_1 < \mathbb{G}_1, \cdots, H_{k-1} < \mathbb{G}_{k-1}).$$

Here $\mathbb{G}_{s+1} = \mathbb{G}_s/H_s$. These induce maps $K_{k-1} \to K_k$ making

$$\mathcal{K}(\mathbb{G}, E_0/u_{i-1}) := (\cdots \to K_{k-1} \to K_k \to \cdots)$$

into a cochain complex. Let $\mathcal{D}(G,R) = R \otimes_{E_0/u_{i-1}} \mathcal{D}(\mathbb{G}, E_0/u_{i-1})$, which is well-defined since all the maps involved are left $u_i^{-1}E_0/u_{i-1}$-linear. Note that we do not always have a theory of power operations over an arbitrary $u_i^{-1}E_0/u_{i-1}$-algebra R since the target maps $t : R \to R \otimes \Gamma^{\vee(i)}[k]$ do not always exist, but we always have source maps given by the usual inclusion $s : R \to R \otimes \Gamma^{\vee(i)}[k]$. Since $\mathcal{D}(G,R)$ can be defined without invoking the target map, it has the expected interpretation in terms of subgroups of G. We define $\mathcal{K}(G,R)$ similarly.

Notation 8.0.2. For a k-algebra R, a surjective map of finite sets $f : S \to T$ induces a map of k-algebras $\prod_T R \to \prod_S R$ where each component R_t maps diagonally into the product of components indexed by the fiber $f^{-1}(t)$. We will denote this map by $\prod_T \Delta$ and refer to it as the "diagonal" map. Under the isomorphism $\prod R \cong \bigoplus R$ (we will use $\prod$ and $\bigoplus$ interchangeably), this map is given by a matrix with entries 0 or 1.

For an integer $h \geq 0$ and a ring R, let $\mathcal{D}(h,R) = \mathcal{D}(G,R)$ for $G = \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^h}$, the height h constant p-divisible group over R. Strictly speaking, R is not necessarily a $u_i^{-1}E_0/u_{i-1}$-algebra but we can still make sense of the complex $\mathcal{D}(G,R)$ by taking $A_k = \mathcal{O}_{\mathrm{Sub}_{\underbrace{1,\cdots,1}_{k}}(G)}$ instead and similarly for B_k. Since $\mathcal{O}_{\mathrm{Sub}_{i_1,\cdots,i_k}(G)} \cong \prod_{\mathrm{Sub}_{i_1,\cdots,i_k}(G)} R$, each term in $\mathcal{D}(G)$ is a direct sum of R indexed by appropriate subgroups of $(\mathbb{Q}_p/\mathbb{Z}_p)^h$ and each map is given by a matrix with integer entries. Thus, $\mathcal{D}(h,R) = R \otimes_{\mathbb{Z}} \mathcal{D}(h,\mathbb{Z})$. We will sometimes drop h or R from the notation when it is clear.

Let $C^{\vee(i)}[\bullet] := \mathcal{K}(\mathbb{G}, E_0/u_{i-1}) = (C^{\vee(i)}[\bullet], \partial^{\vee(i)})$. We can also form the m-twisted complex $C^{\vee(i)}[\bullet]_{\sigma^m} := (C^{\vee(i)}[\bullet], \partial^{\vee(i)}_{\sigma^m})$ whose boundary maps are induced by

$$A_{k-1} \xrightarrow{1^{k-1}\otimes s} A_k = A_1^k \xrightarrow{1^{k-1}\otimes x_k^m} A_1^k = A_k$$

where x_k is the Euler class in the rightmost A_1 factor.

Lemma 8.0.3. *$\mathrm{Ext}^*_{\Gamma^{(i)}}(\omega^m, null)$ is computed by the complex $C^{\vee(i)}[\bullet]_{\sigma^m}$.*

Proof. This is explained in [Rez13, 7.3-7.6], but note [SS25, Remark 2.0.2] on the different handedness conventions. The Koszul complex is a free left $\Gamma^{(i)}$-module resolution for ω^m. Its kth term is $\Gamma^{(i)} \otimes C^{(i)}[k] \otimes \omega^m \cong \Gamma^{(i)} \otimes C^{(i)}[k] \otimes E_0/u_{i-1}$ and its boundary maps are given by $\partial_k = \partial_{k,\mathrm{mult}} - (-1)^k \partial_{k,\mathrm{act}}$, defined as follows. Let $\ell : C^{(i)}[k] \to \Gamma^{(i)}[1] \otimes C^{(i)}[k-1]$ and $r : C^{(i)}[k] \to C^{(i)}[k-1] \otimes \Gamma^{(i)}[1]$. Define

$$\partial_{k,\mathrm{mult}}, \partial_{k,\mathrm{act}} : \Gamma^{(i)} \otimes C^{(i)}[k+1] \otimes \omega^m \to \Gamma^{(i)} \otimes C^{(i)}[k] \otimes \omega^m$$

by

$$\partial_{k,\mathrm{mult}} = (\mathrm{mult} \otimes \mathrm{id}_{C^{(i)}[k]} \otimes \mathrm{id}_{\omega^m}) \circ (\mathrm{id}_{\Gamma^{(i)}} \otimes \ell \otimes \mathrm{id}_{\omega^m})$$
$$\partial_{k,\mathrm{act}} = (\mathrm{id}_{\Gamma^{(i)}} \otimes \mathrm{id}_{C^{(i)}[k]} \otimes \mathrm{act}) \circ (\mathrm{id}_{\Gamma^{(i)}} \otimes r \otimes \mathrm{id}_{\omega^m}).$$

The complex for computing Ext is then given by

$$\mathrm{Hom}_{\Gamma^{(i)}}(\Gamma^{(i)} \otimes C^{(i)}[\bullet], \mathrm{null}) = \mathrm{Hom}_{E_0/u_{i-1}}(C^{(i)}[\bullet], \mathrm{null}) \cong C^{\vee(i)}[\bullet].$$

$\partial_{k,\mathrm{mult}}$ vanishes since $\Gamma^{(i)}$ acts trivially on null, so only $\partial_{k,\mathrm{act}}$ remains. The fact that the action map $\Gamma^{(i)} \otimes E_0/u_{i-1} \to E_0/u_{i-1}$ is left E_0/u_{i-1}-dual to the source map $s : E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]$ lets us identify the boundary maps. $\square$

Remark 8.0.4. [BR19] also mentions that the Ext groups are computed by the dual Koszul complex $C^\vee[\bullet]_{\sigma^m}$. However, they claim [BR19, Thm 10.3] that the boundary maps in the dual Koszul complex are induced by the target instead of the source map i.e. that this is the modular isogeny complex. In private communication with Rezk, we confirm that the boundary maps are indeed induced by the source map. The point is that the action map $\Gamma[1] \otimes E_0 \to E_0$ is E_0-dual to the source and not the target map.

