Power Operations Modulo Lubin-Tate Parameters

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Abstract

We develop a theory of power operations acting on $\pi_*(-/p, \cdots, u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-algebras over Morava E-theory for height h. We show that the analog of the additive operations is Koszul of length $h - i + 1$ i.e. that its Koszul complex has length $h - i + 1$.

1 Introduction

Let E be a Morava E-theory for h. Power operations on $K(h)$-local $\mathbb{E}_\infty$-E-algebras have been studied by Rezk [Rez09, Rez17], building on work of [Str97, Str98, Kas01, HKR00] and others. In [Rez09], Rezk constructed a monad $\mathbb{T}$ on the category Mod_{E_*} of graded E_*-modules which is an algebraic approximation to the free $\mathbb{E}_\infty$-algebra functor on Mod_E, the category of E-module spectra. Its category of algebras $\mathrm{Alg}_{\mathbb{T}}^*$ serves as the natural target for the functor π_* on commutative E-algebra spectra. Up to completion, the degree 0 part $(\mathbb{T}E_*)_0$ parameterizes power operations on π_0 of $K(h)$-local $\mathbb{E}_\infty$-E-algebras, which are natural endomorphisms of the functor π_0. There is a subset of additive operations $\Gamma = \bigoplus_{k \geq 0} \Gamma[k] \subset (\mathbb{T}E_*)_0$, which are the primitives of the Hopf algebra $(\mathbb{T}E_*)_0$ and is a graded E_0-algebra by weight. The E_0-dual $\Gamma^\vee[k]$ is a scheme $\mathrm{Sub}_k(\mathbb{G})$ which classifies degree p^k subgroups of the universal formal group $\mathbb{G}$ over E_0 [Str98]. These make $\Gamma^\vee$ into a (graded) category object in formal affine schemes whose objects are deformations of the formal group $\mathbb{G}_0$ over the base field and whose morphisms are isogenies lifting some power of the Frobenius isogeny [Rez09].

Using the Frobenius isogeny, Rezk gives a congruence criterion for a torsion free Γ-algebra to lift (necessarily uniquely) to a $\mathbb{T}$-algebra. This implies that Γ, together with a single non-additive operation, generate all power operations. At height 1, $\mathrm{Alg}_{\mathbb{T}}$ is the category of δ-rings and the congruence criterion says that a torsion free ring with a homomorphism ψ lifts to a δ-ring if and only if ψ acts as the Frobenius mod p. Thus, we can study power operations by first trying to understand the linear part Γ. [Rez17] shows that Γ is Koszul of length $h + 1$ i.e. that its Koszul complex has length $h + 1$. This supplies us with Koszul (hence finite length) resolutions for computational purposes.

Although operations in Γ are additive, they are not necessarily E_0-linear, so they do not always descend to operations on $\pi_0(-/p, \cdots, u_{i-1})$ for $i \leq h$. For each i, we consider a subquotient $\Gamma^{(i)}$ of Γ which is an $E_0/p, \cdots, u_{i-1}$-algebra of power operations acting on $\pi_0(-/p, \cdots, u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras. They are generated by weight p operations considered in [Hah22], given by the closure of the strict p-torsion points of the universal formal group over $E_0/p, \cdots, u_{i-1}$. If we restrict to a fixed height by inverting u_i, $\mathrm{Spf}(u_i^{-1}\Gamma^{\vee(i)}[k])$ classifies degree p^k étale subgroups. We will show that $\Gamma^{(i)}$ satisfies many of the properties of Γ. Our main result is the following.

Main Theorem. *For $0 \leq i \leq h$, the ring $\Gamma^{(i)}$ is Koszul of length $h - i + 1$.*

As we kill more parameters, $\Gamma^{(i)}$ becomes simpler. $\Gamma^{(h)}$ is just the base field. The first nontrivial case is $\Gamma^{(h-1)}$, which is a twisted free associative algebra on a single generator. This allows us to understand Γ by inductively understanding the simpler $\Gamma^{(i)}$'s, which will be explored further in [SS25].

We construct a mod $p, \cdots, u_{i-1}$ analog $\mathbb{T}^{(i)}$ of the monad $\mathbb{T}$. $\mathbb{T}^{(i)}(E_0/p, \cdots, u_{i-1})$ naturally acts on $\pi_0(-/p, \cdots, u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras and is generated by an additional operation from $\Gamma^{(i)}$. In [SS25], we will give a congruence criterion analogous to Rezk's.

As a step towards proving the chromatic Nullstellensatz, [BSY24, Thm 3.4] exhibits $\pi_0 E(-)$ as the cofree $\mathbb{T}$-algebra functor i.e. the right adjoint in an adjunction between $\mathrm{Alg}_\mathbb{T}$ and Perf_k, the category of perfect k-algebras, where k is the base field. At height 1, this says that Witt vectors are cofree δ-rings. This descends to an adjunction between $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ and Perf_k which exhibits $\pi_0 E(-)/p, \cdots, u_{i-1}$ as the cofree $\mathbb{T}^{(i)}$-algebra.

As an application, we use power operations modulo $p, \cdots, u_{h-2}$ to prove the triviality conjecture 8.0.3 in [HL24], which bounds the $\mathbb{E}_n$-triviality of the algebra of cochains on spheres. Using a technique similar to [Hah22], we show that $\mathbb{E}_{n+1}$-power operations act nontrivially on cochains on the sphere S^n, thus it is not $\mathbb{E}_{n+1}$-trivial.

Outline. In Section 2, we provide background on formal groups and power operations and set up notations. In Section 3, we describe the bar complex in terms of the generalized Tate construction. We use this to construct $\Gamma^{(i)}$ and show that it is Koszul in Section 4. In Section 5, we show that $\Gamma^{(h-1)}$ is a twisted free associative algebra on one generator. Since $\Gamma^{(h-1)}$ is Koszul, it is generated in degree 1. We will show that there are no relations by showing that every subgroup classified by $\Gamma^{(h-1)}$ is cyclic. In Section 6, we consider $\Gamma^{(i)} /\!\!/ u_i$, an inflated version of $\Gamma^{(i+1)}$ consisting of all power operations in $\Gamma^{(i)}$ which descend to operations on $\pi_0(-/p, \cdots, u_i)$. We will show that it is generated by an additional operation from $\Gamma^{(i+1)}$ corresponding to the Frobenius isogeny. We will give an isogeny interpretation of $\Gamma^{(i)} /\!\!/ u_i$ [SS25]. In Section 7, we construct the monad $\mathbb{T}^{(i)}$ and show that the cofreeness statement in [BSY24] works as well modulo $p, \cdots, u_{i-1}$. We thank Allen Yuan for explaining Akhil Mathew's proof. Finally, we prove the remaining cases of the triviality conjecture in Section 8.

Acknowledgements. TODO

2 Preliminaries

All spectra will be implicitly p-localized. Let $E = E(k, \mathbb{G}_0)$ be a Morava E-theory spectrum associated to a perfect field k of characteristic p and a formal group $\mathbb{G}_0$ of height $0 < h < \infty$ over k. It is an even periodic complex orientable spectrum whose associated formal group $\mathbb{G}$ is the universal deformation of G_0. The coefficient ring of E is given by

$$\pi_* E = \mathbb{W}k[[u_1, \cdots, u_{h-1}]][u^{\pm}]$$

where $\mathbb{W}k$ is the Witt vectors on k, $u \in \pi_2 E$, and $u_1, \cdots, u_{h-1} \in \pi_0 E$ are the Lubin-Tate parameters pushed down to degree 0. We also set $u_0 := p$ and write $(-)/u_i$ for the quotient $(-)/p, \cdots, u_i$.

$\mathbb{E}_\infty$-algebras are highly structured ring spectra, with power operations acting on their homotopy groups. Since power operations on $K(h)$-local $\mathbb{E}_\infty$-E-algebras are related to isogenies of formal groups, we first state our conventions on formal groups in Section 2.1. Then, we give a review of power operations in Section 2.2. In Section 2.3, we recall the definition in [Rez17] of being Koszul for a not necessarily commutative graded algebra.

2.1 Formal groups

Following [Str97], a scheme will mean a formal scheme of the form $X = \mathrm{Spf}(A)$ for A a finite product of complete local Noetherian rings. A formal group G over X is a one-dimensional commutative formal group scheme over X.

Definition 2.1.1.   1. A divisor on G is a closed subscheme $D \leq G$ which is finite flat over X.

   2. A finite subgroup of G is a divisor $H < G$ which is also a subgroup scheme.

   3. A point $a \in G(A)$ has exact order n if the divisor $H = [a] + [2a] + \cdots + [na]$ is a subgroup of G. We call H the subgroup generated by a and denote it by $\langle a \rangle$.

   4. An isogeny is a homomorphism of formal groups $G \to G'$ over X with finite kernel. It is an epimorphism of group schemes. We will identify a finite subgroup $H < G$ with the isogeny $G \to G/H$.

   5. The image of a point $a \in \mathbb{G}(A)$ under the isogeny $G \to G'$ is its image in $G'(A)$.

Caution: If X is an $\mathbb{F}_p$-scheme, for all $k \geq 1$, the point $0 \in G(A)$ has exact order p^k and generates the degree p^k subgroup $\ker \varphi^k$.

If t, u are coordinates on formal groups G, G' over X, a homomorphism $G \to G'$ is given by a map $\mathcal{O}_X[[u]] \to \mathcal{O}_X[[t]]$ sending u to some power series $f(t)$. With this notation, for 5., the image of a is $f(a)$. For an isogeny $G \to G/H$, we will put the norm coordinate [Str97, Thm 5.3] on G/H.

Example 2.1.2. For *3.*, if t is a coordinate on G with formal group law F, the norm coordinate for $H = \langle a \rangle$ is given by

$$\prod_{i=1}^{n} (t +_F [i]_F(a)).$$

Definition 2.1.3.   1. Let $\varphi : X \to X$ be the absolute Frobenius.

   2. Let φG be the target of the relative Frobenius $G \to \varphi G$ over X. The relative Frobenius is the Frobenius isogeny whose kernel we denote by $\ker \varphi$.

   3. If $H < G$ is a finite subgroup, let φH be the target of the relative Frobenius $H \to \varphi H$.

The Frobenius isogeny $G \to \varphi G$ is given on coordinates by $\mathcal{O}_X[[t]] \to \mathcal{O}_X[[t]]$, $t \mapsto t^p$. If $H < G$ is a subgroup, there is a commutative diagram

$$
\begin{array}{ccc}
G & \longrightarrow & G/H \\
\varphi \downarrow & & \downarrow \varphi \\
\varphi G & \longrightarrow & \varphi G/\varphi H
\end{array} \quad .
$$

If H corresponds to the isogeny $\sum c_k t^k$, then φH corresponds to $\sum c_k^p t^k$.

2.2 Power operations

Let $\mathcal{O}$ be an operad and E a commutative ring spectrum. Let $F = E \otimes \mathrm{free}_{\mathcal{O}}(-)$ be the free $\mathcal{O}$-E-algebra functor in spectra. It is a direct sum by weight $F = \bigoplus_{m \geq 0} F_m$ where $F_m = E \otimes (\mathcal{O}(m) \otimes_{h\Sigma_m} (-)^{\otimes m})$. Power operations on $\mathcal{O}$-algebras in E-module spectra are natural transformations in the homotopy category

$$\pi_a \to \pi_b : h\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathrm{Set}.$$

By Yoneda's lemma, they are parametrized by $\pi_b F(\mathbb{S}^a) = \bigoplus_{m \geq 0} \pi_b F_m(\mathbb{S}^a)$. Weight m operations are those parameterized by $\pi_b F_m(\mathbb{S}^a)$.

There is a subset of additive operations which are natural transformations $\pi_a \to \pi_b$ considered as functors $h\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathrm{Ab}$. These are parameterized by the equalizer

$$\pi_b F(i_1 + i_2), \pi_b F(i_1) + \pi_b F(i_2) : \pi_b F(\mathbb{S}^a) \rightrightarrows \pi_b F(\mathbb{S}^a \oplus \mathbb{S}^a)$$

where i_1, i_2 are inclusions. This is a direct sum of weight m operations given by the kernel of the transfer maps associated to the inclusions of subgroups $\Sigma_i \times \Sigma_{m-i} \subset \Sigma_m$

$$\ker \left(E_b \left(\mathcal{O}(m)_{h\Sigma_m}^{a\rho_m} \right) \xrightarrow{\mathrm{tr}} \bigoplus_{0 < i < m} E_b \left(\mathcal{O}(m)_{h\Sigma_i \times \Sigma_{m-i}}^{a\rho_m} \right) \right), \tag{1}$$

where ρ_m is the real standard representation for Σ_m and X^V is the Thom spectrum for V over X. This kernel is zero if m is not a power of p [Str98, Rez17].

We will consider power operations on $K(h)$-local $\mathbb{E}_{\infty}$-E-algebras where $E = E(k, G_0)$. These have been studied by Rezk [Rez09, Rez17], building on work of [Str97, Str98, Kas01, HKR00] and others. The material in this section is taken from [Rez09, Rez17, Str97, Str98].

2.2.1 An algebraic approximation: Rezk's monad

The free $\mathbb{E}_{\infty}$-E-algebra on $\mathbb{S}^0$ is given by

$$\mathbb{P}E = \bigoplus_{m \geq 0} \mathbb{P}_m E = \bigoplus_{m \geq 0} E \otimes B\Sigma_m,$$

where $\mathbb{P} = \bigoplus_{m \geq 0} \mathbb{P}_m$ is the free $\mathbb{E}_{\infty}$-algebra monad with $\mathbb{P}_m(-) = (-)_{h\Sigma_m}^{\otimes_E m}$. Rezk [Rez09] constructs an algebraic approximation to $\mathbb{P}$, which is a monad $\mathbb{T}$ on Mod_{E_*} as follows.

Taking homotopy groups $\pi_* : h\mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$ restricts to an equivalence on the full subcategory $h\mathrm{Mod}_E^{\mathrm{ff}} \subset h\mathrm{Mod}_E$ of finite free (i.e. finitely generated and free) E-modules. The functor $\pi_* L_{K(h)} \mathbb{P}_m(-) : h\mathrm{Mod}_E^{\mathrm{ff}} \to \mathrm{Mod}_{E_*}$ lands in finite free E_*-modules. Define $\mathbb{T}_m : h\mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$ to be the left Kan extension of this functor along the inclusion $h\mathrm{Mod}_E^{\mathrm{ff}} \subset h\mathrm{Mod}_E$ and set $\mathbb{T} = \bigoplus_{m \geq 0} \mathbb{T}_m$. Then

$$\mathbb{T}E_* = \bigoplus_{m \geq 0} \mathbb{T}_m E_* = \bigoplus_{m \geq 0} E_*^{\wedge}(B\Sigma_m)$$

where $E_*^{\wedge}(-) = \pi_* L_{K(h)}(E \otimes -)$ is the completed E-homology. $E_*^{\wedge}(B\Sigma_m)$ is a finite free E_*-module dual to $E^*(B\Sigma_m)$. The homotopy groups of $K(h)$-local $\mathbb{E}_{\infty}$-E-algebras naturally land in $\mathrm{Alg}_{\mathbb{T}}^*$.

2.2.2 Power operations on π_0

Up to completion, the degree zero part $(\mathbb{T}E_*)_0$ parameterizes operations on π_0 of $K(h)$-local $\mathbb{E}_\infty$-E-algebras. For R a $K(h)$-local $\mathbb{E}_\infty$-E-algebra (H_∞ suffices), the total weight m power operations is a map

$$P_m : R_0 \to R^0(B\Sigma_m)$$

sending an E-linear map $\alpha : E \to R$ to $P_m(\alpha) : \mathbb{P}_m E \to \mathbb{P}E \to \mathbb{P}R \to R$. Each element $Q \in (\mathbb{T}_m E_*)_0 = \pi_0 L_{K(h)}(\mathbb{P}_m E)$ gives an operation on $\pi_0 R$ by sending α to $P_m(\alpha) \circ Q$. P_m is multiplicative but not additive. It is additive modulo the transfer ideal. The total additive weight m power operations is the composite

$$\overline{P}_m : R_0 \xrightarrow{P_m} R^0(B\Sigma_m) \to R^0(B\Sigma_m)/\mathrm{tr}$$

which is a ring homomorphism. Since the transfer ideal contains a unit if m is not a power of p, $R^0(B\Sigma_m)/\mathrm{tr}$ is only nontrivial if $m = p^k$.

The algebra of additive operations on π_0 is the direct sum $\Gamma = \bigoplus_{k \geq 0} \Gamma[k]$ where $\Gamma[k]$ is the kernel of the transfer (1)

$$\Gamma[k] := \ker\left(E_0^\wedge(B\Sigma_{p^k}) \xrightarrow{\mathrm{tr}} \bigoplus_{0 < i < p^k} E_0^\wedge B(\Sigma_i \times \Sigma_{p^k - i}) \right).$$

Composition and addition of power operations make Γ into an E_0-algebra, but $E_0 = \Gamma[0]$ is not central in Γ. Thus, each $\Gamma[k]$ is both a left and right E_0-module. $\Gamma[k]$ is finite free as a left E_0-module [Str98]. Its left E_0-dual $\Gamma^\vee[k] = E^0(B\Sigma_{p^k})/\mathrm{tr}$ is the cokernel of the transfer.

2.2.3 Power operations in terms of isogenies

Definition 2.2.1.   1. Let $\mathrm{Sub}_k(\mathbb{G})$ be the scheme classifying degree p^k subgroups of $\mathbb{G}$ [Str97, Section 10].

2. Let $s, t : \mathrm{Sub}_k(\mathbb{G}) \to \mathrm{Spf}(E_0)$ be the source and target maps sending a degree p^k subgroup $H \leq \mathbb{G}$ to $\mathbb{G}$ and $\mathbb{G}/H$, respectively.

3. Let $\mathrm{Sub}_{k_1, \cdots, k_q}(\mathbb{G})$ be the iterated pullback

$$\mathrm{Sub}_{k_1, \cdots, k_q}(\mathbb{G}) = \mathrm{Sub}_{k_1}(\mathbb{G})_t \times_s \cdots_t \times_s \mathrm{Sub}_{k_q}(\mathbb{G}),$$

which classifies the following equivalent data.

- a sequence of subgroups $H_1 < \mathbb{G}_1, H_2 < \mathbb{G}_2, \ldots, H_q < \mathbb{G}_q$ where $\deg H_i = p^{k_i}$, $\mathbb{G}_1 = \mathbb{G}$, and $\mathbb{G}_{i+1} = \mathbb{G}_i/H_i$.
- a chain of subgroups $H_1 \leq \cdots \leq H_q < \mathbb{G}$ where $\deg H_i = p^{k_1 + \cdots + k_i}$.
- a sequence $f_1, \ldots, f_{i-1}$ of composable isogenies starting from $\mathbb{G}$ where $\deg f_i = p^{k_i}$.

Notation 2.2.2. $(H_1, \cdots, H_q)$ will denote a chain of subgroups $H_s < \mathbb{G}_s = \mathbb{G}_{s-1}/H_{s-1}$ and $\overline{(H_1, \cdots, H_q)}$ will denote the subgroup of $\mathbb{G}$ which is the kernel of the composition of isogenies given by $H_1, \cdots, H_q$.

Strickland [Str97, Str98] shows that $\mathrm{Spf}(\Gamma^\vee[k]) \cong \mathrm{Sub}_k(\mathbb{G})$. The left and right E_0-module structures on $\Gamma[k]$ correspond to the source and target maps given on rings by $s, t : E_0 \to \Gamma^\vee[k]$ where s is the usual inclusion and t is the total additive power operations map $\overline{P}_{p^k}$.

Since Γ is an E_0-algebra, its (graded) dual $\Gamma^\vee = \bigoplus \Gamma^\vee[k]$ is a co-E_0-algebra as follows. The tensor product $\Gamma^\vee[k_1]_t \otimes_{E_0\, s} \cdots {}_t \otimes_{E_0\, s} \Gamma^\vee[k_q]$ corresponds to the scheme $\mathrm{Sub}_{k_1, \cdots, k_q}(\mathbb{G})$. Unless otherwise specified, we will drop the decorations and write $\otimes$ for ${}_t\otimes_{E_0\, s}$. The multiplicative structure $\Gamma[k_1] \otimes \Gamma[k_2] \to \Gamma[k_1 + k_2]$ on Γ coming from composing power operations corresponds to comultiplication $\Gamma^\vee[k_1 + k_2] \to \Gamma^\vee[k_1] \otimes \Gamma^\vee[k_2]$ on $\Gamma^\vee$ coming from composing isogenies. In fact, these make $\Gamma^\vee$ into a graded category object in formal affine schemes whose objects are deformations of G_0 and whose morphisms are isogenies lifting some power of the Frobenius isogeny [Rez09].

2.2.4 Power operations in nonzero degrees

For $q \in \mathbb{Z}$, there are rings Γ^{-q} and Δ^{-q} [Rez17, Section 3]. Γ^{-q} acts on the degree q part of the underlying E_0-module of a $\mathbb{T}$-algebra, while Δ^{-q} acts on the degree q part of the cotangent space at the augmentation ideal of an augmented $\mathbb{T}$-algebra.

Let $U^{-q} : \mathrm{Alg}_{\mathbb{T}}^* \to \mathrm{Ab}$ be the forgetful functor which sends a $\mathbb{T}$-algebra to its degree q part. Define $\Gamma^{-q} = \mathrm{End}(U^{-q})$ to be the endomorphism ring of U^{-q}. Rezk [Rez17, Prop 3.23] shows that after completion at the maximal ideal $\mathfrak{m} = (p, \cdots, u_{h-1})$, there is an equivalence $(\Gamma^{-q})_{\mathfrak{m}}^\wedge \simeq \mathrm{End}(\pi_q)$ where π_q is considered as a functor $\mathrm{Alg}_{E_*} \to \mathrm{Ab}$. Thus, Γ^{-q} naturally acts on π_q of $K(h)$-local $\mathbb{E}_\infty$-E-algebras. $\Gamma^{-q} = \bigoplus_{k \geq 0} \Gamma^{-q}[k]$ is a graded algebra by weight where

$$\Gamma^{-q}[k] \cong \ker \left(E_q^\wedge B\Sigma_{p^k}^{q\rho_{p^k}} \xrightarrow{\mathrm{tr}} \bigoplus_{0 < i < p^k} E_q^\wedge \left(B\Sigma_i \times \Sigma_{p^k - i} \right)^{q\rho_{p^k}} \right)$$

is the kernel of the transfer (1). Specializing to $q = 0$ recovers Γ.

Let $Q^{-q} : \mathrm{Alg}_{\mathbb{T}/E_*}^* \to \mathrm{Ab}$ be the functor which sends an augmented $\mathbb{T}$-algebra to the degree q part of its indecomposables. Define $\Delta^{-q} = \mathrm{End}(Q^{-q})$, so $\Delta^{-q} = \bigoplus_{k \geq 0} \Delta[k]$ where

$$\Delta^{-q}[k] \cong \mathrm{cok} \left(\bigoplus_{0 < i < p^k} E_q^\wedge B(\Sigma_i \times \Sigma_{p^k - i})^{q\rho_{p^k}} \xrightarrow{\mathrm{res}} E_q^\wedge B\Sigma_{p^k}^{q\rho_{p^k}} \right)$$

is the cokernel of the map induced by the inclusions $\Sigma_i \times \Sigma_{p^k - i} \subset \Sigma_{p^k}$.

Γ^{-q} and Δ^{-q} are the primitives and indecomposables of the Hopf algebra $\mathbb{T}(\pi_* \Sigma^q E)$. [Rez17] shows that all the rings Γ^{-q} and Δ^{-q} are isomorphic. Moreover, there are suspension homomorphisms $\Gamma^{-q} \to \Gamma^{-(q+1)}$ ([Rez17]) induced by the inclusion $1 \oplus q \cdot \rho_m \to (q+1) \cdot \rho_m$. Explicitly, this is the restriction of the map

$$E_q^\wedge((\mathbb{S}^q)_{h\Sigma_m}^{\otimes m}) \simeq E_{q+1}^\wedge(\mathbb{S}^1 \otimes_{h\Sigma_m} (\mathbb{S}^q)^{\otimes m}) \to E_{q+1}^\wedge((\mathbb{S}^{q+1})_{h\Sigma_m}^{\otimes m}).$$

Since the suspension map is an isomorphism if q is odd, we will only consider double suspensions $\Gamma^{-2q} \to \Gamma^{-2(q+1)}$. We will denote this double suspension map by σ, refer to it as the "**suspension map**", and regard it as a homomorphism $\Gamma \hookrightarrow \Gamma$ since all the Γ^q's are isomorphic. For each k, $\sigma : \Gamma^{-2q}[k] \to \Gamma^{-2(q+1)}[k]$ is dual to multiplication $\Gamma^\vee[k] \xrightarrow{\cdot c_{p^k}} \Gamma^\vee[k]$, where c_{p^k} is the Euler class of the reduced complex standard representation of Σ_{p^k} [BHK24, Proposition 4.1].

2.3 Koszulness

Rezk [Rez17] gives the notion of being Koszul for a not necessarily commutative graded ring $A = \bigoplus_{k \geq 0} A[k]$ with $R = A[0]$ a commutative ring which is not necessarily central in A. Let $\overline{\mathcal{B}}(A) = \mathcal{B}(R, A, R)$ be the bar complex where R is viewed as a left or right A-module via the projection $A \to A[0] = R$. Then $\overline{\mathcal{B}}(A) = \bigoplus_{k \geq 0} \overline{\mathcal{B}}(A)[k]$ is a direct sum of its graded pieces where

$$\overline{\mathcal{B}}_q(A)[k] = \bigoplus_{k_1 + \cdots + k_q = k} A[k_1] \otimes_R \cdots \otimes_R A[k_q]$$

with each $k_i > 0$.

Definition 2.3.1 ([Rez17, Definition 4.4])**.** We say that a (not necessarily commutative) graded algebra A is Koszul if $H_q \overline{\mathcal{B}}(A)[k] = 0$ for $q \neq k$. If A is Koszul, we let $C[k] = H_k \overline{\mathcal{B}}(A)[k]$. We say that A is Koszul of length $n + 1$ if $C[k] = 0$ for $k > n$.

If A is Koszul of length $n+1$, any A-module which is flat (resp. projective) over $A[0]$ admits a flat (resp. projective) A-module resolution of length $n + 1$.

Theorem 2.3.2 ([Rez17])**.** *The rings Γ^q and Δ^q are Koszul of length h.*

Sketch proof. Rezk shows that all the Γ^q and Δ^q are Koszul by showing that $\Delta := \Delta^0$ is Koszul. He identifies the bar complex $\overline{\mathcal{B}}(\Delta)[k]$ with a certain transitive E-homology Q_k applied to a partition complex. Since Q_k is a Mackey functor satisfying certain properties, he concludes using a criterion of [ADL16] that $\overline{\mathcal{B}}(\Delta)[k]$ has homology concentrated in degree k. $\square$

Sketch of proof of main theorem. We will adapt Rezk's proof to show that each $\Gamma^{(i)}$ is Koszul of length $h - i + 1$. To show that each graded piece $\overline{\mathcal{B}}(\Gamma^{(i)})[k]$ has homology concentrated in degree k, we first identify its graded dual in terms of the generalized Tate construction with respect to nontransitive subgroups of the symmetric group Σ_k. This allows us to express $\overline{\mathcal{B}}(\Gamma^{(i)})[k]$ in terms of a certain Mackey functor $Q_k^{(i)}$ applied to a partition complex as in [Rez17], and conclude using [ADL16]. We will explain Rezk's proof and our modification in Sections 3 and 4.

Handedness convention. An action of Γ corresponds to a coaction of $\Gamma^\vee$. However, as mentioned in [Rez13, 4.4], the handedness convention on whether Γ acts and $\Gamma^\vee$ coacts on the left or right have not been consistent. Here we use the convention in [Rez13] where actions happen on the left and coactions on the right. This means that in both Γ and $\Gamma^\vee$, the left E_0-module structure is the usual one (source) and the right E_0-module structure is the target. Here are the handedness conventions in our sources:

- [Rez09]: right action, left coaction

- [Rez13] and [BR19]: left action, right coaction

- [Rez17]: left action, left coaction

3 A description in terms of the generalized Tate construction

We saw earlier that $\Gamma^\vee[k]$ can be expressed as the quotient of $E^0(B\Sigma_{p^k})$ by the transfer ideal, which is an algebraic way to correct the non-additivity of power operations. In this section,

we will show that $\Gamma^\vee[k]$ can also be identified with the image of $E^0(B\Sigma_{p^k})$ in a generalized Tate construction compatible with all the structures in Γ. The main results in this section are Propositions 3.4.7 and 3.4.9.

