Stable Differential Forms

University of Cambridge
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0 Introduction

This essay is about stable differential 3-forms on real 6-manifolds. Stability is a natural pointwise non-degeneracy condition on a differential k-form which generalises that of being symplectic for 2-forms.

In the case of interest to us, a stable 3-form α on a 6-manifold M induces a reduction of the structure group to $\mathrm{SL}(3,\mathbb{C})$. We study conditions for the integrability of this structure, and see how this makes M into a complex 3-fold with trivial canonical bundle.

This approach lends itself well to studying deformations of such complex structures. Under suitable conditions—e.g. the complex manifold M admitting Kähler metrics—we find that an open set in $H^3(M,\mathbb{R})$ is a local moduli space for these structures, with the cohomology class $[\alpha]$ of a closed stable 3-form acting as the parameter.

We then go on to discuss connections to the theory of Calabi-Yau 3-folds and Ricci-flat Kähler metrics. These are examples of Riemannian metrics with holonomy in $\mathrm{SU}(3)$ and are hence of general interest in themselves.

In section 1 we go over the linear algebra of stable 3-forms in 6 dimensions. We look at the structure a stable 3-form puts on a vector space V in preparation for the global setting of later sections.

Section 2 sees us go over the details of how a stable differential 3-form α on a real 6-manifold M induces a special geometry on M. In particular, integrability of the resulting almost-complex structure is explored and we see how this leads to the structure of a complex 3-fold on M with trivial canonical bundle.

In section 3 we introduce Hitchin's functional HV on the space of stable differential 3-forms. Critical points of HV when restricted to de Rham cohomology classes correspond exactly to integrable complex structures with trivial canonical bundle as in section 2.

Section 4 involves bringing in techniques from analysis to study the non-degeneracy of the critical points of HV. The idea is that given a non-degenerate critical point of HV on the cohomology class $[\alpha]$, nearby classes should also contain critical points, parametrised by $H^3(M, \mathbb{R})$. This allows us to study deformations of the corresponding complex 3-folds.

Finally, in section 5 we elaborate on the connection with Calabi-Yau 3-folds and their deformations. The only component missing in section 2 is the assumption that the complex 3-fold M admits Kähler metrics. We also go over some of the general theory of holonomy groups and Calabi-Yau manifolds.

Our main reference is Hitchin's paper [4], which we follow for much of the first four sections. Section 5 draws from various sources, [8] for the general theory of holonomy groups and [7] for the Kähler geometry and Calabi-Yau manifold theory.

1 Linear algebra of stable 3-forms in 6 dimensions

Let V be a 6-dimensional real vector space equipped with an orientation. Following the work of Hitchin [4], we define what it means for a 3-form $\alpha \in \bigwedge^3 V^*$ to be stable. There turn out to be two kinds of stable real 3-forms. Focusing on the one of interest to us, we define an associated dual 3-form $\hat{\alpha}$ and complex structure J_α on V.

This will be applied in later sections for α a globally-stable 3-form on a 6-manifold M to obtain an almost-complex structure on M.

1.1 Characterising stability

In this relatively technical section, we define stability of k-forms and present a classification of the stable 3-forms on a 6-dimensional vector space, both over $\mathbb{R}$ and over $\mathbb{C}$.

Definition 1.1. Let V be a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider the (right) action of $\mathrm{GL}(V)$ on $\bigwedge^k V^*$ by pullback. A k-form α is said to be *stable* if its orbit under this action is an open subset of the vector space $\bigwedge^k V^*$.

The action of $\mathrm{GL}(V)$ on $\bigwedge^k V^*$ is by linear isomorphisms, in particular by homeomorphisms. So α is stable if and only if its orbit contains an open neighbourhood of α. That is, nearby forms have the same 'shape'.

In this way, stability gives a measure of non-degeneracy of the k-form α. Indeed, for the case of a 2-form ω on a $2n$-dimensional real vector space U, this definition picks out when ω is a *symplectic* 2-form on U.

Now, for 2-forms we have an easy algebraic criterion for stability. Namely, $\omega \in \bigwedge^2 U^*$ is symplectic if and only if $\omega^n \in \bigwedge^{2n} U^*$ is non-zero, i.e. a volume form. As it turns out, this is equivalent to there being a basis $\{\theta_i, \varphi_i\}_{i=1}^n$ for U^* such that $\omega = \sum_{i=1}^n \theta_i \wedge \varphi_i$.

Of interest to us are similar criteria for the stability of 3-forms in 6 dimensions.

Definition 1.2. Let $\alpha \in \bigwedge^3 V^*$ be a 3-form on V, a 6-dimensional vector space over $\mathbb{K}$. Then K_α is the linear map

$$\begin{aligned} K_\alpha : V &\to \bigwedge^5 V^* \\ v &\mapsto \iota_v \alpha \wedge \alpha, \end{aligned}$$

where $\iota_v \cdot$ denotes interior product along $v \in V$.

Wedge product gives a perfect pairing $V^* \otimes \bigwedge^5 V^* \to \bigwedge^6 V^*$, inducing an isomorphism

$$\bigwedge^5 V^* \cong V \otimes \bigwedge^6 V^*.$$

Hence we may view K_α as a map $V \to V \otimes \bigwedge^6 V^*$, i.e. an endomorphism of V with coefficients in $\bigwedge^6 V^* \cong \mathbb{K}$.

A crucial definition for us will be that of a decomposable 3-form.

Definition 1.3. A 3-form $\eta \in \bigwedge^3 V^*$ is said to be *decomposable* if there exist 1-forms $\zeta_1, \zeta_2, \zeta_3 \in V^*$ such that

$$\eta = \zeta_1 \wedge \zeta_2 \wedge \zeta_3.$$

The map K_α allows us to study the stability of α. To this end, the following result characterises stable 3-forms on V in the complex case.

Lemma 1.4. *Let V be a 6-dimensional vector space over $\mathbb{C}$, and let $\alpha \in \bigwedge^3 V^*$. Then*

$$K_\alpha^2 = (K_\alpha \otimes \mathrm{id}) \circ K_\alpha : V \to V \otimes \left(\bigwedge^6 V^*\right)^{\otimes 2}$$

is multiplication by a scalar

$$\lambda(\alpha) \coloneqq \frac{1}{6} \operatorname{tr} K_\alpha^2 \in \left(\bigwedge^6 V^*\right)^{\otimes 2},$$

and the following are equivalent:

(i) *α is stable,*

(ii) *$\lambda(\alpha) \neq 0$,*

(iii) *$\alpha = \eta + \xi$ for some decomposable 3-forms η, ξ such that $\eta \wedge \xi \neq 0$.*

Moreover the 3-forms η, ξ in (iii) are unique up to reordering.

Proof. The proof begins by studying the shape of the 3-forms satisfying (iii). The key step is showing that these form an *open* $\mathrm{GL}(V)$-orbit in $\bigwedge^3 V^*$. In the process we see that (iii) implies (i), (ii). Then, a certain decomposability condition, which holds on this orbit, is extended to the whole of $\bigwedge^3 V^*$ and used to complete the proof.

So, let α satisfy (iii), say with

$$\begin{aligned} \alpha &= \eta + \xi \\ &= \varphi_1 \wedge \varphi_2 \wedge \varphi_3 + \varphi_4 \wedge \varphi_5 \wedge \varphi_6. \end{aligned}$$

Then $\eta \wedge \xi \neq 0$ implies that $\{\varphi_i\}_{i=1}^6$ is a basis of V^*. In particular any other 3-form satisfying (iii) is in the $\mathrm{GL}(V)$-orbit of α.

Let $\{u_i\}_{i=1}^6$ be the basis of V dual to $\{\varphi_i\}_{i=1}^6$, and let $\epsilon = \varphi_1 \wedge \ldots \wedge \varphi_6$ generate $\bigwedge^6 V^*$. Then one computes

$$K_\alpha(u_i) = \begin{cases} u_i \otimes \epsilon & \text{if } i = 1, 2, 3, \\ -u_i \otimes \epsilon & \text{if } i = 4, 5, 6. \end{cases} \tag{1.1}$$

For example $\iota_{u_1}\alpha\wedge\alpha = \varphi_2\wedge\ldots\wedge\varphi_6$ paired with φ_i on the left gives ϵ if $i=1$, and 0 otherwise, whence $K_\alpha(u_1) = u_1\otimes\epsilon$.

It follows that $K_\alpha^2 = \mathrm{id}\otimes\epsilon^2$, and in particular $\lambda(\alpha)=\epsilon^2\neq 0$.

Considering the effect of $T\in\mathrm{GL}(V)$ on the bases $\{\varphi_i\}_{i=1}^6$ and $\{u_i\}_{i=1}^6$, the above calculation also shows that a general 3-form in the orbit of α, say

$$T^*\alpha = T^*\eta + T^*\xi,$$

is such that

$$K_{T^*\alpha}\left(T^{-1}u_i\right) = \begin{cases} T^{-1}u_i\otimes T^*\epsilon & \text{if } i=1,2,3,\\ -T^{-1}u_i\otimes T^*\epsilon & \text{if } i=4,5,6.\end{cases} \tag{1.2}$$

In particular $\lambda(T^*\alpha) = (T^*\epsilon)^2\neq 0$, and (iii) $\Longrightarrow$ (ii).

Next, to show that α is stable, it will suffice to compute the dimension of its stabiliser $H\subseteq\mathrm{GL}(V)$. Indeed, its orbit is diffeomorphic to the coset space $H\backslash\mathrm{GL}(V)$, and will be open in $\bigwedge^3 V^*$ if it has (complex) dimension $\dim\bigwedge^3 V^* = 20$.

Calculation (1.2) gives the eigenspaces of $K_{T^*\alpha}$. So if $T^*\alpha=\alpha$, it follows that T either preserves or interchanges

$$\mathrm{span}(u_1,u_2,u_3),\quad \mathrm{span}(u_4,u_5,u_6),$$

and hence also

$$\mathrm{span}(\varphi_1,\varphi_2,\varphi_3),\quad \mathrm{span}(\varphi_4,\varphi_5,\varphi_6).$$

If T lies in the identity component of $H=\mathrm{stab}(\alpha)$, then they are preserved. Moreover in this case

$$T^*\eta=\eta,\quad T^*\xi=\xi.$$

So, the identity component of H is isomorphic to

$$\mathrm{SL}(3,\mathbb{C})\times\mathrm{SL}(3,\mathbb{C}).$$

In particular $\dim_{\mathbb{C}} H = 16$, and $\dim_{\mathbb{C}} H\backslash\mathrm{GL}(V) = 36-16=20$. It follows that the orbit of α is open as desired, giving (iii) $\Longrightarrow$ (i).