Lemma 8.0.5. *For $p = 2$ and $m > 0$, there is a cofiber sequence*

$$C^{\vee(i)}[\bullet]_{\sigma^{m-1}} \to C^{\vee(i)}_{\sigma^m}[\bullet] \to C^{\vee(i+1)}_{\sigma^{mp-1}}[\bullet - 1].$$

Proof. The map $C^{\vee(i)}[\bullet]_{\sigma^{m-1}} \to C^{\vee(i)}_{\sigma^m}[\bullet]$ is given levelwise by $\sigma^{\vee^\bullet}$, similar to Definition 4.1.1. Let $R[k]$, $S[k]$, and η_k be as in 8. Proposition 4.3.1 lets us choose an identification $\Delta_k \leftrightarrow \mathbf{p^k}$ such that the induced map $R[k] \to S[k]$ is given on functor of points by $q_1, \cdots, q_k \mapsto q'_k, \cdots, q'_1$. Under this identification, $GL_k(\mathbb{F}_p)$ acts on $R[k]$ by its usual action on $q_1, \cdots, q_k$. Let $B'[k-1] = S[k]/t_k$. There is a commutative diagram

$$\begin{tikzcd} R[k] \arrow[r] & S[k] \arrow[r, two heads] & B'[k-1] \\ R[k-1] \arrow[r] \arrow[u, "\partial^{\vee(i)} = 1^{k-1} \otimes s"] & S[k-1] \arrow[r, two heads] \arrow[u, "\partial^{\vee(i)}"] & B'[k-2] \arrow[u, "\partial^{\vee(i+1)}"'] \end{tikzcd} \tag{15}$$

given on functor of points by

$$\begin{aligned} R[k-1] \to R[k], &\quad q_1, \cdots, q_k \mapsto q_1, \cdots, q_{k-1} \\ S[k-1] \to S[k], &\quad q_1, \cdots, q_k \mapsto q_2, \cdots, q_k \\ S[k] \to B'[k-1], &\quad q_1, \cdots, q_{k-1} \mapsto q_1, \cdots, q_{k-1}, 0. \end{aligned}$$

As before, let $U_3 < GL_k$ be the subgroup of matrices $\left[\begin{array}{ccc|c} & I & & 0 \\ \hline * & \cdots & * & * \end{array}\right]$. Let N be its normalizer and $W = GL_{k-1} \times 1$ its Weyl group. Since U_3 is conjugate to U_2, $GL_k/U_3 \cong GL_k/U_2$, so $GL_k/U_3 \cong \mathrm{Pr}(k, k-1)$ as a left GL_k-set and right W-set.

The map $S[k] \to B'[k-1]$ is N-equivariant and U_3 acts on $B'[k-1]$ trivially. By adjunction, this corresponds to a GL_k-equivariant map $S[k] \to GL_k \otimes_N B'[k-1] \cong GL_k/U_3 \otimes_W B'[k-1] \cong \mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1]$. By Lemma 4.4.7, postcomposing this with the projection $\mathrm{Pr} \otimes_{GL_{k-1}} B'[k-1] \to B'[k-1]$ and taking Steinberg summands gives a map $e_k S[k] \to e_{k-1} B'[k-1]$. By Proposition 4.4.10, it has image $\eta_{k-1} \cdot e_{k-1} B'[k-1]$, which is isomorphic to $e_{k-1} B'[k-1] \cong C^{\vee(i+1)}[k-1]$. This gives the cokernel of $\sigma^{\vee^k} : C^{\vee(i)}[k] \to C^{\vee(i)}[k]$.

It remains to identify the boundary maps, which is the dotted map below.

$$\begin{tikzcd}
R[k] \arrow[r] & S[k] \arrow[r, two heads] & B'[k-1] & B'[k-1] \arrow[l, hook', "\eta_{k-1}"'] \\
R[k] \arrow[r] \arrow[u, "\cdot(-t_k)^{(p-1)m}"] & S[k] \arrow[r, two heads] \arrow[u, "\cdot(\eta_k/\eta_{k-1})^m"] & B'[k-1] \arrow[u, "\cdot(\eta_{k-1}/\eta_{k-2})^{mp}"'] & \\
R[k-1] \arrow[r] \arrow[u, "\partial^{\vee(i)}"] & S[k-1] \arrow[r, two heads] \arrow[u, "\partial^{\vee(i)}"] & B'[k-2] \arrow[u, "\partial^{\vee(i+1)}"'] & B'[k-2] \arrow[l, hook', "\eta_{k-2}"'] \arrow[uu, dashed]
\end{tikzcd}$$

Here η_k/η_{k-1} is the quotient of $\eta_k \in S[k]$ by the image of $\eta_{k-1} \in S[k-1]$ and η_{k-1}/η_{k-2} is defined similarly for B'. It follows that the dotted map must be $\partial^{\vee(i+1)}$ followed by multiplication by $(\eta_{k-1}/\eta_{k-2})^{mp-1}$. $\square$

Lemma 8.0.6 ([Kuh15] Whitehead Conjecture and Tower of S^1 Conjecture)**.** *If $n > 0$, the untwisted complex $(C^\vee[\bullet], \partial^\vee)$ is exact.*

Proof. This is the Whitehead and Tower of S^1 conjectures proven in [Kuh15]. Kuhn considers families of maps

$$S^1 \underset{s_{-1}}{\overset{d_{-1}}{\leftrightarrows}} QL_1(0) \underset{s_0}{\overset{d_0}{\leftrightarrows}} QL_1(1) \underset{s_1}{\overset{d_1}{\leftrightarrows}} QL_1(2) \leftrightarrows \cdots \tag{16}$$

where the s_k's are given by deloopings of the attaching maps of the Goodwillie tower of the identity functor on S^1 and the d_k's are infinite loop maps coming from the symmetric power filtration on the sphere. By [Kuh15, Theorem 1.1], $d_k s_k + s_{k-1} d_{k-1} : QL_1(k) \to QL_1(k)$ is a homotopy equivalence. Here $d_k d_{k-1} = 0$ but $s_k s_{k-1}$ is only null after applying Ω^k.