We build upon the idea of Nikolaus-Scholze [NS18] that the Tate construction is a homotopical way to correct the non-additivity of power operations: $\Gamma^\vee[1] \cong E^0(B\Sigma_p)/\mathrm{tr}$ is the image of $E^0(B\Sigma_p)$ in $\pi_0 E^{t\Sigma_p}$ under the canonical map

$$E^{h\Sigma_p} \to E^{t\Sigma_p}.$$

Thus, $\Gamma^\vee[1]$ is the $C_{p-1} = \mathrm{Aut}(C_p)$ fixed points of the ring corresponding to the closure of the strict p-torsion points $\mathbb{G}[p] - 0$ in the subscheme of p-torsion points $\mathbb{G}[p]$ in $\mathbb{G}$. It follows from Strickland [Str98] (see Remark 3.4.8) that $\Gamma^\vee[k]$ is the image of $E^0(B\Sigma_{p^k})$ in $\pi_0 E^{t\mathcal{T}_k}$, where $(-)^{t\mathcal{T}_k}$ is the Tate construction with respect to non-transitive subgroups of Σ_{p^k} (3.2). In Prop 3.4.7, we show that if $k = k_1 + \cdots + k_q$, then $\Gamma^\vee[k_1] \otimes \cdots \otimes \Gamma^\vee[k_q]$ is the image of $E^0(B\Sigma_{p^{k_q}} \wr \cdots \wr \Sigma_{p^{k_1}})$ in $\pi_0 \left(E \wedge \left(\Sigma_{p^k}/\Sigma_{p^{k_q}} \wr \cdots \wr \Sigma_{p^{k_1}} \right) \right)^{t\mathcal{T}_k}$.

Rezk showed (see 2.3) that Δ is Koszul by identifying the bar complex $\overline{\mathcal{B}}(\Delta)[k]$ with a certain Mackey functor Q_k applied to the partition complex, which we will explain in Section 3.1. In Section 3.2, we recall the generalized Tate construction and construct a variant Q'_k (Def 3.2.2) of Q_k as the image in the Tate construction. In Section 3.3, we identify Q'_k applied to the partition complex with the cobar complex for $\Gamma^\vee$. In fact, as we will see in Section 3.4, this identification upgrades to an isomorphism of cosimplicial E_0-algebras (Proposition 3.4.9). The advantage of working with $\Gamma^\vee$ instead of Γ is that it has a multiplicative structure which will be needed when generalizing to $\Gamma^{(i)}$.

3.1 The bar complex

For a poset X with minimal and maximal elements 0 and 1, following [Rez17, Section 6.1], let $\overline{X}$ denote the quotient $X/(\hat{X} \cup \check{X})$ where $\hat{X}$ and $\check{X}$ are the maximal subposets of X not containing 0 and 1, respectively. Explicitly, this is the poset whose nonbasepoint nondegenerate q-simplices are chains $[0 = x_0 < x_1 < \cdots < x_q = 1]$. Let P_m be the poset of partitions on the set $\underline{m} = \{1, \cdots, m\}$. Rezk showed that Δ is Koszul i.e. $H_q \overline{\mathcal{B}}(\Delta)[k] = 0$ for $q \neq k$, by showing that there is an isomorphism

$$\overline{\mathcal{B}}(\Delta)[k] \xrightarrow{\cong} Q_k(\overline{P}_{p^k}) \tag{2}$$

of simplicial E_0-modules. Here $Q_k : \mathrm{Sp}^{B\Sigma_{p^k}} \to \mathrm{Mod}_{E_0}$ is the transitive E-homology defined by

$$Q_k(X) = \mathrm{cok}\left(E_0^\wedge (X \otimes S)_{h\Sigma_{p^k}} \to E_0^\wedge X_{h\Sigma_m} \right)$$

$$= \mathrm{cok}\left(\bigoplus_{0 < i < p^k} E_0^\wedge \left(X_{h\Sigma_i \times \Sigma_{p^k-i}} \right) \xrightarrow{\mathrm{res}} E_0^\wedge X_{h\Sigma_{p^k}} \right),$$

where S is the Σ_{p^k}-set of nontrivial partitions of $\underline{p}^k$. Since Q_k is a Mackey functor satisfying the properties in [Rez17, Prop 8.3], by a special case of [ADL16, Cor. 1.2], $Q_k(\overline{P}_{p^k})$ has nonzero homology only in degree k.

The simplicial structures on each side of (2) are as follows.

- For $\overline{\mathcal{B}}(\Delta)[k]$, the face maps come from multiplication $\Delta[k_1] \otimes \Delta[k_2] \to \Delta[k_1 + k_2]$ in Δ and the degeneracy maps are identities (inserting $\Delta[0] = E_0$ appropriately).

- The simplicial structure on $Q_k(\overline{P}_{p^k})$ comes from the simplicial structure on P_{p^k}. As a simplicial set, the q-simplices of P_m are chains of partitions $[E_0 \leq \cdots \leq E_q]$ [Rez17, Definition 6.2]. The face and degeneracy maps come from deleting and duplicating an entry E_i, respectively.

  Σ_{p^k} acts on P_{p^k} by permuting the set $\underline{p}^k$. Consider a q-simplex $[E_0 \leq \cdots \leq E_q]$ with stabilizer H. Each face map sends $[E_0 \leq \cdots \leq E_q]$ to a $(q-1)$-simplex whose stabilizer H' contains H. Thus, Σ_{p^k}-equivariantly, the face maps are given by sums of restriction maps $\mathrm{res} : \Sigma_{p^k}/H \to \Sigma_{p^k}/H'$. The degeneracy maps are identities.

  The arguments in [Rez17, Section 6] show that if one of the partitions E_i is not pure, then H is non-transitive, so $Q_k(\Sigma_{p^k}/H) = 0$. Thus, only pure partitions matter. If all the partitions E_i are pure of $\mathrm{mesh}(E_i) = k_1 + \cdots + k_i$ where $k_1 + \cdots + k_q = k$, then $H = \Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}$. So, the face maps in $Q_k(\overline{P}_{p^k})$ are induced by the restriction maps $\Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}} \to \Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_i+k_{i+1}}} \wr \cdots \wr \Sigma_{p^{k_q}}$.

Since all the (co)simplicial objects we will encounter have (co)degeneracy maps given by identities, we will denote a (co)simplicial object simply by $(X_\bullet, \partial)$ when we want to specify the (co)face maps. In this notation, $Q_k(\overline{P}_{p^k}) = (Q_k(\overline{P}_{p^k}), Q_k(\mathrm{res}))$.

We will replace Q_k by another functor Q'_k which agrees with Q_k on the partition complex P_{p^k}, and thus can be used to show Koszulness as well.

3.2 The generalized Tate construction

For a finite group G, a family of subgroups $\mathcal{F}$ (closed under subgroups and conjugation), and a G-spectrum X, there is an associated $\mathcal{F}$-*Tate spectrum* $X^{t\mathcal{F}}$ as follows [GL20, Definition A.1]. Let $\mathcal{O}_G$ be the orbit category i.e. the category of finite transitive G-sets and G-equivariant maps. Let $\mathcal{O}_\mathcal{F} \subset \mathcal{O}_G$ denote the full subcategory spanned by G-sets of the form G/H for some $H \in \mathcal{F}$. A spectrum $X \in \mathrm{Sp}^{BG} = \mathrm{Fun}(BG, \mathrm{Sp})$ determines a functor $\widetilde{X} : \mathcal{O}_G \to \mathrm{Sp}$ which sends G/H to X^{hH} and maps of G-sets to the corresponding transfer maps. The spectrum $X^{t\mathcal{F}}$ is the cofiber

$$\mathrm{colim}_{\mathcal{O}_\mathcal{F}} \widetilde{X} \to X^{hG} \to X^{t\mathcal{F}}.$$

For a subgroup H, let $N = N_G(H)$ be its normalizer and $W = N/H$ its Weyl group.

Lemma 3.2.1. *The spectrum $X^{t\mathcal{F}}$ fits into the cofiber sequence*

$$\mathrm{colim}_{[H] \in \mathcal{F}} (X^{hH})_{hW} \to X^{hG} \to X^{t\mathcal{F}}$$

where the map from the colimit is induced by transfers. Here the colimit ranges over conjugacy classes $[H]$ of subgroups in $\mathcal{F}$ and the only nontrivial maps are between different conjugacy classes (i.e. objects in the category over which the colimit is taken have no nontrivial automorphisms).

Proof. Consider the full subcategory $\mathcal{O}_\mathcal{F}^H \subseteq \mathcal{O}_\mathcal{F}$ spanned by objects isomorphic to G/H. This consists of objects G/gHg^{-1} ranging over all conjugacy classes of H. Pick representatives $g_1, \ldots, g_s$ from each coset in G/N. The objects of $\mathcal{O}_\mathcal{F}^H$ are precisely $G/g_iHg_i^{-1}$ for $i = 1, \ldots, s$. Any map

$$G/g_iHg_i^{-1} \to G/H$$

factors uniquely as the canonical map sending the coset $g_i H g_i^{-1} \in G/g_i H g_i^{-1}$ to $g_i H \in G/H$, followed by an automorphism of G/H. Thus, the groupoid $\mathcal{O}_{\mathcal{F}}^H$ is equivalent to the 1-object category $\{G/H\}$ with automorphism group W.

$\mathcal{O}_{\mathcal{F}} = \operatorname{colim} \mathcal{O}_{\mathcal{F}}^H$ where the colimit ranges over the conjugacy classes $[H]$ of subgroups in $\mathcal{F}$. The same argument shows that $\mathcal{O}_{\mathcal{F}}$ is equivalent to the full subcategory consisting of objects G/H for each conjugacy class $[H]$ in $\mathcal{F}$. Since $\operatorname{colim}_{BW} X^{hH} = (X^{hH})_{hW}$ and colimits can be taken in any order, we get the desired cofiber sequence.

$\qquad\square$

A subgroup $H \leq \Sigma_{p^k}$ is transitive if it acts transitively on $\underline{p}^k = \{1, \cdots, p^k\}$.

Definition 3.2.2. For each k, define $Q'_k : \operatorname{Sp}^{B\Sigma_{p^k}} \to \operatorname{Mod}_{E_0}$ to be the image of $\pi_0(E \wedge X)^{h\Sigma_{p^k}}$ in the Tate construction

$$Q'_k(X) = \operatorname{im}\left(\pi_0(E \wedge X)^{h\Sigma_{p^k}} \to \pi_0(E \wedge X)^{t\mathcal{T}_k} \right),$$

where $\mathcal{T}_k$ is the set of non-transitive subgroups of G.

Since the Tate construction kills spectra induced from subgroups in $\mathcal{T}_k$, Q'_k vanishes on non-transitive $G := \Sigma_{p^k}$-sets. The Tate construction is exact, so Q'_k is additive. In particular, to evaluate Q'_k on finite G-sets, it suffices to evaluate on G/H for subgroups $H \leq G$.

We will use the following fact repeatedly. If $H \leq G$ is a subgroup of index prime to p, then the composite

$$BG_+ \xrightarrow{\operatorname{tr}} BH_+ \xrightarrow{\operatorname{res}} BG_+$$

is an equivalence.

Every non-transitive subgroup of G is contained in some conjugate of a partition subgroup $L = \Sigma_i \times \Sigma_{p^k - i}$, where $i > 0$. Strickland shows that for any L, $\Sigma_{p^{k-1}}^p$ contains a Sylow p-subgroup of L. This, together with the fact that the transfer map factors through subgroup inclusions, gives the image in the Tate construction as the quotient by a single transfer map. We thank Greg Arone for pointing out an incorrect earlier version of the following lemma.

Lemma 3.2.3 ([Str98] Lemma 8.11). *For $X \in \operatorname{Sp}^{B\Sigma_{p^k}}$, the image* $\operatorname{im}\left(\pi_0(X^{h\Sigma_{p^k}}) \to \pi_0(X^{t\mathcal{T}_k}) \right)$ *in the Tate construction is the quotient of $\pi_0(X^{h\Sigma_{p^k}})$ by the transfer from $\pi_0\left(X_{h\Sigma_p}^{h\Sigma_{p^{k-1}}^p} \right)$.*

Corollary 3.2.4. *For $X \in \operatorname{Sp}^{B\Sigma_{p^k}}$, $Q'_k(X)$ is the cokernel of the transfer*

$$Q'_k(X) = \operatorname{cok}\left(E_0\left(X^{h\Sigma_{p^{k-1}}^p} \right)_{h\Sigma_p} \xrightarrow{\operatorname{tr}} E_0 X^{h\Sigma_{p^k}} \right).$$

3.3 Q vs Q'

Applying Q'_k to the partition complex P_{p^k} gives a simplicial E_0-module $(Q'_k(P_{p^k}), Q'_k(\operatorname{res}))$ (3.1). The goal of this section is to prove the following.

Proposition 3.3.1. *There is a natural map $Q_k(P_{p^k}) \to Q'_k(P_{p^k})$ which is an isomorphism of simplicial E_0-modules. Thus, there exists an isomorphism*

$$\overline{\mathcal{B}}(\Delta)[k] \xrightarrow{\cong} Q'_k(\overline{P}_{p^k})$$

of simplicial E_0-modules.

We will prove this by relating Q_k to Q_k' via the norm map. Consider the diagram

$$
\begin{array}{ccc}
\bigoplus_{0<i<p^k} E_0^\wedge\left(X_{h\Sigma_i \times \Sigma_{p^k-i}}\right) & \xrightarrow{\text{res}} E_0^\wedge X_{h\Sigma_{p^k}} \longrightarrow Q_k(X) \\
\Big\downarrow{\scriptstyle\text{Nm}} \qquad\qquad \Big\downarrow{\scriptstyle\text{Nm}} \qquad\qquad \Big\downarrow{\scriptstyle\simeq} \\
\bigoplus_{0<i<p^k} E_0^\wedge\left(X^{h\Sigma_i \times \Sigma_{p^k-i}}\right) & \xrightarrow{\text{tr}} E_0^\wedge X^{h\Sigma_{p^k}} \longrightarrow \text{cok}\,.
\end{array}
$$

The norm map is a $K(h)$-local equivalence under which the restriction map corresponds to the transfer map, so it induces a natural isomorphism on cokernels

$$
\text{Nm}: Q_k(X) \xrightarrow{\simeq} \text{cok}\left(\bigoplus_{0<i<p^k} E_0^\wedge\left(X^{h\Sigma_i \times \Sigma_{p^k-i}}\right) \xrightarrow{\text{tr}} E_0^\wedge\left(X^{h\Sigma_{p^k}}\right)\right)
$$
$$
\cong \text{cok}\left(E_0^\wedge\left(X^{h\Sigma^p_{p^{k-1}}}\right) \xrightarrow{\text{tr}} E_0^\wedge\left(X^{h\Sigma_{p^k}}\right)\right)
$$
$$
\cong \text{cok}\left(E_0\left(X^{h\Sigma^p_{p^{k-1}}}\right) \xrightarrow{\text{tr}} E_0\left(X^{h\Sigma_{p^k}}\right)\right).
$$

Here the second isomorphism is an application of [Str98, Lemma 8.11] and the last one follows from the fact that $E \otimes X^{hH}$ is already $K(h)$-local. This puts us in good shape to compare it to $Q_k'(X)$ given in Corollary 3.2.4.

For $H \leq G$ with Weyl group W, the transfer map factors as

$$
X^{hH} \to X^{hH}_{hW} \to X^{hG}.
$$

Thus, there exists a natural transformation $Q_k \to Q_k'$ of functors $\text{Sp} \to \text{Mod}_{E_0}$ which we will also call the norm map. To show that this is a natural isomorphism, we need to show that killing the transfer from X^{hH} is the same as killing the possibly larger term X^{hH}_{hW}.

Lemma 3.3.2. *For G a finite group and $X \in \text{Sp}^{BG}$, the canonical map*

$$
X \to X_{hG}
$$

is surjective on homotopy groups after inverting p.

Proof. Since $H_*(G; M))$ is killed by the order of G for any G-module M and $* > 0$, the homotopy orbit spectral sequence

$$
H_*(BG; \underline{\pi_* X}) \Rightarrow \pi_* X_{hG}
$$

is given by $\pi_* X$ on the 0th column and is (p-)torsion on positively indexed columns. This gives a short exact sequence

$$
0 \to E_{0,*}^\infty \to \pi_* X_{hG} \to \text{torsion} \to 0,
$$

so the map $E_{0,*}^\infty \to \pi_* X_{hG}$ is an equivalence after inverting p. Precomposing with the projection map tells us that the composite

$$
\pi_* X \twoheadrightarrow E_{0,*}^\infty \to \pi_* X_{hG}
$$

is surjective after inverting p. $\qquad\square$

Since the images of $\pi_* X^{hH}$ and $\pi_* X^{hH}_{hW}$ in $\pi_* X^{hG}$ differ only by torsion, they are the same if $\pi_* X^{hG}$ is torsion free. Applying this to $G = \Sigma_{p^k}$, $H = \Sigma^p_{p^{k-1}}$ and $W = \Sigma_p$ gives the following.

Lemma 3.3.3. *For $X \in \mathrm{Sp}$, the norm map*

$$Q_k \to Q'_k$$

is a natural isomorphism of E_0-modules after inverting p. It is a natural isomorphism (without inverting p) if $E_0\left(X^{h\Sigma_{p^k}}\right)$ is torsion free.

This immediately implies Prop 3.3.1.

Proof of Prop 3.3.1. We are only interested in evaluating Q_k and Q'_k on the partition complex P_{p^k}. For a q-simplex $[E_0 \leq \cdots \leq E_q]$ with stabilizer H,

(i) If one of the partitions E_i is not pure, then H is non-transitive, so $Q'_k(\Sigma_{p^k}/H) = 0 = Q_k(\Sigma_{p^k}/H)$.

(ii) If all the partitions E_i are pure with $\mathrm{mesh}(E_i) = k_1 + \cdots + k_i$ where $k_1 + \cdots + k_q = k$, then $H = \Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}$. The Sylow p-subgroup $C_p \wr \cdots \wr C_p$ of H is good [HKR00, Theorem 7.3], so $E_0^\wedge(\Sigma_{p^k}/H)^{h\Sigma_{p^k}} = E^0(BH)$ is torsion free and the map $Q_k(\Sigma_{p^k}/H) \to Q'_k(\Sigma_{p^k}/H)$ is an isomorphism.

In both cases, the map from Q_k to Q'_k is an isomorphism. Since the norm map $Q_k \to Q'_k$ is a natural transformation, $Q_k(P_m) \to Q'_k(P_m)$ is an isomorphism of simplicial E_0-modules. $\square$

We record the following, which we will need later.

Proposition 3.3.4. *Let $k = k_1 + \cdots + k_q$ with each $k_i > 0$. Then $Q'_k\left(\Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}\right)$ is the quotient of $E^0\left(B\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}\right)$ by transfers from*

$$E^0 B\left(\left(\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_{i-1}}}\right)^{p^{k_i}} \rtimes \Sigma^p_{p^{k_i-1}} \wr \cdots \wr \Sigma_{p^{k_q}}\right)$$

for $i = 1, \ldots, q$.

Proof. Let $G = \Sigma_{p^k}$, $H = \Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}$, $K = \Sigma^p_{p^k-1}$, and $K^i = \left(\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_{i-1}}}\right)^{p^{k_i}} \rtimes \Sigma^p_{p^{k_i-1}} \wr \cdots \wr \Sigma_{p^{k_q}}$ for $1 \leq i \leq q$. Since $E^0(BH)$ is torsion free, by Lemma 3.2.3, $Q'(G/H)$ is the quotient of $E^0(BH)$ by the image of the transfer from $E_0^\wedge(G/H)^{hK}$. As a K-set, $G/H = \bigsqcup K_g$ where $K_g = gHg^{-1} \cap K$, so $E_0^\wedge(G/H)^{hK} = \bigoplus E^0(BK_g)$. Transfers from K^i are clearly killed in the Tate construction. We will show that killing these kills transfers from each K_g.

We induct on q. For $q = 1$, Strickland [Str98, Lemma 8.11] shows that for all k, $Q'_k(G/H)$ is the cokernel of a single transfer from K. Before proving the general case, we will do the case $q = 2$. We claim that for each $g \in G$, $K_g = \prod_\alpha K_\alpha$ is a product of subgroups whose Sylow p-subgroups are contained in K^1 or K^2.

K is the set of permutations on $\underline{p^k}$ which preserves each block $B_s = \{sp^{k-1} + 1, \cdots, (s+1)p^{k-1}\}$ ($1 \leq s \leq p$) of size p^{k-1}. For each g, gHg^{-1} is the set of permutations on a set of p^k elements partitioned into blocks C_t ($1 \leq t \leq p^{k_2}$) of size p^{k_1}, which preserve elements in the same block but are allowed to permute the blocks. The blocks B_s partition each block C_t into

12

p (possibly trivial) parts. Elements in $gHg^{-1} \cap K$ can only send a block C_t to another block of the same partition type. So, $K_g = \prod_\alpha K_\alpha$ where the product ranges over all partition types which occur for some block C_t. For a partition type $(r_1, \cdots, r_p)$ which occurs for some C_t, if there are m blocks of this type, then the permutation group acting on these m blocks is

$$K_\alpha = \left(\Sigma_{r_1} \times \cdots \times \Sigma_{r_p} \right) \wr \Sigma_m.$$

If there is only one nonzero $r_i = p^k$, then $m \leq p^{k_2-1}$, so $K_\alpha \leq K_2$. Otherwise, $\Sigma_{r_1} \times \cdots \times \Sigma_{r_p}$ is a nontransitive subgroup of $\Sigma_{p^{k_1}}$, so its Sylow p-subgroup is contained in $\Sigma^p_{p^{k_1-1}}$. In either case, killing transfers from K_1 or K_2 kills the transfer from K_α. Thus, killing transfers from K_1 and K_2 kills transfers from each K_g. This gives the case $q = 2$.

We will now prove the statement for general q, assuming that it is true for $q - 1$ (and all k). We will use the same notation as before, where now C_t's are blocks of size p^{k-k_q} since $H \leq \Sigma_{p^{k-k_q}} \wr \Sigma_{p^{k_q}}$. As before, $K_g = \prod_\alpha K_\alpha$ where the product ranges over all partition types which occur for some block C_t. For a partition type $(r_1, \cdots, r_p)$, if there are m blocks of this type, then the permutation group acting on these m blocks is

$$K_\alpha = \left(\left(\Sigma_{r_1} \times \cdots \times \Sigma_{r_p} \right) \cap \left(\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_{q-1}}} \right) \right) \wr \Sigma_m.$$

If there is only one nonzero $r_i = p^k$, then $m \leq p^{k_q-1}$, so $K_\alpha \leq K^q$. Otherwise, $\left(\Sigma_{r_1} \times \cdots \times \Sigma_{r_p} \right) \cap \left(\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_{q-1}}} \right)$ is a nontransitive subgroup of $\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_{q-1}}}$. By the induction hypothesis, its Sylow p-subgroup is contained in some K^i for $1 \leq i \leq q-1$. Thus, to kill transfers from each K_g, it suffices to kill transfers from each K^i for $1 \leq i \leq q$, which concludes the proof. $\square$

3.4   Dualizing: an isomorphism of algebras

Let $m = p^k$. Since $\Gamma = \bigoplus_{k \geq 0} \Gamma[k]$ is a graded E_0-algebra, $\Gamma^\vee = \bigoplus_{k \geq 0} \Gamma^\vee[k]$ is a graded co-E_0-algebra. Let $\overline{\mathcal{B}}(\Gamma^\vee)[k]$ be the cobar complex for $\Gamma^\vee$ which is the cosimplicial E_0-module dual (as left E_0-modules) to the bar complex $\overline{\mathcal{B}}(\Gamma)[k]$.

The partition complex P_m is a simplicial spectrum via the inclusion of finite Σ_m-sets into spectra, so its Spanier-Whitehead dual $P_m^\vee$ is a cosimplicial spectrum and $(Q'_k(P_m^\vee), Q'_k(\mathrm{res}^\vee))$ is a cosimplicial E_0-module. Since finite G-sets are self dual and restriction maps $G/K_+ \xrightarrow{\mathrm{res}} G/H_+$ are dual to transfers $G/H_+ \simeq G/H_+^\vee \xrightarrow{\mathrm{tr}} G/K_+^\vee \simeq G/K_+$, we have $(Q'_k(P_m^\vee), Q'_k(\mathrm{res}^\vee)) \simeq (Q'_k(P_m), Q'_k(\mathrm{tr}))$. In fact, $(Q'_k(P_m), Q'_k(\mathrm{res}))$ and $(Q'_k(P_m), Q'_k(\mathrm{tr}))$ are the covariant and contravariant parts of the Mackey functor Q' applied to the partition complex P_m.

The goal of this section is to show that $\overline{\mathcal{B}}(\Gamma^\vee)[k]$ and $Q'_k(\overline{P}_m^\vee)$ admit the structure of cosimplicial E_0-algebras and are isomorphic as such (Proposition 3.4.9). Indeed, the qth term

$$\overline{\mathcal{B}}_q(\Gamma^\vee)[k] = \bigoplus_{\substack{k_1 + \cdots + k_q = k \\ k_i > 0}} \Gamma^\vee[k_1] \otimes \cdots \otimes \Gamma^\vee[k_q]$$

is a direct sum of rings classifying chains of subgroups of $\mathbb{G}$. The coface maps are ring maps corresponding to composition of isogenies. On the other hand, each term of $Q'_k(P_m^\vee)$ is a quotient of the ring $E^0(BH)$ by a transfer ideal so it inherits the cohomology ring structure. A corollary of the following lemma shows that the face maps are E_0-algebra maps (Cor 3.4.2).

Since Q'_k vanishes on non-transitive G-sets, we will implicitly assume that all subgroups H in sight are stabilizers of pure partitions, so $E_0^\wedge(BH)$ and $E^0(BH)$ are finite free left E_0-modules dual to each other. We will also identify the category of finite G-sets as a subcategory of Sp^{BG}. Let $K \leq H \leq G$. There are restriction maps $G/K_+ \xrightarrow{\mathrm{res}} G/H_+$ and $BK_+ \xrightarrow{\mathrm{res}} BH_+$, and transfer maps $G/H_+ \xrightarrow{\mathrm{tr}} G/K_+$ and $BH_+ \xrightarrow{\mathrm{tr}} BK_+$.

Lemma 3.4.1. *For $G = \Sigma_{p^k}$ and $K \leq H \leq G$ stabilizers of pure partitions, the maps $Q'_k(\mathrm{tr}) : Q'_k(G/H) \to Q'_k(G/K)$ and $Q'_k(\mathrm{res}) : Q'_k(G/K) \to Q'_k(G/H)$ are induced by $E^0(\mathrm{res})$ and $E^0(\mathrm{tr})$, respectively, and are dual as maps of left E_0-modules.*

Proof. We claim that
$$E_0^\wedge(\mathrm{res}^{hG}) : E^0 BK \simeq E_0^\wedge(G/K_+)^{hG} \to E_0^\wedge(G/H_+)^{hG} \simeq E^0 BH$$
is $E^0(\mathrm{tr})$. To see this, note that we have a commutative diagram

$$
\begin{array}{ccc}
E_0^\wedge BK & \xrightarrow{E_0^\wedge(\mathrm{res})} & E_0^\wedge BH \\
{\scriptstyle \mathrm{Nm}}\downarrow & & \downarrow{\scriptstyle \mathrm{Nm}} \\
E^0 BK & \xrightarrow{E^0(\mathrm{tr})} & E^0 BH.
\end{array}
$$

Since the $K(n)$-local norm map is a natural isomorphism
$$E_0^\wedge(-)_{hG} \to \pi_0(E \wedge -)^{hG},$$
applying it to the map $G/K_+ \xrightarrow{\mathrm{res}} G/H_+$ gives
$$E_0^\wedge(\mathrm{res}^{hG}) \circ \mathrm{Nm} = \mathrm{Nm} \circ E_0^\wedge(\mathrm{res}_{hG}),$$
so $E_0^\wedge(\mathrm{res}^{hG}) = E^0(\mathrm{tr})$ as claimed. Similarly,
$$E_0^\wedge(\mathrm{tr}^{hG}) : E^0 BH \simeq E_0^\wedge(G/H_+)^{hG} \to E_0^\wedge(G/K_+)^{hG} \simeq E^0 BK$$
is $E^0(\mathrm{res})$.

This shows that $Q'_k(\mathrm{res})$ and $Q'_k(\mathrm{tr})$ are quotients of $E_0^\wedge(\mathrm{res}^{hG}) = E^0(\mathrm{tr})$ and $E_0^\wedge(\mathrm{tr}^{hG}) = E^0(\mathrm{res})$, respectively. Since $E_0^\wedge BK$ and $E_0^\wedge BH$ are free, $E^0(\mathrm{tr})$ and $E^0(\mathrm{res})$ are dual to each other as maps of E_0-modules. Since $Q'_k(G/H) \cong Q_k(G/H)$ is free, $Q'_k(G/H)$ is a direct summand of $E^0 BH$, similarly for K. Thus, the duality between $E^0(\mathrm{tr})$ and $E^0(\mathrm{res})$ induces a duality on quotients between $Q'_k(\mathrm{res})$ and $Q'_k(\mathrm{tr})$. $\square$

Since the restriction map is a ring map in cohomology, we obtain the following.