Now observe that for α above, given a choice of square root for $\lambda(\alpha)=\epsilon^2$, both

$$K_\alpha^*\alpha\pm\alpha\otimes\lambda(\alpha)^{3/2} \tag{1.3}$$

are decomposable $\left(\bigwedge^6 V^*\right)^{\otimes 3}$-valued 3-forms. Recall K_α maps $V\longrightarrow V\otimes\left(\bigwedge^6 V^*\right)$, so that the pullback $K_\alpha^*\alpha$ is an element of $\bigwedge^3 V^*\otimes\left(\bigwedge^6 V^*\right)^{\otimes 3}$. And indeed, using (1.1),

$$K_\alpha^*\alpha = (\varphi_1\wedge\varphi_2\wedge\varphi_3 - \varphi_4\wedge\varphi_5\wedge\varphi_6)\otimes\epsilon^3,$$

so both $K_\alpha^*\alpha\pm\alpha\otimes\epsilon^3$ are decomposable.

Decomposability is an *algebraic* condition on 3-forms: there is a set of quadratic polynomials in the coordinates of $\bigwedge^3 V^*$ whose common zero locus is precisely the subset of decomposable 3-forms. These are the Plücker equations, and for example may be found as (3.4.10) of Jacobson's [6].

This means that the decomposability of (1.3), which holds on an open subset of $\bigwedge^3 V^*$, in fact holds for all α. Similarly, the conditions that K_α^2 is a scalar, and $\mathrm{tr}\,K_\alpha=0$, also extend to all $\alpha\in\bigwedge^3 V^*$.

Then for arbitrary α we have decomposable $\left(\bigwedge^6 V^*\right)^{\otimes 3}$-valued 3-forms $\tilde\eta,\tilde\xi$ such that

$$\alpha\otimes\lambda(\alpha)^{3/2} = \tilde\eta+\tilde\xi.$$

If $\lambda(\alpha) \neq 0$, then we can use $\lambda(\alpha)^{1/2}$ to trivialise $\bigwedge^6 V^*$, and find honest decomposable 3-forms η, ξ with $\alpha = \eta + \xi$. Moreover it must be that $\eta \wedge \xi \neq 0$, else we can find $v \in V$ such that $\iota_v \alpha = 0$, contradicting $\lambda(\alpha) \neq 0$. Thus (ii) $\Longrightarrow$ (iii).

In particular, the orbit of 3-forms satisfying (iii) is an open dense subset of $\bigwedge^3 V^*$, being the complement of the quartic hypersurface $\lambda(\alpha) = 0$. So if α is a stable 3-form, i.e. its orbit under $\mathrm{GL}(V)$ is open, then α lies in this open dense orbit. This gives (i) $\Longrightarrow$ (iii).

Finally, the claim about uniqueness of η, ξ follows from their being determined by (1.3) once a square root for $\lambda(\alpha)$ is chosen. □

Lemma 1.4 will mostly be of interest to us in its consequences for the shape of stable *real* 3-forms. This is the content of the next theorem.

Theorem 1.5. *Let V be a 6-dimensional vector space over $\mathbb{R}$, and let $\alpha \in \bigwedge^3 V^*$. Then*

$$K_\alpha^2 = (K_\alpha \otimes \mathrm{id}) \circ K_\alpha : V \to V \otimes \left(\bigwedge^6 V^*\right)^{\otimes 2}$$

is multiplication by a scalar

$$\lambda(\alpha) \coloneqq \frac{1}{6} \operatorname{tr} K_\alpha^2 \in \left(\bigwedge^6 V^*\right)^{\otimes 2},$$

and α is stable if and only if $\lambda(\alpha) \neq 0$.

Moreover,

- *if $\lambda(\alpha) > 0$, then $\alpha = \eta + \xi$ where η, ξ are real decomposable and satisfy $\eta \wedge \xi \neq 0$, and*
- *if $\lambda(\alpha) < 0$, then $\alpha = \eta + \bar{\eta}$ where η is complex decomposable and satisfies $\eta \wedge \bar{\eta} \neq 0$.*

Proof. Let us begin by clarifying some notation. Let L be a 1-dimensional real vector space. Then for $u \in L \otimes L$, we write $u > 0$ if there is a $x \in L \setminus \{0\}$ such that $u = x \otimes x$, and we write $u < 0$ if there is a $x \in L \setminus \{0\}$ such that $u = -x \otimes x$.

Now suppose $\alpha \in \bigwedge^3 V^*$ has $\lambda(\alpha) \neq 0$. Taking the complexification of V, lemma 1.4 gives complex decomposable 3-forms η, ξ with $\eta \wedge \xi \neq 0$ and $\alpha = \eta + \xi$. The condition $\alpha = \bar{\alpha}$ and uniqueness of η, ξ leaves two possibilities:

- either $\eta = \bar{\eta}$ and $\xi = \bar{\xi}$,
- or $\eta = \bar{\xi}$ and $\xi = \bar{\eta}$.

In the first case, we find a (real) basis $\{\varphi_i\}_{i=1}^6$ of V^* such that

$$\begin{aligned} \alpha &= \eta + \xi \\ &= \varphi_1 \wedge \varphi_2 \wedge \varphi_3 + \varphi_4 \wedge \varphi_5 \wedge \varphi_6. \end{aligned}$$

The 3-forms of this shape form an orbit under the action of $\mathrm{GL}(V)$.

The proof of lemma 1.4 also gave $\lambda(\alpha) = (\eta \wedge \xi)^2$. So, since in this case η, ξ are real, we have $\lambda(\alpha) > 0$.

In the latter case, we find complex covectors $\{\zeta_i\}_{i=1}^3$ such that $\{\zeta_i, \bar{\zeta}_i\}_{i=1}^3$ form a basis of $V^* \otimes \mathbb{C}$, and with

$$\begin{aligned} \alpha &= \eta + \bar{\eta} \\ &= \zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3. \end{aligned}$$

Splitting the ζ_i into their real and imaginary parts gives a basis $\{\varphi_i\}_{i=1}^6$ of V^* such that

$$\alpha = 2 \operatorname{Re}(\varphi_1 + i\varphi_2) \wedge (\varphi_3 + i\varphi_4) \wedge (\varphi_5 + i\varphi_6).$$

Again, the 3-forms of this shape form an orbit under the action of $\mathrm{GL}(V)$.

And since $\eta \wedge \bar{\eta} = -\bar{\eta} \wedge \eta$ is pure imaginary, in this case $\lambda(\alpha) = (\eta \wedge \bar{\eta})^2 < 0$.

So the real vector space $\bigwedge^3 V^*$ contains the two orbits $U_+ = \{\lambda > 0\}$ and $U_- = \{\lambda < 0\}$, whose union is open dense. From this it follows that if α is stable then $\lambda(\alpha) \neq 0$, completing the proof. $\square$

1.2 Structures associated to a stable 3-form

From now on we shall be concerned only with stable real 3-forms α such that $\lambda(\alpha) < 0$. As we will see in this section, the structure such a 3-form puts on V is such that the stabiliser of α in $\mathrm{GL}(V) \cong \mathrm{GL}(6, \mathbb{R})$ is isomorphic to $\mathrm{SL}(3, \mathbb{C})$.

This will later permit a reduction of the structure group of a real 6-manifold equipped with a stable differential 3-form to $\mathrm{SL}(3, \mathbb{C})$.

So fix an orientation on V, and a stable 3-form $\alpha \in U_-$. Let $\{\varphi_i\}_{i=1}^6$ and $\{\zeta_i\}_{i=1}^3$ be as in the proof of theorem 1.5, i.e.

$$\alpha = \zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3$$

with real and imaginary parts

$$\zeta_1 = \varphi_1 + i\varphi_2, \quad \zeta_2 = \varphi_3 + i\varphi_4, \quad \zeta_3 = \varphi_5 + i\varphi_6.$$

Definition 1.6. Choose a square root $\sqrt{-\lambda(\alpha)} \in \bigwedge^6 V^*$ compatible with the orientation on V. Then the *complex structure associated to α* is the endomorphism of V given by

$$J_\alpha = -\frac{1}{\sqrt{-\lambda(\alpha)}} \cdot K_\alpha$$

By theorem 1.5, $J_\alpha^2 = -\mathrm{id}_V$ and we indeed obtain a complex structure on V. The minus sign in the definition is a matter of convenience, ensuring that proposition 1.7 holds and that the orientation determined by J_α agrees with the original one on V.

Suppose without loss of generality that $\varphi_1, \ldots, \varphi_6$ is a positively oriented basis of V^*.

Proposition 1.7. *The complex 1-forms ζ_i are of type $(1,0)$ with respect to J_α.*

Proof. The proof is an exercise in the linear algebra of complex structures.

If $\epsilon = \zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3$, then we compute

$$\begin{aligned} \epsilon &= -\zeta_1 \wedge \bar{\zeta}_1 \wedge \zeta_2 \wedge \bar{\zeta}_2 \wedge \zeta_3 \wedge \bar{\zeta}_3 \\ &= -(-2i)^3 \, \varphi_1 \wedge \cdots \wedge \varphi_6 \\ &= -8i \, (\varphi_1 \wedge \cdots \wedge \varphi_6). \end{aligned}$$

So we take as square root for $-\lambda(\alpha) = -\epsilon^2$

$$\sqrt{-\lambda(\alpha)} = 8 \, \varphi_1 \wedge \cdots \wedge \varphi_6 = i\epsilon.$$

Now if $\{u_i\}_{i=1}^6$ is the dual basis to $\{\varphi_i\}_{i=1}^6$, and

$$z_1 = \tfrac{1}{2}(u_1 - iu_2), \quad z_2 = \tfrac{1}{2}(u_3 - iu_4), \quad z_3 = \tfrac{1}{2}(u_5 - iu_6),$$

then computation (1.1) shows that

$$\begin{aligned} K_\alpha(z_i) &= z_i \otimes \epsilon, \\ K_\alpha(\bar{z}_i) &= -\bar{z}_i \otimes \epsilon. \end{aligned}$$

Hence $J_\alpha(z_i) = iz_i$. And by construction of the z_i ,

$$J_\alpha(\zeta_i) = i\zeta_i,$$

completing the proof. □

Remark. The proposition also serves as an equivalent definition of J_α. Indeed, the condition that the ζ_i be of type (1,0) determines J_α via

$$\bigwedge^{1,0} V^* = \{\theta \in V^* \otimes \mathbb{C} : \theta \wedge \zeta_1 \wedge \zeta_2 \wedge \zeta_3 = 0\}.$$

Definition 1.8. Given a stable 3-form $\alpha = \eta + \bar{\eta} \in U_-$ such that $i\eta \wedge \bar{\eta}$ is positively oriented, the *associated dual form* is

$$\hat{\alpha} = -i(\eta - \bar{\eta}).$$

The form $\hat{\alpha}$ is characterised by being itself stable, verifying that the complex 3-form

$$\alpha + i\hat{\alpha} = 2\eta = 2\,\zeta_1 \wedge \zeta_2 \wedge \zeta_3$$

is decomposable and of type (3,0) with respect of J_α, and with $\alpha \wedge \hat{\alpha}$ positively oriented.