Let Φ_n denote the height n Bousfield-Kuhn functor. Applying $E_0^\wedge \Phi_n \Omega^m$ (for $m \geq n$) to (16) results in a chain complex, so $d_k s_k + s_{k-1} d_{k-1}$ gives a contracting chain homotopy. Since Φ_n preserves limits and E is 2-periodic, the sequence

$$0 \to E_0^\wedge \Phi_n(S^1) \to E_0^\wedge \Phi_n L_1(0) \to E_0^\wedge \Phi_n L_1(1) \to \cdots \to E_0^\wedge \Phi_n L_1(n) \to 0$$

is exact. [BR19] identifies the Koszul complex $(C^\vee[\bullet], \partial^\vee)$ with the Goodwillie tower $E_0^\wedge \Phi_n L_1(\bullet)$. Thus, the lemma follows since

$$\Phi_n(S^1) = \Phi_n(B\mathbb{Z}) = \Phi_n \Omega^\infty(\Sigma H\mathbb{Z}) = L_{K(n)} \Sigma H\mathbb{Z} = 0$$

if $n > 0$. $\square$

However, by dimension counting, $C^{\vee(i)}[\bullet]$ is not exact for $i > 0$. In fact, it is not exact in the worst possible way.

Lemma 8.0.7. *For $p = 2$, $i > 0$, and $0 < k \leq n - i$, $u_i^{-1} H^k(C^{\vee(i)}[\bullet], \partial^{\vee(i)})$ is nonzero, thus $H^k(C^{\vee(i)}[\bullet], \partial^{\vee(i)})$ is nonzero.*

Note that $H^0 = 0$ since the zeroth boundary map is injective. To prove that $H^k \neq 0$, it suffices to show that $H^k \neq 0$ after a flat base change. As mentioned in the proof of [SS25, Prop 4.5.3], Stapleton constructed a ring C_i over which $\mathbb{G}$ splits as $\mathbb{G}^0 \oplus \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^{n-i}}$. Let $R = C_i/I_i$ where $I_i = (p, \cdots, u_{i-1})$. R is a faithfully flat $u_i^{-1}E_0/u_{i-1}$-algebra. We will make the flat base change $E_0/u_{i-1} \to u_i^{-1}E_0/u_{i-1} \to R$ over which everything will be p-divisible groups. We first show that $H^1 \neq 0$ and use it to show that $H^k \neq 0$. From now on we set $p = 2$.

Lemma 8.0.8. *For all $i \geq 0$, over R,*

$$H^1(C^{\vee(i)}[\bullet], \partial^{\vee(i)}) \cong (R[x]/x^{p^i})/R$$

as R-modules.

The idea is that for $i = 0$, any two degree p étale subgroups are contained in some degree p^2 étale subgroup, so $H^1 = 0$. However, this fails if $i > 0$ since there are degree p étale subgroups which differ by some $\ker \varphi^i$ (see proof of [SS25, Prop 4.5.3]). Any degree p^2 subgroup containing both of them must also contain $\ker \varphi^i$, hence cannot be étale.

Let $\Lambda = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$. Over R, the connected-étale sequence $0 \to \mathbb{G}^0 \to \mathbb{G} \to \mathbb{G}' \to 0$ for $\mathbb{G}$ splits and $\mathbb{G}' \simeq \underline{\Lambda^{n-i}}$. The p-series $\langle p \rangle^{(i)}(t)$ is equal to $g(x^{p^i})$ for some g, where $x = \prod_{i=1}^{p-1}[i]t$. Let f be the Weierstrass polynomial of g.

To compute H^1, we consider the following parts of $\mathcal{D}(\mathbb{G}, R)$ and $\mathcal{K}(\mathbb{G}, R)$.

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})} \arrow[r] & \mathcal{O}_{\mathrm{Sub}_{11}^{(i)}(\mathbb{G})} \arrow[r] & C^{\vee(i)}[2] \\
& \mathcal{O}_{\mathrm{Sub}_1^{(i)}(\mathbb{G})} \arrow[u] \arrow[r, equal] & C^{\vee(i)}[1] \arrow[u, "\partial_1^{\vee(i)}"'] \\
& R \arrow[u] \arrow[r, equal] & C^{\vee(i)}[0]. \arrow[u, "\partial_0^{\vee(i)}"']
\end{tikzcd}$$

$H^1 = \ker \partial_1^{\vee(i)} / \operatorname{im} \partial_0^{\vee(i)} \cong D/R$ where D is the pullback

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})} \arrow[r] & \mathcal{O}_{\mathrm{Sub}_{11}^{(i)}(\mathbb{G})} \\
D \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & \mathcal{O}_{\mathrm{Sub}_1^{(i)}(\mathbb{G})}. \arrow[u]
\end{tikzcd} \tag{17}$$

Since $\mathbb{G}' \simeq \underline{\Lambda^{n-i}}$, everything is discrete hence splits over R. Write

$$[p]_{\mathbb{G}'}(t) = \prod_{c \in \mathcal{I}} (t - c)$$
$$f(x) = \prod_{a \in \mathcal{J}} (x - a).$$

Then

$$\mathcal{O}_{\mathrm{Sub}_1(\mathbb{G}')} \cong R[x_1]/f(x_1) \cong \prod_a R[x_1]/x_1 - a \cong \prod_{\mathrm{Sub}_1(\Lambda^{n-i})} R$$
$$\mathcal{O}_{\mathrm{Sub}_{11}(\mathbb{G}')} \cong R[x_1, x_2]/f(x_1), f^{(P)}(x_2) \cong \prod_{a_1, a_2 \in \mathcal{J}} R[x_1, x_2]/x_1 - a_1, x_2 - P(a_2) \cong \prod_{\mathrm{Sub}_{11}(\Lambda^{n-i})} R$$
$$\mathcal{O}_{\mathrm{Sub}_2(\mathbb{G}')} \cong \prod_{\mathrm{Sub}_2(\Lambda^{n-i})} R.$$

Lemma 8.0.9. *For all integers $h \geq 0$ and any ring S, the pullback $D(h,S)$ of the diagram (17) corresponding to $\underline{\Lambda^h}$ over S is isomorphic to S. Let D' be the pullback of the diagram (17) corresponding to $\mathbb{G}'$. Then $D' \cong R$.*

In particular, $D(n-i,R) \cong R$.