Corollary 3.4.2. *The cosimplicial E_0-module $(Q'_k(P_m^\vee), Q'_k(\mathrm{tr})) \simeq (Q'_k(P_m), E^0(\mathrm{res}))$ is a cosimplicial E_0-algebra with the cohomology ring structure. As a cosimplicial E_0-module, it is left E_0-dual to the simplicial E_0-module $(Q'_k(P_m), Q'_k(\mathrm{res})) \simeq (Q'_k(P_m), E^0(\mathrm{tr}))$.*

By Proposition 3.3.1, this implies that there is an isomoprhism $(Q'_k(P_m), E^0(\mathrm{res})) \to \overline{\mathcal{B}}(\Delta^\vee)[k] \simeq \overline{\mathcal{B}}(\Gamma^\vee)[k]$ of cosimplicial E_0-modules since $\Delta \simeq \Gamma$, but this isomorphism is non-canonical. We will use Prop 3.4.6 to produce a natural map $(Q'_k(P_m), E^0(\mathrm{res})) \to \overline{\mathcal{B}}(\Gamma^\vee)[k]$ and show that it is an isomorphism of cosimplicial E_0-algebras (Prop 3.4.9).

Following [Rez09], for P, Q bimodules over a ring R, we write

$$P_R \otimes_R Q = P \otimes Q/(pr \otimes q \sim p \otimes rq)$$
$$_R P \otimes_R Q = P \otimes Q/(rp \otimes q \sim p \otimes rq).$$

As in [Rez09, Section 6], let $P = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{T}}^*}(\mathbb{T}E_*, \mathbb{T}E_*)$ be the monoid of endomorphisms of the forgetful functor $\mathrm{Alg}_{\mathbb{T}} \to \mathrm{Set}$ to the degree 0 part. There is a bijection

$$\pi : P \to \mathbb{T}E_0$$

given by evaluation on the canonical generator $\iota \in \mathbb{T}E_0$. P is a monoid under composition of morphisms

$$\circ : P \times P \to P, \quad (f, g) \mapsto g \circ f.$$

If $f, g \in P$ corresponds to maps $\tilde{f}, \tilde{g} : E_0 \to \mathbb{T}E_0$ of E_0-modules, then $g \circ f$ corresponds to

$$E_0 \xrightarrow{\tilde{f}} \mathbb{T}E_0 \xrightarrow{\mathbb{T}\tilde{g}} \mathbb{T}\mathbb{T}E_0 \xrightarrow{\mu} \mathbb{T}E_0.$$

Note that this monoid product is defined in the reverse order to [Rez09] in order to be consistent with [Rez17] where the left E_0-module structure on Γ is the usual one (see 2.3).

$\Gamma \subset P$ is the subset of additive power operations, which is the set of endomorphisms of the forgetful functor $U^0 : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{Ab}$. The map $E_0 \to P \cong \mathbb{T}E_0$ given by $r \mapsto r \cdot 1$ factors through $E_0 \to \Gamma$, and the monoid product on P descends to a multiplication map $\Gamma_{E_0} \otimes_{E_0} \Gamma \to \Gamma$. Under the bijection π, the left E_0-module structure on $\Gamma \subset \mathbb{T}E_0$ is the usual one

$$E_0 \otimes \Gamma \to \Gamma, \quad r \otimes f(\iota) \mapsto rf(\iota),$$

since it is given by the map $\iota \mapsto r$ followed by $\iota \mapsto f(\iota)$.

The following is dual to the linearization in [Rez17, Section 5]. Let $F : \mathcal{A} \to \mathcal{B}$ be a reduced functor from an additive to an abelian category. We define the linearization $\mathcal{L}_F : \mathcal{A} \to \mathcal{B}$ of F together with a natural transformation $\epsilon : \mathcal{L}_F \to F$ by setting $\mathcal{L}_F(X)$ to be the equalizer of

$$F(X) \xrightarrow[F(i_1)+F(i_2)]{F(i_1+i_2)} F(X \oplus X),$$

where i_1 and i_2 are the two inclusions.

Lemma 3.4.3. *[Rez17, Proposition 5.2] The functor $\mathcal{L}_F$ is additive, and any natural transformation $G \to F$ from an additive functor G factors uniquely through $\epsilon : \mathcal{L}_F \to F$.*

Proof. If $f, g : Y \to X$ are two maps in $\mathcal{A}$, then

$$\epsilon \mathcal{L}_F(f + g) = F(p_1 + p_2)F(f, g)\epsilon \mathcal{L}_F(i_1 + i_2) = F(p_1 + p_2)F(f, g)(F(i_1) + F(i_2))$$
$$= F(p_1 + p_2)(F(f, 0) + F(0, g)) = F(f) + F(g)$$

as a map $\epsilon \mathcal{L}_F(Y) \to F(X)$, where p_i for $i = 1, 2$ are projections. This shows that $\mathcal{L}_F(f + g) = \mathcal{L}_F(f) + \mathcal{L}_F(g)$, thus $\mathcal{L}_F$ is additive. The second part follows from the fact that $\mathcal{L}_g \simeq G$ when G is additive. $\square$

In particular, given functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ where $\mathcal{A}$ is additive and $\mathcal{B}$ and $\mathcal{C}$ are abelian, there is a natural transformation

$$\mathcal{L}_G \circ \mathcal{L}_F \to \mathcal{L}_{G \circ F}.$$

Lemma 3.4.4. *[Rez17, Proposition 5.7] Let $m = p^k$ and let M be a free left E_0-module. Then there is a natural isomorphism*

$$\Gamma[k]_{E_0} \otimes_{E_0} M \to \mathcal{L}_{\mathbb{T}_m}(M).$$

Proof. If $M = E_0$, we have the identity map $\Gamma[k] \to \mathcal{L}_{\mathbb{T}_m}(E_0)$ since $\Gamma[k]$ is the kernel of the transfer. Since $\Gamma[k]_{E_0} \otimes_{E_0} -$ and $\mathcal{L}_{\mathbb{T}_m}$ are additive fuctors, we can extend this to finite free modules. Since $\mathbb{T}$ preserves filtered colimits, so does $\mathcal{L}_{\mathbb{T}_m}$, thus the isomorphism can be extended to all free modules. $\square$

For $m_i = p^{k_i}$, this gives a map of left E_0-modules

$$\Gamma[k_1]_{E_0} \otimes \cdots \otimes_{E_0} \Gamma[k_q] \to \mathcal{L}_{\mathbb{T}_{m_1}} \cdots \mathcal{L}_{\mathbb{T}_{m_q}}(E_0) \to \mathcal{L}_{\mathbb{T}_{m_1} \cdots \mathbb{T}_{m_q}}(E_0) \to \mathbb{T}_{m_1} \cdots \mathbb{T}_{m_q} E_0 \qquad (3)$$

whose composite we denote by j.

Lemma 3.4.5. *For a finite free left E_0-module M, the composite*

$$j : \Gamma[k]_{E_0} \otimes_{E_0} M \xrightarrow{\simeq} \mathcal{L}_{\mathbb{T}_m}(M) \to \mathbb{T}_m M$$

is an inclusion of a direct summand.

Proof. For $M = E_0$, $\Gamma[k] \subset \mathbb{T}_m E_0$ is a direct summand Rezk ([Rez09, Proposition 4.14]) shows that

$$\mathbb{T}(X \oplus Y) \cong \mathbb{T}(X) \otimes \mathbb{T}(Y)$$

for any E_0-modules X and Y. Let $d = \operatorname{rank} M$ and let $e_1, \ldots, e_d$ be a basis for M. This gives an isomorphism

$$\mathbb{T}_m M \cong \mathbb{T}_m \left(\bigoplus^d E_0 \right) \cong \bigoplus_{i_1 + \cdots + i_d = m} E_0^{\wedge} \left(B\Sigma_{i_1} \times \cdots \times \Sigma_{i_d} \right),$$

which has summands

$$E_0^{\wedge} \left(B\Sigma_0^{i-1} \times \Sigma_m \times \Sigma_0^{d-i} \right) \cong \mathbb{T}_m E_0$$

for each i given by the inclusion $e_i : E_0 \to M \cong \bigoplus^d E_0$ into the ith factor.

$\mathcal{L}_F(M) \cong \bigoplus^d \mathcal{L}_F(E_0)$ includes into $\mathbb{T}_m M$ by including each $\mathcal{L}_F(E_0)$ into the corresponding summand $\mathbb{T}_m E_0$. Thus, it is a direct summand. $\square$

Explicitly, for $i = 1, \cdots, d$ and $f \in \Gamma[k]$, $j(f \otimes e_i)$ is given by

$$e_i \circ f : E_0 \xrightarrow{f} \Gamma[k] \subset \mathbb{T}_m E_0 \xrightarrow{\mathbb{T}_m e_i} \mathbb{T}_m M.$$

Since $\Gamma[k]$ is the linearization of $\mathbb{T}_m E_0$, $\mathbb{T}_m(g_1 + g_2) : \Gamma[k] \to \mathbb{T}_m M$ is equal to $\mathbb{T}_m(g_1) + \mathbb{T}_m(g_2)$. Moreover, $\mathbb{T}_m(r \cdot g)$ is equal to the composite

$$\mathbb{T}_m g \circ \mathbb{T}_m r : \mathbb{T}_m E_0 \xrightarrow{\mathbb{T}_m r} \mathbb{T}_m \mathbb{T}_1 E_0 \xrightarrow{\simeq} \mathbb{T}_m E_0 \xrightarrow{\mathbb{T}_m g} \mathbb{T}_m M,$$

whose precomposition with $f : E_0 \to \Gamma[k] \subset \mathbb{T}_m E_0$ is $j(f \cdot r \otimes g)$. Since j is well-defined, $j(f \otimes r \cdot g) = j(f \cdot r \otimes g)$ is the composite

$$E_0 \xrightarrow{f} \Gamma[k] \subset \mathbb{T}_m E_0 \xrightarrow{\mathbb{T}_m(r \cdot g)} \mathbb{T}_m M.$$

Thus, for all $f \in \Gamma[k]$ and $g \in M$, $j(f \otimes g)$ is given by

$$E_0 \xrightarrow{f} \Gamma[k] \subset \mathbb{T}_m E_0 \xrightarrow{\mathbb{T}_m g} \mathbb{T}_m M.$$

In particular, the map $\Gamma[k_1]_{E_0} \otimes \cdots \otimes_{E_0} \Gamma[k_q] \to \mathbb{T}_{m_1} \cdots \mathbb{T}_{m_q} E_0$ is given by $j(f_1 \otimes \cdots f_q) = f_q \circ \cdots \circ f_1$ for $f_i \in \Gamma[k_i]$.

Proposition 3.4.6. *For $m_i = p^{k_i}$, the map $j^\vee$ (3) factors through the quotient $Q'_k \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ of $(\mathbb{T}_{m_q} \cdots \mathbb{T}_{m_1} E_0)^\vee = E^0 \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ and induces an isomorphism*

$$Q'_k \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right) \to \Gamma^\vee[k_q]_t \otimes_{E_0} \cdots \otimes_{E_0\, s} \Gamma^\vee[k_1]$$

of left E_0-modules.

Proof. By definition, $Q'_k \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ is a quotient of $E^0 \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ since it is the image in the Tate construction. We will show that j factors through $Q'_k \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)^\vee$. By Proposition 3.3.4, $Q'_k \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)^\vee$ includes into $E_0^\wedge \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ as the kernel of certain transfer maps. We will show that the composition of j with each of these transfer map is zero. For simplicity of notation, we will do the case $q = 2$.

Since $\Gamma[k_i] \subset \mathbb{T}_{m_i} E_0$ is the kernel of the transfer, the composite

$$E_0 \xrightarrow{f_i} \mathbb{T}_{m_i} E_0 \xrightarrow{\mathrm{tr}} E_0^\wedge \left(B\Sigma_{\Sigma_{m_i-1}}^p \right)$$

is zero. and so is

$$E_0 \xrightarrow{f_2} \mathbb{T}_{m_2} E_0 \xrightarrow{\mathbb{T}_{m_2} f_1} \mathbb{T}_{m_2} \mathbb{T}_{m_1} E_0 \xrightarrow{\mathbb{T}_{m_2} \mathrm{tr}} \mathbb{T}_{m_2} \left(E_0^\wedge B\Sigma_{m_1-1}^p \right) = E_0^\wedge \left(B\Sigma_{m_1-1}^p \wr \Sigma_{m_2} \right)$$

is zero. By naturality of the transfer map, the diagram

$$
\begin{array}{ccc}
E_0 \xrightarrow{ f_2 } \mathbb{T}_{m_2} E_0 \xrightarrow{ \mathbb{T}_{m_2} f_1 } & \mathbb{T}_{m_2} \mathbb{T}_{m_1} E_0 \\
{\scriptstyle \mathrm{tr}} \downarrow & \downarrow {\scriptstyle \mathrm{tr}} \\
E_0^\wedge B\Sigma_{m_2-1}^p \longrightarrow E_0^\wedge \left(B\Sigma_{m_1}^{m_2} \right)_{h\Sigma_{m_2-1}^p} = E_0^\wedge \left(B\Sigma_{m_1}^{m_2} \rtimes \Sigma_{m_2-1}^p \right)
\end{array}
$$

commutes, where the transfers come from $\mathbb{T}_{m_2}$ and the bottom map is induced by f_1. Since $f_2 \in \Gamma[k_2]$, $\mathrm{tr} \circ f_2 = 0$, so $\mathrm{tr} \circ \mathbb{T}_{m_2} f_1 \circ f_2 = 0$.

Thus, $j(f_2 \otimes f_1) \in E_0^\wedge \left(B\Sigma_{m_1} \wr \Sigma_{m_2} \right)$ is in the kernel of transfers to $E_0^\wedge \left(B\Sigma_{m_1-1}^p \wr \Sigma_{m_2} \right)$ and $E_0^\wedge \left(B\Sigma_{m_1}^{m_2} \rtimes \Sigma_{m_2-1}^p \right)$. But $Q'_k \left(B\Sigma_{m_1} \wr \Sigma_{m_2} \right)^\vee$ is precisely the kernel of these transfer maps, so j factors as $\Gamma[k_2]_{E_0} \otimes_{E_0} \Gamma[k_1] \to Q'_k \left(B\Sigma_{m_1} \wr \Sigma_{m_2} \right)^\vee \to \mathbb{T}_{m_2} \mathbb{T}_{m_1} E_0$. The same holds for general q.

Dualizing gives the commutative diagram

$$
\begin{array}{ccc}
\Gamma^\vee[k_q]_t \otimes \cdots \otimes_s \Gamma^\vee[k_1] & \longleftarrow & (\mathbb{T}_{m_q} \cdots \mathbb{T}_{m_1} E_0)^\vee = E^0 \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right) \\
\uparrow & \swarrow & \\
Q'_k \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right) & &
\end{array}
\quad .
$$

The top horizontal map is surjective since it is a projection onto a direct summand by Lemma 3.4.5, hence so is the vertical map. The source and target of the vertical map are both isomorphic to $\Delta[k_q]_{E_0} \otimes \cdots \otimes_{E_0} \Delta[k_1]$, so it is a surjection of finite free E_0-modules of the same rank, therefore an isomorphism. $\square$

Proposition 3.4.7. *For $m_i = p^{k_i}$, the projection from $(\mathbb{T}_{m_q} \cdots \mathbb{T}_{m_1} E_0)^\vee = E^0 \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ induces an isomorphism*

$$Q'_k \left(\Sigma_m / \Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right) \xrightarrow{\simeq} \Gamma^\vee[k_q]_t \otimes_{E_0} \cdots \otimes_{E_0\,s} \Gamma^\vee[k_1]$$

of E_0-algebras.

Proof. $Q'_k \left(\Sigma_m / \Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ is a quotient ring of $E^0 \left(B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q} \right)$ with the cohomology ring structure coming from the space level diagonal $X \to X \times X$ where $X = B\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_q}$. The ring structure on $\Gamma^\vee[k_q]_t \otimes \cdots \otimes_s \Gamma^\vee[k_1]$ is given as a tensor product of rings $\Gamma^\vee[k_i]$ classifying subgroups of $\mathbb{G}$. We claim that these two ring structures are the same.

The case $q = 1$ is clear since both $Q'_k(\Sigma_m / \Sigma_m)$ and $\Gamma^\vee[k]$ are quotients of $E^0(B\Sigma_m)$ by the transfer. We will do the case $q = 2$. The general case is analogous.

The cohomology and tensor product ring structures are induced by the diagonal Δ and $\Delta_{k_1}, \Delta_{k_2}$, respectively. They fit into the commutative diagram below, where the right diagonal map is an equivalence.

$$B\Sigma_{m_1} \wr \Sigma_{m_2} = B(\Sigma_{m_1})^{m_2}_{h\Sigma_{m_2}} \xrightarrow{\quad\quad\quad\Delta\quad\quad\quad} B(\Sigma_{m_1})^{m_2}_{h\Sigma_{m_2}} \times B(\Sigma_{m_1})^{m_2}_{h\Sigma_{m_2}}$$

with the maps $\Delta_{k_1}, \Delta_{k_2}$ going to $(B\Sigma_{m_1} \times B\Sigma_{m_1})^{m_2}_{h\Sigma_{m_2} \times \Sigma_{m_2}}$.

Thus, these two ring structures are the same and we get an isomorphism of E_0-algebras. □

Remark 3.4.8. In fact, since the map $E^0(B\Sigma_{p^k}) \to \pi_0(E^{t\mathcal{T}_k})$ is the localization map inverting the Euler class c_{p^k}, this is consistent with [Str98, Thm 8.6] which shows that $\Gamma^\vee[k]$ is the quotient of $E^0(B\Sigma_{p^k})$ by the annihilator of c_{p^k}.

Proposition 3.4.9. *There is an isomorphism*

$$Q'_k(\overline{P}{}^\vee_m) \xrightarrow{\simeq} \overline{\mathcal{B}}(\Gamma^\vee)[k]$$

of cosimplicial E_0-algebras.

Proof. By Proposition 3.4.7, the two cosimplicial objects are componentwise isomorphic. Since the codegeneracy maps are identities, it remains to show that these isomorphisms are compatible with the coface maps. The multiplication $\Gamma[k_1] \otimes \Gamma[k_2] \to \Gamma[k_1 + k_2]$ is induced by $\mathbb{T}_{p^{k_2}} \mathbb{T}_{p^{k_1}} E \to \mathbb{T}_{p^{k_1 + k_2}} E$, which is $E^\wedge_0(\mathrm{res}) : E^\wedge_0 B(\Sigma_{p^{k_1}} \wr \Sigma_{p^{k_2}}) \to E^\wedge_0 B\Sigma_{p^{k_1 + k_2}}$. Taking duals tells us that the coface maps of $\overline{\mathcal{B}}(\Gamma^\vee)[k]$ are given by $E^0(\mathrm{res})$, which are also the coface maps of $Q'_k(\overline{P}{}^\vee_m)$. □

Remark 3.4.10. Propositions 3.4.7 and 3.4.9 can be modified for Γ^{-q} by

$$Q'_k \left((\mathbb{S}^q)^{\otimes p^k} \otimes \Sigma_{p^k} / \Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}} \right) \xrightarrow{\simeq} (\Gamma^{-q})^\vee[k_q] \otimes \cdots \otimes (\Gamma^{-q})^\vee[k_1]$$

and

$$Q'_k((\mathbb{S}^q)^{\otimes p^k} \otimes \overline{P}{}^\vee_m) \xrightarrow{\simeq} \overline{\mathcal{B}}((\Gamma^{-q})^\vee)[k].$$

Notation 3.4.11. Since we will no longer invoke Rezk's transitive E-homology Q_k, from now on, we will set $Q_k := Q'_k$ (Def 3.2.2).

4 Power operations modulo Lubin-Tate parameters

We will construct an analog $\Gamma^{(i)}$ of Γ which act on $\pi_0(-/u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras. The naive definition Γ/u_{i-1} does not work since Γ parameterizes operations acting on π_0 which are additive but not E_0-linear in general. However, some of these operations do descend to operations on $\pi_0(-/u_{i-1})$. In [Hah22], Hahn considers a subset of weight p operations which are defined modulo $p, \cdots, u_{i-1}$. We will define $\Gamma^{(i)}$ to be the algebra of operations generated by these. This is a reasonable definition since $\Gamma = \Gamma^{(0)}$ is Koszul so it is generated by the weight p operations $\Gamma[1]$.

We will first define $\Gamma^{\vee(i)} = \bigoplus_{k \geq 0} \Gamma^{\vee(i)}[k]$ and show that each $\Gamma^{\vee(i)}[k]$ is a finite free left E_0/u_{i-1}-module. We then set $\Gamma^{(i)} = \bigoplus_{k \geq 0} \Gamma^{(i)}[k]$ to be its left E_0/u_{i-1}-dual. As in the case of $\Gamma^\vee$, $\Gamma^{\vee(i)}$ can also be described as the image in the Tate construction. This will allow us to replicate Rezk's proof (2.3) to show that $\Gamma^{(i)}$ is Koszul of length $h - i + 1$.

4.1 Constructing $\Gamma^{\vee(i)}$

We will define $\Gamma^{(i)}[1]$ to consist of certain operations in $\Gamma[1]$ considered in [Hah22] which are defined mod $p, \cdots, u_{i-1}$ and set $\Gamma^{(i)}$ to be the subquotient of Γ generated by these.

We introduce some notations which will be used throughout this paper. We fix a p-typical complex orientation on E and let $t \in E^0(\mathbb{CP}^\infty)$ be the orientation class pushed down to degree 0. The Euler class of the reduced complex standard representation of Σ_p is given by

$$x = \prod_{i=1}^{p-1} [i](t) \equiv \prod_{i=1}^{p-1} [\omega^i](t) \mod [p](t) = (-1)^p t^{p-1}$$

where ω is a primitive $(p-1)$st root of unity. We denote by $\langle p \rangle(t) = [p](t)/t$ the reduced p-series, so $E^0(BC_p)/\mathrm{tr} \cong E_0[[t]]/\langle p \rangle(t)$. Since the logarithm of a p-typical formal group law only has nonzero terms in degrees which are powers of p, by the Lagrange Inversion Formula, its exponential only has nonzero terms in degrees congruent to $1 \mod (p-1)$, hence its reduced p-series only has nonzero terms in degrees divisible by $p - 1$. Since $[p](t) \mod p, \cdots, u_{i-1}$ is of the form $g(t^{p^i})$, the p-series satisfies

$$[p](t) \equiv u_i t^{p^i} \mod p, \cdots, u_{i-1}, t^{p^i+1}.$$

Definition 4.1.1. For $0 \leq i \leq h$, let $\langle p \rangle^{(i)}(t)$ denote the power series such that $t^{p^i} \langle p \rangle^{(i)}(t)$ is obtained from $[p](t)$ by setting $p, \cdots, u_{i-1} = 0$ [Hah22, Definition 2.6]. Let $f^{(i)}(x)$ denote the power series such that $\langle p \rangle^{(i)}(t) = f^{(i)}(x)$, so $f^{(i)}(x)$ is of the form $g(x^{p^i})$. Let $f(x) = f^{(0)}(x)$, so $\Gamma^\vee[1] = E^0(B\Sigma_p)/\mathrm{tr} \cong E_0[[x]]/f(x)$.

Theorem 3.3 in [Hah22] and unpublished work by David Jongwon Lee shows that for a $K(h)$-local $\mathbb{E}_\infty$-E-algebra R, the total additive power operations descends to a map on $\pi_0(R/p, \cdots, u_{i-1})$

$$
\begin{array}{ccc}
\pi_0 R & \xrightarrow{\quad P \quad} & \pi_0\left(R^{BC_p}/\langle p \rangle(t)\right) \\
\downarrow & & \downarrow \\
\pi_0(R/p, \cdots, u_{i-1}) & \xrightarrow{\quad P \quad} & \pi_0\left(R^{BC_p}/p, \cdots, u_{i-1}, \langle p \rangle^{(i)}(t)\right).
\end{array}
\tag{4}
$$

There is an analogous diagram with C_p replaced by Σ_p, through which (4) factors.

Definition 4.1.2. Let $\Gamma^{\vee(i)}[0] = E_0/u_{i-1}$. Let

$$\Gamma^{\vee(i)}[1] = \pi_0\left(R^{B\Sigma_p}/u_{i-1}, f^{(i)}(x)\right) = E_0[[x]]/u_{i-1}, f^{(i)}(x).$$

Thus, $\Gamma^{\vee(i)}[1]$ is a quotient of $\Gamma^{\vee}[1] = E_0[[x]]/f(x)$ and parametrizes a subset of power operations which are defined modulo $p, \cdots, u_{i-1}$. The source and target maps $s, t : E \to \Gamma^{\vee}[1]$ descend to maps $E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]$, which will also be denoted by s and t. Indeed, these are the inclusion and power operation map P in (4) which give the left and right E_0/u_{i-1}-module structures on $\Gamma^{\vee(i)}[1]$.

Consider

$$\Gamma^{\vee(i)}[1]^{\otimes k} = \Gamma^{\vee(i)}[1]_t \otimes_s \cdots _t \otimes_s \Gamma^{\vee(i)}[1],$$

where the tensor products are taken over E_0/u_{i-1}.

Definition 4.1.3. For $k \geq 1$, let $\Gamma^{\vee(i)}[k]$ be the image of $\Gamma^{\vee}[k]$ under the composite of the restriction and projection maps

$$\Gamma^{\vee(i)}[k] = \mathrm{im}\left(\Gamma^{\vee}[k] \hookrightarrow \Gamma^{\vee}[1]^{\otimes k} \twoheadrightarrow \Gamma^{\vee(i)}[1]^{\otimes k}\right).$$

Since both the restriction and projection maps are maps of E_0-algebras, $\Gamma^{\vee(i)}[k]$ is an E_0-algebra. In fact, as a subalgebra of $\Gamma^{\vee(i)}[1]^{\otimes k}$, it is an E_0/u_{i-1}-algebra. The co-E_0-algebra structure on $\Gamma^{\vee}$ descends to a co-E_0/u_{i-1}-algebra structure on $\Gamma^{\vee(i)} = \bigoplus_{k \geq 0} \Gamma^{\vee(i)}[k]$.

4.2 $\Gamma^{\vee(i)}$ via the generalized Tate construction

In order to use Rezk's proof to show that $\Gamma^{(i)}$ is Koszul, we need to express $\Gamma^{(i)}$ in terms of a Mackey functor $Q^{(i)}$. As in Section 3, this will be done using the generalized Tate construction. The Tate construction $(E \wedge X)^{t\mathcal{T}_k}$ has an E-module structure induced by the usual E-module structure on $(E \wedge X)^{h\Sigma_{p^k}}$. This allows us to take quotients by elements of E.

Definition 4.2.1. For each k, let $Q_k^{(i)} : \mathrm{Sp}^{B\Sigma_{p^k}} \to \mathrm{Mod}_{E_0/u_{i-1}}$ be the image of $\pi_0(E \wedge -)^{h\Sigma_{p^k}}$ in the Tate construction mod $p, \cdots, u_{i-1}$

$$Q_k^{(i)}(X) = \mathrm{im}\left(\pi_0(E \wedge X)^{h\Sigma_{p^k}} \to \pi_0\left((E \wedge X)^{t\mathcal{T}_k}/u_{i-1}\right)\right).$$

Note that $Q_k^{(0)} = Q_k$ and that $Q_k^{(i)}$ is equipped with the structure of a Σ_{p^k} Mackey functor. We will proof an analogs of Propositions 3.4.7 and 3.4.9 for $\Gamma^{\vee(i)}$ and $Q^{(i)}$.

Lemma 4.2.2. *The isomorphism $Q_1(\Sigma_p/\Sigma_p) \xrightarrow{\cong} \Gamma^{\vee}[1]$ of Prop 3.4.7 induces an isomorphism*

$$Q_1^{(i)}(\Sigma_p/\Sigma_p) \xrightarrow{\cong} \Gamma^{\vee(i)}[1]$$

of E_0/u_{i-1}-algebras.

Proof. Since the canonical map

$$E_0[[x]]/f(x) \cong E^0(B\Sigma_p)/\mathrm{tr} \to \pi_0 E^{t\Sigma_p} \cong x^{-1}E_0[[x]]/f(x)$$

is a localization, $\pi_0 E^{t\Sigma_p}$ is flat over E_0, so $\pi_0(E^{t\Sigma_p}/u_{i-1}) \cong (\pi_0 E^{t\Sigma_p})/u_{i-1}$. Thus, $Q_1^{(i)}(\Sigma_p/\Sigma_p) \cong \Gamma^{\vee(i)}[1]$. $\square$

Lemma 4.2.3. *Let $k = k_1 + k_2$. Under the composite*

$$E^0(B\Sigma_{p^k}) \to E^0\left(B\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}}\right) \to \Gamma^\vee[k_1]_t \otimes_s \Gamma^\vee[k_2],$$

the Euler class c_{p^k} maps to $c_{p^{k_1}} \otimes c_{p^{k_2}}$.