In the global setting, the (3,0)-form $\alpha + i\hat{\alpha}$ will allow us to trivialise the canonical bundle of the resulting complex 3-folds.

In section 3 the following lemma will be useful.

Lemma 1.9. *The derivative of*

$$\begin{aligned} \phi : U_- &\to \bigwedge^6 V^* \\ \alpha &\mapsto \sqrt{-\lambda(\alpha)} \end{aligned}$$

at α_0 is the linear map $\beta \mapsto -\hat{\alpha}_0 \wedge \beta$.

Proof. Let $\alpha(t)$ be a smooth one-parameter variation of α_0 with $\dot{\alpha}(0) = \beta$.

Since stability is an open condition, we obtain a smooth curve $\eta(t)$ of complex decomposable forms with $\alpha(t) = \eta(t) + \overline{\eta(t)}$.

It is a fact that such a curve may be written as $\eta(t) = \zeta_1(t) \wedge \zeta_2(t) \wedge \zeta_3(t)$ for smooth $\zeta_i(t)$, c.f. the proof of proposition 2.2. And then the derivative at $t = 0$ is

$$\dot{\eta} = \dot{\zeta}_1 \wedge \zeta_2 \wedge \zeta_3 + \zeta_1 \wedge \dot{\zeta}_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_2 \wedge \dot{\zeta}_3.$$

So $\eta \wedge \dot{\eta} = 0$ and hence

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \phi(\alpha(t)) &= \left.\frac{d}{dt}\right|_{t=0} i\eta(t) \wedge \overline{\eta(t)} \\ &= i(\dot{\eta} \wedge \bar{\eta} + \eta \wedge \dot{\bar{\eta}}) \\ &= i(\eta - \bar{\eta}) \wedge (\dot{\eta} + \dot{\bar{\eta}}) \\ &= -\hat{\alpha}_0 \wedge \beta, \end{aligned}$$

as desired. □

We state a further result to be used in section 3.

Lemma 1.10. *Let $\alpha_0 \in U_-$ and consider the derivative of*

$$\begin{aligned} X : U_- &\to U_- \\ \alpha &\mapsto \hat{\alpha} \end{aligned}$$

at α_0, regarded as a linear map $T : \bigwedge^3 V^* \to \bigwedge^3 V^*$. Complexifying, and taking the type decomposition of $\bigwedge^3 V^* \otimes \mathbb{C}$ with respect to J_{α_0}, we have that

- *T acts as multiplication by $-i$ on $\bigwedge^{3,0} V^* \oplus \bigwedge^{2,1} V^*$,*
- *T acts as multiplication by i on $\bigwedge^{1,2} V^* \oplus \bigwedge^{0,3} V^*$.*

Proof. Our hands-on approach differs from that of proposition 6 in section 3.3 of [4].

Consider a one-parameter variation of α_0,

$$\alpha(t) = \eta(t) + \overline{\eta(t)},$$

given by

$$\eta(t) = \zeta_1(t) \wedge \zeta_2(t) \wedge \zeta_3(t)$$

as in the previous lemma. Note that the $\dot{\zeta}_i(0)$ are arbitrary elements of $V^* \otimes \mathbb{C}$.

We have that

$$X\big(\alpha(t)\big) = i\big(\bar{\eta}(t) - \eta(t)\big),$$

from which

$$T\big(\dot{\eta} + \dot{\bar{\eta}}\big) = i\big(\dot{\bar{\eta}} - \dot{\eta}\big). \tag{1.4}$$

Multiplying each $\dot{\zeta}_i$ by i has the effect of multiplying $\dot{\eta}$ by i as well. So

$$T\big(i(\dot{\eta} - \dot{\bar{\eta}})\big) = \dot{\bar{\eta}} + \dot{\eta},$$

and combining this with (1.4) we find

$$T(\dot{\eta}) = -i\,\dot{\eta}.$$

To finish off, note

$$\dot{\eta} = \dot{\zeta}_1 \wedge \zeta_2 \wedge \zeta_3 + \zeta_1 \wedge \dot{\zeta}_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_2 \wedge \dot{\zeta}_3$$

can be made to be equal each of

$$\zeta_1 \wedge \zeta_2 \wedge \zeta_3, \quad \bar{\zeta}_1 \wedge \zeta_2 \wedge \zeta_3, \quad \zeta_1 \wedge \bar{\zeta}_2 \wedge \zeta_3, \quad \zeta_1 \wedge \zeta_2 \wedge \bar{\zeta}_3$$

via judicious choice of $\dot{\zeta}_i$, from which the result follows as these form a basis of

$$\bigwedge^{3,0} V^* \oplus \bigwedge^{2,1} V^*$$

given the complex structure J_{α_0}, by proposition 1.7. □

It is useful to know how to forget about the stable 3-form α and record only the complex structure J_α on V. This is done as follows.

Proposition 1.11. *There is an action of $\mathbb{C}^*$ on U_- via*

$$(x + iy) \bullet \alpha = x\alpha - y\hat{\alpha}.$$

The orbit space $U_-/\mathbb{C}^*$ parametrises the complex structures on V compatible with its orientation.

Proof. If $\alpha = \eta + \bar{\eta}$, then

$$(x + iy) \bullet \alpha = (x + iy)\eta + (x - iy)\bar{\eta}, \tag{1.5}$$

which by theorem 1.5 is again in U_-.

And now $\hat{\hat{\alpha}} = -\alpha$ implies we get a well-defined $\mathbb{C}^*$-action on U_-.

Recall how $\alpha = \eta + \bar{\eta}$ determines J_α via

$$\bigwedge^{1,0} V^* = \{\theta \in V^* \otimes \mathbb{C} : \theta \wedge \eta = 0\}.$$

So (1.5) implies that the complex structure J_α is invariant under the $\mathbb{C}^*$-action.

Finally, given a complex structure J on V, choosing a non-zero element $\eta \in \bigwedge^{3,0} V^*$ and letting $\alpha = \eta + \bar{\eta}$, we get $\alpha \in U_-$ with $J_\alpha = J$. Since the choice of η is unique up to the action of $\mathbb{C}^*$, we get that the space of complex structures on V is precisely $U_-/\mathbb{C}^*$. □

2 Stable differential 3-forms on real 6-manifolds

We move on now to the global situation. Our discussion mirrors section 5 of Hitchin [4].

Fix a real oriented 6-manifold M with a stable differential 3-form $\alpha \in \Omega^3(M)$.

Stability of α is understood pointwise. That is, for each $p \in M$, the form $\alpha|_p \in \bigwedge^3 T_p^*M$ is stable with respect to the action of $\mathrm{GL}(T_pM)$. The functional λ of theorem 1.5 gives us a smooth section $\lambda(\alpha)$ of $\left(\bigwedge^6 T^*M\right)^{\otimes 2}$. We demand our stable 3-forms satisfy $\lambda(\alpha) < 0$ pointwise.

We apply the formalism of section 1 to the pair (M, α). The form α picks out an almost-complex structure on M, and we study conditions for its integrability. In good cases, this gives M the structure of a Calabi-Yau 3-fold—see definition 2.5.

Definition 2.1. We define $J_\alpha \in \mathrm{End}(TM)$ to be given pointwise as in definition 1.6, i.e.

$$J_\alpha = -\frac{1}{\sqrt{-\lambda(\alpha)}} \cdot K_\alpha.$$

Then J_α is smooth and $J_\alpha^2 = -\mathrm{id}_{TM}$, giving an almost-complex structure on M.

Proposition 2.2. *Let (M, α) be as above.*

Then there is a unique locally-decomposable $\eta \in \Omega^3(M, \mathbb{C})$ such that $\alpha = \eta + \bar{\eta}$ and $i\eta \wedge \bar{\eta}$ is positively-oriented.

Proof. The proof of lemma 1.4 gives us a unique $\eta \in \Omega^3(M, \mathbb{C})$ satisfying the last two conditions. Indeed, (1.3) expresses η in terms of α, so in particular the η we obtain is smooth.

Moreover η is pointwise decomposable, i.e. for each $p \in M$ there are covectors $\zeta_1, \zeta_2, \zeta_3$ in $T_p^*M \otimes \mathbb{C}$ such that $\eta|_p = \zeta_1 \wedge \zeta_2 \wedge \zeta_3 \neq 0$. Let us show this is enough to deduce local decomposability of η, i.e. that the ζ_i can be extended to local smooth complex 1-forms verifying $\eta = \zeta_1 \wedge \zeta_2 \wedge \zeta_3$.

Indeed, the smooth bundle map

$$\begin{gathered}\Omega^1(M, \mathbb{C}) \to \Omega^4(M, \mathbb{C}) \\ \theta \mapsto \theta \wedge \eta\end{gathered}$$

has as kernel a smooth sub-bundle of $\Omega^1(M, \mathbb{C})$ of (complex) rank 3. Letting $\zeta_1, \zeta_2, \zeta_3$ be a local smooth frame for this bundle, we have that η and $\zeta_1 \wedge \zeta_2 \wedge \zeta_3$ differ by a smooth nowhere-zero complex function. After modifying ζ_1 say, the result follows. □

Remark. This last proposition gives an equivalent way of understanding J_α. Indeed, if we locally write $\eta = \zeta_1 \wedge \zeta_2 \wedge \zeta_3$, then J_α is uniquely determined in terms of $\zeta_1, \zeta_2, \zeta_3$ spanning the space of (1,0)-forms.

Definition 2.3. Let (M, α) be as above, and let η be as in proposition 2.2. Then we define $\hat{\alpha} \in \Omega^3(M)$ by

$$\hat{\alpha} = -i(\eta - \bar{\eta}).$$

We now give necessary and sufficient conditions for the integrability of J_α.

Theorem 2.4. *Let (M, α) be a real 6-manifold equipped with a stable 3-form as above, with corresponding almost-complex structure J_α. Let $\alpha = \eta + \bar{\eta}$ as in proposition 2.2. Then J_α is integrable if and only if $d\eta$ has no $(2,2)$-component.*

Proof. Here integrability means that M admits the structure of a complex manifold for which the canonical almost-complex structure is J_α. Since η is a (3,0)-form with respect to J_α, necessity is then immediate.