Proof. By Lemma 8.0.6, for $\mathbb{G}_h$ the universal formal group over the height h Morava E-theory $E' := E_h$, the complex $\mathcal{K}(\mathbb{G}_h, E'_0)$ is exact. Let C_0 be Stapleton's ring ([Sta13, Corollary 2.18]) over which $C_0 \otimes \mathbb{G}_h \simeq \underline{\Lambda^h}$. The diagrams $\mathcal{D}(h,C_0)$ and $\mathcal{K}(h,C_0)$ are obtained by tensoring the corresponding diagrams for $\mathbb{G}_h/E'$ with C_0. In particular, $\mathcal{K}(h,C_0)$ is exact, so $D(h,C_0) \cong \operatorname{im}\partial_0^\vee = C_0$, where $D(h,C_0)$ is the pullback of the diagram (17) corresponding to $\underline{\Lambda^h}$ over C_0. This also implies that

$$\begin{tikzcd} \mathrm{Sub}_2(\Lambda^{n-i}) \arrow[d] & \mathrm{Sub}_{11}(\Lambda^{n-i}) \arrow[l] \arrow[d] \\ * \arrow[ur, phantom, "\urcorner"] & \mathrm{Sub}_1(\Lambda^{n-i}) \arrow[l] \end{tikzcd}$$

is a pushout of sets. Thus, for any ring S, the pullback $D(h,S) \cong S$ and $H^1\mathcal{K}(h,S) = 0$. $\square$

Since $[p]_{\mathbb{G}}(t) = [p]_{\mathbb{G}'}(t^{p^i})$ splits, the analogous formulas for $\mathcal{O}_{\mathrm{Sub}_1^{(i)}(\mathbb{G})}$, $\mathcal{O}_{\mathrm{Sub}_{11}^{(i)}(\mathbb{G})}$, and $\mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})}$ are obtained from the formulas for $\mathbb{G}'$ by replacing x by x^{p^i}. Since $\mathbb{G} \simeq \mathbb{G}^0 \oplus \mathbb{G}'$, there exists a section $\mathbb{G}' \to \mathbb{G}$, so all c, a, and $b := P(a)$ have p^ith roots in R, denoted by $\mathfrak{c}, \mathfrak{a}, \mathfrak{b}$. For each $r \in R$, there is an isomorphism of R-algebras

$$R[x]/x^{p^i} \to R[x]/x^{p^i} - r^{p^i}, \quad x \mapsto x - r. \tag{18}$$

Thus, there are isomorphisms

$$\begin{tikzcd} \prod_{\mathrm{Sub}_{11}(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i}, x_2^{p^i} \arrow[r, "\simeq"] & \prod_{\mathrm{Sub}_{11}(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i} - a, x_2^{p^i} - b \cong \mathcal{O}_{\mathrm{Sub}_{11}^{(i)}(\mathbb{G})} \\ \prod_{\mathrm{Sub}_1(\Lambda^{n-i})} R[x_1]/x_1^{p^i} \arrow[r, "\simeq"] \arrow[u, "\prod_{\mathrm{Sub}_1(\Lambda^{n-i})}(\Delta \circ i_1)"] & \prod_{\mathrm{Sub}_1(\Lambda^{n-i})} R[x_1]/x_1^{p^i} - a \cong \mathcal{O}_{\mathrm{Sub}_1^{(i)}(\mathbb{G})} \arrow[u, "1 \otimes s"'] \end{tikzcd} \tag{19}$$

where Δ are "diagonal" maps (Notation 8.0.2) induced by $\mathrm{Sub}_{11}(\Lambda^{n-i}) \to \mathrm{Sub}_1(\Lambda^{n-i})$ and i_1 the inclusion $R[x_1]/x_1^{p^i} \to R[x_1,x_2]/x_1^{p^i}, x_2^{p^i}$.

Lemma 8.0.10. *There is an isomorphism of R-algebras*

$$\mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})} \cong \prod_{\mathrm{Sub}_2(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i}, x_2^{p^i}$$

compatible with the inclusion

$$\begin{tikzcd} \mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})} \arrow[r, "\prod \Delta"] & \mathcal{O}_{\mathrm{Sub}_{11}^{(i)}(\mathbb{G})} \\ \prod_{\mathrm{Sub}_2(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i}, x_2^{p^i} \arrow[r] \arrow[u, "\simeq"'] & \prod_{\mathrm{Sub}_{11}(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i}, x_2^{p^i} \arrow[u, "\simeq"] \end{tikzcd}$$

where the top horizontal map is $\prod_{\mathrm{Sub}_2(\Lambda^{n-i})} \Delta$ induced by $\mathrm{Sub}_{11}(\Lambda^{n-i}) \to \mathrm{Sub}_2(\Lambda^{n-i})$.

Proof. Let $\widetilde{R} = R[x_1, x_2]/x_1^{p^i}, x_2^{p^i}$, $S = \mathrm{Spec}(R)$, and $\widetilde{S} = \mathrm{Spec}(\widetilde{R})$. Consider the diagram

$$\begin{tikzcd} \mathrm{Sub}_2^{(i)}(\mathbb{G}) \arrow[d] & \mathrm{Sub}_{11}^{(i)}(\mathbb{G}) \arrow[l] \arrow[d] \\ \mathrm{Sub}_2(\mathbb{G}') & \mathrm{Sub}_{11}(\mathbb{G}'). \arrow[l] \end{tikzcd} \tag{20}$$

For $K' \in \mathrm{Sub}_2(\Lambda^{n-i})$, let $\mathrm{Sub}_{K',2}^{(i)}(\mathbb{G}), \mathrm{Sub}_{K',11}^{(i)}(\mathbb{G}), \mathrm{Sub}_{K'_11}(\mathbb{G}')$ be the inverse images in $\mathrm{Sub}_2^{(i)}(\mathbb{G}), \mathrm{Sub}_{11}^{(i)}(\mathbb{G}), \mathrm{Sub}_{11}(\mathbb{G}')$ of $\mathrm{Sub}_{K',2}(\mathbb{G}')$, the component in $\mathrm{Sub}_2(\mathbb{G}')$ indexed by K'. Then (20) is the disjoint union of diagrams

$$\begin{tikzcd} \mathrm{Sub}_{K',2}^{(i)}(\mathbb{G}) \arrow[d] & \mathrm{Sub}_{K',11}^{(i)}(\mathbb{G}) \arrow[l] \arrow[d] \\ S & \mathrm{Sub}_{K',11}(\mathbb{G}'). \arrow[l] \end{tikzcd} \tag{21}$$

over all K.