Proof. The restriction of the reduced standard representation $\bar{\rho}_k$ of Σ_{p^k} to $\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}}$ splits as $N\bar{\rho}_{k_2} \oplus \bar{\rho}_{k_1}$, so $c_{p^k} = Nc_{p^{k_2}} \cdot c_{p^{k_1}}$ in $E^0\left(B\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}}\right)$. Here N is the norm $N_{\Sigma_{p^{k_2}}}^{\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}}}$ and $\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}}$ acts on $\bar{\rho}_{k_1}$ by the projection to $\Sigma_{p^{k_1}}$. We will show that for all $f \in \Gamma[k_1]$ and $g \in \Gamma[k_2]$, $(Nc_{p^{k_2}} \cdot c_{p^{k_1}})(g \circ f) = (c_{p^{k_1}} \otimes c_{p^{k_2}})(g \circ f)$.

Recall (2.3) that our notation swaps the left and right E_0-module structures on Γ of [Rez09]. As the dual to a graded bialgebra, $\Gamma^\vee$ has comultiplication

$$c^* : \Gamma^\vee[k] \to \Gamma^\vee[k_1]_t \otimes_s \Gamma^\vee[k_2].$$

We need to show that

$$c^*(c_{p^k})(g \circ f) = (Nc_{p^{k_2}} \cdot c_{p^{k_1}})(g \circ f) = (c_{p^{k_1}} \otimes c_{p^{k_2}})(g \circ f) = c_{p^{k_1}}\left(c_{p^{k_2}}(g) \circ f\right).$$

Note that for $f \in \Gamma[k]$ and $\alpha \in \Gamma^\vee[k]$, $\alpha(f)$ is the composite

$$\alpha(f) : E_0 \xrightarrow{\tilde{f}} \mathbb{T}_m E_0 \xrightarrow{\alpha} E_0.$$

- $c_{p^{k_1}}\left(c_{p^{k_2}}(g) \circ f\right)$ is the composite

$$E_0 \xrightarrow{\tilde{f}} \mathbb{T}_{m_1} E_0 \xrightarrow{\mathbb{T}_{m_1} \tilde{g}} \mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \xrightarrow{\mathbb{T}_{m_1} c_{p^{k_2}}} \mathbb{T}_{m_1} E_0 \xrightarrow{c_{p^{k_1}}} E_0.$$

- $c_{p^{k_1}} \in E^0(B\Sigma_{p^{k_2}} \wr \Sigma_{p^{k_1}})$ is $\mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \to \mathbb{T}_{m_1} E_0 \xrightarrow{c_{p^{k_1}}} E_0$, where the first map is projection and the second is $c_{p^{k_1}} \in E^0(\Sigma_{p^{k_1}})$. For a complex representation V of a group G, the Σ_p-norm of the G-equivariant Euler class $e_V : S^0 \to S^V$ is the $G \wr \Sigma_p$-equivariant Euler class $Ne_V : S^0 \to S^{V \otimes p}$ with the usual action of $G \wr \Sigma_p$ on $V \otimes p$. Since E is complex oriented, there is an isomorphism $E \otimes S^{V \otimes p} \simeq E \otimes S^{p|V|}$ of $G \wr \Sigma_p$-spectra, so the normed Euler class is the $G \wr \Sigma_p$-equivariant map $Ne_V : S^0 \to S^{p|V|}$. The corresponding G-equivariant map factors through the counit

$$(S^0)^{\otimes p}_{h\Sigma_p} \to S^{p|V|}_{h\Sigma_p} \to S^{p|V|},$$

hence the corresponding nonequivariant map factors as

$$\left((S^0)^{\otimes p}_{h\Sigma_p}\right)_{hG} \to S^{p|V|}_{h\Sigma_p} \to S^{p|V|}.$$

In our case, since E is 2-periodic, this implies that $Nc_{p^{k_2}}$ is the composite

$$Nc_{p^{k_2}} : \mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \xrightarrow{\mathbb{T}_{m_1} c_{p^{k_2}}} \mathbb{T}_{m_1} E_0 \to E_0.$$

Since the product in $(\mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0)^\vee$ is given by the diagonal, $Nc_{p^{k_2}} \cdot c_{p^{k_1}}$ is

$$\mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \xrightarrow{\Delta} \mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \otimes \mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \xrightarrow{\mathbb{T}_{m_1} c_{p^{k_2}} \otimes \text{proj}} \mathbb{T}_{m_1} E_0 \otimes \mathbb{T}_{m_1} E_0 \xrightarrow{\text{proj} \otimes c_{p^{k_1}}} E_0$$

where proj corresponds to the unit $1 \in E^0(B\Sigma_{m_i})$. This is equal to

$$\mathbb{T}_{m_1} \mathbb{T}_{m_2} E_0 \xrightarrow{\mathbb{T}_{m_1} c_{p^{k_2}}} \mathbb{T}_{m_1} E_0 \xrightarrow{c_{p^{k_1}}} E_0.$$

$\square$

Lemma 4.2.4. *For each k, the isomorphism $Q_k(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p) \xrightarrow{\cong} \Gamma^\vee[1]^{\otimes k}$ induces an isomorphism*

$$Q_k^{(i)}(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_k) \xrightarrow{\cong} \Gamma^{\vee(i)}[1]^{\otimes k}$$

of E_0/u_{i-1}-algebras.

Proof. For $H \leq G = \Sigma_{p^k}$, the map into the Tate construction

$$E^0(BH) \to \pi_0(E \wedge G/H)^{t\mathcal{T}_k}$$

is the localization map inverting the restriction to $E^0(BH)$ of the Euler class $c_{p^k} \in E^0(BG)$ of the reduced complex standard representation of Σ_{p^k}. In particular, it is flat, so

$$\pi_0\left((E \wedge G/H)^{t\mathcal{T}_k}/u_{i-1} \right) \cong \left(\pi_0 (E \wedge G/H)^{t\mathcal{T}_k} \right)/u_{i-1}. \tag{5}$$

Under the isomorphism $Q_k'(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p) \xrightarrow{\cong} \Gamma^\vee[1]^{\otimes k}$, (the restriction of) c_{p^k} corresponds to $c_p^{\otimes k}$ (Lemma 4.2.3), where c_p is the Euler class of the reduced complex standard representation of Σ_p. Under this identification, the map from $Q_k'(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)$ into the Tate construction corresponds to the localization map inverting $c_p^{\otimes k}$. Explicitly, we have the top half of the following commutative diagram in E_0-algebras, where the dotted map is an isomorphism.

$$
\begin{array}{ccccc}
E^0(B\Sigma_p \wr \cdots \wr \Sigma_p) & \longrightarrow & Q_k'(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p) & \longhookrightarrow & \pi_0\left((E \wedge \Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)^{t\mathcal{T}_k}/u_{i-1} \right) \\
& & \simeq\downarrow & & \vdots\downarrow \\
& & \Gamma^\vee[1]^{\otimes k} & \longhookrightarrow & (\pi_0 E^{t\Sigma_p})^{\otimes k} \\
& & \downarrow & & \downarrow \\
& & \Gamma^{\vee(i)}[1]^{\otimes k} & \longhookrightarrow & (\pi_0 E^{t\Sigma_p}/u_{i-1})^{\otimes k}
\end{array}
$$

By (5),

$$
\begin{aligned}
\pi_0\left((E \wedge \Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)^{t\mathcal{T}_k}/u_{i-1} \right) &\cong \left(\pi_0(E \wedge \Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)^{t\mathcal{T}_k} \right)/u_{i-1} \\
&\cong (\pi_0 E^{t\Sigma_p})^{\otimes k}/u_{i-1} \\
&\cong (\pi_0 E^{t\Sigma_p}/u_{i-1})^{\otimes k},
\end{aligned}
$$

where the last isomorphism is Corollary 4.2.6 below. The map $\Gamma^{\vee(i)}[1]^{\otimes k} \to (\pi_0 E^{t\Sigma_p}/u_{i-1})^{\otimes k}$ is injective since it is a tensor product of injective maps where $\Gamma^{\vee(i)}[1]$ is a flat left and right E_0/u_{i-1}-module (Lemma 4.3.3) and $\pi_0 E^{t\Sigma_p}/u_{i-1}$ is a flat left E_0/u_{i-1}-module. These give the bottom half of the diagram.

Thus, the image of $Q_k'(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)$ in $\pi_0\left((E \wedge \Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)^{t\mathcal{T}_k}/u_{i-1} \right)$ is the same as its image in $(\pi_0 E^{t\Sigma_p}/u_{i-1})^{\otimes k}$. But the map from $Q_k'(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)$ to the latter factors through the surjection $\Gamma^\vee[1]^{\otimes k} \twoheadrightarrow \Gamma^{\vee(i)}[1]^{\otimes k}$, so its image is isomorphic to $\Gamma^{\vee(i)}[1]^{\otimes k}$. $\square$

Lemma 4.2.5.
$$\pi_0 E^{t\Sigma_p}/u_{i-1} \cong \pi_0 E^{t\Sigma_p}/\!\!/ u_{i-1}$$

where $/\!\!/$ is the quotient with respect to the right E_0-module structure.

Proof. The source and target maps $s, t : E_0 \to E^0(B\Sigma_p)$ give two E_0-algebra structures on $\pi_0 E^{t\Sigma_p}$ via the canonical map $E^{h\Sigma_p} \to E^{t\Sigma_p}$. The target map t is the composite

$$E_0 \xrightarrow{P} E^0(B\Sigma_p) \to x^{-1}E^0(B\Sigma_p) = \pi_0 E^{t\Sigma_p} \to t^{-1}E^0(BC_p) = \pi_0 E^{tC_p},$$

which is π_0 of the Nikolaus-Scholze Frobenius, denoted by φ in [Hah22].

[Hah22, Lemma 2.5] shows that

$$\varphi(u_i) \equiv \text{unit} \cdot u_i \mod p, \cdots, u_{i-1}$$

in $t^{-1}E^0(BC_p)$, hence also in $x^{-1}E^0(B\Sigma_p)$. Thus, these two E_0-algebra structures differ by units and it makes no difference whether we take the quotient by $p, \cdots, u_{i-1}$ with respect to s or t. $\qquad\square$

This lets us pull the quotient inside the tensor product.

Corollary 4.2.6.
$$(\pi_0 E^{t\Sigma_p})^{\otimes_k}/u_{i-1} \cong (\pi_0 E^{t\Sigma_p}/u_{i-1})^{\otimes_k}.$$

We obtain an analog of Prop 3.4.7.

Lemma 4.2.7. *For $k_1 + \cdots + k_q = k$, the isomorphism $Q_k(\Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}) \xrightarrow{\simeq} \Gamma^\vee[k_q]_t \otimes \cdots \otimes_s \Gamma^\vee[k_1]$ induces an isomorphism*

$$Q_k^{(i)}(\Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}}) \xrightarrow{\simeq} \Gamma^{\vee(i)}[k_q]_t \otimes \cdots \otimes_s \Gamma^{\vee(i)}[k_1]$$

of E_0/u_{i-1}-algebras. In particular, $\Gamma^{\vee(i)}[k] \cong Q_k^{(i)}(\Sigma_{p^k}/\Sigma_{p^k})$.

Proof. By definition, $\Gamma^{\vee(i)}[k]$ is the image of $\Gamma^\vee[k]$ in $\Gamma^{\vee(i)}[1]^{\otimes_k}$. This implies that $\Gamma^{\vee(i)}[k_q]_t \otimes \cdots \otimes_s \Gamma^{\vee(i)}[k_1]$ is the image of $\Gamma^\vee[k_q]_t \otimes \cdots \otimes_s \Gamma^\vee[k_1]$ in $\Gamma^{\vee(i)}[1]^{\otimes_k}$. $Q_k^{(i)}(\Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}})$ is the image of $E^0(B\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}})$ in $\pi_0\left((E \wedge \Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}})^{t\mathcal{T}_k}/u_{i-1}\right)$. We will show that this is the same as its image in $\Gamma^{\vee(i)}[1]^{\otimes_k}$.

For any Mackey functor M, the restriction and transfer maps exhibit $M(\Sigma_{p^k}/\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_q}})$ as a direct summand of $M(\Sigma_{p^k}/\Sigma_p \wr \cdots \wr \Sigma_p)$. Thus, the result follows from the case $k_1 = \cdots = k_q = 1$, which is Lemma 4.2.4. $\qquad\square$

Putting these together gives the following analog of Prop 3.4.9.

Proposition 4.2.8. *The isomorphism $Q_k(\overline{P}_m^\vee) \xrightarrow{\simeq} \overline{\mathcal{B}}(\Gamma^\vee)[k]$ induces an isomorphism*

$$Q_k^{(i)}(\overline{P}_m^\vee) \xrightarrow{\simeq} \overline{\mathcal{B}}(\Gamma^{\vee(i)})[k]$$

of cosimplicial E_0/u_{i-1}-algebras.

4.3 Some facts about $\Gamma^{\vee(i)}$

We collect some facts about $\Gamma^{\vee(i)}$. In particular, we will show that $\Gamma^{\vee(i)}[k]$ is finite free as a left and right E_0/u_{i-1}-module, so taking duals does no harm.

Lemma 4.3.1. *In $\Gamma^{\vee(i)}[1]$, we have*

$$x^{p^i-1}P(u_i) = u_i.$$

23

Proof. By [Hah22, Lemma 2.5],

$$\varphi(u_i) \equiv \left(\frac{\varphi(u)}{u} \right)^{p^i-1} u_i \mod p, \cdots, u_{i-1}$$

in $t^{-1}E^0(BC_p)$, where φ is the Nikolaus-Scholze Frobenius $\varphi : E_0 \to t^{-1}E^0(BC_p)$. [Hah22, Proposition 4.5] shows that

$$\varphi(u) = \frac{u}{x}.$$

Thus, $x^{p^i-1}\varphi(u_i) \equiv u_i \mod p, \cdots, u_{i-1}$ in $t^{-1}E^0(BC_p)$, so $x^{p^i-1}P(u_i) = u_i$ in $\Gamma^{\vee(i)}[1]$. $\qquad\square$

Lemma 4.3.2. *The quotient with respect to the right E_0/u_{i-1}-module structure*

$$\Gamma^{\vee(i)}[1] /\!\!/ u_{i-1} := \Gamma^{\vee(i)}[1]_t \otimes_{E_0/u_{i-1}} {}_s E_0/u_i = \Gamma^{\vee(i)}[1]/x\tilde{f}^{(i+1)}(x)$$

for some power series $\tilde{f}^{(i+1)}(x) \equiv f^{(i+1)}(x) \mod p, \cdots, u_i$.

Here the source map $s : E_0/u_{i-1} \to E_0/u_i$ is the usual quotient map.

Proof. Modulo $p, \cdots, u_{i-1}$, $\langle p \rangle(t) = g(t^{p^i})$ for some power series g, so $f^{(i)}(x) = \tilde{g}(x^{p^i}) = u_i - x^{p^i}\tilde{f}^{(i+1)}(x)$ for some $\tilde{g}$ and $\tilde{f}^{(i+1)}$. So, in $\Gamma^{\vee(i)}[1]$, $u_i = x^{p^i}\tilde{f}^{(i+1)}(x)$. Since $f^{(i)}(x) \equiv x^{p^i}f^{(i+1)}(x) \mod p, \cdots, u_i$,

$$\tilde{f}^{(i+1)}(x) \equiv f^{(i+1)}(x) \mod p, \cdots, u_i.$$

By Lemma 4.3.1, in $\Gamma^{\vee(i)}[1]$, $x^{p^i-1}P(u_i) = u_i = x^{p^i}\tilde{f}^{(i+1)}(x)$. Since $\Gamma^{\vee(i)}[1]$ is an integral domain (in fact a regular local ring),

$$P(u_i) = x\tilde{f}^{(i+1)}(x). \tag{6}$$

Thus,

$$\Gamma^{\vee(i)}[1] /\!\!/ u_i = \Gamma^{\vee(i)}[1]/x\tilde{f}^{(i+1)}(x).$$

$\qquad\square$

Lemma 4.3.3. *The target map $t : E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]$ is finite flat, thus $\Gamma^{\vee(i)}[1]$ is a finite free right E_0/u_{i-1}-module.*

Proof. Strickland [Str97, Proposition 13.1]) shows that $t : E_0 \to \Gamma^{\vee}[1]$ is finite flat, thus $t : E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]$ is a finite map and it remains to show that it is flat. Suppose by induction that $t : E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]$ is flat. We will show that $t : E_0/u_i \to \Gamma^{\vee(i+1)}[1]$ is flat.
  Let $R = \Gamma^{\vee(i)}[1]$ and $f = \tilde{f}^{(i+1)}$. Then, $R/x = E_0/u_i$ and $R/f = \Gamma^{\vee(i+1)}[1]$. The map $t : E_0/u_i \to R/f$ factors through the pushout $\Gamma^{\vee(i)}[1]_t \otimes_{E_0/u_{i-1}} {}_s E_0/u_i = R/xf$

$$t : E_0/u_i \xrightarrow{\tilde{t}} R/xf \to R/f.$$

Consider R/xf and R/f as E_0/u_i-modules via $\tilde{t}$ and t, respectively. Since $\tilde{t}$ is the base change of a flat map, R/xf is a flat E_0/u_i-module.
  Consider $R/x \cong E_0/u_i$ as an E_0/u_i-module with the module structure coming from the Frobenius map $\varphi : e \mapsto e^p$. Since $E_0/u_i \cong k[[u_{i+1}, \cdots, u_{n-1}]]$ is free over $k[[u_{i+1}^p, \cdots, u_{n-1}^p]] \cong$

$\varphi(k)[[u_{i+1}^p, \cdots, u_{n-1}^p]]$, R/x is a flat E_0/u_i-module. By [Rez09, Proposition 3.25], the composite of the total power operation map followed by projection

$$E_0 \to E^0(B\Sigma_p) \to E_0$$

is the Frobenius. Passing to quotients, we get that the composite

$$E_0/u_i \xrightarrow{\tilde{t}} R/xf \to R/x$$

is also the Frobenius.

Thus, there is a short exact sequence of E_0/u_i-modules.

$$0 \to R/f \xrightarrow{\cdot x} R/xf \to R/x \to 0.$$

Indeed, since $\tilde{t}(e) - t(e) \equiv 0 \mod f$, $\tilde{t}(e)x = t(e)x$ in R/xf, so the map $R/f \to R/xf$ given by multiplication by x is a map of E_0/u_i-modules. Since R/xf and R/x are flat, so is R/f. $\square$

Lemma 4.3.4. *For $k = k_1 + k_2$, the map $\Gamma^{\vee(i)}[k] \to \Gamma^{\vee(i)}[k_1] \otimes \Gamma^{\vee(i)}[k_2]$ is an inclusion of a direct summand both as a left and right E_0/u_{i-1}-module.*

Proof. Let $R = \Gamma^{\vee(i)}[k]$ and $S = \Gamma^{\vee(i)}[k_1]_t \otimes_s \Gamma^{\vee(i)}[k_2]$. Since $\Gamma^{\vee(i)}$ can be defined in terms of the Mackey functor $Q_k^{(i)}$, the restriction and transfer maps exhibit R as an abelian group direct summand of S. We will show that these are left and right E_0/u_{i-1}-module maps, thus R is a left and right E_0/u_{i-1}-module direct summand of S. Since the source and target maps to S factor through the source and target maps to R followed by the restriction res : $R \to S$, res is a ring map which is both a map of left and right E_0/u_{i-1}-modules.

The transfer tr : $S \to R$ is a map of R-modules [Ada78, page 128] with the R-module structure on S given by res, thus it is a map of left E_0/u_{i-1}-modules. Since $\mathrm{tr}(t(e)s) = t(e)\mathrm{tr}(s)$ as $t(e) \in R \subset S$, tr is also a map of right E_0/u_{i-1}-modules. $\square$

Corollary 4.3.5. *For all $k \geq 0$, the source and target maps $s, t : E_0/u_{i-1} \to \Gamma^{\vee(i)}[k]$ are finite flat, thus $\Gamma^{\vee(i)}[k]$ is finite free as both a left and right E_0/u_{i-1}-module.*

Proof. The source and target maps $E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]^{\otimes k}$ are finite flat since they are compositions of finite flat maps. The statement follows from Lemma 4.3.4 above since a direct summand of a finite free module is finite free. $\square$

Lemma 4.3.6. *Let $u \in (E_0/u_{h-2})^\times$ be the coefficient of t^{p^h} in the p-series $[p](t)$. Then*

$$\Gamma^{\vee(h-1)}[1] \cong (E_0/u_{h-2})[[x]]/u_{h-1} - ux^{p^{h-1}}$$

is a regular local ring and $P(u_{h-1}) = ux$. Moreover,

$$\Gamma^{\vee(h-1)}[1]^{\otimes k} \cong \mathbb{F}_p[[u_{h-1}]][[x_1, \cdots, x_k]]/u_{h-1} - ux_1^{p^{h-1}}, x_1 - x_2^{p^{h-1}}, \cdots, x_{k-1} - x_k^{p^{h-1}}$$

is a regular local ring with maximal ideal generated by x_k.

Proof. Since $\Gamma^{\vee(h-1)}[1]$ is a E_0/u_{h-2}-algebra, its formal group has height $\geq h-1$ and $[p](t) = g(t^{p^{h-1}}) \mod p, \cdots, u_{h-2}$ for some power series $g(t) = u_{h-1}t + t^2(\cdots)$. Since we chose a p-typical coordinate, $[p](y)/y$ only has nonzero terms in degrees divisible by $p-1$, so

$$g(t) = u_{h-1}t + ut^p + t^{2p-1}(\cdots) = (u_{h-1}t + ut^p)(1 + t^{2p-2}(\cdots))$$

25

for some unit u and

$$f^{(h-1)}(x) = u_{h-1} - ux^{p^{h-1}},$$

giving the formula for $\Gamma^{\vee(h-1)}[1]$. Note that $x = t$ if $p = 2$ and $x = -t^{p-1}$ if $p > 2$ but the signs do not matter when $p = 2$ since things are only nontrivial when $h \geq 2$ in which case E_0/u_{h-2} is an algebra over a field of characteristic p.

By Lemma 4.3.1, $x^{p^{h-1}-1}P(u_{h-1}) = u_{h-1} = ux^{p^{h-1}}$. Since $\Gamma^{\vee(h-1)}[1]$ is a regular local ring hence an integral domain, $P(u_{h-1}) = ux$. The last statement follows from these. $\square$

Lemma 4.3.7. $\Gamma^{\vee(i)}[k]$ *is an integral domain.*

Proof. $k = 0$ is trivial. For $k = 1$, $\Gamma^{\vee(i)}[1] \cong E_0/u_{i-1}[[x]]/f^{(i)}(x)$ is a regular local Noetherian ring hence an integral domain. For general k, there are inclusions of rings

$$\Gamma^{\vee(i)}[k] \hookrightarrow \Gamma^{\vee(i)}[1]^{\otimes k} \hookrightarrow (\pi_0 E^{t\Sigma_p})^{\otimes k}/u_{i-1},$$

so it suffices to show that the latter is an integral domain.

A corollary to Theorem 23.9 in [Mat87] states that if A, B are Noetherian local rings and $A \to B$ is a flat local homomorphism, then B is normal if A and $B/\mathfrak{m}_A B$ are normal. We apply this to the source map $s : E_0/u_{i-1} \to (\pi_0 E^{t\Sigma_p})^{\otimes k}/p, \cdots, u_{i-1}$ which is flat since it is a localization of the flat map $s : E_0/u_{i-1} \to \Gamma^{\vee(i)}[1]^{\otimes k}$. $A = E_0/u_{i-1}$ and

$$B/\mathfrak{m}_A B = (\pi_0 E^{t\Sigma_p})^{\otimes k}/p, \cdots, u_{h-1} \cong \left(x^{-1}\Gamma^{\vee(n-1)}[1]\right)^{\otimes k}$$

are regular local rings, hence normal ($x^{-1}\Gamma^{\vee(h-1)}[1]$ is a field by Lemma 4.3.6). Thus, B is normal, hence an integral domain. $\square$

4.4 Defining $\Gamma^{(i)}$

We can now dualize $\Gamma^{\vee(i)}$ to define $\Gamma^{(i)}$. By Corollary 4.3.5, $\Gamma^{\vee(i)}[k]$ is a finite free left and right E_0/u_{i-1}-module. We define $\Gamma^{(i)}[k]$ to be its left E_0/u_{i-1}-dual and set $\Gamma^{(i)} = \bigoplus_{k \geq 0} \Gamma^{(i)}[k]$.

For an R-bimodule M, its left R-module dual $M^\vee$ has a left R-module structure given by $(r \cdot f)(m) = f(rm)$ and a right R-module structure given by $(f \cdot r)(m) = f(mr)$. Thus, $\Gamma^{(i)}[k]$ is an E_0/u_{i-1}-bimodule with left and right module structures induced by s and t, respectively. $\Gamma^{(i)}[k_1] \otimes \cdots \Gamma^{(i)}[k_q]$ is dual to $\Gamma^{\vee(i)}[k_1] \otimes \cdots \otimes \Gamma^{\vee(i)}[k_q]$ as left E_0/u_{i-1}-modules, where the tensor products are with respect to the left and right module structures. Composition of power operations

$$\Gamma^{(i)}[k_1] \otimes \cdots \otimes \Gamma^{(i)}[k_q] \to \Gamma^{(i)}[k_1 + \cdots k_q]$$

is given dually by

$$\Gamma^{\vee(i)}[k_1 + \cdots k_q] \to \Gamma^{\vee(i)}[k_1] \otimes \cdots \Gamma^{\vee(i)}[k_q],$$

which is induced by the corresponding map on $\Gamma^\vee$.

Proposition 4.4.1. $\Gamma^{(i)} = \bigoplus_{k \geq 0} \Gamma^{(i)}[k]$ *is a E_0/u_{i-1}-algebra of power operations acting on* $\pi_0(-/u_{i-1})$ *of $K(h)$-local $\mathbb{E}_\infty$-E-algebras. It is generated by weight p operations parameterized by $\Gamma^{\vee(i)}[1]$. An element $Q \in \Gamma^{(i)}[k] = (\Gamma^{\vee(i)}[k])^\vee$ gives an operation*

$$E_0/u_{i-1} \xrightarrow{P} \Gamma^{\vee(i)}[k] \xrightarrow{Q} E_0/u_{i-1}$$

by postcomposing with the total power operations map P. For each $k \geq 0$, $\Gamma^{\vee(i)}[k]$ is finite free as both a left and right E_0/u_{i-1}-module.

Remark 4.4.2. As in Remark 3.4.10, we can modify Definition 4.2.1 for nonzero degrees to obtain analogs $\Gamma^{(i)-q}$ of Γ^{-q}, which act on $\pi_q(-/u_{i-1})$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras.

4.4.1 Interpretation in terms of isogenies

Definition 4.4.3. For the universal formal group $\mathbb{G}$, let

$$\mathrm{Sub}_k^{(i)}(\mathbb{G}) = \mathrm{Spf}(\Gamma^{\vee(i)}[k]),$$

which is a closed subscheme of $\mathrm{Sub}_k(\mathbb{G})$. If R is an E_0/u_{i-1}-algebra via $\phi : E_0 \to E_0/u_{i-1} \to R$,

$$\mathrm{Sub}_k^{(i)}(\mathbb{G}_R) = \phi^*\mathrm{Sub}_k^{(i)}(\mathbb{G})$$

where $\mathbb{G}_R = \phi^*\mathbb{G}$. We say that a degree p^k subgroup $H < \mathbb{G}_R$ avoids $\ker \varphi^i$ if it is classified by $\mathrm{Sub}_k^{(i)}(\mathbb{G}_R)$.

This definition is consistent with the $i = 0$ case since $\mathrm{Sub}_k(\mathbb{G}_R) = \phi^*\mathrm{Sub}_k(\mathbb{G})$ for R an E_0-algebra [Str97]. The situation is simpler if we restrict to a fixed height by inverting u_i.

Lemma 4.4.4. $u_i^{-1}\mathrm{Sub}_k^{(i)}(\mathbb{G}) = \mathrm{Spf}(u_i^{-1}\Gamma^{\vee(i)}[k])$ *classifies degree p^k étale subgroups over* $u_i^{-1}E_0/u_{i-1}$-*algebras.*

Proof. If $i = 0$, inverting p makes everything étale, so this is clear. Thus, we will assume $i \geq 1$. After inverting u_i, our rings are no longer local, so we will need to work with p-divisible groups instead of formal groups.