By the Newlander-Niremberg theorem, integrability is equivalent to the exterior derivative respecting the type decomposition with respect to J_α, in that

$$d\left(\Omega^{1,0}(M)\right) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M). \tag{2.1}$$

Write $\eta = \zeta_1 \wedge \zeta_2 \wedge \zeta_3$ locally as in the proof of proposition 2.2, and suppose $\Pi^{2,2}(d\eta) = 0$. We have

$$d\eta = d\zeta_1 \wedge \zeta_2 \wedge \zeta_3 - \zeta_1 \wedge d\zeta_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_2 \wedge d\zeta_3. \tag{2.2}$$

By construction of J_α, condition (2.1) is equivalent to each of $d\zeta_1, d\zeta_2, d\zeta_3$ having zero (0,2)-component. Taking the (2,2) component of (2.2) gives

$$\Pi^{2,2}(d\eta) = \Pi^{0,2}(d\zeta_1) \wedge \zeta_2 \wedge \zeta_3 - \zeta_1 \wedge \Pi^{0,2}(d\zeta_2) \wedge \zeta_3 + \zeta_1 \wedge \zeta_2 \wedge \Pi^{0,2}(d\zeta_3).$$

Since $\{\zeta_1 \wedge \zeta_2,\ \zeta_1 \wedge \zeta_3,\ \zeta_2 \wedge \zeta_3\}$ are linearly independent, the vanishing of $\Pi^{2,2}(d\eta)$ implies each $\Pi^{0,2}(d\zeta_i)$ vanishes too, whence the result. □

Remark. The complex manifolds obtained this way admit a global smooth (3,0)-form, namely η, so their canonical bundle K_M is *smoothly* trivial. However at this stage K_M needn't be holomorphically trivial—we don't ask for $d\eta = 0$.

Remark. In analogy which proposition 1.11, there is an action of $C^\infty(M, \mathbb{C}^*)$ on the space of stable 3-forms which leaves the (almost-)complex structure unchanged. Quotienting by this action forgets the choice of a particular α.

2.1 The Calabi-Yau condition

The theory of stable differential forms lends itself well to working with *Calabi-Yau* 3-folds. In section 3, we see how the condition that η be a holomorphic volume form on the complex manifold M can be cleanly stated in terms of Hitchin's volume functional [4].

We take the following definition. Non-equivalent ones exist in the literature.

Definition 2.5. A *Calabi-Yau manifold* M is a compact complex manifold M admitting Kähler metrics and with trivial canonical bundle.

That is, M compact Kähler is Calabi-Yau if it admits a holomorphic volume form, that is a global nowhere-zero (3,0)-form η such that $\bar\partial\eta = 0$.

Proposition 2.6. *Let (M, α) be a real 6-manifold equipped with a stable differential 3-form, and let η be as in proposition 2.2. If $d\alpha = d\hat\alpha = 0$, then M admits the structure of a complex 3-fold on which η is a holomorphic volume form.*

Conversely if M is a complex 3-fold equipped with a holomorphic volume form η then the real 3-form $\alpha = \eta + \bar\eta$ is stable and satisfies $d\alpha = d\hat\alpha = 0$.

Proof. Suppose $d\alpha = d\hat\alpha = 0$. Theorem 2.4 gives M the structure of a complex 3-fold, on which η is a smooth (3,0)-form. But now

$$\bar\partial\eta = d\eta = d(\alpha + i\hat\alpha) = 0,$$

so we have a holomorphic volume form as desired.

The converse follows from the same calculation. □

Remark. Condition $d\alpha = d\hat{\alpha} = 0$ has the virtue of not directly referencing the almost-complex structure J_α, whereas $\Pi^{2,2}(d\eta) = 0$ in theorem 2.4 uses the induced type decomposition on $\Omega^*(M, \mathbb{C})$.

Remark. It should be noted that even if M is a compact complex 3-fold with trivial canonical bundle, it need not be the case that M admits Kähler metrics. See [10], where such a complex structure is constructed on the connected sum $M = \#^{25}(S^3 \times S^3)$. That this fails to admit Kähler metrics follows from $H^2(M) = 0$.

So in general our complex manifolds need not be Calabi-Yau.

Remark. Assuming that M is compact and α verifies $d\alpha = d\hat{\alpha} = 0$, then the possible holomorphic volume forms on the resulting complex manifold are all multiples of η.

Then as in proposition 1.11, there is a $\mathbb{C}^*$-action on the space of stable 3-forms which leaves the resulting complex manifold unchanged. Quotienting by this action forgets the choice of α. This will become important when comparing this approach to deformations of Calabi-Yau 3-folds with the general results of Tian and Todorov.

3 Hitchin's volume functional

In this section we introduce Hitchin's volume functional HV on the space of stable 3-forms on a real 6-manifold [4]. The critical points of HV when restricted to a de Rham cohomology class give us integrable complex structures with trivial canonical bundle as seen in proposition 2.6. We continue to follow section 5 of Hitchin [4].

We go on to discuss further properties of HV, including a form of non-degeneracy for its critical points which will later enable us to produce a moduli space for such complex structures.

Definition 3.1. Given a stable 3-form α on an (oriented) real 6-manifold M, define

$$HV(\alpha) = \int_M \sqrt{-\lambda(\alpha)}.$$

If $\alpha = \eta + \bar{\eta}$, then we can equivalently write this as

$$HV(\alpha) = \int_M i\eta \wedge \bar{\eta} = \frac{1}{2}\int_M \alpha \wedge \hat{\alpha}.$$

Remark. The very definition of stable differential forms means that the 3-forms close to α (with respect to the C^0 norm, say) are again stable. Hence it makes sense to consider HV as a functional and to perform calculus of variations on it.

Theorem 3.2. *Let (M, α) be as above, and suppose $d\alpha = 0$. Then $d\hat{\alpha} = 0$ (and hence we obtain a complex 3-fold with trivial canonical bundle as in proposition 2.6) if and only if α is a critical point of HV when restricted to its de Rham cohomology class.*

Proof. By lemma 1.9, the first variation of HV at α is

$$\delta HV_\alpha(\beta) = -\int_M \hat{\alpha} \wedge \beta.$$

Taking $\beta = d\gamma$ and applying Stokes' theorem,

$$\begin{aligned}\delta HV_\alpha(d\gamma) &= -\int_M \hat{\alpha} \wedge d\gamma \\ &= -\int_M d\hat{\alpha} \wedge \gamma.\end{aligned}$$

Since we are considering variations with arbitrary γ, the result follows. □

An important property of HV is its diffeomorphism invariance.

Proposition 3.3. *Let $f : M \to M$ be an orientation-preserving diffeomorphism. Then if α is stable, $HV(f^*\alpha) = HV(\alpha)$.*

Proof. All of the structure associated to α gets pulled back along with it. For example, $f^*\eta$ remains a locally-decomposable complex 3-form, and verifies both $f^*\alpha = f^*\eta + \overline{f^*\eta}$ and that $if^*\eta \wedge \overline{f^*\eta} = f^*(i\eta \wedge \bar{\eta})$ is positively oriented.

Consequently,

$$\begin{aligned} HV(f^*\alpha) &= \int_M if^*\eta \wedge \overline{f^*\eta} \\ &= \int_M f^*(i\eta \wedge \bar{\eta}) \\ &= \int_M i\eta \wedge \bar{\eta} \\ &= HV(\alpha), \end{aligned}$$

where we use that f is orientation-preserving. □

This invariance is related to the fact that if α produces a complex 3-fold with trivial canonical bundle, then so does $f^*\alpha$, and moreover the resulting complex manifolds are isomorphic. Here the isomorphism is within the following category.

Definition 3.4. Let TrivCx^3 be the category where:

- the objects are pairs (M, η) where M is a complex 3-fold and η a holomorphic volume form on M, and
- the morphisms from (M, η) to (M', η') are those holomorphic maps $f : M \to M'$ such that $f^*\eta' = \eta$.

For the case of $(M, f^*\alpha)$ and (M, α) above, the isomorphism in TrivCx^3 is furnished by the map f itself.

A consequence of this diffeomorphism invariance is that if α is a critical point of HV when restricted to its cohomology class, then this critical point is very far from being non-degenerate.

Then the natural question to ask is whether α is a non-degenerate critical point *in the direction transverse to the orbit of α under* $\mathrm{Diff}(M)$. This turns out to be the case, at least when the complex 3-fold associated to (M, α) satisfies the $\partial\bar{\partial}$-lemma.

This is a relatively mild condition, and for example holds as soon as (M, α) admits a Kähler metric. When we come to discuss deformations of Calabi-Yau 3-folds, this will come for free as a starting assumption.

Theorem 3.5. *Let M be a real 6-manifold equipped with a stable 3-form α giving M the structure of a complex 3-fold with trivial canonical bundle, and satisfying the $\partial\bar{\partial}$-lemma.*

Then α is a non-degenerate critical point of HV in the direction transverse to the orbit of α under $\mathrm{Diff}(M)$. *That is, if $\delta^2 HV_\alpha$ denotes the second variation of HV at α, and the exact 3-form $d\gamma$ is such that for all exact 3-forms $d\mu$*

$$\delta^2 HV_\alpha(d\gamma, d\mu) = 0,$$

then $d\gamma$ lies in the tangent space at α to its orbit under $\mathrm{Diff}(M)$.

Proof. To compute $\delta^2 HV$ we refer back to the map T of lemma 1.10. We get

$$\begin{aligned}\delta^2 HV(d\gamma, d\mu) &= -\int_M T d\gamma \wedge d\mu \\ &= -\int_M dT d\gamma \wedge \mu,\end{aligned}$$

again by Stokes' theorem.

So if $d\gamma$ is as in the statement, then $dTd\gamma = 0$.

Next, the tangent space at α to the orbit under $\mathrm{Diff}(M)$ consists of those 3-forms of the shape $\mathcal{L}_v\alpha$, where $\mathcal{L}_v$ is the Lie derivative with respect to the vector field v.

By Cartan's magic formula, and since α is closed, these are

$$d\iota_v\alpha = d(\iota_v\eta + \iota_v\bar{\eta}).$$

With v being arbitrary and η a (3,0)-form, $\iota_v\eta$ runs over the (2,0)-forms, hence $\iota_v\eta + \iota_v\bar{\eta}$ hits precisely the real sections in $\Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$.

So it remains to prove that if $dTd\gamma = 0$ for $d\gamma$ an exact real 3-form, then $d\gamma = d\nu$ for some real ν of type (2,0)+(0,2).