If $K' \cong \mathbb{Z}/p^2$, since a degree p^2 cyclic subgroup has a unique degree p subgroup, the map $\mathrm{Sub}_{K',11}^{(i)}(\mathbb{G}) \to \mathrm{Sub}_{K',2}^{(i)}(\mathbb{G})$ is an isomorphism. The isomorphism $\mathcal{O}_{\mathrm{Sub}_{K',11}^{(i)}(\mathbb{G})} \cong \prod \widetilde{R}$ (where the product is over the appropriate subset of $\mathrm{Sub}_{11}(\Lambda^{n-i})$) identifies this with the identity map.

If $K' \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, then $\mathrm{Sub}_{K',11}(\mathbb{G}') = \coprod \mathrm{Spec}(R[x_1, x_2]/x_1 - a, x_2 - b) \simeq \coprod S$ and $\mathrm{Sub}_{K',11}^{(i)}(\mathbb{G}) = \coprod \mathrm{Spec}(R[x_1, x_2]/x_1^{p^i} - a, x_2^{p^i} - b) \simeq \coprod \widetilde{S}$, where both disjoint unions are over all $a, b \in \mathcal{J}$ which generate K'. Each such pair a, b, gives a section $S \to \mathrm{Sub}_{K',11}(\mathbb{G}')$ classifying a given pair $H' < K'$ and lifts to a map $\widetilde{S} \to \mathrm{Sub}_{K',11}^{(i)}(\mathbb{G})$ under the isomorphism 18. These form a diagram

$$\begin{tikzcd} T := \mathrm{Sub}_{K',2}^{(i)}(\mathbb{G}) \arrow[d] & \mathrm{Sub}_{K',11}^{(i)}(\mathbb{G}) = \coprod \widetilde{S} \arrow[l] \arrow[d] & \widetilde{S} \arrow[l] \arrow[d] \\ \mathrm{Sub}_{K',2}(\mathbb{G}') = S & \mathrm{Sub}_{K',11}(\mathbb{G}') = \coprod S \arrow[l] & S \arrow[l] \end{tikzcd}$$

where the bottom composite is the identity.

We will show that the top composite is an isomorphism by showing that they classify the same data. Indeed, T classifies degree p^2 étale subgroups $K < \mathbb{G}$ which are lifts of $K' < \mathbb{G}'$, $\widetilde{S}$ classifies pairs $H < K$ which map to $H' < K'$, and the map $\widetilde{S} \to T$ sends $H < K$ to K. Pick a level structure $f : \mathbb{Z}/p \oplus \mathbb{Z}/p \to \mathbb{G}$ on K. This gives a level structure $f' : \mathbb{Z}/p \oplus \mathbb{Z}/p \to \mathbb{G}'$ on K'. Let $i : \mathbb{Z}/p \to \mathbb{Z}/p \oplus \mathbb{Z}/p$ be the inclusion such that $f' \circ i$ is a level structure on H'. Then $f \circ i$ is a level structure which forgets to a subgroup $H < K$. We claim that H does not depend on the choice of level structure f. Let f_2 be another level structure with f_2', i_2, and H_2 defined analogously. Since H and H_2 both map to the same étale subgroup H', $f' \circ i$ and $f_2' \circ i_2$ must differ by an automorphism of $\mathbb{Z}/p$. Let $\sigma \in \mathrm{Aut}(\mathbb{Z}/p)$ be such that $i = i_2 \circ \sigma$, so i and $i_2 \circ \sigma$ are lifts of the same level structure on $\mathbb{G}'$, hence differ by some $h \in \mathrm{Hom}(\mathbb{Z}/p, \mathbb{G}^0)$ since $\mathbb{G} \simeq \mathbb{G}^0 \oplus \mathbb{G}'$ splits. Since $H, H_2 < K$, the image of h is contained in K. But K is étale, so its formal part $K \cap \mathbb{G}^0$ is trivial, hence $h = 0$, $i_2 \circ \sigma = i$, and $H_2 = H$. Thus, each K classified by T corresponds uniquely to a filtration $H < K$ classified by $\widetilde{S}$. So, each such pair a, b gives a map $\widetilde{S} \to \mathrm{Sub}_{K',11}^{(i)}(\mathbb{G}) \to T$ whose composite is an isomorphism.

Let $\mathrm{Level}_{K',11}(\mathbb{G}')$ be the scheme classifying level structures on K'. The transitive $GL_2(\mathbb{F}_p)$ action on $\mathrm{Level}_{K',11}(\mathbb{G}')$ lifts to a transitive $GL_2(\mathbb{F}_p)$ action on its inverse image $\mathrm{Level}_{K',11}(\mathbb{G})$ in $\mathrm{Level}_{11}(\mathbb{G})$. This action descends to an action on $\mathrm{Sub}_{K',11}(\mathbb{G})$ compatible with the map to $\mathrm{Sub}_{K',2}(\mathbb{G})$ and sections $\widetilde{S} \xrightarrow{\simeq} \widetilde{S}_{a,b} \subset \mathrm{Sub}_{K',11}(\mathbb{G})$ for each pair a, b. So, the isomorphisms $\widetilde{S} \xrightarrow{\simeq} \widetilde{S}_{a,b} \subset \mathrm{Sub}_{K',11}(\mathbb{G}) \to T$ does not depend on the choice of a, b, giving an identification of T with $\widetilde{S}$.

Taking the disjoint union over all $K' \in \mathrm{Sub}_2(\Lambda^{n-i})$ gives the lemma.