The localization map $\Gamma^{\vee(i)}[k] \to u_i^{-1}\Gamma^{\vee(i)}[k]$ classifies the universal degree p^k subgroup H over the universal p-divisible group $\mathbb{G}$ over $u_i^{-1}\Gamma^{\vee(i)}[k]$. By Lemma 4.4.7, $c_{p^k} \mid u_i$, so c_{p^k} is invertible in $u_i^{-1}\Gamma^{\vee(i)}[k]$. Since H has equation $f_H(y) = c_{p^k}y + y^2(\cdots)$ with c_{p^k} a unit,

$$\mathcal{O}_H = A[y]/f_H(y) \cong A[y]/y \times A[y]/(f_H(y)/y)$$

where $A = u_i^{-1}\Gamma^{\vee(i)}[k]$, so the connected part of H is trivial, hence H is étale. Since étaleness is preserved under base change, the pullback of H under a map $u_i^{-1}\Gamma^{\vee(i)}[k] \to R$ is a degree p^k étale subgroup $H_R < \mathbb{G}_R$.

Conversely, we will show that any degree p^k étale subgroup $H_R < \mathbb{G}_R$ is classified by a map from $u_i^{-1}\Gamma^{\vee(i)}[k]$. H_R is classified by a map $\Gamma^\vee[k] \to R$ which factors through $u_i^{-1}\Gamma^\vee[k]/u_{i-1}$. $\Gamma^{\vee(i)}[k]$ is the image of of $\Gamma^\vee[k]/u_{i-1}$ in the Tate construction, which is the map inverting c_{p^k}. Since $c_{p^k} \mid u_i$ in $\Gamma^{\vee(i)}[k]$, inverting u_i inverts c_{p^k}. The following diagram summarizes the relationships between inverting u_i and c_{p^k}.

$$\begin{array}{ccccc}
\Gamma^\vee[k]/u_{i-1} & \longrightarrow\!\!\!\!\!\rightarrow & \Gamma^{\vee(i)}[k] & \xrightarrow{c_{p^k}^{-1}} & \pi_0 E^{t\mathcal{T}_k}/u_{i-1} \\
{\scriptstyle u_i^{-1}}\downarrow & & {\scriptstyle u_i^{-1}}\searrow & & \downarrow{\scriptstyle u_i^{-1}} \\
u_i^{-1}\Gamma^\vee[k]/u_{i-1} & \xrightarrow{\quad c_{p^k}^{-1} \quad} & & & u_i^{-1}\Gamma^{\vee(i)}[k].
\end{array} \qquad (7)$$

In particular, $u_i^{-1}\Gamma^{\vee(i)}[k] = u_i^{-1}c_{p^k}^{-1}\Gamma^\vee[k]/u_{i-1} = c_{p^k}^{-1}\left(u_i^{-1}\Gamma^\vee[k]/u_{i-1}\right)$. Since H is étale, the Kahler differentials

$$\Omega^1_{R[y]/f_H(y)} \simeq R[y]/(f_H, f_H')$$

27

vanish. Since $y \mid f_H(y)$, f'_H is a unit modulo y, so c_{p^k} is a unit in R. Thus, the map $u_i^{-1}\Gamma^\vee[k]/u_{i-1} \to R$ factors through $u_i^{-1}\Gamma^{\vee(i)}[k]$.

$\square$

Lemma 4.4.5. *Let $R \hookrightarrow S$ be a ring map splitting $S = R \oplus T$ as R-modules. Let $c, u \in R$. If $c|u$ in S, then $c|u$ in R.*

Proof. Let $u = cv$ in S. Since $S = R \oplus T$, $v = r + t$ for some $r \in R$ and $t \in T$, so $u = cr + ct$. $ct \in T$ since T is an R-module. Since $u, cr \in R$, $ct \in R$. Thus, $ct \in R \cap T = 0$, so $u = cr \in R$, hence $c|u$ in R. $\square$

Notation 4.4.6. If $f(t) = \sum c_i t^i \in R[[t]]$ is a power series and $\phi : R \to S$ is a ring map, we denote by f^ϕ the power series

$$f^\phi(t) = \sum \phi(c_i)t^i \in S[[t]]$$

obtained by applying ϕ to the coefficients of f.

Lemma 4.4.7. *For $i \geq 1$, c_{p^k} divides u_i in $\Gamma^{\vee(i)}[k]$.*

Proof. By Lemma 4.4.5, it suffices to show that c_{p^k} divides u_i in

$$\Gamma^{\vee(i)}[1]^{\otimes k} \cong E_0/u_{i-1}[[x_1, \cdots, x_k]]/f(x_1), f^P(x_2), \cdots, f^{P^{k-1}}(x_k).$$

From Lemma 4.3.1, $P^{s-1}(u_i) = x_s^{p^i-1}P^s(u_i)$ for $1 \leq s \leq k$, so

$$u_i = (x_1 \cdots x_k)^{p^i-1}P^k(u_i) = c_{p^k}{}^{p^i-1}P^k(u_i) \tag{8}$$

and $c_{p^k}|u_i$ in $\Gamma^{\vee(i)}[k]$ since $i \geq 1$. $\square$

Remark 4.4.8.
$$\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G}) = \mathrm{Sub}^{(i)}_{k_1}(\mathbb{G})_t \times_s \cdots_t \times_s \mathrm{Sub}^{(i)}_{k_q}(\mathbb{G})$$

classifies a sequence of subgroups $(H_1, \cdots, H_q)$ where $H_1 < \mathbb{G}_1 = \mathbb{G}$ and $H_j < \mathbb{G}_j = \mathbb{G}_{j-1}/H_{j-1}$, such that each $H_j < \mathbb{G}_j$ is a subgroup of degree p^{k_j} avoiding $\ker \varphi^i$. Since there exists a map $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G}) \to \mathrm{Sub}^{(i)}_{k_1+\cdots+k_q}(\mathbb{G})$ coming from comultiplication on $\Gamma^{\vee(i)}$, the composition of subgroups avoiding $\ker \varphi^i$ also avoids $\ker \varphi^i$. The fact that the power operations map descends to $E_0/u_{i-1} \to \Gamma^{\vee(i)}[k]$ says that if $H < \mathbb{G}$ is a subgroup avoiding $\ker \varphi^i$ ($\mathbb{G}$ has height $\geq i$), then $\mathbb{G}/H$ has height $\geq i$. For $i = 0$, we know that $\mathrm{Sub}_{k_1,\cdots,k_q}(\mathbb{G})$ classifies a sequence of subgroups $(H_1, \cdots, H_q)$, which is equivalent to a filtration $(H_1 < \cdots < H_q)$ (Def 2.2.1). In the case where we have quotiented out by $p, \cdots, u_{i-1}$, $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G})$ classifies a sequence of subgroups avoiding $\ker \varphi^i$. However, we do not know if this is equivalent to a filtration by subgroups avoiding $\ker \varphi^i$, which is classified by the closed subscheme of the product $\prod_{i=1}^q \mathrm{Sub}^{(i)}_{k_1+\cdots+k_i}(\mathbb{G})$ cut out by the relation that each subgroup contains the previous one, and admits a map from $\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}(\mathbb{G})$. These two agree once we invert u_i since $u_i^{-1}\mathrm{Sub}^{(i)}_{k_1,\cdots,k_q}$ classifies a sequence of étale subgroups, which is equivalent to a filtration consisting of étale subgroups.

4.5 Power operations mod $p, \cdots, u_{i-1}$ are Koszul

As mentioned in Section 2.3, we now have the ingredients to follow the proof in Section 8 of [Rez17] to show that $\Gamma^{(i)}$ is Koszul. Proposition 4.2.8 gives an isomorphism

$$Q_k^{(i)}(\overline{P}_m^{\vee}) \xrightarrow{\cong} \overline{\mathcal{B}}(\Gamma^{\vee(i)})[k]$$

of cosimplicial E_0/u_{i-1}-algebras. Dualizing gives an isomorphism

$$\overline{\mathcal{B}}(\Gamma^{(i)})[k] \xrightarrow{\cong} Q_k^{(i)}(\overline{P}_m)$$

of simplicial left E_0/u_{i-1}-modules.

The Mackey functor $Q_k^{(i)}$ clearly satisfies conditions (i) and (ii) in [Rez17, Prop 8.3]. Applying [Rez17, Prop 8.3 and Cor 8.5], [ADL16, Cor 1.2] to $Q_k^{(i)}$, we get that

$$H_*\overline{\mathcal{B}}(\Gamma^{(i)})[k] = H_* Q_k^{(i)}(\overline{P}_m) \cong \begin{cases} 0 & \text{if } * \neq k \\ Q_k^{(i)}(\Sigma_m/\Delta_k) \otimes_R \mathrm{St}_k & \text{if } * = k, \end{cases}$$

where $\Delta_k \leq \Sigma_m$ is a subgroup isomorphic to $\mathbb{F}_p{}^k$, $R = \mathbb{Z}_{(p)}[\mathrm{Aut}(\Delta_k)] = \mathbb{Z}_{(p)}[GL_k(\mathbb{F}_p)]$, and St_k is the Steinberg representation of $GL_k(\mathbb{F}_p)$.

Theorem 4.5.1. $\Gamma^{(i)}$ is Koszul for $0 \leq i \leq h$.

Definition 4.5.2. Let $C^{(i)}[k] = H_k\overline{\mathcal{B}}(\Gamma^{(i)})[k]$.

Since each $\Gamma^{(i)}[k]$ is a finite free E_0/u_{i-1}-module, by [Rez17, Proposition 4.6], $C^{(i)}[k]$ is a finitely generated projective left E_0/u_{i-1}-module, hence finite free.

Proposition 4.5.3. The ranks of $C^{(i)}[k]$ are given by

$$\sum_{k=0}^{\infty} \mathrm{rank}\, C^{(i)}[k] \cdot T^k = (1 + p^i T) \cdots (1 + p^{h-1} T).$$

Thus, $\Gamma^{(i)}$ is Koszul of length $h - i + 1$.

To prove the proposition, we will first compute the ranks of $\Gamma^{(i)}[k]$. [Str97] shows that $\Gamma[k]$ is a free left module over E_0, with rank given by the Gaussian binomial coefficient

$$d(k, h) := \begin{bmatrix} h + k - 1 \\ h - 1 \end{bmatrix}_p = \prod_{j=1}^{h-1} \frac{p^{k+j} - 1}{p^j - 1} = |\{ \text{ subgroups of } (\mathbb{Q}_p/\mathbb{Z}_p)^h \text{ of degree } p^k \}|,$$

where $d(k, h) = 0$ if $k > h$.

Proposition 4.5.4. $\Gamma^{(i)}[k]$ is a free left module over E_0/u_{i-1} of rank $d(k, h-i)p^{ki}$.

Proof. Since $\Gamma^{\vee(i)}[k]$ is a finite free left E_0/u_{i-1}-module, we can compute its rank over any base change. We will compute the rank of $u_i^{-1}\Gamma^{\vee(i)}[k]$ over $u_i^{-1}E_0/u_{i-1}$. The universal formal group $\mathbb{G}$ over E_0/u_{i-1} gives rise to the p-divisible group $\mathrm{colim}_k \mathbb{G}[p^k]$. For R a $u_i^{-1}E_0/u_{i-1}$-algebra, let $\mathbb{G}_R$ denote the p-divisible group over R induced by the universal formal group over E_0/u_{i-1}.

Since $[p^k](t) = t^{p^{ki}} g(t^{p^{ki}})$ where $g(t^{p^{ki}})$ is a power series in $t^{p^{ki}}$ with constant term a power of u_i, after inverting u_i, $\mathbb{G}[p^k]$ splits as

$$\mathbb{G}[p^k] \cong \operatorname{Spec}\left(u_i^{-1}E_0/u_{i-1}[t]/t^{p^{ki}}\right) \bigsqcup \operatorname{Spec}\left(u_i^{-1}E_0/u_{i-1}[t]/g_w(t^{p^{ki}})\right),$$

where g_w is the Weierstrass polynomial for g. Since $\ker \varphi^{ki} = \operatorname{Spf}\left(u_i^{-1}E_0/u_{i-1}[t]/t^{p^{ki}}\right)$ is the connected component of the identity, it is the connected part, so the connected-étale sequence for $\mathbb{G}[p^k]$ is

$$0 \to \ker \varphi^{ki} \to \mathbb{G}[p^k] \to \mathbb{G}'[p^k] \to 0,$$

where $\mathbb{G}'[p^k] = \mathbb{G}[p^k]/\ker\varphi^{ki} \simeq \operatorname{Spec}\left(u_i^{-1}E_0/u_{i-1}[t]/g_w(t)\right)$. Taking colimits gives a connected-étale sequence of p-divisible groups

$$0 \to \mathbb{G}^0 \to \mathbb{G} \to \mathbb{G}' \to 0, \tag{9}$$

for $\mathbb{G}$ the universal p-divisible group over $u_i^{-1}E_0/u_{i-1}$. Since $\mathbb{G}$ has height h and $\mathbb{G}^0$ has height i, the étale part $\mathbb{G}'$ has height $h - i$ and is pro-étale locally isomorphic to the constant scheme $\underline{\Lambda^{h-i}}$ where $\Lambda = \mathbb{Q}_p/\mathbb{Z}_p$.

By Lemma 4.4.4, $\operatorname{Sub}^{(i)}_{p^k}(\mathbb{G}) \simeq \operatorname{Spec}(u_i^{-1}\Gamma^{\vee(i)}[k])$ classifies degree p^k étale subgroups. Since a subgroup of $\mathbb{G}$ is étale if and only if it is isomorphic to its image in $\mathbb{G}'$, there is a natural map $f : \operatorname{Sub}^{(i)}_{p^k}(\mathbb{G}) \to \operatorname{Sub}_{p^k}(\mathbb{G}')$. Over $u_i^{-1}E_0/u_{i-1}$, we have a sequence of maps

$$\operatorname{Sub}^{(i)}_k(\mathbb{G}) \xrightarrow{f} \operatorname{Sub}_k(\mathbb{G}') \xrightarrow{g} \operatorname{Spec}(u_i^{-1}E_0/u_{i-1}) \tag{10}$$

where g is the canonical map.

In [Sta13a, Corollary 2.18], Stapleton shows that for $t \leq h$, there exists an $L_t := \pi_0 L_{K(t)} E$-algebra C_t which is initial among L_t-algebras over which $\mathbb{G}$ splits as $\mathbb{G} \simeq \mathbb{G}^0 \oplus \underline{\Lambda^{h-t}}$. By [Sta13b, Proposition 3.6], $L_t/I_t \to C_t/I_t$ is faithfully flat, where $I_t = (p, \cdots, u_{t-1})$. We apply this to $t = i$, so $L_t/I_t = u_i^{-1}E_0/u_{i-1}$. Let $C = C_i/I_i$. After a faithfully flat base change to C, $\mathbb{G}'_C$ is isomorphic to the constant scheme $\underline{\Lambda^{h-i}}$ and the sequence (10) becomes

$$\operatorname{Sub}^{(i)}_k(\mathbb{G}_C) \xrightarrow{C \otimes f} \underline{\operatorname{Sub}_k(\Lambda^{n-i})} \xrightarrow{C \otimes g} \operatorname{Spec}(C). \tag{11}$$

Since $C \otimes g$ is faithfully flat, so is g. Since $C \otimes (f \circ g)$ is flat (it is the base change of the flat map $f \circ g$) and $\underline{\operatorname{Sub}_k(\Lambda^{n-i})}$ is a constant scheme, $C \otimes f$ is flat. The canonical section $\mathbb{G}'_C \to \mathbb{G}_C \simeq \mathbb{G}^0{}_C \oplus \mathbb{G}'_C$ mapping into the étale part of $\mathbb{G}_C$ induces a map $\underline{\operatorname{Sub}_k(\Lambda^{h-i})} \to \operatorname{Sub}_k(\mathbb{G}_C)$. Since this map sends a subgroup $H < \underline{\Lambda^{h-i}}$ to an étale subgroup of $\mathbb{G}_C$, it factors through $\operatorname{Sub}^{(i)}_k(\mathbb{G}_C)$. This gives a section for $C \otimes f$, so $C \otimes f$ is surjective hence faithfully flat. Thus, f and g are faithfully flat. Since $u_i^{-1}\Gamma^{\vee(i)}[k]$ is an integral domain and faithfully flat maps of rings are injective, $\mathcal{O}_{\operatorname{Sub}_k(\mathbb{G}')}$ is an integral domain. Thus, f and g are dominant morphisms of integral schemes, so $\deg(f \circ g) = \deg f \cdot \deg g$.

$\deg g = \deg(C \otimes g) = |\operatorname{Sub}_k(\Lambda^{h-i})| = d(k, h - i)$. $\deg f = \deg(C \otimes f)$ is the dimension of the fiber of f, which we claim to be p^{ki}. Over C,

$$\underline{\operatorname{Sub}_k(\Lambda^{h-i})} = \bigsqcup_{|A|=p^k} \underline{\operatorname{Sub}_A(\Lambda^{h-i})}$$

30

where the disjoint union ranges over all abelian groups A of order p^k and $\underline{\mathrm{Sub}_A(\Lambda^{h-i})} = \underline{\mathrm{Level}_A(\Lambda^{h-i})}/\mathrm{Aut}(A)$ is the scheme classifying subgroups of Λ^{h-i} isomorphic to A. Thus, we can cover $\underline{\mathrm{Sub}_{p^k}(\Lambda^{h-i})}$ by level structures $\underline{\mathrm{Level}_A(\Lambda^{h-i})}$.

Let $\mathrm{Level}_A^{(i)}(\mathbb{G})$ be the pullback of $\mathrm{Level}_A(\mathbb{G})$ along the closed embedding $\mathrm{Sub}_{p^k}^{(i)}(\mathbb{G}) \to \mathrm{Sub}_{p^k}(\mathbb{G})$ over $u_i^{-1}E_0/u_{i-1}$. Since closed embeddings are preserved under pullbacks, $\mathrm{Level}_A^{(i)}(\mathbb{G})$ is a closed subscheme of $\mathrm{Level}_A(\mathbb{G})$ classifying level-A structures which map to étale subgroups. Since $H < \mathbb{G}$ is étale if and only if it is isomorphic to its étale part which is its image in $\mathbb{G}'$, a level structure on H is the same as a level structure on its image. So, for each A, $\mathrm{Level}_A^{(i)}(\mathbb{G})$ is the pullback

$$
\begin{array}{ccc}
\mathrm{Level}_A^{(i)}(\mathbb{G}) & \longrightarrow & \mathrm{Level}_A(\mathbb{G}') \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{Sub}_k^{(i)}(\mathbb{G}) & \longrightarrow & \mathrm{Sub}_k(\mathbb{G}')
\end{array}
$$

over $u_i^{-1}E_0/u_{i-1}$. Thus, $\mathrm{Sub}_k^{(i)}(\mathbb{G})$ is covered by $\bigsqcup_A \mathrm{Level}_A^{(i)}(\mathbb{G})$ and $\deg f$ is equal to the degree of $\mathrm{Level}_A^{(i)}(\mathbb{G}) \to \mathrm{Level}_A(\mathbb{G}')$ for any A.

We can compute this degree after base changing to C. The fiber of $C \otimes f$ consists of all lifts of a level structure on $\mathbb{G}'_C$ to a level structure on $\mathbb{G}_C$. Since $\mathbb{G}_C$ splits into its connected and étale part, any level structure on $\mathbb{G}'_C$ has a lift to a level structure on $\mathbb{G}$. Any two such lifts differ by a homomorphism $A \to \mathbb{G}_C^0$ (not necessarily a level structure on $\mathbb{G}_{0C}$). Thus, the fiber is equivalent to $\mathrm{Hom}(A, \mathbb{G}_C^0)$. If $A \cong \mathbb{Z}/p^{k_1} \oplus \cdots \oplus \mathbb{Z}/p^{k_q}$ with $k_1 + \cdots + k_q = k$,

$$
\mathrm{Hom}(A, \mathbb{G}_C^0) \simeq \prod_{j=1}^q \mathbb{G}_C^0[p^{k_j}] = \prod_{j=1}^q \ker \varphi^{k_j i}
$$

is free of rank $\prod_{j=1}^q p^{k_j i} = p^{ki}$.

Thus,

$$
\mathrm{rank}\,\Gamma^{(i)}[k] = \mathrm{rank}\,\Gamma^{\vee(i)}[k] = \mathrm{rank}\,u_i^{-1}\Gamma^{\vee(i)}[k] = \deg f \cdot \deg g = d(k, h-i)p^{ki}.
$$

$\square$

Proof of Proposition 4.5.3. [Rez17, Proposition 3.21] shows that the ranks of $\Gamma[k]$ are given by the generating series

$$
\sum_{k \geq 0} d(k, h) \cdot T^k = \prod_{j=0}^{h-1} (1 - p^j T)^{-1}.
$$

By Proposition 4.5.4, $\Gamma^{(i)}[k]$ has rank $d(k, h-i)p^{ki}$, so the generating series is

$$
\sum_k d(k, h-i) \cdot (p^i T)^k = \prod_{j=0}^{h-i-1} (1 - p^{i+j}T)^{-1} = \prod_{j=i}^{h-1} (1 - p^j T)^{-1}.
$$

By [Rez17, Proposition 4.6], the ranks of $C^{(i)}[k]$ are given by

$$
\sum_k \mathrm{rank}\,C^{(i)}[k] \cdot T^k = \left(\sum_k (-1)^k d(k, h-i)p^{ki}T^k \right) = \prod_{j=i}^{h-1} (1 + p^j T).
$$

$\square$

31

5 Explicit description of $\Gamma^{(h-1)}$

$\Gamma^{(h)}$ is the base field k. The first nontrivial case is $\Gamma^{(h-1)}$, which turns out to be a twisted free associative algebra on one generator. Indeed, since $\Gamma^{\vee(h-1)}$ is Koszul, it is generated by elements in degree 1. We will show that there are no relations between the generators (Prop 5.0.10). This is done using an algebro-geometric argument which shows that every finite subgroup of the universal formal group over $\mathbb{E}_0/u_{h-2}$ is cyclic. Tracing through the commutation relations will show that it is in fact generated by a single element.

Proposition 5.0.1. *Let* $u \in (E_0/u_{h-2})^\times \cong (k[[u_{h-1}]])^\times$ *be the coefficient of* t^{p^h} *in the p-series* $[p](t)$. *Then*

$$\Gamma^{(h-1)} \cong k[[u_{h-1}]]\langle Q \rangle / \left(Q u_{h-1}^{p^{h-1}} \sim u^{p^{h-1}-1} u_{h-1} Q \right).$$

is a twisted free associative algebra on one generator Q.

For height 1, this recovers $\Gamma \simeq \mathbb{Z}_p[\psi]$ where ψ is the pth Adams operation. Throughout, let S be an E_0/u_{i-1}-algebra and $\mathbb{G}$ a formal group over S.

Lemma 5.0.2. *If* $H < \mathbb{G}$ *avoids* $\ker \varphi^i$, *then* H *is étale after inverting* u_i. *If* $H < \mathbb{G}$ *is classified by a map* $\Gamma^\vee[k] \to S$ *such that* c_{p^k} *maps to a nonzero divisor in S, then H avoids* $\ker \varphi^i$ *if and only if it is étale after inverting u_i.*

In particular, the condition that c_{p^k} is not a zero divisor in S holds if u_i is not a zero divisor in S.

Proof. If $H < \mathbb{G}$ avoids $\ker \varphi^i$, then by Lemma 4.4.4, H is étale after inverting u_i. For the second part, suppose that H is étale after inverting u_i. $\Gamma^\vee[k] \to S$ factors through $\Gamma^\vee[k]/u_{i-1}$ which extends to a map $u_i^{-1}\Gamma^{\vee(i)}[k] \to u_i^{-1}S$. We need to show that the map $\Gamma^\vee[k]/u_{i-1} \to S$ factors through $\Gamma^{\vee(i)}[k]$ i.e. vanishes on $\ker(\Gamma^\vee[k]/u_{i-1} \to \Gamma^{\vee(i)}[k])$. By (7), there is a sequence of localizations

$$\Gamma^\vee[k]/u_{i-1} \to c_{p^k}^{-1}\Gamma^\vee[k]/u_{i-1} \hookrightarrow (u_i c_{p^k})^{-1}\Gamma^\vee[k]/u_{i-1} = u_i^{-1}\Gamma^{\vee(i)}[k],$$

so $\ker(\Gamma^\vee[k]/u_{i-1} \to \Gamma^{\vee(i)}[k]) = \ker(\Gamma^\vee[k]/u_{i-1} \to u_i^{-1}\Gamma^{\vee(i)}[k]) = \ker(\Gamma^\vee[k]/u_{i-1} \to c_{p^k}^{-1}\Gamma^\vee[k]/u_{i-1})$. Since the map $\Gamma^\vee[k]/u_{i-1} \to S$ extends to a map $c_{p^k}^{-1}\Gamma^\vee[k] \to c_{p^k}^{-1}S$ and the localization map $S \to c_{p^k}^{-1}S$ is injective, the map $\Gamma^\vee[k]/u_{i-1} \to S$ vanishes on $\ker(\Gamma^\vee[k]/u_{i-1} \to c_{p^k}^{-1}\Gamma^\vee[k]/u_{i-1})$ as desired. $\square$

The following follows from Definitions 2.1.3 and 4.4.3.

Lemma 5.0.3. *The relative Frobenius on* $\mathrm{Sub}_k^{(i)}(\mathbb{G})$ *over* $\mathrm{Spf}(E_0/u_{i-1})$ *is the map* $\mathrm{Sub}_k^{(i)}(\mathbb{G}) \to \mathrm{Sub}_k^{(i)}(\varphi\mathbb{G})$ *sending* $H < \mathbb{G}$ *to* $\varphi H < \varphi\mathbb{G}$. *Thus, if H avoids $\ker \varphi^i$, so does φH.*

Lemma 5.0.4. *Over a reduced base, if* $H < \mathbb{G}$ *is a degree p subgroup such that φH avoids* $\ker \varphi^i$, *then so does H.*

Proof. The relative Frobenius on $\mathrm{Sub}_1^{(i)}(\mathbb{G})$ over $\mathrm{Spf}(E_0/u_{i-1})$ is the E_0/u_{i-1}-algebra map

$$(E_0/u_{i-1})_\varphi \otimes \Gamma^{\vee(i)}[1] \cong (E_0/u_{i-1})[[x_2]]/f^{(i)\varphi}(x_2) \to \Gamma^{\vee(i)}[1] \cong (E_0/u_{i-1})[[x_1]]/f^{(i)}(x_1), \quad x_2 \mapsto x_1^p.$$

The notation $(E_0/u_{i-1})_\varphi \otimes -$ means that E_0/u_{i-1} is given a right E_0/u_{i-1}-module structure by the Frobenius φ and $f^{(i)\varphi}$ is Notation 4.4.6. Let H be a degree p subgroup ver a ring S such that φH avoids $\ker \varphi^i$. Then H and φH are classified by the maps labeled H and φH in the diagram below making the outer part commute.

Since the horizontal maps send x_2 to x_1^p, $x_2 = x_1^p$ in S. So, $(f^{(i)}(x_1))^p = f^{(i)\varphi}(x_1^p) = f^{(i)\varphi}(x_2) = 0$ in S. If S is reduced, this means that $f^{(i)}(x_1) = 0$, so the dotted map exists and H avoids $\ker \varphi^i$. $\qquad \square$

Lemma 5.0.5. *Let $H < \mathbb{G}$ be a degree p subgroup over an E_0/u_{i-1}-algebra S. If H is classified by a map $\Gamma^\vee[1] \to S$ such that the Euler class c_p is not a zero divisor in S, then H avoids $\ker \varphi^i$.*

Proof. Since S is a E_0/u_{i-1}-algebra, the map classifying H factors through $\Gamma^\vee[1]/u_{i-1} \cong E_0/u_{i-1}[[x]]/f(x)$. Since

$$f(x) \equiv x^{\frac{p^i-1}{p-1}} f^{(i)}(x) \mod p, \cdots, u_{i-1}$$

and $c_p = x$ is not a zero divisor in S, $f^{(i)}(x)$ must map to 0 in S. Thus, the map classifying H factors through $\Gamma^{\vee(i)}[1]$ and H avoids $\ker \varphi^i$. $\qquad \square$

Corollary 5.0.6. *Let H be a degree p subgroup contained in a degree p^k subgroup $K < \mathbb{G}$ over an E_0/u_{i-1}-algebra S. If K is classified by a map $\Gamma^\vee[k] \to S$ such that c_{p^k} is is not a zero divisor in S, then H avoids $\ker \varphi^i$.*

Proof. In this case, the map classifying $K < \mathbb{G}$ lifts to a map $\Gamma^\vee[1] \otimes \Gamma^\vee[k-1] \to S$ classifying $H < K < \mathbb{G}$. Since c_p divides c_{p^k} in $\Gamma^\vee[1] \otimes \Gamma^\vee[k-1]$, c_p is not a zero divisor in S and we conclude by Lemma 5.0.5. $\qquad \square$

In view of proving Proposition 5.0.1, we specialize to the case $i = h - 1$.