First let $\gamma = \kappa + \bar{\kappa}$ for κ a (2,0)-form. By lemma 1.10, since $d\kappa$ is of type (3,0)+(2,1),

$$dTd\gamma = -id(d\kappa - d\bar{\kappa}) = 0.$$

Hence we may assume without loss of generality that γ is of type (1,1). Applying the $\partial\bar{\partial}$-lemma to $\partial\gamma, \bar{\partial}\gamma$, and using that $i\partial\bar{\partial}$ is a real operator, we find a real 1-form ρ such that

$$d\gamma = i\partial\bar{\partial}\rho.$$

Writing $\rho = \psi + \bar{\psi}$ with ψ of type (1,0) gives

$$d\gamma = i\partial\bar{\partial}(\psi + \bar{\psi}) = id(\bar{\partial}\bar{\psi} - \partial\psi).$$

Since $i(\bar{\partial}\bar{\psi} - \partial\psi)$ is real of type (2,0)+(0,2) we are done. □

Non-degeneracy of the critical point suggests that, when we quotient out by the action of $\mathrm{Diff}(M)$, cohomology classes near $[\alpha]$ will themselves contain critical points of HV by an implicit function theorem argument. Moreover the resulting complex manifolds will be non-isomorphic, in the sense that they give different points of TrivCx^3.

The aim of the next section is to make this rigorous. The main ingredient is an application of the Banach space implicit function theorem. This forces us to work in a Sobolev completion $L^2_k(\bigwedge^3)$ of the space of smooth differential 3-forms $\Omega^3(M, \mathbb{R})$.

Remark. Indeed, when suitably topologised, spaces of smooth functions are merely *Fréchet spaces* and in this generality the inverse function theorem sadly fails to hold.

4 A moduli space

In this section we go over the analysis necessary to prove one of the main results of Hitchin [4], theorem 4.4. The upshot is that an open set in the vector space $H^3(M, \mathbb{R})$ is a local moduli space for complex structures on M with trivial canonical bundle together with a choice of holomorphic volume form η.

We follow section 6 of [4]. For the functional analysis background on Sobolev spaces and elliptic differential operators the reader is invited to check appendix A or the references mentioned therein.

4.1 Extending HV

Fix a smooth stable 3-form α_0 satisfying the hypotheses of theorem 3.5. To start off we wish to extend the domain of definition of HV from a neighbourhood of α_0 in $\Omega^3(M,\mathbb{R})$ to one in a Sobolev completion $L^2_k\left(\bigwedge^3\right)$.

Fix a Hermitian metric on the complex manifold given by (M,α_0) and use the corresponding J_{α_0}-invariant Riemannian metric on M.

In order that 3-forms in a Sobolev neighbourhood of α_0 be stable, in the sense that $\lambda < 0$ almost-everywhere, we want k such that there is a Sobolev embedding of $L^2_k\left(\bigwedge^3\right)$ into $C^0\left(\bigwedge^3\right)$. In view of the embedding theorem A.14, we take $k \geq 4$.

Moreover for these values of k the Sobolev space $L^2_k(M)$ of functions is a Banach algebra under pointwise multiplication. Namely, for all $f,g \in L^2_k(M)$, the product fg, defined almost-everywhere, is again in $L^2_k(M)$, and there is a constant $C > 0$ independent of f,g such that

$$\|fg\|_{L^2_k} \leq C\|f\|_{L^2_k}\|g\|_{L^2_k}.$$

This is a consequence of the Sobolev embedding theorem, see result 5.23 in [1].

Consequently pointwise multiplication gives a smooth map *of Banach spaces*

$$L^2_k(M) \times L^2_k(M) \to L^2_k(M).$$

The upshot of this Banach algebra property is that Hitchin's functional HV extends to a smooth real-valued function on a Sobolev neighbourhood of α_0.

4.2 The auxiliary space E

We wish to study the behaviour of HV in the direction transverse to the $\mathrm{Diff}(M)$-orbit through α_0. The aim is to restrict HV to a slice

$$\alpha_0 + E \coloneqq \{\alpha_0 + \beta : \beta \in E\}$$

transverse to this orbit, with E a suitable Banach subspace of $L^2_k\left(\bigwedge^3\right)$.

The tangent space to the orbit of α_0 consisted of those dv with v a real form of type (2,0)+(0,2). Recall the type decomposition is taken here with respect to J_{α_0}. Using the Riemannian metric on (M,α_0), we take E to be the L^2-orthogonal complement of this tangent space.

Via use of the adjoint d^* we have

$$\int_M \langle \beta, dv\rangle \,\mathrm{vol}_g = \int_M \langle d^*\beta, v\rangle \,\mathrm{vol}_g,$$

whence the orthogonality condition is $\Pi^{2,0}(d^*\beta) = 0$. This condition extends naturally to the Sobolev space $L^2_k\left(\bigwedge^3\right)$.

And since we are interested in critical points of HV on restriction to de Rham classes, we also take our β to be closed. So set

$$E = \{\beta \in L^2_k\left(\textstyle\bigwedge^3\right) : d\beta = 0,\ \Pi^{2,0}(d^*\beta) = 0\}.$$

Since the space E is constructed using the L^2 inner product, one might expect to obtain an orthogonal projection $L^2_k\left(\bigwedge^3\right) \to E$ respecting the Sobolev norm. This will be important when we come to look at the critical points of HV, and is the content of the following lemma.

Lemma 4.1. *There is a continuous linear map*

$$P : L^2_k\left(\textstyle\bigwedge^3\right) \to E$$

such that

- $P|_E = \mathrm{id}_E$,
- $\ker P$ *is* L^2*-orthogonal to* E.

Proof. Recall the Laplacian Δ is an elliptic operator c.f. example A.9. So by elliptic regularity we have a Green's operator for Δ as in definition A.11

$$G : L^2_*\left(\bigwedge^3\right) \to L^2_{*+2}\left(\bigwedge^3\right).$$

If $\beta \in L^2_k\left(\bigwedge^3\right)$, then G allows us to perform Hodge decomposition on β. Namely

$$\beta_{\mathcal{H}} := \beta - d(d^*G\beta) - d^*(Gd\beta) \tag{4.1}$$

is harmonic, since

$$\begin{aligned}\Delta\beta_{\mathcal{H}} &= dd^*\beta + d^*d\beta - dd^*d(d^*G\beta) - d^*dd^*(Gd\beta) \\ &= dd^*\beta + d^*d\beta - dd^*\Delta G\beta - d^*\Delta Gd\beta \\ &= dd^*(\beta - \Delta G\beta) + d^*(d\beta - \Delta Gd\beta),\end{aligned}$$

and now we note $1 - \Delta G$ lands in the harmonic forms.

Then we have that

$$\beta \mapsto \beta - d^*(Gd\beta)$$

is the L^2-orthogonal projection onto the subspace of $L^2_k\left(\bigwedge^3\right)$ of closed forms, well-behaved with respect to the Sobolev norm.

Now we claim that there is a $\kappa \in L^2_{k+1}\left(\bigwedge^{2,0}\right)$ such that

$$\Pi^{2,0}d^*(\beta - d(\kappa + \bar{\kappa})) = 0, \tag{4.2}$$

and the map P we seek is given by

$$P(\beta) = \beta - d^*(Gd\beta) - d(\kappa + \bar{\kappa}).$$

Write $d^*G\beta = \psi + \gamma + \bar{\psi}$ for ψ of type (2,0) and γ real of type (1,1). Note this type decomposition is pure linear algebra so respects the Sobolev norm on $L^2_{k+1}\left(\bigwedge^2\right)$.

By the Hodge decomposition (4.1),

$$d^*\beta = d^*d(d^*G\beta),$$

and so condition (4.2) becomes

$$(\partial^*\partial + \bar{\partial}^*\bar{\partial})(\psi - \kappa) + \bar{\partial}^*\partial\gamma = 0.$$

The final ingredient we need is ellipticity of

$$Q := \partial^*\partial + \bar{\partial}^*\bar{\partial} : \Omega^{2,0}(M) \to \Omega^{2,0}(M).$$

Take this as a given for now. Do observe that Q is self-adjoint. Then using the associated Green's operator G_Q, the lemma will follow from

$$\bar{\partial}^*\partial\gamma \perp \ker Q,$$

as given by theorem A.17.

Now since

$$(Q\rho, \rho)_{L^2} = \|\partial\rho\|^2_{L^2} + \|\bar{\partial}\rho\|^2_{L^2},$$

the kernel of Q consists of those ρ with $\partial\rho = 0$ and $\bar{\partial}\rho = 0$. So if $\rho \in \ker Q$, then

$$(\bar{\partial}^*\partial\gamma, \rho)_{L^2} = (\partial\gamma, \bar{\partial}\rho)_{L^2} = 0,$$

as desired.

It remains to prove that Q is indeed elliptic.

If $\xi = \theta + \bar\theta \in T^*_pM$ is a real covector, then the symbol of Q gives

$$\sigma_2(Q)(\xi) = -\iota_{\theta^\sharp}(\theta \wedge \cdot) - \iota_{\bar\theta^\sharp}(\bar\theta \wedge \cdot),$$

for $\theta^\sharp \in T^{1,0}_pM$ the metric dual of θ.

Suppose $\xi \neq 0$ and that $\omega \in \bigwedge^{2,0} T^*_pM$ is in the kernel of $\sigma_2(Q)(\xi)$. Then we have, for any $v_1, v_2 \in T_pM \otimes \mathbb{C}$

$$0 = -2|\theta|^2_g\, \omega(v_1, v_2) + \theta(v_1)\, \omega(\theta^\sharp, v_2) - \theta(v_2)\, \omega(\theta^\sharp, v_1). \tag{4.3}$$

Letting $v_1 = \theta^\sharp$ gives

$$0 = -|\theta|^2_g\, \omega(\theta^\sharp, v)$$

for all $v \in T_pM \otimes \mathbb{C}$. Since $\theta \neq 0$, now (4.3) implies $\omega = 0$.

Hence Q is indeed elliptic and we are done. □

Note as well that E contains the finite-dimensional space of harmonic forms $\mathcal{H}^3$ as a subspace. By Hodge decomposition, we have an L^2-orthogonal splitting

$$E = \mathcal{H}^3 \oplus E_2,$$

where E_2 is the subspace of exact forms in E. Note that using the ideas in the first part of the above proof, the projection P_2 onto E_2 is well-behaved.

4.3 Characterising the critical points of HV

By construction, the slice $\alpha_0 + E$ is transverse to the $\mathrm{Diff}(M)$-orbit through α_0. So it remains transverse to the orbit through nearby $\alpha \in \alpha_0 + E$.

Since HV is diffeomorphism-invariant, $\alpha \in \alpha_0 + E$ is a critical point of HV when restricted to the (suitably completed) cohomology class $[\alpha]$ if and only if the first variation δHV_α kills the exact forms in just E. That is, if and only if for all $d\gamma \in E_2$,

$$\delta HV_\alpha(\gamma) = \int_M \hat\alpha \wedge d\gamma = 0.$$

Using the Hodge star, this condition is equivalent to $*\hat\alpha \perp E_2$ which in terms of the projection P_2 is

$$P_2(*\hat\alpha) = 0.$$

4.4 The main result

We are almost ready to apply the Banach space implicit function theorem. We have a smooth map of Banach spaces

$$\begin{aligned} F : \mathcal{H}^3 \oplus E_2 &\longrightarrow E_2 \\ \alpha - \alpha_0 &\longmapsto P_2(*\hat\alpha) \end{aligned}$$

with $F(0,0) = 0$.