$\square$

Proof of Lemma 8.0.8. By (19) and the lemma above, the pullback D of (17) is isomorphic to the pullback D_0 of the diagram

$$\begin{tikzcd}
\prod_{\mathrm{Sub}_2(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i}, x_2^{p^i} \arrow[r] & \prod_{\mathrm{Sub}_{11}(\Lambda^{n-i})} R[x_1,x_2]/x_1^{p^i}, x_2^{p^i} \\
 & \prod_{\mathrm{Sub}_1(\Lambda^{n-i})} R[x_1]/x_1^{p^i} \arrow[u]
\end{tikzcd}$$

which contains $R[x_1]/x_1^{p^i}$. In fact, $D_0 = R[x_1]/x_1^{p^i}$ since (8) is a pushout of sets. $\square$

Proof of Lemma 8.0.7. To compute H^k, consider the following part of $\mathcal{D}(\mathbb{G}, R)$ and $\mathcal{K}(\mathbb{G}, R)$.

$$\begin{tikzcd}
B_{k+1} = (B_k \otimes A_1) \oplus (A_{k-1} \otimes \mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})}) \arrow[r] & A_{k+1} \arrow[r] & C^{\vee(i)}[k+1] \\
B_k \arrow[u] \arrow[r] & A_k \arrow[u] \arrow[r] & C^{\vee(i)}[k] \arrow[u] \\
 & A_{k-1} \arrow[u] &
\end{tikzcd}$$

Then $H^k \cong (\operatorname{im} D_k)/(B_k + A_{k-1})$ where D_k is the pullback

$$\begin{tikzcd}
(B_k \otimes A_1) \oplus (A_{k-1} \otimes \mathcal{O}_{\mathrm{Sub}_2^{(i)}(\mathbb{G})}) \arrow[r] & A_{k+1} \\
D_k \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & A_k \arrow[u]
\end{tikzcd}$$

and $\operatorname{im} D_k$ is its image in A_k. By the same argument as the one for H^1 in the proof of Lemma 8.0.8, this diagram is isomorphic to

$$\begin{tikzcd}
(B_k \otimes A_1) \oplus \left(\prod_{\mathrm{Sub}_{1\cdots 12}(\Lambda^{n-i})} \widetilde{R}_{k+1}\right) \arrow[r] & A_{k+1}(n-i, \widetilde{R}_{k+1}) \\
D_k \arrow[u] \arrow[r] & A_k(n-i, \widetilde{R}_k) \arrow[u]
\end{tikzcd}$$

whose pullback we also denote by D_k. Here $\widetilde{R}_k = R[x_1, \cdots, x_k]/x_1^{p^i}, \cdots, x_k^{p^i}$ and $i_k : \widetilde{R}_k \to \widetilde{R}_{k+1}$ is the usual inclusion.

$\operatorname{im} D_k$ contains $\supseteq A_k \cap \operatorname{im}(B_k \otimes A_1) + A_k \cap (A_{k-1} \otimes \Gamma^{\vee(i)}[2])$. Since the first term contains $A_k \cap \operatorname{im} B_k = \operatorname{im} B_k$ and the second term is isomorphic to $A_{k-1} \otimes D$ (where D is the pullback of 17), there is a containment

$$\operatorname{im} D_k \supseteq \operatorname{im} B_k + A_{k-1} \otimes D \supseteq \operatorname{im} B_k + A_{k-1}.$$

To show $H^k \neq 0$, it suffices to show that the second containment is strict.

Suppose by contradiction that $\operatorname{im} B_k + A_{k-1} \otimes D = \operatorname{im} B_k + A_{k-1}$ in A_k. Note that $A_{k-1} \cong A_{k-1}(n-i, \widetilde{R}_{k-1}) \cong \prod \widetilde{R}_{k-1}$, $A_{k-1} \otimes D \cong A_{k-1}(n-i, \widetilde{R}_k) \cong \prod \widetilde{R}_k$, and the map $A_{k-1} \to A_k$ induces a map $A_{k-1} \otimes D \to A_k$. Consider $(x_k, 0, 0, \cdots) \in A_{k-1} \otimes D \cong \prod \widetilde{R}_k$ which maps to $(\Delta(x_k), 0, 0, \cdots) \in \operatorname{im} B_k + A_{k-1} \otimes D \subseteq A_k$. Since $\operatorname{im} B_k + A_{k-1} \otimes D = \operatorname{im} B_k + A_{k-1}$, there exists $(f_1(\hat{x_k}), f_2(\hat{x_k}), \cdots) \in A_{k-1}$ such that

$$(\Delta(x_k), 0, 0, \cdots) + (\Delta f_1(\hat{x_k}), \Delta f_2(\hat{x_k}), \cdots) \in \operatorname{im} B_k,$$

where $f_j(\hat{x_k})$ are polynomials in $x_1, \cdots, x_{k-1}$ and each Δ is the diagonal corresponding to each component of $A_{k-1} \otimes D \cong \prod \widetilde{R}_k$ (equivalently, components of $A_{k-1} \cong \prod R$). By definition of B_k, there exists

$$\vec{g_j}(x_1, \cdots, x_k) = (g_{j_1}(\vec{x}), g_{j_2}(\vec{x}), \cdots) \in \operatorname{im} \prod_{\mathrm{Sub}_{1\cdots 2\cdots 1}(\Lambda^{n-i})} \widetilde{R}_k$$

(with the 2 in the subscript in the jth position) such that

$$(\Delta(x_k), 0, 0, \cdots) + (\Delta f_1(\hat{x_k}), \Delta f_2(\hat{x_k}), \cdots) = \sum_{j=1}^{k-1} \vec{g_j}(x_1, \cdots, x_k).$$

Since $B_k \to A_k$ is a direct sum of "diagonal" maps, $\vec{g_j}(x_1, \cdots, x_{k-1}, 0) \in \operatorname{im} \prod_{\mathrm{Sub}_{1\cdots 2\cdots 1}(\Lambda^{n-i})}$ and

$$\sum_{j=1}^{k-1} \vec{g_j}(x_1, \cdots, x_{k-1}, 0) = (0, 0, \cdots) + (\Delta f_1(\hat{x_k}), \Delta f_2(\hat{x_k}), \cdots) \in \operatorname{im} B_k.$$

Thus, $(\Delta(x_k), 0, 0, \cdots) \in \operatorname{im} B_k$.