Lemma 5.0.7. *Let S be an E_0/u_{h-2}-algebra and $K < \mathbb{G}$ a degree p^k subgroup. If K is classified by a map $\Gamma^\vee[k] \to S$ such that c_{p^k} is not a zero divisor in S, then the subgroup of p-torsion points $K[p]$ is a degree p subgroup avoiding $\ker \varphi^{h-1}$.*

Proof. If $k = 0$, there is nothing to prove, so we may assume $k \geq 1$. Since $c_{p^k} \neq 0$ and $K[p] = K \cap \mathbb{G}[p] \leq K$ is a subgroup of degree $\geq p$, it suffices to show that $K[p]$ has degree p as Corollary 5.0.6 will then imply that it avoids $\ker \varphi^{h-1}$.

Since $\mathbb{G}$ has height $\geq h - 1$, $[p](y) = g(y^{p^{h-1}})$ for some g. Applying the Weierstrass preparation theorem to g implies that $[p](y) = w(y^{p^{h-1}})u(y^{p^{h-1}})$ where w is a monic polynomial

of degree p with no constant term and u a unit. Since $K[p] < \mathbb{G}[p]$ is a subgroup, the p isogeny on $\mathbb{G}$ factors as $K[p]$ followed by some isogeny with kernel K', so

$$f_{K'} \circ f_{K[p]}(y) = w(y^{p^{n-1}}).$$

The proof of Theorem 9.2 in [Str98] shows that the equation for the universal degree p^k subgroup over $\Gamma^\vee[k]$ is given by $f_K(y) = c_{p^k} y + y^2(\cdots)$. The equation for any degree p^k subgroup over any ring is obtained by pulling back from this. Since $f_{K[p]}$ divides f_K and f_K is not divisible by y^2, neither is $f_{K[p]}$. So, $f_{K'}(y) = y^{p^{n-1}}(\cdots)$ has degree $\geq p^{h-1}$. Since $\mathbb{G}[p]$ has degree p^h, $K[p]$ and K' must have degrees p and p^{h-1}, respectively.

$\square$

Remark 5.0.8. In fact, the above proof shows that $K' = \ker \varphi^{h-1}$, so $\mathbb{G}[p]$ is the composite of a degree p subgroup $H_1 = K[p] < \mathbb{G}$ avoiding $\ker \varphi^{h-1}$ followed by φ^{h-1}. By Lemma 5.0.3, $\varphi^{h-1} H_1 < \varphi^{h-1} \mathbb{G}$ avoids $\ker \varphi^{h-1}$, so $\mathbb{G}[p]$ is also the composite of φ^{h-1} followed by a degree p subgroup $H_2 = \varphi^{h-1} H_1$ avoiding $\ker \varphi^{h-1}$.

Lemma 5.0.9. *Let S be an E_0/u_{h-2}-algebra and $K < \mathbb{G}$ a degree p^k subgroup. If K is classified by a map $\Gamma^\vee[k] \to S$ such that c_{p^k} is not a zero divisor in S, then K can be uniquely written as the composition of degree p subgroups avoiding $\ker \varphi^{h-1}$. In fact, K is cyclic i.e. $K[p] < K[p^2] < \cdots < K[p^k] = K$ is the unique length k filtration of K.*

Proof. With notation 2.2.2, let $\mathbb{G}_1 = \mathbb{G}$, $K_{\leq 1} = K_1 = K[p]$, $\mathbb{G}_j = \mathbb{G}_{j-1}/K_{j-1}$, $K_j = (K/K_{\leq j-1})[p]$, and $K_{\leq j} = \overline{(K_1, \cdots, K_j)}$, so $K = K_{\leq q}$ for some $q \leq k$.

By Lemma 5.0.7 above, $K_1 < \mathbb{G}_1$ is a degree p subgroup avoiding $\ker \varphi^{h-1}$. Suppose inductively that $K_1, \cdots, K_{j-1}$ are degree p subgroups avoiding $\ker \varphi^{h-1}$, so their composite $K_{\leq j-1}$ is a degree p^{j-1} subgroup avoiding $\ker \varphi^{h-1}$ and $\mathbb{G}_j = \mathbb{G}/K_{\leq j-1}$ has height $\geq h-1$. K_j is only nontrivial if $j \leq k$. The map classifying K factors through $\Gamma^\vee[k] \to \Gamma^\vee[j-1] \otimes \Gamma^\vee[k-j+1]$ which sends c_{p^k} to $c_{p^{j-1}} c_{p^{k-j+1}}$. Since $c_{p^k} \neq 0$ in S, neither is $c_{p^{k-j+1}}$. This allows us to apply Lemma 5.0.7 to $K/K_{\leq j-1}$ to get that K_j is a degree p subgroup avoiding $\ker \varphi^{h-1}$, completing the induction. Thus, $K = \overline{(K_1, \cdots, K_k)}$ is the composition of degree p subgroups avoiding $\ker \varphi^{n-1}$.

To show uniqueness, suppose $\overline{(K_1', \cdots, K_k')}$ is another decomposition of K into degree p subgroups avoiding $\ker \varphi^{h-1}$. Since K_1' has degree p, it must be contained in the degree p subgroup $K[p] = K_1$ (Lemma 5.0.7), so $K_1' = K_1$. Suppose that $K_q' = K_q$ for all $1 \leq q < j$. Similarly, since $K_j' \leq (K/K_{\leq j-1}')[p] = (K/K_{\leq j-1})[p] = K_j$ is an inclusion of subgroups of the same rank, they must be equal. Thus, $(K_1', \cdots, K_k') = (K_1, \cdots, K_k)$.

For the last part, since $K_{\leq 1} = K[p]$, by induction,

$$K_{\leq j}/K_{\leq j-1} = K_{\leq j}/K[p^{j-1}] = K_j = (K/K[p^{j-1}])[p] = K[p^j]/K[p^{j-1}],$$

so $K_{\leq j} = K[p^j]$. Thus, $K[p] < \cdots < K[p^k] = K$ is the unique length k filtration on K. $\square$

Since the E_0/u_{i-1}-algebras $\Gamma^{\vee(i)}[k]$ are integral domains with $c_{p^k} \neq 0$, we may apply the above results to $S = \Gamma^{\vee(i)}[k]$.

Proposition 5.0.10 (No Adem relations)**.**

$$\Gamma^{\vee(h-1)}[k] \xrightarrow{\cong} \Gamma^{\vee(h-1)}[1]^{\otimes k}$$

is an isomorphism for all $k \geq 0$.

Proof. Let K denote the universal degree p^k subgroup over $\Gamma^{\vee(h-1)}[k]$. Let f denote the inclusion $\Gamma^{\vee(h-1)}[k] \hookrightarrow \Gamma^{\vee(h-1)}[1]^{\otimes k}$. Lemma 5.0.9 applied to $S = \Gamma^{\vee(h-1)}[k]$ gives a map $g : \Gamma^{\vee(h-1)}[1]^{\otimes k} \to \Gamma^{\vee(h-1)}[k]$ classifying a sequence of subgroups $(K_1, \cdots, K_k)$ whose composite is K, so $g \circ f = \mathrm{id}$. The composite of the universal sequence $(K_{u1}, \cdots, K_{uk})$ over $\Gamma^{\vee(h-1)}[1]^{\otimes k}$ is f^*K. $f \circ g$ classifies the sequence $(f^*K_1, \cdots, f^*K_k)$ of degree p subgroups avoiding $\ker \varphi^{h-1}$ over $\Gamma^{\vee(h-1)}[1]^{\otimes k}$. Since $c_{p^k} = c_p^k$ is not a zero divisor in $\Gamma^{\vee(h-1)}[1]^{\otimes k}$, the uniqueness of decompositions of f^*K implies that $(f^*K_1, \cdots, f^*K_k) = (K_{u1}, \cdots, K_{uk})$, so $f \circ g = \mathrm{id}$. Thus, f and g are inverse isomorphisms $\Gamma^{\vee(h-1)}[k] \xrightarrow{\cong} \Gamma^{\vee(h-1)}[1]^{\otimes k}$. $\square$

Proof of Proposition 5.0.1. Since $\Gamma^{\vee(h-1)}$ is Koszul, it is generated by the weight p operations $Q_0, \cdots, Q_{p^{h-1}-1}$ dual to the basis $1, x, \cdots, x^{p^{h-1}-1}$ of $\Gamma^{\vee(h-1)}[1] \cong k[[u_{h-1}]][[x]]/u_{h-1}-ux^{p^{h-1}}$. Proposition 5.0.10 shows that there are no relations between the generators, so they freely generate $\Gamma^{(h-1)}$ as an associative E_0/u_{h-2}-algebra. Note that E_0/u_{h-2} is not central in $\Gamma^{(h-1)}$, so we need to identify the commutation relations.

For $y \in E_0/u_{h-1}$, $P(y) = Q_0(y) + Q_1(y)x + \cdots Q_{p^{h-1}-1}(y)x^{p^{h-1}-1}$. Here we have abused notation and also denote the operation $E_0/u_{h-1} \xrightarrow{P} \Gamma^{\vee(h-1)}[1] \xrightarrow{Q_j} E_0/u_{h-1}$ by Q_j. Since P is multiplicative, $P(u_{h-1}y) = P(u_{h-1})P(y) = uxP(y)$ where

$$\text{LHS} = Q_0(u_{h-1}y) + Q_1(u_{h-1}y)x + \cdots + Q_{p^{h-1}-1}(u_{h-1}y)x^{p^{h-1}-1}$$
$$\text{RHS} = uQ_0(y)x + uQ_1(y)x^2 + \cdots + uQ_{p^{h-1}-2}(y)x^{p^{h-1}-1} + uQ_{p^{h-1}-1}(y)x^{p^{h-1}}$$
$$= u_{h-1}Q_{p^{h-1}-1}(y) + uQ_0(y)x + \cdots + uQ_{p^{h-1}-2}(y)x^{p^{h-1}-1}.$$

Comparing coefficients gives $Q_j u_{h-1} = uQ_{j-1}$ for $1 \le j \le p^{h-1}-1$ and $Q_0 u_{h-1} = u_{h-1}Q_{p^{h-1}-1}$. Since u is a unit, $Q, Qu_{h-1}, \cdots, Qu_{h-1}^{p^{h-1}-1}$, where $Q = Q_{p^{h-1}-1}$, also generates $\Gamma^{(h-1)}$. Thus,

$$\Gamma^{(h-1)} \cong k[[u_{h-1}]]\langle Q \rangle / Qu_{h-1}^{p^{h-1}} \sim u^{p^{h-1}-1}u_{h-1}Q$$

as desired. $\square$

6 $\quad \Gamma^{(i)} /\!\!/ u_i$

$\Gamma^{(i+1)}$ consists of certain power operations in $\Gamma^{(i)}$ which are defined modulo $p, \cdots, u_i$. $\Gamma^{\vee(i+1)}[k]$ is given by the image of $\Gamma^\vee[k]$ in the Tate construction $\pi_0(E^{t\mathcal{T}_k}/u_i)$ and classifies degree p^k subgroups avoiding $\ker \varphi^{i+1}$ (Definition 4.4.3). In this section, we consider an algebra of operations $\Gamma^{(i)} /\!\!/ u_i$ consisting of all operations in $\Gamma^{(i)}$ which are defined modulo u_i. For $i = 0$, this is just Γ/p since every additive operations descends to an operation on $\pi_0(-/p)$. We will see in [SS25] that in terms of isogenies, $\mathrm{Sub}_k^{(i)}(\mathbb{G}) /\!\!/ u_i := \mathrm{Spf}(\Gamma^{\vee(i)}[k] /\!\!/ u_i)$ is the union of closed subschemes

$$\mathrm{Sub}_k^{(i)} /\!\!/ u_i = \bigcup_{s=0}^{k} \mathrm{Sub}_s^{(i+1)}(\varphi^{k-s}\mathbb{G}). \tag{12}$$

In unpublished work (see [BSY24, 3.5]), Rezk shows that the additive operations Γ together with a single non-additive operation $\theta \in \mathbb{T}(E_0)$ of weight p generates all power operations. θ acts as a p-derivation, so $\psi(x) = x^p + p\theta(x)$ is an additive lift of Frobenius. We will see that $\Gamma^{(i)} /\!\!/ u_i$ is an algebra of power operations which differs from $\Gamma^{(i+1)}$ by an additional operation witnessing the lift of Frobenius. The result of this section is summarized in Prop 6.0.8.

Definition 6.0.1. Let

$$\Gamma^{(i)}[k]/\!\!/ u_i := \Gamma^{(i)}[k]_t \otimes_s E_0/u_i, \qquad \Gamma^{\vee(i)}[k]/\!\!/ u_i := \Gamma^{\vee(i)}[k]_t \otimes_s E_0/u_i = \Gamma^{\vee(i)}[k]/P(u_i)$$

$$\Gamma^{(i)}/\!\!/ u_i := \bigoplus_{k \geq 0} \Gamma^{(i)}[k]/\!\!/ u_i, \qquad\qquad \Gamma^{\vee(i)}/\!\!/ u_i := \bigoplus_{k \geq 0} \Gamma^{\vee(i)}[k]/\!\!/ u_i.$$

Here we are considering E_0/u_i as a quotient of $\Gamma^{\vee(i)}[0] = E_0/u_{i-1}$, where the source and target maps $s, t : E_0/u_{i-1} \to \Gamma^{\vee(i)}[0]$ are the identity. Weight p^k operations in $\Gamma^{(i)}$ which descend to operations on $\pi_0(-/u_i)$ are parameterized by $\Gamma^{\vee(i)}[k]/\!\!/ u_i$, which is the pushout of the total power operations map $P : E_0/u_{i-1} \to \Gamma^{\vee(i)}[k]$ along the projection $E_0/u_{i-1} \to E_0/u_i$.

A priori, we do not know whether $\Gamma^{(i)}[k]/\!\!/ u_i$ is the left E_0/u_i-dual of $\Gamma^{\vee(i)}[k]/\!\!/ u_i$, but we will show (Cor 6.0.5) that this is the case. We claim that killing u_i on the right also kills u_i on the left.

Lemma 6.0.2. *(1) $u_i = 0$ in $\Gamma^{\vee(i)}[k]/\!\!/ u_i$.*

(2) For all $k \geq 0$, $\Gamma^{\vee(i)}[k]/\!\!/ u_i$ is finite free as left and right E_0/u_i-modules.

Proof. The first statement is a result of the fact that $x^{p^i-1}P(u_i) = u_i$ in $\Gamma^{\vee(i)}[1]$ (4.3.1). Since $\Gamma^{\vee(i)}[1]/\!\!/ u_i = \Gamma^{\vee(i)}[1]/P(u_i)$ where P is the total power operations map, $u_i = 0$ in $\Gamma^{\vee(i)}[1]/\!\!/ u_i$. Thus,

$$\Gamma^{\vee(i)}[1]^{\otimes k}/\!\!/ u_i = \Gamma^{\vee(i)}[1]^{\otimes(k-1)}{}_t \otimes_s E_0/u_{i_t} \otimes_s \Gamma^{\vee(i)}[1]_t \otimes_s E_0/u_i = (\Gamma^{\vee(i)}[1]/\!\!/ u_i)^{\otimes k}$$

and there is a pushout diagram

$$
\begin{array}{ccccc}
E_0/u_{i-1} & \xrightarrow{\ t\ } & \Gamma^{\vee(i)}[k] & \lhook\joinrel\longrightarrow & \Gamma^{\vee(i)}[1]^{\otimes k} \\
\downarrow & & \downarrow & & \downarrow \\
E_0/u_i & \longrightarrow & \Gamma^{\vee(i)}[k]/\!\!/ u_i & \lhook\joinrel\longrightarrow & (\Gamma^{\vee(i)}[1]/\!\!/ u_i)^{\otimes k}.
\end{array}
$$

Since it is a base change of the (left and right module) direct summand inclusion $\Gamma^{\vee(i)}[k] \to \Gamma^{\vee(i)}[1]^{\otimes k}$ (Lemma 4.3.4), the map $\Gamma^{\vee(i)}[k]/\!\!/ u_i \to (\Gamma^{\vee(i)}[1]/\!\!/ u_i)^{\otimes k}$ is a direct summand inclusion hence an injective map of E_0/u_i-algebras. Since $u_i = 0$ in the target, it is also zero in the source.

For (2), the source and target maps $s, t : E_0/u_i \to \Gamma^{\vee(i)}[k]/\!\!/ u_i$ are finite flat since they are base changes of finite flat maps $s, t : E_0/u_{i-1} \to \Gamma^{\vee(i)}[k]$. Thus, $\Gamma^{\vee(i)}[k]/\!\!/ u_i$ is a finite free left and right E_0/u_i-module. $\qquad\square$

The total power operations maps fit into the diagram

$$
\begin{array}{ccc}
E_0/u_{i-1} & \xrightarrow{\ P\ } & \Gamma^{\vee(i)}[k] \\
\downarrow & & \downarrow \\
E_0/u_i & \xrightarrow{\ P\ } & \Gamma^{\vee(i)}[k]/\!\!/ u_i \\
& \searrow^{P} & \downarrow \\
& & \Gamma^{\vee(i+1)}[k].
\end{array}
$$

By definition, $\Gamma^{\vee(i)}/\!\!/ u_i$ parametrizes the largest collection of power operations in $\Gamma^{(i)}$ which are defined modulo u_i. $\Gamma^{\vee(i)}/\!\!/ u_i$ inherits a coalgebra structure as a quotient of $\Gamma^{\vee(i)}$, thus

its (graded) left E_0/u_i-dual $\left(\Gamma^{\vee(i)}/\!\!/u_i\right)^\vee$ is an algebra of power operations. We will see that $\Gamma^{(i)}/\!\!/u_i$ has an E_0/u_i-algebra structure as a quotient of $\Gamma^{(i)}$, and is isomorphic to $\left(\Gamma^{\vee(i)}/\!\!/u_i\right)^\vee$. The E_0/u_{i-1}-algebra structure on $\Gamma^{(i)}$ is E_0/u_{i-1}-dual to the coalgebra structure on $\Gamma^{\vee(i)}$. Since we are dealing with E_0/u_i-duals in $\Gamma^{(i)}/\!\!/u_i$ and E_0/u_{i-1}-duals in $\Gamma^{(i)}$, the description of $\Gamma^{(i)}/\!\!/u_i$ as a quotient of $\Gamma^{(i)}$ does not work well with dualizing. This will be solved by working in the derived setting.

For a ring R, we denote by DMod_R the derived category of left R-modules. Let $-\otimes_R^{\mathbb{L}} -$, RHom_R, and $D := D_R = \mathrm{RHom}_R(-, R)$ denote the derived tensor product, derived Hom, and derived dual, respectively.

Lemma 6.0.3. *Let $u \in R$ be a nonzero divisor. There is a natural isomorphism of functors on $\mathrm{DMod}_{R/u}$*

$$\Sigma D \simeq D_{R/u},$$

where $D = \mathrm{RHom}_R(-, R)$ is considered as a functor on $\mathrm{DMod}_{R/u}$ via the forgetful functor $\mathrm{DMod}_{R/u} \to \mathrm{DMod}_R$.

Proof. The forgetful functor $\mathrm{DMod}_{R/u} \to \mathrm{DMod}_R$ has a left adjoint $R/u \otimes_R^{\mathbb{L}} -$. Since the forgetful functor preserves all small colimits, it has a right adjoint G. We have

$$G(R) \simeq \mathrm{RHom}_{R/u}(R/u, G(R)) \simeq \mathrm{RHom}_R(R/u, R) \simeq \Sigma^{-1} R/u.$$

For a derived R/u-module N,

$$\mathrm{RHom}_R(N, R) \simeq \mathrm{RHom}_{R/u}(N, G(R)) \simeq \mathrm{RHom}_{R/u}(N, \Sigma^{-1} R/u) \simeq \Sigma^{-1} \mathrm{RHom}_{R/u}(N, R/u).$$

$\square$

Lemma 6.0.4. *Let M be an R-bimodule. The collection of all $N \in \mathrm{DMod}_R$ for which the map $DM \otimes_R^{\mathbb{L}} DN \to D(M \otimes_R^{\mathbb{L}} N)$ is an equivalence forms a thick subcategory containing R. In particular, it is an equivalence for $N = \bigoplus_{j=0}^m R/r_j$ a finite direct sum of quotients of R by an element.*

Corollary 6.0.5.   *1. There are isomorphisms*

$$\Gamma^{(i)}[k]/\!\!/u_i \simeq \Sigma D(\Gamma^{\vee(i)}[k]/\!\!/u_i) \simeq D_{E_0/u_i}(\Gamma^{\vee(i)}[k]/\!\!/u_i) \simeq \left(\Gamma^{\vee(i)}[k]/\!\!/u_i\right)^\vee$$

of (derived) left E_0/u_{i-1}-modules. Here $(-)^\vee$ denotes the underived left E_0/u_i-dual and a derived E_0/u_i-module is considered as a derived E_0/u_{i-1}-module via the forgetful functor.

   2. $\Gamma^{(i)}[k]/\!\!/u_i$ and $\Gamma^{\vee(i)}[k]/\!\!/u_i$ are dual as left E_0/u_i-modules. In particular, $\Gamma^{(i)}[k]/\!\!/u_i$ is a finite free left E_0/u_i-module.

Proof. By the lemma above, since $\Gamma^{\vee(i)}[k]$ is a finite free right E_0/u_{i-1}-module,

$$\Sigma D(\Gamma^{\vee(i)}[k]/\!\!/u_i) = \Sigma D(\Gamma^{\vee(i)}[k] \otimes E/u_i) \simeq \Sigma D(\Gamma^{\vee(i)}[k] \otimes^{\mathbb{L}} E/u_i) \simeq \Gamma^{(i)}[k] \otimes^{\mathbb{L}} E/u_i.$$

Lemma 6.0.3 shows that this is equivalent to $D_{E_0/u_i}(\Gamma^{\vee(i)}[k]/\!\!/u_i)$, which is discrete and equivalent to the underived left E_0/u_i-dual of $\Gamma^{\vee(i)}[k]/\!\!/u_i$ since $\Gamma^{\vee(i)}[k]/\!\!/u_i$ is a finite free E_0/u_i-module. Thus, $\Gamma^{(i)}[k] \otimes^{\mathbb{L}} E/u_i$ is discrete and equivalent to the underived tensor product $\Gamma^{(i)}[k]/\!\!/u_i$, which is thus E_0/u_i-dual to $\Gamma^{\vee(i)}[k]/\!\!/u_i$. $\square$

Lemma 6.0.6. $\Gamma^{(i)}/\!\!/u_i$ *is an* E_0/u_i*-algebra dual to the co-*E_0/u_i*-algebra* $\Gamma^{\vee(i)}/\!\!/u_i$ *and contains* $\Gamma^{(i+1)}$ *as a subalgebra.*

Proof. Since $\Gamma^{(i)}/\!\!/u_i$ is already a E_0/u_i-module, it remains to produce a multiplication structure. The co-multiplication $\Gamma^{\vee(i)} \to \Gamma^{\vee(i)} \otimes_{E_0/u_{i-1}} \Gamma^{\vee(i)}$ is a map of E_0/u_{i-1}-bimodules (since it is dual to the multiplication map $\Gamma^{(i)} \otimes \Gamma^{(i)} \to \Gamma^{(i)}$ which is a map of E_0/u_{i-1}-bimodules), so it descends to a map $\Gamma^{\vee(i)}/\!\!/u_i \to \left(\Gamma^{\vee(i)} \otimes_{E_0/u_{i-1}} \Gamma^{\vee(i)}\right)/\!\!/u_i$. By Lemma 6.0.2,

$$\left(\Gamma^{\vee(i)} \otimes_{E_0/u_{i-1}} \Gamma^{\vee(i)}\right)/\!\!/u_i \cong \Gamma^{\vee(i)} \otimes (\Gamma^{\vee(i)}/\!\!/u_i) \cong \Gamma^{\vee(i)}{}_t \otimes_s \left(E_0/u_{i_t} \otimes_s \Gamma^{\vee(i)}/\!\!/u_i\right)$$

$$\cong (\Gamma^{\vee(i)}/\!\!/u_i) \otimes (\Gamma^{\vee(i)}/\!\!/u_i),$$

so we get a co-multiplication on $\Gamma^{\vee(i)}/\!\!/u_i$. Applying ΣD gives a multiplication on $\Gamma^{(i)}/\!\!/u_i$. Indeed, since $(\Gamma^{\vee(i)}/\!\!/u_i) \otimes (\Gamma^{\vee(i)}/\!\!/u_i) \in \mathrm{DMod}_{E_0/u_i}$, by Lemma 6.0.3 and Cor 6.0.5,

$$\Sigma D\left((\Gamma^{\vee(i)}/\!\!/u_i) \otimes (\Gamma^{\vee(i)}/\!\!/u_i)\right) \simeq D_{E_0/u_i}\left((\Gamma^{\vee(i)}/\!\!/u_i) \otimes (\Gamma^{\vee(i)}/\!\!/u_i)\right)$$

$$\simeq D_{E_0/u_i}(\Gamma^{\vee(i)}/\!\!/u_i) \otimes D_{E_0/u_i}(\Gamma^{\vee(i)}/\!\!/u_i)$$

$$\simeq (\Gamma^{(i)}/\!\!/u_i) \otimes (\Gamma^{(i)}/\!\!/u_i).$$

Similarly, since the comultiplication on $\Gamma^{\vee(i+1)}$ is a map in DMod_{E_0/u_i}, applying ΣD gives a multiplication on $\Sigma D\Gamma^{\vee(i+1)}$, which is isomorphic to $\Gamma^{(i+1)}$ as E_0/u_i-algebras. Here the algebra structure on $\Gamma^{(i+1)}$ is the usual one coming from addition and multiplication of power operations. Applying ΣD to the quotient of algebras $\Gamma^{\vee(i)}/\!\!/u_i \twoheadrightarrow \Gamma^{\vee(i+1)}$ gives an inclusion of algebras

$$\Gamma^{(i+1)} \simeq \Sigma D\Gamma^{\vee(i+1)} \hookrightarrow \Sigma D\Gamma^{\vee(i)}/\!\!/u_i \simeq \Gamma^{(i)}/\!\!/u_i.$$

$\square$

Lemma 6.0.7. *The projection map* $\Gamma^{(i)} \twoheadrightarrow \Gamma^{(i)}/\!\!/u_i$ *is a map of* E_0/u_{i-1}*-algebras.*

Proof. Consider the (shifted) cofiber sequence $\Gamma^{\vee(i)} \to \Gamma^{\vee(i)}/\!\!/u_i \to \Sigma\Gamma^{\vee(i)}$ in $\mathrm{DMod}_{E_0/u_{i-1}}$. Since $\Gamma^{\vee(i)}$ is a free right E_0/u_{i-1}-module, the tensor product is the same as the derived tensor product. Comultiplication on $\Gamma^{\vee(i)}$ and $\Gamma^{\vee(i)}/\!\!/u_i$ induce a map of cofiber sequences

$$
\begin{array}{ccc}
\Gamma^{\vee(i)} & \longrightarrow & \Gamma^{\vee(i)} \otimes \Gamma^{\vee(i)} \\
\downarrow & & \downarrow \\
\Gamma^{\vee(i)}/\!\!/u_i & \longrightarrow & \Gamma^{\vee(i)} \otimes \Gamma^{\vee(i)}/\!\!/u_i \\
\downarrow & & \downarrow \\
\Sigma\Gamma^{\vee(i)} & \longrightarrow & \Gamma^{\vee(i)} \otimes \Sigma\Gamma^{\vee(i)}
\end{array}
$$

where the top square is a pushout. Applying ΣD gives the diagram

$$
\begin{array}{ccc}
\Sigma\Gamma^{(i)} & \longleftarrow & \Sigma\Gamma^{(i)} \otimes \Gamma^{(i)} \\
\uparrow & & \uparrow \\
\Gamma^{(i)}/\!\!/u_i & \longleftarrow & \Gamma^{(i)} \otimes \Gamma^{(i)}/\!\!/u_i \\
\uparrow & & \uparrow \\
\Gamma^{(i)} & \longleftarrow & \Gamma^{(i)} \otimes \Gamma^{(i)}.
\end{array}
$$

The map $\Gamma^{(i)} \otimes \Gamma^{(i)} /\!/ u_i \simeq (\Gamma^{(i)} /\!/ u_i) \otimes (\Gamma^{(i)} /\!/ u_i) \to \Gamma^{(i)} /\!/ u_i$ is by definition the multiplication map on $\Gamma^{(i)} /\!/ u_i$. Since the first and last rows in the top diagram are the usual comultiplication maps, the first and last rows in the bottom diagram are the usual multiplication maps. Thus, $\Gamma^{(i)} \to \Gamma^{(i)} /\!/ u_i$ is compatible with the algebra structures on the source and target, hence is a map of E_0/u_{i-1}-algebras. $\qquad\square$

We would like to emphasize that although $\Gamma^{(i)} /\!/ u_i$ is a quotient algebra of $\Gamma^{(i)}$, elements of $\Gamma^{(i)} /\!/ u_i$ should not be thought of as equivalence classes of power operations acting on E_0/u_{i-1}. Rather, they should be interpreted as power operations acting on E_0/u_i via the dual $\Gamma^{\vee(i)} /\!/ u_i$. With this interpretation, we will see that $\Gamma^{(i)} /\!/ u_i$ is an algebra of operations generated by one additional operation from $\Gamma^{(i+1)}$.