If we can show that the differential $D_2F : E_2 \longrightarrow E_2$ at $(0,0)$ is a linear isomorphism, then we will get a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{H}^3$ and a smooth map $g : \mathcal{U} \to E_2$ with $g(0) = 0$ such that $(x, g(x))$ is a non-degenerate zero of F restricted to the slice $x \oplus E_2$.

From here on we impose the condition that the complex manifold furnished by (M, α_0) satisfy the $\partial\bar\partial$-lemma. For example it may be that (M, α_0) is Calabi-Yau.

Proposition 4.2. *The differential D_2F at $(0,0)$ is injective.*

Proof. This is essentially a consequence of theorem 3.5 and the construction of E_2.

Firstly by lemma 1.10, the differential D_2F is

$$D_2F(d\gamma) = P_2(*Td\gamma),$$

where the linear map T is as given in the lemma.

Then if $d\gamma \in E_2$ is in the kernel of D_2F, we get that for all $d\mu$ in E_2,

$$\delta^2 HV_{\alpha_0}(d\gamma, d\mu) = \int_M Td\gamma \wedge d\mu = 0.$$

But $\delta^2 HV_{\alpha_0}(d\gamma, d\mu) = 0$ also for $d\mu$ in the tangent space to the $\mathrm{Diff}(M)$-orbit through α_0 by diffeomorphism invariance of HV. So

$$\int_M Td\gamma \wedge d\mu = 0$$

for *all* exact $d\mu$ and by Stokes' theorem we get $dTd\gamma = 0$.

The proof of theorem 3.5 now shows that $d\gamma$ is tangent to the $\mathrm{Diff}(M)$-orbit through α_0, with the caveat that we need the $\partial\bar{\partial}$-lemma to hold in the Sobolev setting.

But $d\gamma \in E_2$ by assumption. So $d\gamma = 0$ and we are done. □

Proposition 4.3. *The differential D_2F at $(0,0)$ is surjective.*

Proof. The proof is a relatively involved calculation. Let $d\gamma \in E_2$.

It suffices to find an exact $d\mu \in L^2_k\left(\bigwedge^3\right)$ with $P_2(*Td\mu) = d\gamma$. Indeed as seen in the proof of proposition 4.2, $P_2(*Tdv) = 0$ for dv orthogonal to E_2, so given $d\mu$ we just take its orthogonal projection onto E_2.

Given the description of the tangent space to the $\mathrm{Diff}(M)$-orbit through α_0 as consisting of dv for v real of type (2,0)+(0,2), without loss of generality $\gamma \in L^2_{k+1}\left(\bigwedge^{1,1}\right)$ and is real.

By definition of E_2, we have $\Pi^{2,0}(d^* d\gamma) = 0$. The assumption on the type of γ means we can write this as $\bar{\partial}^*\partial\gamma = 0$.

In the next part of the proof, we put $d\gamma$ in an even nicer form. We claim there exist a co-closed $\rho \in L^2_k\left(\bigwedge^3\right)$ and a $\sigma \in L^2_{k+1}\left(\bigwedge^{2,2}\right)$ such that

$$d\gamma = \rho + \partial^*\sigma + \bar{\partial}^*\bar{\sigma}.$$

The Laplacian $\Delta_{\bar{\partial}}$ acting on (2,1)-forms is elliptic c.f. example A.10. So there is a Green's operator

$$G : L^2_*\left(\bigwedge^{2,1}\right) \to L^2_{*+2}\left(\bigwedge^{2,1}\right),$$

and analogously to (4.1), a decomposition

$$\partial\gamma = \lambda + \bar{\partial}^* G\bar{\partial}\partial\gamma \tag{4.4}$$

with $\Delta_{\bar{\partial}}\lambda = 0$. Notice there is no $\bar{\partial}$-exact term since $\partial\gamma$ is $\bar{\partial}^*$-closed.

Since $\bar{\partial}^*\lambda = 0$, we have

$$d{*\lambda} = \bar{\partial}*\lambda,$$

where $*$ is the Hodge star. Applying the $\partial\bar{\partial}$-lemma to $d{*\lambda}$, we find $\tilde{v}$ such that

$$d{*\lambda} = \partial\bar{\partial}\tilde{v}$$

and hence $*\lambda - \partial\tilde{v}$ is closed. Applying $*^{-1}$, this gives $v \in L^2_{k+1}\left(\bigwedge^{2,2}\right)$ such that

$$\psi \coloneqq \lambda - \bar{\partial}^* v$$

is co-closed $d^*\psi = 0$.

And now setting $\bar\sigma = \nu + G\bar\partial\partial\gamma$, (4.4) gives

$$\partial\gamma = \psi + \bar\partial^*\bar\sigma.$$

Then with $\rho = \psi + \bar\psi$, and noting γ is real, we get

$$d\gamma = \rho + \partial^*\sigma + \bar\partial^*\bar\sigma$$

as desired.

Split σ into its real and imaginary parts $\sigma = u + iv$, both again of type (2,2). Then

$$d\gamma = \rho + (\partial^* + \bar\partial^*)u + i(\partial^* - \bar\partial^*)v. \tag{4.5}$$

By lemma 1.10,

$$Td^*v = T(\partial^* + \bar\partial^*)v = i(\partial^* - \bar\partial^*)v,$$

and hence (4.5) becomes

$$d\gamma = \rho + d^*u + Td^*v. \tag{4.6}$$

Now by construction $d^*(\rho + d^*u) = 0$, so in particular $\rho + d^*u$ is L^2-orthogonal to E_2, itself consisting of exact forms. Applying the projection P_2 to (4.6),

$$d\gamma = P_2(Td^*v) = -P_2(T{*}d{*}v)$$

and hence setting $d\mu = -d{*}v$ completes the proof. □

The previous two propositions together with the Inverse Mapping Theorem for Banach spaces imply that the differential D_2F at (0,0) is a linear isomorphism. We are a short step away from completing the proof of our main theorem.

Theorem 4.4. *Let M be a compact oriented real 6-manifold, and $\alpha_0 \in \Omega^3(M,\mathbb{R})$ a smooth stable 3-form giving M the structure of a complex 3-fold with trivial canonical bundle.*

Suppose (M,α_0) satisfies the $\partial\bar\partial$-lemma, and recall that α_0 is closed.

Then an open neighbourhood of the de Rham class $[\alpha_0]$ in $H^3(M,\mathbb{R})$ is a moduli space for such structures on M, in the sense that it parametrises objects close to (M,α_0) in the category TrivCx^3 *of definition 3.4.*

Proof. Propositions 4.2 and 4.3, together with the implicit function theorem for Banach spaces (see 10.2.1 of [3] for a reference) give us a neighbourhood $\mathcal{U}$ of α_0 in $\mathcal{H}^3$ and a smooth map $g : \mathcal{U} \to E_2$ such that for $\alpha_{\mathcal{H}} \in \mathcal{U}$ the 3-form

$$\alpha = \alpha_{\mathcal{H}} + g(\alpha_{\mathcal{H}}) \in L^2_k(\textstyle\bigwedge^3)$$

is a critical point of HV when restricted to its cohomology class.

Now if we take $k \geq 5$, the Sobolev embedding theorem together with the form of the Newlander-Niremberg theorem presented in [12]—theorem II to be precise—ensure we can apply the formalism of section 2 to produce the structure of a complex 3-fold on M on which α is the real part of a holomorphic volume form.

Moreover the auxiliary space E being complementary to the $\mathrm{Diff}(M)$-orbit through α_0 means the resulting (M,α) give non-isomorphic objects in TrivCx^3, and that they run over a whole neighbourhood of (M,α_0).

Finally since the space of harmonic forms $\mathcal{H}^3$ is naturally identified with $H^3(M,\mathbb{R})$, we obtain the desired moduli space and the proof is complete. □

Remark. Our moduli space is for pairs (M, η) of a compact complex 3-fold M with a choice of holomorphic volume form η. As in proposition 1.11, there is an action of $\mathbb{C}^*$ on the moduli space which changes the form η to some non-zero multiple of it.

Quotienting by the $\mathbb{C}^*$-action recovers a local moduli space for compact complex 3-folds with trivial canonical bundle but no preferred choice of trivialisation.

5 Applications to the theory of Calabi-Yau 3-folds

The results of Hitchin [4] presented above gives an efficient approach to study deformations of Calabi-Yau 3-folds. In this section, we first go over some properties of interest of Calabi-Yau manifolds, and then go on to describe what theorem 4.4 tells us about the moduli space of Calabi-Yau structures on a real 6-manifold M.

5.1 Riemannian holonomy groups

Calabi-Yau manifolds are an instance of Riemannian manifolds with holonomy in $\mathrm{SU}(n)$. This is in contrast to the generic Kähler case, where the holonomy group is $\mathrm{U}(n)$. For completeness we go over the basics of Riemannian holonomy groups.

Definition 5.1. Let (M, g) be a connected Riemannian manifold of dimension n and fix a point $p \in M$. For each smooth loop γ in M based at p, the Levi-Civita connection induces an automorphism τ_γ of the tangent space T_pM, given by parallel transporting along γ.

Then the *holonomy group of* (M, g) *based at* p is

$$\mathrm{Hol}_p(M, g) = \{\tau_\gamma : \gamma \text{ a smooth loop based at p}\} \subset \mathrm{GL}(T_pM).$$

The *restricted holonomy group of* (M, g) *based at* p is

$$\mathrm{Hol}^0_p(M, g) = \{\tau_\gamma : \gamma \text{ a smooth, nullhomotopic loop based at p}\} \subset \mathrm{GL}(T_pM).$$

Some first properties of holonomy groups include the following.

Lemma 5.2. *Let* (M, g) *be as above. Then*

(i) $\mathrm{Hol}^0_p(M, g)$ *is a connected Lie subgroup of* $\mathrm{Isom}(T_pM) \cong \mathrm{O}(n)$,

(ii) $\mathrm{Hol}^0_p(M, g)$ *is a normal subgroup of* $\mathrm{Hol}_p(M, g)$*, and the quotient*

$$\mathrm{Hol}_p(M, g)/\mathrm{Hol}^0_p(M, g)$$

is countable, and

(iii) *if* $p, q \in M$ *then* $\mathrm{Hol}_p(M, g)$ *and* $\mathrm{Hol}_q(M, g)$ *are conjugate, likewise for the reduced holonomy groups.*

Proof. We refer to section 4 in part II of [8]. □

The last item of the lemma allows us to speak of *the* holonomy group of (M, g) as a subgroup of $\mathrm{O}(n)$, well-defined up to conjugation.