In fact, the same argument shows that all $(0, \cdots, 0, \Delta(x_k), 0, \cdots) \in \operatorname{im} B_k + A_{k-1} \otimes D$ are in $\operatorname{im} B_k$. Since $\operatorname{im} B_k$ is symmetric in $x_1, \cdots, x_k$, $(0, \cdots, 0, \Delta(x_s), 0, \cdots) \in \operatorname{im} B_k$ for all s, so $\operatorname{im} B_k \supseteq A_{k-1} \otimes D \supseteq A_{k-1}$, a contradiction by the lemma below. $\square$

Lemma 8.0.11. *For R as above, $A_{k-1}(\mathbb{G}, R) \not\subseteq \operatorname{im} B_k(\mathbb{G}, R)$ if $k \leq n-i$.*

Proof. Note that $A_k(\mathbb{G}, R) \cong A_k(n-i, \widetilde{R}_k)$ and $B_k(\mathbb{G}, R) \cong B_k(n-i, \widetilde{R}_k)$. Since $\operatorname{im} B_k(\mathbb{G}, R)$ is symmetric in $x_1, \cdots, x_k$, if $\operatorname{im} B_k(\mathbb{G}, R) \supseteq A_{k-1}(\mathbb{G}, R) = A_{k-1}(n-i, \widetilde{R}_{k-1})$, then $\operatorname{im} B_k(\mathbb{G}, R) \supseteq A_{k-1}(n-i, \widetilde{R}_k)$, which is a contradiction by the lemma below. $\square$

Lemma 8.0.12. *For all integers $h \geq 0$ and any ring S, $A_{k-1}(h, S) \not\subseteq \operatorname{im} B_k(h, S)$ if $k \leq h$. Equivalently, $\partial_{k-1}^{\vee} : K_{k-1} \to K_k$ is nonzero.*

Proof. This is true for $h = 0$ and 1. Suppose that it is true for $h = n-1$. Let $\Lambda = \mathbb{Q}_p/\mathbb{Z}_p$. For $h = n$, since $\Lambda^n \cong \Lambda^{n-1} \times \Lambda$,

$$\mathrm{Sub}_{i_1, \cdots, i_q}(\Lambda^n) = \mathrm{Sub}_{i_1, \cdots, i_q}(\Lambda^{n-1}) \sqcup \mathrm{Sub}_{i_1, \cdots, i_q}(\Lambda^{n-1})^{\mathrm{c}}$$

is the disjoint union of the set of increasing chain of subgroups all contained in Λ^{n-1} and its complement. This splits each $A_k(n)$ into $A_k(n-1) \times A'_k \cong A_k(n-1) \oplus A'_k$, and similarly for $B_k(n)$. The map $B_k(n) \to A_k(n)$ is the direct sum of maps $B_k(n-1) \to A_k(n-1)$ and $B'_k \to A'_k$, so $K_k(n) = A_k(n-1)/B_k(n-1) \oplus A'_k/B'_k = K_k(n-1) \oplus K'_k$. Since the map $A_{k-1}(n) \to A_k(n)$ is the direct sum of maps $A_{k-1}(n-1) \to A_k(n) \oplus A'_k$ and $A'_{k-1} \to A'_k$, the boundary map $\partial^\vee_{k-1}(n) : K_{k-1}(n) \to K_k(n)$ is the direct sum of maps $K_{k-1}(n-1) \to K_k(n-1) \oplus K'_k$ which is $\partial^\vee_{k-1}(n-1)$ on the first component, and $K'_{k-1} \to K'_k$. If $k \leq n-1$, $\partial^\vee_{k-1}(n-1)$ is nonzero, so $\partial^\vee_{k-1}(n)$ is nonzero.

It remains to show that $\partial^\vee_{n-1}(n)$ is nonzero. The inclusions of chains of p-torsion subgroups $\mathrm{Sub}^{p-tors}_{i_1,\cdots,i_q}(\Lambda^n) \subseteq \mathrm{Sub}_{i_1,\cdots,i_q}(\Lambda^n)$ induce splittings $A_k := A_k(n) \cong A_k^{p-tors} \oplus A'_k$ (analogously for B_k and K_k), hence a diagram

$$\begin{tikzcd}
 & B_n^{p-tors} \arrow[dr] & & \\
B_n \arrow[r] \arrow[ur, two heads] & A_n \arrow[r, two heads] & A_n^{p-tors} \arrow[r, two heads] & K_n^{p-tors} = A_n^{p-tors}/B_n^{p-tors} \\
 & A_{n-1} \arrow[u] \arrow[r, two heads] & A_{n-1}^{p-tors} \arrow[u, "\simeq"'] \arrow[r, two heads] & K_{n-1}^{p-tors}. \arrow[u, "\partial^\vee_{n-1}"']
\end{tikzcd}$$

The map $A_{n-1}^{p-tors} \to A_n^{p-tors}$ is an isomorphism since the last subgroup in a p-torsion chain of length n must be $G[p]$. To show $\partial^\vee_{n-1}$ is nonzero, it suffices to show that K_n^{p-tors} is nonzero.

We first claim that $K_k(S) := K_k(n,S) \neq 0$ if $k \leq n$. To see this, note that $K_k(\mathbb{Z})$ is finitely generated since it is a quotient of the finitely generated free abelian group $A_k(\mathbb{Z})$. Let $\mathbb{G}$ be the universal formal group over $E = E_n$. Let C_0 be Stapleton's ring ([Sta13, Corollary 2.18]), which is faithfully flat over $p^{-1}E_0$ and over which $\mathbb{G} \simeq \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^n}$. Then $C_0 \otimes K_k(\mathbb{Z}) = K_k(C_0) = C_0 \otimes_{E_0} K_k(E_0) = C_0 \otimes_{E_0} C^\vee[k]$ is a nonzero finite free C_0-module of rank $d = \mathrm{rank}_{E_0} C^\vee[k]$. So, $K_k(\mathbb{Z})$ contains a nonzero direct sum $\mathbb{Z}^{\oplus d}$ hence $K_k(S) = S \otimes K_k(\mathbb{Z})$ is nonzero.