Since $u_i = 0$ in $\Gamma^{\vee(i)}[k] /\!/ u_i$, Lemma 4.3.2 shows that

$$\Gamma^{\vee(i)}[1] /\!/ u_i \cong \Gamma^{\vee(i)}[1]/u_i, x f^{(i+1)}(x), \qquad (13)$$

so $\Gamma^{(i)}[1] /\!/ u_i$ has rank 1 greater than $\Gamma^{(i+1)}[1]$. Thus, $\Gamma^{(i)} /\!/ u_i$ is generated by operations in $\Gamma^{(i+1)}[1]$ and one additional operation. Let $N = \operatorname{rank} \Gamma^{\vee(i+1)}[1]$. Let $Q_0, \cdots, Q_N$ be operations dual to the basis $1, x, \cdots, x^N$ of $\Gamma^{\vee(i)}[1] /\!/ u_i$. Since the comultiplication map $\Gamma^{\vee(i)}[k] /\!/ u_i \to \Gamma^{\vee(i)}[1]^{\otimes k} /\!/ u_i$ is injective, these operations generate $\Gamma^{(i)} /\!/ u_i$. For $y \in E_0/u_i$, write $P(y) = Q_0(y) + Q_1(y)x + \cdots + Q_N(y)x^N \in \Gamma^{\vee(i)}[1] /\!/ u_i$. Let

$$P(y) = Q_0'(y) + Q_1'(y)x + \cdots + Q_{N-1}'(y)x^{N-1} \in \Gamma^{\vee(i+1)}[1]$$

denote its projection onto $\Gamma^{\vee(i+1)}[1]$. With this notation, $\Gamma^{(i+1)}$ is the subalgebra of $\Gamma^{(i)} /\!/ u_i$ generated by $Q_0', \cdots, Q_{N-1}'$. Since the Weierstrass polynomial of $f^{(i+1)}(x)$ has degree N, $Q_j' = Q_j + c_j Q_N$ for $0 \le j < N$ and some coefficient c_j. Thus, $Q_0', \cdots, Q_{N-1}', Q_N$ generate $\Gamma^{(i)} /\!/ u_i$.

In fact, we can pick the additional operation Q_N to witness a lift of Frobenius. The inclusion of basepoint $i : * \to B\Sigma_p$ induces a map $i^* : E^0(B\Sigma_p) \to E_0$ which is the ring map sending $x = c_p$ to 0. [Rez09, Prop 3.25] shows that the composite of the total power operations map with i^*

$$E_0 \xrightarrow{P} E^0(B\Sigma_p) \xrightarrow{i^*} E_0$$

is the map $y \mapsto y^p$. By (13), passing to quotients gives

$$E_0/u_i \xrightarrow{P} \Gamma^{\vee(i)}[1] /\!/ u_i \xrightarrow{i^*} E_0/u_i, \quad y \mapsto y^p.$$

Since i^* is the ring map sending x to 0, i^* is the operation Q_0, so $Q_0(y) = y^p$. Since the constant term of the Weierstrass polynomial of $f^{(i+1)}(x)$ is a unit multiple $u \cdot u_{i+1}$ of u_{i+1},

$$Q_0'(y) = Q_0(y) + u_{i+1}(-uQ_N)(y) = y^p \mod u_{i+1}.$$

To summarize, we have the following.

Proposition 6.0.8. *1. $\Gamma^{(i)} /\!/ u_i$ is an E_0/u_i-algebra of power operations (generated by weight p operations) acting on $\pi_0(-/u_i)$ of $K(h)$-local $\mathbb{E}_\infty$-E-algebras and contains $\Gamma^{(i+1)}$ as a subalgebra.*

2. It is generated by operations in $\Gamma^{(i+1)}$ together with an additional weight p operation θ not in $\Gamma^{(i+1)}$, such that the operation Q defined by

$$Q(y) = y^p + u_i\theta(y) \equiv y^p \mod u_{i+1}$$

is in $\Gamma^{(i+1)}[1]$.

7 The monad $\mathbb{T}^{(i)}$

In this section, we will define a mod $p, \cdots, u_{i-1}$ analog $\mathbb{T}^{(i)}$ (Def 7.0.7) of Rezk's monad $\mathbb{T}$. We also give a mod $p, \cdots, u_{i-1}$ version of cofreeness of Lubin-Tate theory (Thm 7.0.10).

Let R be a commutative ring and let Mod_R denote the category of left R-modules. For an R-bimodule P, let $H_P : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be the functor given by $H_P(M) = \mathrm{Mod}_R(P, M)$, where $\mathrm{Mod}_R(P, M)$ is the set of left R-module maps $P \to M$. Using our handedness convention in Remark 2.3, [Rez09, Section] shows that if $(\Gamma, \epsilon, \Delta, \eta, \mu)$ is a (graded) twisted commutative R-bialgebra, then H_Γ is a comonad (using η and μ) which is lax symmetric monoidal (using ϵ and Δ).

The category Mod_Γ of Γ-modules is the category of coalgebras for H_Γ. This is the same as an R-module with a left action of Γ. Mod_Γ is symmetric monoidal with tensor product given in [Rez09, 5.4]. The category Alg_Γ of Γ-algebras is the category of commutative monoid objects in Mod_Γ. Since H_Γ is lax symmetric monoidal, the cofree and forgetful adjunction $U_\Gamma \colon \mathrm{Mod}_\Gamma \rightleftarrows \mathrm{Mod}_R \colon H_\Gamma$ restricts to

$$U_\Gamma \colon \mathrm{Alg}_\Gamma \rightleftarrows \mathrm{Alg}_R \colon H_\Gamma.$$

The forgetful functor $U_\Gamma : \mathrm{Alg}_\Gamma \to \mathrm{Alg}_R$ is plethyistic. The free Γ-algebra on an R-module M is the free symmetric algebra $\mathrm{Sym}_R(\Gamma_R \otimes M)$. Here the symmetric powers are taken with respect to the left R-module structure on Γ and the tensor product $\Gamma_R \otimes -$ is using the right R-module structure on Γ. If Γ is a (graded) twisted commutative R-bialgebra, then its left R-dual $\Gamma^\vee$ is a (graded) category object in formal affine schemes.

Rezk shows that the algebra of additive power operations Γ is a (graded) twisted commutative E_0-bialgebra where η and μ are the unit and multiplication maps giving the E_0-algebra structure on Γ, $\epsilon : \Gamma \to E_0$ is the augmentation map, and Δ is induced by comultiplication $\Delta^\times$ on $\mathbb{T}(E_0)$ as follows. Since $\mathbb{T}(E_0)$ corepresents the forgetful functor $U_\mathbb{T} : \mathrm{Alg}_\mathbb{T} \to \mathrm{Alg}_{E_0}$ to E_0-algebras (and not just sets), $\mathbb{T}(E_0)$ admits the structure of a co-E_0-algebra with coaddition Δ^+, comultiplication $\Delta^\times$, and co-E_0-unit ϵ. The monoid of endomorphisms $P = \mathrm{Alg}_\mathbb{T}(\mathbb{T}(E_0), \mathbb{T}(E_0))$ of $U_\mathbb{T}$ has a left E_0-module structure by addition and scaling of natural endomorphisms. Evaluation at the canonical generator $\iota \in \mathbb{T}(E_0)$ defines an isomorphism $P \to \mathbb{T}(E_0)$ of left E_0-modules. $\Gamma \subset P$ is the subset of additive natural endomorphisms of $U_\mathbb{T}$, and the inclusion $\Gamma \subset P \simeq \mathbb{T}(E_0)$ are of left E_0-modules. Γ is the set of primitive elements with respect to Δ^+, and $\Delta^\times$ and ϵ restrict to maps Δ and ϵ on Γ. Explicitly, $\Delta^\times$ and ϵ come from the E_0-algebra structures of the cohomology rings $E^0(B\Sigma_{p^k})$.

[Rez09, Proposition 7.25] shows that each of the forgetful functor

$$\mathrm{Alg}_\mathbb{T} \xrightarrow{U} \mathrm{Alg}_\Gamma \xrightarrow{U_\Gamma} \mathrm{Alg}_{E_0}$$

is plethyistic (hence monadic and comonadic) [Rez09, 4.22], and $U_\Gamma \circ U = U_\mathbb{T}$. $U_\mathbb{T}$ has free and cofree functors $F_\mathbb{T} = \mathbb{T}(-)$ and $W_\mathbb{T} = \mathrm{Alg}_{E_0}(\mathbb{T}(E_0), -)$ [BSY24, Proposition 3.17]. The right adjoint of U_Γ is given by $H_\Gamma = \mathrm{Mod}_{E_0}(\Gamma, -) = \mathrm{Alg}_{E_0}(\mathrm{Sym}_{E_0}(\Gamma), -)$. Note that unlike the case of $\mathbb{T}$, Γ is not an element in Alg_Γ since it is not commutative, so we need to take the free symmetric algebra.

We claim that all the above go through for $\Gamma^{(i-1)} /\!\!/ u_{i-1}$ and $\Gamma^{(i)}$.

Proposition 7.0.1. $\Gamma^{(i-1)} /\!\!/ u_{i-1}$ *and* $\Gamma^{(i)}$ *are (graded) twisted commutative* E_0 / u_{i-1}*-bialgebras. Thus,* $\mathrm{Sub}^{(i-1)} /\!\!/ u_{i-1}$ *and* $\mathrm{Sub}^{(i)}$ *are (graded) category objects in formal affine schemes.*

Proof. We check (a)-(d) in [Rez09, 5.3]. $\Gamma^{(i-1)}/\!\!/u_{i-1}$ and $\Gamma^{(i)}$ are associative E_0/u_{i-1}-algebras with unit η and multiplication μ, giving (a) and (b). ϵ is the augmentation map, which is projection onto the weight 0 part, so the condition in (c) is satisfied. $\Gamma^{\vee(i)}[k]$ is a quotient ring of $E^0(B\Sigma_{p^k})$ and $\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1} = \Gamma^{\vee(i-1)}[k]/P(u_{i-1})$ is the quotient of the ring $\Gamma^{\vee(i-1)}[k]$ by the ideal generated by $P(u_{i-1})$. Thus, the product structure on the cohomology ring $E^0(B\Sigma_{p^k})$ descends to $\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1}$ and $\Gamma^{\vee(i)}[k]$, giving a map Δ on the duals which is coassociative and cocommutative with counit ϵ.

We check that Δ is a 2-multimorphism. Since the left E_0-module structure on $E^0(B\Sigma_{p^k})$ is given by its usual E_0-algebra structure (i.e. source map), the multiplication map on $E^0(B\Sigma_{p^k})$ is a map of left E_0-modules. The right E_0-module structure is given by the target map t which is a ring map. Since multiplication on $E^0(B\Sigma_{p^k})$ is commutative, for all $e \in E_0$ and $p, q \in E^0(B\Sigma_{p^k})$,

$$pt(e) \cdot q = pqt(e) = p \cdot qt(e),$$

so the multiplication map is a map of right E_0-modules with respect to both of the right module structures on $E^0(B\Sigma_{p^k})$ (see [Rez09, 5.1]).

The condition (d) on Δ is precisely the fact the product on $\Gamma^\vee[k]_t \otimes_s \Gamma^\vee[\ell]$ as a quotient of the cohomology ring $E^0(B\Sigma_{p^\ell} \wr \Sigma_{p^k})$ is the same as the componentwise product on $\Gamma^\vee[k]$ and $\Gamma^\vee[\ell]$. This condition and the condition of being a 2-multimorphism are satisfied by $\Gamma^\vee[k]$. Since $\Gamma^{\vee(i-1)}[k]/\!\!/u_{i-1}$ and $\Gamma^{\vee(i)}[k]$ are its quotient rings, they satisfy these conditions as well, giving (d). $\qquad\square$

Definition 7.0.2. For $\Gamma = \Gamma^{(i)}$ or $\Gamma^{(i-1)}/\!\!/u_{i-1}$, the category Mod_Γ of modules over Γ is defined to be the category of coalgebras for the comonad H_Γ. The category Alg_Γ of Γ-algebras is the category of commutative monoid objects in Mod_Γ.

Corollary 7.0.3. *For $\Gamma = \Gamma^{(i-1)}/\!\!/u_{i-1}$ or $\Gamma^{(i)}$, the forgetful functor $U_\Gamma : \mathrm{Alg}_\Gamma \to \mathrm{Alg}_{E_0/u_{i-1}}$ is monadic and comonadic. $U_{\Gamma^{(i-1)}/\!\!/u_{i-1}}$ factors as*

$$\mathrm{Alg}_{\Gamma^{(i-1)}/\!\!/u_{i-1}} \xrightarrow{U} \mathrm{Alg}_{\Gamma^{(i)}} \xrightarrow{U_{\Gamma^{(i)}}} \mathrm{Alg}_{E_0/u_{i-1}}$$

and each of U, $U_{\Gamma^{(i)}}$, and $U_{\Gamma^{(i-1)}/\!\!/u_{i-1}}$ is plethyistic.

Proof. The proof follows from Rezk. Since $\Gamma^{(i)} \subset \Gamma^{(i-1)}/\!\!/u_{i-1}$ is a subalgebra, $U_{\Gamma^{(i-1)}/\!\!/u_{i-1}}$ factors through $U_{\Gamma^{(i)}}$. The argument in [Rez09, Proposition 7.25] gives the plethyistic part. $\quad\square$

An analog $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ of $\mathrm{Alg}_{\mathbb{T}}$ should consist of objects acted on by $\Gamma^{(i-1)}/\!\!/u_{i-1}$ satisfying some "Frobenius congruence". We will define this to be a certain subcategory of $\mathrm{Alg}_{\Gamma^{(i-1)}/\!\!/u_{i-1}}$. To simplify notation, let $\widetilde{\Gamma}^{(i)} = \Gamma^{(i-1)}/\!\!/u_{i-1}$. Since $H_{\widetilde{\Gamma}^{(i)}}$ has a left adjoint, it is an accessible comonad, so the category $\mathrm{Mod}_{\widetilde{\Gamma}^{(i)}}$ of coalgebras over $H_{\widetilde{\Gamma}^{(i)}}$ is locally presentable. Thus, the category $\mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$ of commutative monoid objects is also locally presentable.

Recall that $Q_0 \in \Gamma^{(i-1)}[1]/\!\!/u_{i-1}$ is the operation corresponding to the ring map $\Gamma^{\vee(i-1)}[1]/\!\!/u_{i-1} \to E_0/u_{i-1}, x \mapsto 0$ which classifies $\ker \varphi$.

Definition 7.0.4. Let $\mathrm{Alg}_{\mathbb{T}^{(0)}} = \mathrm{Alg}_{\mathbb{T}}$. For $i \geq 1$, let $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ be the full subcategory of $\mathrm{Alg}_{\Gamma^{(i-1)}/\!\!/u_{i-1}}$ consisting of objects A such that $Q_0 : A \to A$ is equal to the ring Frobenius on A.

Lemma 7.0.5. $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ *is locally presentable. Thus, by the adjoint functor theorem, the inclusion* $\mathrm{Alg}_{\mathbb{T}^{(i)}} \to \mathrm{Alg}_{\Gamma^{(i-1)}/\!\!/u_{i-1}}$ *has left and right adjoints, hence is plethyistic.*

Proof. Since $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ is a full subcategory of the locally presentable category $\mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$, it suffices to show that it is closed under limits and colimits. Since the forgetful functor $U_{\widetilde{\Gamma}^{(i)}}$ perserves limits and colimits, limits and colimits in $\mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$ can be computed in $\mathrm{Alg}_{E_0/u_{i-1}}$.

Let $A = \lim_{\alpha \in I} A_\alpha$ be a limit in $\mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$ with each $A_\alpha \in \mathrm{Alg}_{\mathbb{T}^{(i)}}$. Let $(a_\alpha)_I \in A$. For each $\alpha \in I$, Q_0 commutes with the projection $A \to A_\alpha$ since all maps are maps of $\widetilde{\Gamma}^{(i)}$-algebras. Since Q_0 is the ring Frobenius on A_α, $Q_0((a_\alpha)_I) = (Q_0(a_\alpha))_I = (a_\alpha^p)_I$, so Q_0 is the ring Frobenius on A, hence $A \in \mathrm{Alg}_{\mathbb{T}^{(i)}}$.

To show that $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ is closed under colimits, it suffices to show that it is closed under finite coproducts, pushouts, and filtered colimits. The coproduct of A and B in $\mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$ is the tensor product $A \otimes B$ over E_0/u_{i-1} [Rez09, 5.4]. Given $a \otimes b \in A \otimes B$, $Q_0(a \otimes b)$ is given by

$$E_0/u_{i-1} \xrightarrow{Q_0} \widetilde{\Gamma}^{(i)} \xrightarrow{\Delta}_s \widetilde{\Gamma}^{(i)} \otimes_s \widetilde{\Gamma}^{(i)} \xrightarrow{P_A(a) \otimes P_B(b)} A \otimes B,$$

where P_A, P_B are the $H_{\widetilde{\Gamma}^{(i)}}$-algebra structures on A and B. Since $Q_0 : \widetilde{\Gamma}^{\vee(i)}[1] \to E_0/u_{i-1}$ is a ring map, the diagram

$$\begin{array}{ccc}
\widetilde{\Gamma}^{\vee(i)}[1]_s \otimes_s \widetilde{\Gamma}^{\vee(i)}[1] & \longrightarrow & \widetilde{\Gamma}^{\vee(i)}[1] \\
{\scriptstyle Q_0 \otimes Q_0} \downarrow & & \downarrow {\scriptstyle Q_0} \\
E_0/u_{i-1} \otimes E_0/u_{i-1} & \xrightarrow{\cong} & E_0/u_{i-1}
\end{array}$$

commutes, where the top horizontal map is the cohomology ring product dual to Δ. Dualizing gives $Q_0(a \otimes b) = Q_0(a) \otimes Q_0(b)$. Thus, if $A, B \in \mathrm{Alg}_{\mathbb{T}^{(i)}}$, so is their coproduct $A \otimes B$. The case of pushouts follows from the fact that the projection $A \otimes B \to A \otimes_C B$ is a surjective map in $\mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$, hence respects power operations.

For filtered colimits, since each element of the filtered colimit $A = \mathrm{colim}_{\alpha \in I} A_\alpha$ comes from some A_α, the action of Q_0 on it is determined by the action of Q_0 on A_α, so $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ is closed under filtered colimits. Thus, $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ is a full subcategory closed under all limits and colimits, hence locally presentable and the inclusion $\mathrm{Alg}_{\mathbb{T}^{(i)}} \to \mathrm{Alg}_{\widetilde{\Gamma}^{(i)}}$ is plethyistic by the adjoint functor theorem. $\square$

Together with Corollary 7.0.3, this implies the following.

Corollary 7.0.6 (The monad $\mathbb{T}^{(i)}$). *The first three functors in the composite*

$$\mathrm{Alg}_{\mathbb{T}^{(i)}} \to \mathrm{Alg}_{\Gamma^{(i-1)} /\!\!/ u_{i-1}} \to \mathrm{Alg}_{\Gamma^{(i)}} \to \mathrm{Alg}_{E_0/u_{i-1}} \to \mathrm{Mod}_{E_0/u_{i-1}} \tag{14}$$

are plethyistic and the last one is monadic.

Definition 7.0.7. Let $\mathbb{T}^{(i)}$ be the monad on $\mathrm{Mod}_{E_0/u_{i-1}}$ coming from the composition of adjunctions (14).

With this definition, $\mathrm{Alg}_{\mathbb{T}^{(i)}}$ is the category of algebras for $\mathbb{T}^{(i)}$.

Lemma 7.0.8. *(1) If $A \in \mathrm{Alg}_{\Gamma^{(i)}}$, then $A/u_i \in \mathrm{Alg}_{\Gamma^{(i)} /\!\!/ u_i}$.*

(2) If $A \in \mathrm{Alg}_{\mathbb{T}^{(i)}}$, then $A/u_i \in \mathrm{Alg}_{\mathbb{T}^{(i+1)}}$.

Proof. Let $\Gamma = \Gamma^{(i)}, \Gamma^{(i-1)} /\!\!/ u_{i-1}$, or $\mathbb{T}^{(i)}$. Since Alg_Γ is symmetric monoidal with unit E_0/u_{i-1}, for each $A \in \mathrm{Alg}_\Gamma$, the unit map $E_0/u_{i-1} \to A$ is a map of Γ-algebras. For (1), given $A \in$

$\text{Alg}_{\Gamma^{(i)}}$, consider the diagram

$$
\begin{array}{ccccc}
E_0/u_{i-1} & \longrightarrow & A & \xrightarrow{\;P_A\;} & A \otimes_s \Gamma^{\vee(i)}[k] \\
\downarrow & & \downarrow & & \downarrow \\
E_0/u_o & \longrightarrow & A/u_i & \dashrightarrow & A \otimes_s \Gamma^{\vee(i)}[k] /\!\!/ u_i \cong A/u_i \otimes_s \Gamma^{\vee(i)}[k] /\!\!/ u_i
\end{array}
$$

Since the left square is a pushout and the outer diagram commutes, the dotted map exists, giving the $\Gamma^{(i)}/\!\!/u_i$-algebra structure on A/u_i. In fact, the outer square is a pushout hence so is the right square.

For (2), if $A \in \text{Alg}_{\mathbb{T}^{(i)}}$, then $A \in \text{Alg}_{\Gamma^{(i)}}$ so $A/u_i \in \text{Alg}_{\Gamma^{(i)}/\!\!/u_i}$ by (1). Let $\widetilde{Q}_0, Q_0$ be the operations corresponding to $\ker \varphi$ in $\Gamma^{\vee(i-1)}/\!\!/u_{i-1}$ and $\Gamma^{\vee(i)}/\!\!/u_i$, respectively. The actions of $\widetilde{Q}_0$ on A and Q_0 on A/u_i are given by the top and bottom rows of the commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & A \otimes_s \Gamma^{\vee(i-1)}[1]/\!\!/u_{i-1} & \xrightarrow{1 \otimes \widetilde{Q}_0} & A \\
\downarrow & & \downarrow & & \downarrow \\
A/u_i & \longrightarrow & A \otimes_s \Gamma^{\vee(i)}[1]/\!\!/u_i & \xrightarrow{1 \otimes Q_0} & A/u_i.
\end{array}
$$

Since $A \to A/u_i$ is surjective and $\widetilde{Q}_0$ is the ring Frobenius, so is Q_0. Thus, $A/u_i \in \text{Alg}_{\mathbb{T}^{(i+1)}}$. $\quad\square$

In [SS25], we will give a mod $p, \cdots, u_{i-1}$ version of the congruence criterion [Rez09, Thm A] using the decomposition (12).

A key idea in proving the Chromatic Nullstellensatz is to exhibit $\pi_0 E(-)$ as the cofree $\mathbb{T}$-algebra. This generalizes to arbitrary heights the height 1 phenomena of the Witt vectors being cofree δ-rings.

Theorem 7.0.9 ([BSY24, Theorem 3.4] Cofreeness of Lubin-Tate theory)**.** *Let k be a perfect field of characteristic p and let Perf_k denote the category of perfect k-algebras. For $A \in \text{Perf}_k$, let $E(A)$ denote any height h Lubin-Tate theory associated to A. There is an adjunction*

$$(-/\mathfrak{m})^{\text{perf}} : \text{Alg}_{\mathbb{T}} \rightleftarrows \text{Perf}_k : \pi_0 E(-)$$

where the right adjoint is fully faithful. Here $\mathfrak{m} = (p, \cdots, u_{h-1})$ is the maximal ideal and $(-)^{\text{perf}}$ is the colimit perfection, which is the colimit along the Frobenius map.

The same proof gives the mod $p, \cdots, u_{i-1}$ analog.

Theorem 7.0.10. *There is an adjunction*

$$(-/\mathfrak{m})^{\text{perf}} : \text{Alg}_{\mathbb{T}^{(i)}} \rightleftarrows \text{Perf}_k : \pi_0 E(-)/u_{i-1}$$

where the right adjoint is fully faithful.

We first recall the proof of [BSY24, Theorem 3.4], then explain the changes needed for the mod $p, \cdots, u_{i-1}$ case. The following is a proof due to Akhil Mathew and explained to us by Allen Yuan.

Proof of [BSY24, Theorem 3.4]. Let $\Psi \in \Gamma$ denote the operation corresponding to the $[p]$ isogeny. Since $[p]$ commutes with every isogeny, Ψ is central in Γ, so for all $\mathbb{T}$-algebras R, $\Psi : R \to R$ is a map of $\mathbb{T}$-algebras. Define a $\mathbb{T}$-algebra R to be perfect if Ψ is an isomorphism on R. For $R \in \mathrm{Alg}_{\mathbb{T}}$, let $R^{\mathrm{perf}} = \mathrm{colim}(R \xrightarrow{\Psi} R \xrightarrow{\Psi} \cdots)$ be its the colimit perfection. Since Ψ is $\mathfrak{m}$-adically continuous, the derived $\mathfrak{m}$-completion of a perfect $\mathbb{T}$-algebra is perfect, so we have an adjunction

$$(\hat{-})^{\mathrm{perf}} : \mathrm{Alg}_{\mathbb{T}} \rightleftarrows \mathrm{PerfAlg}_{\mathbb{T}}^{\wedge} : \mathrm{fgt}$$

where $\mathrm{PerfAlg}_{\mathbb{T}}^{\wedge}$ is the category of perfect (derived) $\mathfrak{m}$-complete $\mathbb{T}$-algebras.

We claim that there is an equivalence of categories $(-)/\mathfrak{m} : \mathrm{PerfAlg}_{\mathbb{T}}^{\wedge} \to \mathrm{Perf}_k$. Essential surjectivity is clear. To see fully faithfulness, by Lemma 7.0.11 below, every perfect (derived) $\mathfrak{m}$-complete $\mathbb{T}$-algebra is of the form $E_0(A)$, and we have an inclusion

$$\mathrm{Alg}_{\mathbb{T}}(E_0(A_1), E_0(A_2)) \subseteq \mathrm{Alg}_{E_0}(E_0(A_1), E_0(A_2)) = \mathrm{Alg}_k(A_1, A_2)$$

where the last equality is by formal étaleness of $E_0(k) \to E_0(A_1)$. This inclusion has a section since any k-algebra map $A_1 \to A_2$ induces a map $E(A_1) \to E(A_2)$. $\qquad\square$

Lemma 7.0.11. *The functor* $(\hat{-})^{\mathrm{perf}} : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{PerfAlg}_{\mathbb{T}}^{\wedge}$ *sends R to $E_0(A)$ for some perfect k-algebra A.*

Proof. View $R \in \mathrm{Alg}_{\mathbb{T}}$ as an animated $\mathbb{T}$-algebra and $(-)/\!/\mathfrak{m} : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{Alg}_k^{\mathrm{ani}}$ as a functor to animated k-algebras, where $(-)/\!/\mathfrak{m}$ denotes the derived quotient. Then $\hat{R}^{\mathrm{perf}}/\!/\mathfrak{m} \simeq (\hat{R}/\!/\mathfrak{m})^{\mathrm{perf}} =: A$ since Ψ acts as the (animated) Frobenius modulo $\mathfrak{m}$. But perfect animated rings are discrete and $\hat{R}^{\mathrm{perf}}$ is an $\mathfrak{m}$-complete $E_0(k)$-algebra, so one can deduce that $\hat{R}^{\mathrm{perf}} \simeq E_0(A)$ as $E_0(k)$-algebras.