One motivation for studying holonomy groups of Riemannian manifolds is their connection with *parallel tensors.* This is made precise via the following result (which is 2.5.2 in [7]).

Proposition 5.3. *Let (M, g) be as above, and set $H = \mathrm{Hol}_p(M, g)$. Write E for the vector bundle $\bigwedge^k TM \otimes \bigwedge^l T^*M$ of tensors of type (k, l).*

If $S_p \in E_p$ is a tensor at p which is invariant under the action of $H \subset \mathrm{GL}(T_pM)$, then there is a unique $S \in C^\infty(E)$ such that $\nabla S = 0$ and $S|_p = S_p$.

Conversely if a tensor $S \in C^\infty(E)$ has $\nabla S = 0$ then $S|_p \in E_p$ is invariant under H.

Proof sketch. One recalls how the Levi-Civita connection ∇ on M induces a connection on the associated bundle E, and how parallel transport along γ is related for TM and E.

In the first case, one defines $S|_q$ to be the parallel transport of S_p along a smooth curve γ joining p to q. Invariance under the action of H implies $S|_q$ is well-defined independently of the choice of γ, and moreover that $S|_q$ depends smoothly on q.

The other direction is more straightforward. □

5.2 Kähler metrics of holonomy SU(n)

We go over some of the properties of Calabi-Yau manifolds. If M is Calabi-Yau, triviality of the canonical bundle together with Yau's proof of the Calabi conjecture produces a family of Ricci-flat Kähler metrics on M. Such metrics have holonomy in $\mathrm{SU}(n)$.

A reference for the majority of the material in this section is Joyce's text [7].

First let us show that a Kähler metric has holonomy in $\mathrm{U}(n)$. This is a consequence of the following lemma.

Lemma 5.4. *Let (M, J, g) be a Kähler manifold of complex dimension n, with corresponding fundamental form ω.*

Then if ∇ is the Levi-Civita connection for g, we have both

$$\nabla J = 0 \quad \textit{and} \quad \nabla\omega = 0.$$

Proof. A routine calculation. See the proof of theorem 4.17 in [2]. □

By proposition 5.3, the holonomy group H of a Kähler manifold preserves both J and ω. And now to deduce that H is a subgroup of $U(n)$, recall how the intersection of any two of

$$\mathrm{O}(2n), \qquad \mathrm{GL}(n, \mathbb{C}), \qquad \mathrm{Sp}(2n, \mathbb{R})$$

inside $\mathrm{GL}(2n, \mathbb{R})$ is the unitary group $\mathrm{U}(n)$.

Of convenience to us will be the so-called Ricci form of a Kähler metric.

Definition 5.5. Let (M, J, g) be a Kähler manifold. Let R be the Riemann curvature tensor of the metric g, and Ric the resulting Ricci curvature. Then the *Ricci form* of M is the tensor given by

$$\rho(X, Y) = \mathrm{Ric}(JX, Y),$$

for X, Y vector fields.

Recall that Ric is symmetric. Moreover since $\nabla J = 0$ on a Kähler manifold, we have

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}(JX, JY)$$

for all X, Y. So Ric satisfies the same formal properties as a J-invariant metric, from which ρ is a real differential form of type (1,1) just like the Kähler form ω.

The following proposition summarises a few basic properties of ρ.

Proposition 5.6. *Let (M, J, g) be a Kähler manifold with Ricci form ρ. Then*

(i) *the curvature of the Chern connection on the canonical bundle K_M can be naturally identified with ρ,*

(ii) $d\rho = 0$, *and*

(iii) *the cohomology class $[\rho] \in H^2(M, \mathbb{R})$ depends only on the complex structure on M, and equals*

$$[\rho] = 2\pi c_1(M)$$

where $c_1(M)$ is the first Chern class of M.

Proof sketch. Part (i) involves a computation in local coordinates, which for example may be found in section 4.6 of [7]. Then (ii) and (iii) follow from properties of the Chern connection on a line bundle. □

The next theorem is key in connecting Calabi-Yau manifolds to the theory of reduced holonomy Riemannian metrics.

Theorem 5.7. *Let M be a Calabi-Yau manifold, and let ω be the Kähler form of some Kähler metric g on M. Then there exists a unique Kähler metric g' on M with vanishing Ricci curvature and for which the Kähler form ω' is in the same cohomology class as ω,*

$$[\omega'] = [\omega] \quad \textit{in } H^{1,1}(M).$$

Proof. Follows from Yau's proof of the Calabi conjecture—see section 5 of [7]. □

Definition 5.8. Let M be a complex manifold. The *Kähler cone* $\mathcal{K}_M$ is the set of all possible Kähler classes on M

$$\mathcal{K}_M = \{[\omega] : \omega \text{ a Kähler form on } M\} \subset H^{1,1}(M) \cap H^2(M, \mathbb{R}).$$

Recall that the Kähler cone $\mathcal{K}_M$ is an open convex cone in the real vector space

$$H^{1,1}(M, \mathbb{R}) = H^{1,1}(M) \cap H^2(M, \mathbb{R}).$$

Corollary 5.9. *Let M be a Calabi-Yau manifold. Then the space of Ricci-flat Kähler metrics on M is naturally identified with $\mathcal{K}_M$, and hence carries the structure of a smooth real manifold of dimension $h^{1,1}(M)$.* □

And finally we have the following result on $\mathrm{SU}(n)$-holonomy metrics.

Lemma 5.10. *Let M be a Calabi-Yau manifold and g some choice of Ricci-flat Kähler metric on M. Then if η is a global holomorphic volume form on M, we have*

$$\nabla\eta = 0$$

where ∇ is induced by the Levi-Civita connection of g.

In particular $\mathrm{Hol}(M, g) \subseteq \mathrm{SU}(n)$.

Proof. Involves using the so-called *Weitzenbock formula*, see proposition 6.2.4 in [7]. □

Remark. The assumption that η be globally-defined in lemma 5.10 is essential.

Remark. The proof of the weaker statement that the reduced holonomy group $\mathrm{Hol}^0(M, g)$ of a Ricci-flat Kähler metric g is contained in $\mathrm{SU}(n)$ is more straightforward. Since the curvature of the Chern connection on K_M vanishes, one can find local flat sections of K_M. To finish off this proof one uses a certain factorisation lemma for null-homotopic curves, see for example appendix 7 of [8].

5.3 Deformations of Calabi-Yau 3-folds

In this section we apply theorem 4.4 of Hitchin to the theory of deformations of Calabi-Yau manifolds. We also go over how the Ricci-flat Kähler metrics of the previous section behave under deformation. Finally we compare our approach to deformations of Calabi-Yau 3-folds with the more general results of Tian and Todorov.

So let M be a Calabi-Yau 3-fold. If $\eta \in \Omega^{3,0}(M)$ is a holomorphic volume form on M, then the real 3-form $\alpha = \eta + \bar{\eta}$ is stable and theorem 4.4 tells us an open neighbourhood of $[\alpha]$ in $H^3(M, \mathbb{R})$ is a local moduli space for complex 3-folds with a distinguished holomorphic volume form.

We can go further, thanks to the following result—see theorem 15 of [9].

Theorem 5.11. *Let $\{M_t : t \in \mathcal{U}\}$ be a smooth family of compact complex manifolds, with t ranging over some open subset $\mathcal{U}$ of a Euclidean space. Then if M_{t_0} admits a Kähler metric, then so do M_t for t in a sufficiently small neighbourhood of t_0.*

Moreover we can choose the Kähler metric on M_t to depend smoothly on t.

In view of this result, theorem 4.4 gives us a neighbourhood $\mathcal{U}$ of (M, η) in the moduli space of *Calabi-Yau* 3-folds together with a choice of holomorphic volume form.

Now corollary 5.9 says the Calabi-Yau 3-fold M admits a smooth family of Ricci-flat Kähler metrics. We briefly elaborate on how these metrics on M behave under deformation.

Let ω be the Kähler form of a Ricci-flat Kähler metric on $M = M_{t_0}$. Theorem 5.11 gives a smooth family of Kähler forms $\{\omega_t\}$ on the nearby Calabi-Yau 3-folds in the moduli space $\{M_t : t \in \mathcal{U}\}$, with $\omega_{t_0} = \omega$.

By theorem 5.7, we can modify each ω_t uniquely within its Kähler class to yield a Ricci-flat Kähler metric on M_t. So without loss of generality the ω_t are Ricci-flat and we have constructed a deformation $\{\omega_t : t \in \mathcal{U}\}$ of the Ricci-flat Kähler metric ω on M to Ricci-flat Kähler metrics on the nearby $\{M_t : t \in \mathcal{U}\}$.

Remark. One may also study an enhanced moduli space for triples (M, η, ω) with M a complex 3-fold, η a global holomorphic volume form on M, and ω the Kähler form of a Ricci-flat Kähler metric on M.

It is a fact that the Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ of a Kähler manifold are constant under deformation—see proposition 9.20 in [15]. In particular $h^{1,1}$ is constant, and hence this enhanced moduli space carries the structure of a smooth manifold with a submersion to our original moduli space given by forgetting the choice of Kähler metric.

Remark. Our approach to the deformation of Calabi-Yau 3-folds fits within a more general framework. Kodaira's direct approach to the deformation of complex structures is used by Tian [13] and independently Todorov [14] to show that the space of deformations of a Calabi-Yau n-fold M carries the structure of a complex manifold of dimension $h^{1,n-1}(M)$.

This agrees with theorem 4.4, since for a Calabi-Yau 3-fold the Betti number b_3 is

$$b_3 = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} = 2(1 + h^{1,2})$$

and this gives the right real dimension once we quotient by the $\mathbb{C}^*$-action to forget the choice of holomorphic volume form.

A Introduction to the theory of elliptic operators

A result of analysis which is invaluable in differential geometry is *elliptic regularity*. Let E, E' be two $\mathbb{K}$-vector bundles over a compact Riemannian manifold M. As usual here $\mathbb{K}$ is one of $\mathbb{R}$ or $\mathbb{C}$. Elliptic regularity concerns the class of *elliptic differential operators*

$$Q : C^\infty(E) \to C^\infty(E').$$

The slogan is that the kernel of an elliptic operator Q is very well behaved, and more generally if u solves the equation $Q(u) = v$, and the order of Q is k say, then u admits k more derivatives than v does—hence the name elliptic *regularity*. We also have easy to check conditions for when the differential equation $Q(u) = v$ has a solution u. We will make all this precise as we go on.