Since $K_n \cong K_n^{p-tors} \oplus A'_n/B'_n \neq 0$, to show that $K_n^{p-tors} \neq 0$, it suffices to show that $A'_n/B'_n = 0$. Given a chain of subgroups $H_1 < \cdots < H_n$ of Λ^n not all of which are p-torsion, consider the minimal k such that H_{k+2}/H_k is not p-torsion, so $H_{k+2}/H_k \simeq \mathbb{Z}/p^2$ has a unique degree p subgroup. Thus, $H_1 < \cdots < H_n$ is uniquely determined by $H_1 < \cdots < H_k < H_{k+2} < \cdots < H_n \in \mathrm{Sub}_{1\cdots 2\cdots 1}(\Lambda^n)$. Therefore, $B'_n \to A'_n$ is surjective, $A'_n/B'_n = 0$, and $\partial^\vee_{n-1} \neq 0$. $\square$

Corollary 8.0.13. *For $i > 0$, $m \geq 0$, and $0 < k \leq n-i$, $u_i^{-1} H^k(C^{\vee(i)}[\bullet]_{\sigma^m})$ is nonzero, thus $H^k(C^{\vee(i)}[\bullet]_{\sigma^m})$ is nonzero.*

Proof. Since $C^{\vee(i+1)}[\bullet]_{\sigma^{mp-1}}$ is a complex of E_0/u_i-modules and u_i is invertible in $R_i := C_i/I_i$, $C^{\vee(i+1)}[\bullet]_{\sigma^{mp-1}}$ becomes 0 after base changing to R_i. Thus, by Lemma 8.0.5, the map $C^{\vee(i)}[\bullet]_{\sigma^{m-1}} \to C^{\vee(i)}[\bullet]_{\sigma^m}$ becomes an equivalence after base changing to R_i and the corollary follows from Lemma 8.0.7. $\square$

Proposition 8.0.1 follows by applying the above corollary to the long exact sequence of Ext groups from the cofiber sequence in Lemma 8.0.5.

9 Tor non-vanishing

Since the Tor groups with Γ on the left vanish in nonzero degrees, one might ask if the same is true for Tor groups with Γ on the right. In Example 9.0.3 below, we show that this is not true at $p = 2$ height 2, which leads us to conjecture the following.

Conjecture 9.0.1. $\mathrm{Tor}^*_\Gamma(E_0, {}_{\sigma^m}\Gamma)$ *is nonzero if* $0 \le * < n$.

The Koszul resolution implies that the Tor groups vanish if $* > n$.

Proposition 9.0.2. *For $p = 2$, $0 \le i \le n$, and $m > 0$, there is a cofiber sequence*

$$E_0/u_{i-1} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} {}_{\sigma^m}\Gamma^{(i)} \to E_0/u_{i-1} \otimes^{\mathbb{L}}_{\Gamma^{(i)}} {}_{\sigma^{m-1}}\Gamma^{(i)} \to \left(E_0/u_i \otimes^{\mathbb{L}}_{\Gamma^{(i+1)}} {}_{\sigma^{mp-1}}\Gamma^{(i)}/u_i\right)[1].$$

Here $\Gamma^{(i)}/u_i := \bigoplus_{k\ge 0} \Gamma^{(i)}[k]/u_i$ is the quotient of $\Gamma^{(i)}$ by u_i with respect to the left E_0/u_{i-1}-module structure and is a left $\Gamma^{(i+1)}$-module as follows. $Q \in \Gamma^{(i+1)}$ is an operation on π_0/u_i and $Q' \in \Gamma^{(i)}/u_i$ is an operation $\pi_0/u_{i-1} \to \pi_0/u_i$, so their composite $QQ' := Q \circ Q'$ is an operation $\pi_0/u_{i-1} \to \pi_0/u_i$.

Proof. The proof is similar to the proof of Lemma 8.0.5, with the maps in (15) replaced by direct sums over $\ell \ge 0$ of

$$\begin{aligned}
R[k-1] \otimes \Gamma^{\vee(i)}[\ell+1] \to R[k] \otimes \Gamma^{\vee(i)}[\ell], &\quad (q_1, \cdots, q_k, H) \mapsto (q_1, \cdots, q_{k-1}, \langle q_k\rangle + H)\\
S[k-1] \otimes \Gamma^{\vee(i)}[\ell+1] \to S[k] \otimes \Gamma^{\vee(i)}[\ell], &\quad (q_1, \cdots, q_k, H) \mapsto (q_2, \cdots, q_k, \langle q_1\rangle + H)\\
S[k] \otimes \Gamma^{\vee(i)}[\ell] \to B'[k-1] \otimes \Gamma^{\vee(i)}[\ell], &\quad (q_1, \cdots, q_{k-1}, H) \mapsto (q_1, \cdots, q_{k-1}, 0, H).
\end{aligned}$$

∎

The proposition will imply the conjecture if

$$\mathrm{Tor}^*_{\Gamma^{(i+1)}}(E_0/u_i, {}_{\sigma^m}\Gamma^{(i)}/u_i) \ne 0 \tag{22}$$

for $i \ge 0$, $* < n - i$, and $m > 0$.

Example 9.0.3. At the prime $p = 2$ and height 2, [Rez08] gives a description of Γ by generators and relations. Using this, we see that $\Gamma/p = \Gamma /\!\!/ p$ is generated by Q_0, Q_1, Q_2 and $\Gamma^{(1)}$ is generated by $Q_0' = Q_0 + u_1 Q_2, Q_1$. The relations in Γ/p are the mod p Adem relations

$$\begin{aligned}
Q_1Q_0 &= 0\\
Q_2Q_0 &= Q_0Q_1 + u_1Q_0Q_2.
\end{aligned}$$

The cofiber $E_0/p \otimes^{\mathbb{L}}_{\Gamma^{(1)}} {}_{\sigma^m}\Gamma/p$ is computed by the complex

$$\Gamma^{(1)}[1] \otimes \Gamma/p = (E_0/p)\{q_0', q_1\} \otimes \Gamma/p \xrightarrow{\partial\circ(\sigma^m\otimes 1)} \Gamma/p = \Gamma^{(1)}[0] \otimes \Gamma/p.$$

If $m > 0$, the boundary map is not injective since $\sigma(q_0') = u_1q_1$, $\sigma(q_1) = q_0'$, and $\partial(q_1 \otimes Q_0) = 0$, hence $\mathrm{Tor}^1_{\Gamma^{(1)}}(E_0/p, {}_{\sigma^m}\Gamma/p) \ne 0$. Clearly $\mathrm{Tor}^0 \ne 0$. Thus, by Proposition 9.0.2, if $m > 0$, $\mathrm{Tor}^*_\Gamma(E_0, {}_{\sigma^m}\Gamma) \ne 0$ for $i = 0$ and 1.

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