We still need to show that $\hat{R}^{\mathrm{perf}} \simeq E_0(A)$ as $\mathbb{T}$-algebras. Let $(-)^{\flat} = \lim_{\phi}(-)/p$ denote the Tilt, where ϕ is the ring Frobenius. By the Witt-tilt adjunction, there is a map of rings $W(A) \to \hat{R}^{\mathrm{perf}}$ corresponding to the identity map $A \to A = E_0(A)^{\flat} \simeq (\hat{R}^{\mathrm{perf}})^{\flat}$. By Lemma 7.0.12, for any $a \in A$, the Teichmüller element $[a] \in W(A)$ are acted on trivially by power operations i.e. $P([a]) = [a]^p$ where P is the total additive power operation. This, together with the map of $\mathbb{T}$-algebras $E_0(k) \to \hat{R}^{\mathrm{perf}}$ given by the unit, uniquely determines how power operations act on $\hat{R}^{\mathrm{perf}}$ since $\hat{R}^{\mathrm{perf}} \simeq E_0(A) \simeq W(A) \otimes_{W(k)} E_0(k)$ as $E_0(k)$-algebras. These also uniquely determine the $\mathbb{T}$-algebra structure on $E_0(A)$, hence they agree. $\qquad\square$

Lemma 7.0.12. *Let S be an $\mathfrak{m}$-complete $\mathbb{T}$-algebra. If $x \in S$ has infinite pth power roots, then $P(x) = x^p$.*

Proof. The total (additive) power operations is a map $P : S \to S[[t]]/\langle p \rangle(t)$ with $P(t) = t^p + t(\cdots)$. Since P is a ring map, $P(y^{p^N}) = (y^p + y(\cdots))^{p^N}$. Since $\langle p \rangle(t) \equiv t^{p^n} \mod \mathfrak{m}$ in $E_0[[t]]$, $t^{p^n} \in \mathfrak{m}$ in $S[[t]]/\langle p \rangle(t)$. Using this and the fact that $p \in \mathfrak{m}$, we see that for any $d \geq 0$, there exists $N \gg 0$ such that $P(y^{p^N}) \equiv y^{p^N} \mod \mathfrak{m}^d$. Since x has infinite pth power roots, $P(x) = P(y^{p^N}) \equiv y^{p^N} = x \mod \mathfrak{m}^d$ for all d, thus $P(x) = x$ by completeness of S. $\qquad\square$

The proof of Theorem 7.0.10 follows word by word from the above. We check the following.

Lemma 7.0.13. *The operation Ψ corresponding to the $[p]$ isogeny is an element of $\Gamma^{(i-1)}[h]/\!/u_{i-1}$.*

It is crucial that we use $\Gamma^{(i-1)}/\!/u_{i-1}$ since Ψ is not an element of $\Gamma^{(i)}$.

Proof. Modulo $p, \cdots, u_{i-1}$, $\mathbb{G}[p] = \overline{(\ker \varphi^i, H)}$ is the composite of $\ker \varphi^i$ followed by a degree p^{h-i} subgroup $H < \varphi^i \mathbb{G}$. Over $u_i^{-1} E_0/u_{i-1}$, $u_i^{-1} H$ is the étale part of $\mathbb{G}[p]$. Since u_i is not a zero divisor in E_0/u_{i-1}, by Lemma 5.0.2, H avoids $\ker \varphi^i$, so $H \in \mathrm{Sub}_{h-i}^{(i)}(\varphi^i \mathbb{G})$. The lemma then follows by the decomposition 12. $\square$

Lemma 7.0.14. *For $A \in \mathrm{Alg}_{\mathbb{T}^{(i)}}$, $\Psi : A \to A$ is the ring Frobenius ϕ^h modulo $\mathfrak{m}$.*

Proof. Let $i > 0$, so everything is in characteristic p. By definition, the operation Q_0 corresponding to the Frobenius isogeny agrees with the ring Frobenius ϕ on A. Since $[p]$ is a lift of the h-fold Frobenius isogeny on $\mathbb{G}/\mathfrak{m}$, the map $\Gamma^\vee[h] \to E_0$ classifying $\mathbb{G}[p]$ agrees modulo $\mathfrak{m}$ with the map $\Gamma^\vee[h] \to E_0/p$ classifying $\ker \varphi^h$, which acts on A as $Q_0^h = \phi^h$. $\square$

Lemma 7.0.11, which show that $(\hat{-})^{\mathrm{perf}} : \mathrm{Alg}_{\mathbb{T}^{(i)}} \to \mathrm{PerfAlg}_{\mathbb{T}}^\wedge$ sends R to $E_0(A)/u_{i-1}$ is even simpler in the case $i > 0$: since we are in characteristic p, we do not have to deal with Teichmüller lifts. For the last part of the proof, we have inclusions

$$\mathrm{Alg}_{\mathbb{T}^{(i)}}(E_0(A_1)/u_{i-1}, E_0(A_2)/u_{i-1}) \subseteq \mathrm{Alg}_{E_0/u_{i-1}}(E_0(A_1)/u_{i-1}, E_0(A_2)/u_{i-1}) = \mathrm{Alg}_k(A_1, A_2)$$

where the last equality is again by formal étaleness. This inclusion has a section given by

$$\mathrm{Alg}_k(A_1, A_2) \to \mathrm{Alg}_{\mathbb{T}}(E_0(A_1), E_0(A_2)) \to \mathrm{Alg}_{\mathbb{T}^{(i)}}(E_0(A_1)/u_{i-1}, E_0(A_2)/u_{i-1}).$$

8 Cochains on spheres and the triviality conjecture

In this section, we apply the theory of power operations modulo $p, \cdots, u_{h-2}$ to prove the remaining cases of the Triviality Conjecture 8.0.3 in [HL24].

Let $\mathcal{C}$ be a stably $\mathbb{E}_n$-monoidal ∞-category with unit $\mathbf{1}$. Let $\mathcal{O}$ be an operad in $\mathcal{C}$ which is nonunital ($\mathcal{O}(0) = 0$) and reduced ($\mathcal{O}(1) = \mathbf{1}$). Such operads come with an augmentation map $\epsilon : \mathcal{O} \to \mathbf{1}$ to the trivial operad $\mathbf{1}$, where $\mathbf{1}(1) = \mathbf{1}$ and $\mathbf{1}(n) = 0$ for $n \neq 1$ (this is also the non-unital $\mathbb{E}_0$-operad). The trivial $\mathcal{O}$-algebra functor $\mathrm{tr}_{\mathcal{O}}$ is the forgetful functor $\mathrm{tr}_{\mathcal{O}} := \epsilon^* : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We say that an augmented $\mathbb{E}_n$-algebra in $\mathcal{C}$ is $\mathbb{E}_n$-trivial if it is in the image of $\mathrm{tr}_{\mathbb{E}_n}$.

For R an $\mathbb{E}_{n+1}$-ring spectrum, we say that an augmented $\mathbb{E}_n$-R-algebra is $\mathbb{E}_n$-trivial over R if it is $\mathbb{E}_n$-trivial as an element of $\mathcal{C} = \mathrm{Mod}_R(\mathrm{Sp})$. If X is a space, the R-valued cochains $C^*(X; R)$ on X is the limit $\lim_X R$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}_R)$. If Y is a pointed space, we write R^Y for the mapping spectrum $\mathrm{map}_{Sp}(\Sigma^\infty Y, R)$. For X a space, $C^*(X; R) \simeq R^{X_+}$. If X is pointed, then $C^*(X; R)$ is an augmented $\mathbb{E}_n$-R-algebra.

Theorem 8.0.1 (Theorem A [HL24])**.** *Let R be an $\mathbb{E}_{n+1}$-ring spectrum and X a pointed space. Then $C^*(\Sigma^n X; R)$ is $\mathbb{E}_n$-trivial over R.*

This is a special case of Theorem B in [HL24], using the fact that $C^*(\Sigma^n X; R) \simeq \Omega^n C^*(X; R)$ is an n-fold loop object.

Theorem 8.0.2 (Theorem B [HL24])**.** *Let $\mathcal{C}$ be a stably $\mathbb{E}_n$-monoidal ∞-category. If A is an augmented $\mathbb{E}_n$-algebra in $\mathcal{C}$, then $\Omega^n A$ is $\mathbb{E}_n$-trivial.*

They conjecture that this bound is sharp and prove it in some cases as follows.

Conjecture 8.0.3 (Triviality Conjecture [HL24])**.** *Let $n \geq 1$ and let R be an $\mathbb{E}_{n+2}$-algebra spectrum. Then $C^*(S^n; R)$ is $\mathbb{E}_{n+1}$-trivial over R if and only if R is a $\mathbb{Q}$-algebra.*

Theorem 8.0.4 (Theorem C [HL24])**.** *The Triviality Conjecture holds*

(1) when $n \leq 2$,

(2) when R is bounded below,

(3) when R is the underlying $\mathbb{E}_{n+2}$-algebra of an $\mathbb{E}_{\infty}$-algebra, and

(4) when there exists a prime p such that $R \otimes \mathbb{F}_p$ or $R \otimes KU/p$ are non-zero.

We will resolve the remaining cases and show that the conjecture is true for all R. To prove the triviality conjecture for the case $R \otimes KU/p \neq 0$, [HL24, Prop 4.22] uses height 1 power operations. Our proof will reduce to Lemma 8.0.8, which is similar to this, but we instead consider power operations modulo $p, \cdots, u_{h-2}$.

We first make a reduction. By Theorem C, it suffices to consider the case where $R \neq 0$ and $R \otimes F_p = 0$ for all primes p. If $L_{K(0)}R \neq 0$ and for all primes p and $h \geq 0$ we have $L_{K(h)}R = 0$, then $R \otimes \mathrm{End}(\mathbb{S}/p)$ is $K(h)$-acyclic for all $0 \leq h \leq \infty$. By the Nilpotence theorem, $R \otimes \mathrm{End}(\mathbb{S}/p) = 0$, so every prime is invertible in R, hence R is rational [HL24, Lemma 4.13].

Thus, we may assume $L_{K(h)}R \neq 0$ for some prime p and some $1 \leq h < \infty$. This implies that $L_{K(h)}(E_h \otimes R) \neq 0$, so we may further assume that R is a nonzero $K(h)$-local $\mathbb{E}_{n+2}$-E_h-algebra.

Notation 8.0.5. • Let $E = E_h$.

- Let ρ (resp. $\bar{\rho}$) denote the real (resp. reduced) standard representation of Σ_p.

- For $n \geq 1$, let
$$X_{n,k} := \mathbb{E}_n(p) \otimes_{h\Sigma_p} \mathbb{S}^{k \otimes p} = \mathbb{E}_n(p)_{h\Sigma_p}^{k\rho}$$
  be the Thom spectrum for $k\rho$ over $\mathbb{E}_n(p)_{h\Sigma_p}$ and let $\overline{X_{n,k}} := \mathbb{E}_n(p)_{h\Sigma_p}^{k\bar{\rho}} = \Sigma^{-k} X_{n,k}$. Then $E^0(\overline{X_{n,k}})$ parametrizes weight p power operations acting on π_k of $K(h)$-local $\mathbb{E}_n$-E-algebras. As in Section 2, there are suspension maps $\overline{X_{n,k}} \to \overline{X_{n,k+1}}$ induced by the inclusion of representations $k\bar{\rho} \subset (k+1)\bar{\rho}$. These relate power operations acting on π_k to power operations acting on π_{k+1}.

- Let tr_k be the ideal in $E^0(B\Sigma_p^{k\rho})$ generated by transfers from nontransitive subgroups of Σ_p. Let $\mathrm{tr} = \mathrm{tr}_0$. If we write $E^0(B\Sigma_p) \cong E_0[[x]]/xf(x)$, then $E^0(B\Sigma_p)/\mathrm{tr} \cong E_0[[x]]/f(x)$.

We first recall how power operations work. If R is an $\mathbb{E}_n$-E-algebra, there is a commutative square

$$
\begin{array}{ccc}
E_0 & \xrightarrow{P} E^0(B\Sigma_p) \longrightarrow & E^0(X_{n,0}) \\
\downarrow & & \downarrow \\
\pi_0 R & \xrightarrow{\qquad P \qquad} & R^0(X_{n,0})
\end{array}
\tag{15}
$$

where the total power operations map P sends an element $\mathbb{S} \to R$ in $\pi_0 R$ to $X_{n,0} \to \mathbb{E}_n(p) \otimes_{h\Sigma_p} R^{\otimes p} \to R$. If $y \in \pi_0 R$ is in the image of E_0, then $P(y)$ lands in the subset $\pi_0 R \otimes_{E_0} E^0(X_{n,0})$ of $R^0(X_{n,0})$.

If X is a spectrum, the total power operations map $R^{-q}(X) \to R^0(\mathbb{E}_n(p) \otimes_{h\Sigma_p} X^{\otimes p})$ sends an element $\Sigma^q X \to R$ in $R^{-q}(X)$ to $\mathbb{E}_n(p) \otimes_{h\Sigma_p} X^{\otimes p} \to \mathbb{E}_n(p) \otimes_{h\Sigma_p} R^{\otimes p} \to R$. If $X = \mathbb{S}^j$ is a sphere, we can further postcompose this with suspension maps to get

$$R^{-q}(S^j) \to R^0(X_{n,j+q}) \to R^{i-j-q}(X_{n,i}).$$

Lemma 8.0.6. *For R a $K(h)$-local E-algebra, $R^0(X_{2n+1,0})$ and $R^0(X_{2n+2,0})$ has $R^0(B\Sigma_p)/x^{n+1}$ as a quotient ring. In particular, x^n is nonzero in both of these rings.*

Proof. For R a $K(h)$-local E-algebra, $R^0(B\Sigma_p) \cong \pi_0 R \otimes_{E_0} E^0(B\Sigma_p)$ since $E^0(B\Sigma_p)$ is a finite free E_0-module [Rez09, 3.23]. As a CW complex, $B\Sigma_p$ has cells in dimensions 0, $2(p-1)-1$, $2(p-1)$, $4(p-1)-1$, $4(p-1)$, $\cdots$. Since $X_{n,0} = \mathbb{E}_n(p)_{h\Sigma_p}$ is $B\Sigma_p$ with only the first n cells, $E^0(X_{2n-1,0}) \cong E^0(B\Sigma_p)/x^n$ and $R^0(X_{2n-1,0}) \cong R^0(B\Sigma_p)/\!\!/x^n$, where $/\!\!/$ denotes the derived quotient, giving the odd case. The even case follows from the surjections $R^0(X_{n+1,k}) \twoheadrightarrow R^0(X_{n,k})$ and the fact that $R^0(X_{n,0}) \supseteq \pi_0 R \otimes_{E_0} E^0(X_{n,0})$ for all n. $\qquad\square$

Recall that $\Gamma^{\vee(i)}[1] \cong \pi_0\left(E^{B\Sigma_p}\right)/(p, \cdots, u_{i-1}, f^{(i)}(x))$ where $f^{(i)}(x)$ is the power series such that $f(x) \equiv x^{p^i} f^{(i)}(x) \mod p, \cdots, u_{i-1}$.

Definition 8.0.7. For R an E-algebra and $a \geq 1$, let

$$\Gamma_R^{\vee(i)}[1] = R^0(B\Sigma_p)/(p, \cdots, u_{i-1}, f^{(i)}(x)) = \pi_0 R \otimes_{E_0} \Gamma^{\vee(i)}[1]$$
$$\Gamma_{n,R}^{\vee(i)}[1] = R^0(X_{n,0})/(p, \cdots, u_{i-1}, f^{(i)}(x)).$$

The total power operations map P descends to a map $(\pi_0 R)/p, \cdots, u_{i-1} \to \Gamma_{n,R}^{\vee(i)}[1]$ and there are analogs of the square (15) for all i. Note that while $P : (\pi_0 R)/u_{i-1} \to \Gamma_{n,R}^{\vee(i)}[1]$ is well-defined and multiplicative, it is only additive on the image of E_0/u_{i-1}, which is contained in the subset $\pi_0 R \otimes_{E_0} E^0(X_{n,0})/(p, \cdots, u_{i-1}, f^{(i)}(x))$.

Proof of the Triviality conjecture. Since all $\mathbb{E}_{n+1}$-power operations vanish on (the augmentation ideal of) trivial $\mathbb{E}_{n+1}$-algebras, to prove the conjecture, it suffices to show that there exists a nonvanishing $\mathbb{E}_{n+1}$-power operation on $C^*(S^n; R)$, equivalently, that the total $\mathbb{E}_{n+1}$-power operations map is nonzero. Let $A = R^{S^n}$ be the augmentation ideal of $R^{S^n_+}$.

   (i) If $n+1 = 2r+1$ is odd, the total power operations map $\pi_0 A \to A^0(X_{n+1,0})$ is given by

$$\pi_0 A = R^0(\mathbb{S}^n) \to R^0(X_{n+1,n}) \to R^{-n}(X_{n+1,0}) = A^0(X_{n+1,0}).$$

Since $n = 2r$ is even, $n\rho$ can be identified with r copies of the complex standard representation of Σ_p. This gives a Thom isomorphism $R^0(X_{n+1,n}) \cong R^{-n}(X_{n+1,0})$ under which the last map can be identified with multiplication by the Euler class x^r on $R^0(X_{n+1,0})$, where we have used periodicity of E to identify R^{-n} with R^0. Since $R^0(\mathbb{S}^n) \cong \pi_0 R$ by periodicity, the power operations map on $\pi_0 A$ can be identified with

$$\pi_0 R \xrightarrow{P} R^0(X_{n+1,0}) \xrightarrow{\cdot x^r} R^0(X_{n+1,0})$$

where P is the total power operations map on $\pi_0 R$. This descends to

$$(\pi_0 R)/u_{h-2} \xrightarrow{P} \Gamma_{n,R}^{\vee(h-1)}[1] \xrightarrow{\cdot x^r} \Gamma_{n,R}^{\vee(h-1)}[1].$$

   (ii) If $n+1 = 2r+2$ is even, the power operations map $\pi_1 A \to A^0(X_{n+1,1})$ is given by

$$\pi_1 A = R^{-1}(\mathbb{S}^n) \to R^0(X_{n+1,n+1}) \to R^{-n}(X_{n+1,1}) = A^{-1}(X_{n+1,1}). \qquad (16)$$

[Rez09] shows that if k is odd, the suspension map $E^0(B\Sigma_p^{(k+1)\overline{\rho}})/\mathrm{tr}_{k+1} \to E^0(B\Sigma_p^{k\overline{\rho}})/\mathrm{tr}_k$ after modding out by the transfer is an isomorphism. Together with periodicity and Thom isomorphisms, this identifies the top map in the following diagram as multiplication by x^r.

$$
\begin{array}{ccc}
E^0(B\Sigma_p)/\mathrm{tr} & \xrightarrow{\;\;\cdot x^r\;\;} & E^0(B\Sigma_p)/\mathrm{tr} \\
\simeq \uparrow & & \uparrow \simeq \\
E^0(B\Sigma_p^{(n+1)\rho})/\mathrm{tr}_{n+1} & \longrightarrow & E^{-n}(B\Sigma_p^{\rho})/\mathrm{tr}_1 \\
\downarrow & & \downarrow \\
E^0(X_{n+1,n+1})/\mathrm{tr}_{n+1} & \longrightarrow & E^{-n}(X_{n+1,1})/\mathrm{tr}_1 \\
\downarrow & & \uparrow \simeq \\
E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1} & \longrightarrow & E^{-n}(X_{n+1,1})
\end{array}
$$

Since $n + 1 = 2r + 2$ is even, $E^{-n}(X_{n+1,1}) \cong E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1}$ by [BHK24, Theorem 6.10]. Since both $E^{-n}(B\Sigma_p^{\rho})/\mathrm{tr}_1$ and $E^{-n}(X_{n+1,1})$ are quotients of $E^{-n}(B\Sigma_p^{\rho})$, this implies that the middle right map factors through $E^{-n}(X_{n+1,1})$, so $\mathrm{tr}_1 = 0$ in $E^{-n}(X_{n+1,1})$, giving the bottom right isomorphism. The bottom left surjection comes from the fact that $E^0(X_{n+1,n+1})$ surjects onto $E^0(X_{n,n+1}) \cong E^0(B\Sigma_p)/x^{r+1}$. Thus, the map $E^0(X_{n+1,n+1})/\mathrm{tr}_{n+1} \to E^{-n}(X_{n+1,1})/\mathrm{tr}_1$ is equal to the composite

$$E^0(X_{n+1,n+1})/\mathrm{tr}_{n+1} \to E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1} \xrightarrow{\cdot x^r} E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1}, \qquad (17)$$

hence the composite $E^{-1}(\mathbb{S}^n) \to E^0(X_{n+1,n+1})/\mathrm{tr}_{n+1} \to E^{-n}(X_{n+1,1})/\mathrm{tr}_1$ is equal to the composition along the top row of the commutative diagram

$$
\begin{array}{ccccc}
E_0 & \xrightarrow{\;P\;} & E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1} & \xrightarrow{\;\cdot x^r\;} & E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1} \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0 R & \xrightarrow{\;P\;} & R^0(B\Sigma_p)/\mathrm{tr}, x^{r+1} & \xrightarrow{\;\cdot x^r\;} & R^0(B\Sigma_p)/\mathrm{tr}, x^{r+1}.
\end{array}
\qquad (18)
$$

Here we have identified $E^0(B\Sigma_p)/\mathrm{tr}, x^{r+1}$ with $E^0(X_{n,0})/\mathrm{tr}$ and P is the total power operations map on π_0.

Strictly speaking, power operations on $R^{-1}(\mathbb{S}^n)$ fits into the commutative diagram

$$
\begin{array}{ccccc}
E^{-1}(\mathbb{S}^n) & \longrightarrow & E^0(X_{n+1,n+1}) & \longrightarrow & E^{-n}(X_{n+1,1}) \\
\downarrow & & \downarrow & & \downarrow \\
R^{-1}(\mathbb{S}^n) & \longrightarrow & R^0(X_{n+1,n+1}) & \longrightarrow & R^{-n}(X_{n+1,1})
\end{array}
\qquad (19)
$$

coming from the map $E^{S_+^n} \to R^{S_+^n}$ of $\mathbb{E}_{n+1}$-E-algebras. However, we will only be interested in the action of these power operations on elements in the image of E_0 in $\pi_0 R$, which is determined by (18). Moreover, since $R^{-n}(X_{n+1,1}) \supseteq \pi_0 R \otimes_{E_0} E^{-n}(X_{n+1,1}) \cong R^0(B\Sigma_p)/\mathrm{tr}, x^{r+1}$, the bottom composite in (19) is nonzero if the bottom composite of (18) is nonzero. Thus, for our purposes, we can replace (19) by (18).

Thus, by Lemma 8.0.6, in both (i) and (ii), we have reduced the problem to showing that the bottom composite in the following commutative diagram, which sends 1 to x^r, is nonzero.

$$
\begin{array}{ccccc}
E_0/u_{h-2} & \xrightarrow{P} & \Gamma^{\vee(h-1)}[1]/x^{r+1} & \xrightarrow{\cdot x^r} & \Gamma^{\vee(h-1)}[1]/x^{r+1} \\
\downarrow & & \downarrow & & \downarrow \\
(\pi_0 R)/u_{h-2} & \xrightarrow{P} & \Gamma^{\vee(h-1)}_R[1]/x^{r+1} & \xrightarrow{\cdot x^r} & \Gamma^{\vee(h-1)}_R[1]/x^{r+1}.
\end{array}
$$

Since $P(u_{h-1})$ is a unit times x, this is equivalent to showing that u_{h-1}^r is sent to a nonzero element under the power operations map $(\pi_0 R)/u_{h-2} \xrightarrow{P} \Gamma^{\vee(h-1)}_R[1]/x^{r+1}$, so we are done by Lemma 8.0.8 below. $\qquad\square$

The proof of the following lemma is essentially the same as [Hah22, Corollary 4.5.2]. The difference is that since we are in the $\mathbb{E}_{n+1}$ case with $n < \infty$, at each step, there are fewer power operations available, but this turns out to be just enough.

Lemma 8.0.8. *Let R be an E-algebra respecting $\mathbb{E}_{2r+1}$-power operations. If $L_{K(h)}R \neq 0$, the total power operations map*

$$
\pi_0 R \xrightarrow{P} \Gamma^{\vee(h-1)}_R[1]/x^{r+1}
$$

sends u_{h-1}^r to a nonzero element.

Proof. The case $h = 1$ is resolved in the proof of Prop 4.22 in [HL24]. Thus, we may assume that $h \geq 2$.

Recall (Lemma 4.3.6) that $\Gamma^{\vee(h-1)}[1] \cong E_0/u_{h-2}[x]/u_{h-1} - ux^{p^{h-1}}$ and $P(u_{h-1}) = ux$, where u is a unit. Let $r = r_1 p^{h-1} + s_1$ with $0 \leq s_1 < p^{h-1}$, so $x^r = (u^{-1}u_{h-1})^{r_1}x^{s_1}$. Let $E_0/u_{h-1}^d := E_0/p, \cdots, u_{h-2}, u_{h-1}^d$. Then

$$
\Gamma^{\vee(h-1)}[1]/x^{r+1} \cong E_0/u_{h-1}^{r+1}\{1, x, \cdots, x^{s_1}\} \oplus E_0/u_{h-1}^{r_1}\{x^{s_1+1}, \cdots, x^{p^{h-1}-1}\},
$$

where $S\{y\}$ denotes the free S-module on y, so there exists an $\mathbb{E}_{n+1}$-power operation $Q_1 : E_0/u_{h-2} \to E_0/u_{h-1}^{r_1+1}, u_{h-1}^r \mapsto u_{h-1}^{r_1}$. Since $P(u_{h-1}^{r_1+1}) = (u^{-1}x)^{r_1+1}$, the diagram

$$
\begin{array}{ccc}
E_0/u_{h-2} & \xrightarrow{P} & \Gamma^{\vee(h-1)}[1]/x^{r+1} \\
\downarrow & & \downarrow \\
E_0/u_{h-1}^{r_1+1} & \xrightarrow{P} & \Gamma^{\vee(h-1)}[1]/x^{r_1+1}
\end{array}
$$

is a pushout, giving power operations on $E_0/u_{h-1}^{r_1+1}$. Let $r_1 = r_2 p^{h-1} + s_2$. Since

$$
\Gamma^{\vee(h-1)}[1]/x^{r_1+1} \cong E_0/u_{h-1}^{r_2+1}\{1, \cdots, x^{s_2} \oplus E_0/u_{h-1}^{r_2}\{x^{s_1+1}, \cdots, x^{p^{h-1}-1}\}\},
$$

there exists an operation $Q_2 : E_0/u_{h-1}^{r_1+1} \to E_0/u_{h-1}^{r_2+1}, u_{h-1}^{r_1} \mapsto u_{h-1}^{r_2}$.

Since $h \geq 2$, continuing this way, at some point we have $r_d < p^{h-1}$. At this point, we have obtained a sequence of operations

$$
E_0/u_{h-2} \xrightarrow{Q_1} E_0/u_{h-1}^{r_1+1} \xrightarrow{Q_2} E_0/u_{h-1}^{r_2+1} \to \cdots \xrightarrow{Q_d} E_0/u_{h-1}^{r_d+1}
$$

49

sending $u_{h-1}^r \mapsto u_{h-1}^{r_1} \mapsto \cdots \mapsto u_{h-1}^{r_d}$. As before, P descends to

$$E_0/u_{h-1}^{r_d+1} \xrightarrow{P} \Gamma^{\vee(h-1)}[1]/x^{r_d+1} \cong E_0/u_{h-1}\{1, \cdots, x^{r_d}\},$$

so there exists an operation $Q_{d+1} : E_0/u_{h-1}^{r_d+1} \to E_0/u_{h-1}$ sending $u_{h-1}^{r_d}$ to 1.

By commutativity of

$$
\begin{array}{ccc}
E_0/u_{h-2} & \xrightarrow{\quad P \quad} & \Gamma^{\vee(h-1)}[1]/x^{r+1} \\
\downarrow & & \downarrow \\
(\pi_0 R)/u_{h-2} & \xrightarrow{\quad P \quad} & \Gamma_R^{\vee(h-1)}[1]/x^{r+1} \cong \pi_0 R \otimes_{E_0} \Gamma^{\vee(h-1)},
\end{array}
$$

we have a sequence of operations

$$R_0/u_{h-2} \xrightarrow{Q_1} R_0/u_{h-1}^{r_1+1} \xrightarrow{Q_2} \cdots \xrightarrow{Q_{d+1}} R_0/u_{h-1}$$

whose composite sends u_{h-1}^r to 1. Since $R \neq 0$ and is $\mathfrak{m} = (p, \cdots, u_{h-1})$-complete, $R_0/u_{h-1} \neq 0$, so $1 \neq 0$ in R_0/u_{h-1}. Thus, there is a sequence of operations whose composite sends u_{n-1}^r to a nonzero unit, so $P(u_{h-1}^r)$ must be nonzero in $\Gamma_R^{\vee(h-1)}[1]/x^{r+1}$. $\qquad\square$

Corollary 8.0.9. *If $L_{K(h)}R \neq 0$ and $u_{h-1}^r = 0$ in R, then R does not admit the structure of an $\mathbb{E}_{2r+1}$-algebra.*

[Bur22] shows that for each h and n, there exists generalized Moore spectra of type h which admit an $\mathbb{E}_n$-algebra structure. The above corollary allows us to give a lower bound on the u_{h-1}-torsion power of such Moore spectra. In particular, the generalized Moore spectrum $\mathbb{S}/(p^{i_0}, \cdots, u_{h-2}^{i_{h-2}}, u_{h-1}^r)$ is not $\mathbb{E}_{2r+1}$.

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