References for this material include section 10 of Nicolaescu [11] and, for a quicker presentation, section 1.3 of Joyce [7]. Proofs are omitted.

A.1 Partial differential operators on manifolds

Definition A.1. Let E, E' be $\mathbb{K}$-vector bundles of ranks d, d' over M a smooth manifold. Then we say a $\mathbb{K}$-linear map $L : C^\infty(E) \to C^\infty(E')$ is a *partial differential operator of order at most* k, written $L \in PDO^{(k)}(E, E')$, if for each $p \in M$ there is a neighbourhood U of p which is the domain of a chart $\{x^i\}$ and over which E, E' trivialise—so that sections of E, E' over U are identified with functions from some open set in $\mathbb{R}^n$ to a Euclidean space $\mathbb{R}^d, \mathbb{R}^{d'}$—such that L is given as

$$L(u)_i(x) = \sum_{\substack{|\alpha| \le k \\ 1 \le j \le d}} \phi_{\alpha,i,j}(x)(\partial^\alpha u_j)(x)$$

for $i = 1, \ldots, d'$, where α runs over the multi-indices of size $\le k$ and the $\phi_{\alpha,i,j}$ are smooth $\mathbb{K}$-valued functions on U.

Remark. An equivalent definition which avoids taking coordinates is motivated (as many things often are) by the Leibniz rule, and involves considering the interaction between L and the $C^\infty(M)$-module structures on $C^\infty(E)$ and $C^\infty(E')$.

So for example $L \in PDO^{(0)}(E, E')$ is the same as L being a smooth section of $\mathrm{Hom}(E, E')$, or equivalently L commuting with multiplication by any $f \in C^\infty(M)$, i.e.

$$[L, f]u \coloneqq L(fu) - fL(u) = 0.$$

Then one defines $PDO^{(k)}$ inductively as the set of operators L such that

$$\mathrm{ad}(f)(L) \coloneqq [L, f] \in PDO^{(k-1)}(E, E')$$

for all $f \in C^\infty(M)$. Hopefully the below examples illustrate this idea.

Example A.2. The exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is in $PDO^{(1)}$.

Indeed it satisfies

$$[d, f]\, \omega = d(f\omega) - f d\omega = df \wedge \omega$$

so that $[d, f]$ is a bundle morphism $\Omega^p(M) \to \Omega^{p+1}(M)$.

Example A.3. If E is a vector bundle over M equipped with a connection, then covariant differentiation $\nabla : C^\infty(E) \to C^\infty(E) \otimes T^*M$ is in $PDO^{(1)}$.

This time $[\nabla, f]$ is tensoring with df.

The above are the most basic examples of partial differential operators, but the following is particularly important.

Example A.4. Let M be a compact, oriented, Riemannian manifold. Then if d^* is the formal adjoint to d, defined using the Hodge star $*$, then the *Hodge Laplacian*

$$\Delta : \Omega^p(M) \to \Omega^p(M)$$

defined by $\Delta = dd^* + d^*d$ is in $PDO^{(2)}$.

To see this one recalls that $d^* = \pm * \circ d \circ *$, and checks that composition of partial differential operators behaves as expected. Then d^* is in $PDO^{(1)}$ and Δ is in $PDO^{(2)}$.

On a complex manifold we also have access to ∂ and $\bar{\partial}$ acting on complex differential forms, and on a Hermitian manifold to the associated Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$.

Definition A.5. Let $L \in PDO^{(k)}(E, E')$. The *symbol* of L, denoted $\sigma_k(L)$, assigns to each $\xi \in T_p^*M$ a linear map $\sigma_k(L)(\xi) : E|_p \to E'|_p$ as follows.

Working in coordinates as in definition A.1, so that

$$L(u)_i(x) = \sum_{\substack{|\alpha| \le k \\ 1 \le j \le d}} \phi_{\alpha,i,j}(x)(\partial^\alpha u_j)(x),$$

and if $\{e_j\}_{j=1}^d$, $\{e_i'\}_{i=1}^{d'}$ are the frames trivialising E, E' at p, we set

$$\sigma_k(L)(\xi)\left(\sum_{j=1}^d a_j e_j\right) = \sum_{\substack{|\alpha| = k \\ 1 \le j \le d \\ 1 \le i \le d'}} a_j\, \phi_{\alpha,i,j}(p)\, \xi^\alpha\, e_i',$$

where ξ is identified with its components with respect to $\{dx^i|_p\}_{i=1}^n$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ is the usual multi-index notation.

Remark. Once again there is an equivalent definition which avoids taking coordinates. Given $\xi \in T_p^*M$, find $f \in C^\infty(M)$ such that $df|_p = \xi$. Since $L \in PDO^{(k)}(E, E')$ the iterated commutator

$$\frac{1}{k!} \underbrace{\mathrm{ad}(f) \cdots \mathrm{ad}(f)}_{k \text{ times}} L$$

is a bundle morphism $E \to E'$. Then set $\sigma_k(L)(\xi)$ to be the resulting linear map between the fibers $E|_p \to E'|_p$.

One checks this definition depends only on ξ and not on the choice of f, and that it agrees with the above.

The symbol of L is a homogeneous polynomial on each cotangent space, and captures the highest-order behaviour of L in an invariant way. We can now make the key definition.

Definition A.6. Let $L \in PDO^{(k)}(E, E')$. Then L is an *elliptic operator of order* k if for all $p \in M$ and all $\xi \in T_p^*M \setminus \{0\}$, the symbol of L is such that $\sigma_k(L)(\xi)$ is an isomorphism between the fibers $E|_p \to E'|_p$.

Example A.7. The symbol of exterior differentiation $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is

$$\sigma_1(d)(\xi) = \xi \wedge \cdot,$$

so d is far from being elliptic.

Example A.8. On an oriented Riemannian manifold, the symbol of d^* is (up to a sign) interior product along the metric dual

$$\sigma_1(d^*)(\xi) = -\iota_{\xi^\sharp} \cdot \ .$$

Here $\xi^\sharp$ is the image of ξ under the isomorphism $T_p^*M \to T_pM$ induced by the metric. Again, d^* is far from being elliptic.

Example A.9. Next consider the Laplacian $\Delta : \Omega^p(M) \to \Omega^p(M)$. Since this maps sections of $\Omega^p(M)$ back to sections of $\Omega^p(M)$, unlike d or d^* this has a chance of being elliptic.

To compute $\sigma_2(\Delta)$, one can show that the symbol of a composition of two partial differential operators is the composition of their symbols. And then using the previous two examples one calculates

$$\sigma_2(\Delta)(\xi) = -\xi(\xi^\sharp) \cdot ,$$

that is $\sigma_2(\Delta)(\xi)$ is scalar multiplication by $-\xi(\xi^\sharp) = -|\xi|^2$.

So the Laplacian Δ is elliptic! It is this fact together with elliptic regularity which is at the heart of the Hodge decomposition theorem.

Example A.10. Let M be a Hermitian manifold. Then analogously to the above, we have that

- $\sigma_1(\partial)(\xi) = \Pi^{1,0}(\xi) \wedge \cdot$ is wedging against the (1,0) part of ξ,
- $\sigma_1(\partial^*)(\xi) = -\iota_{\Pi^{1,0}(\xi)^\sharp} \cdot$ is (the negative of) interior product along the metric dual of the (1,0) part of ξ,
- $\sigma_2(\Delta_\partial)(\xi) = -|\Pi^{1,0}(\xi)|^2$.

Since a real covector $\xi \in T_p^*M$ is zero if and only if its (1,0) part is zero, we find that the ∂-Laplacian Δ_∂ is also elliptic.

Similarly for $\bar\partial$ and $\Delta_{\bar\partial}$, mutatis mutandis.

Ellipticity of Δ_∂ and $\Delta_{\bar\partial}$ are responsible for Hodge decomposition in the context of (p,q)-forms and Dolbeault cohomology.

A final and important definition is that of the adjoint of a differential operator.

Definition A.11. Suppose E, E' are vector bundles over M a compact Riemannian manifold, equipped with metrics on the fibers, and let $L \in PDO^{(k)}(E, E')$. Then there is a unique *formal adjoint* $L^* \in PDO^{(k)}(E', E)$ verifying

$$\int_M \langle v, L(u)\rangle \operatorname{vol}_g = \int_M \langle L^*(v), u\rangle \operatorname{vol}_g$$

for all $u \in C^\infty(E), v \in C^\infty(E')$.

A.2 Sobolev Spaces

So far our operators have been mapping between spaces of *smooth* sections of vector bundles. We will now work on a compact Riemannian manifold M and introduce the Sobolev spaces of sections.

The metric g on M induces a finite Radon measure μ on M with its Borel σ-algebra. This μ is such that if $f \in C^\infty(M)$, then

$$\int_M f\, d\mu = \int_M f \operatorname{vol}_g,$$

where on the right we have the usual integral of a smooth differential form.

Definition A.12. Let E be a vector bundle over M equipped with a metric on the fibers, and let $1 \leq p < \infty$. Then $L^p(E)$ denotes the Banach space of measurable, almost-everywhere defined sections u of E for which the norm

$$\|u\|_{L^p} \coloneqq \left(\int_M |u|^p \, d\mu \right)^{1/p}$$

is finite.

Measurability is understood by working locally in trivialisations of E, so that u becomes a function from M to some Euclidean space.

Definition A.13. Let E be a vector bundle over M equipped with a metric on the fibers and a connection ∇_E, let $1 \leq p < \infty$ and let $k \geq 0$ be an integer.

Then the *Sobolev space* $L^p_k(E)$ of sections of E is the Banach space consisting of those $u \in L^p(E)$ admitting k weak derivatives in L^p, i.e. such that for $i = 1, \ldots, k$ there are $v_i \in L^p\left(E \otimes (T^*M)^{\otimes i}\right)$ with

$$\int_M \langle \varphi, v_i \rangle \, d\mu = \int_M \left\langle (\nabla^i)^* \varphi, u \right\rangle \, d\mu$$

for all $\varphi \in C^\infty\left(E \otimes (T^*M)^{\otimes i}\right)$, where

$$\nabla^i \in PDO^{(i)}(E, E \otimes (T^*M)^{\otimes i})$$

is formed using ∇_E and the Levi-Civita connection on M and $(\nabla^i)^*$ is its formal adjoint.

Write $\nabla^i u$ for the weak derivative v_i.

The norm on $L^p_k(E)$ is

$$\|u\|_{L^p_k} = \sum_{i=0}^{k} \|\nabla^i u\|_{L^p}.$$

The traditional Sobolev spaces for domains in $\mathbb{R}^n$ are fundamental in the study of PDEs, and here we have their natural generalisation to the setting of vector bundles over compact manifolds.

The famous Sobolev embedding theorem extends to this situation. A reference for the following is 10.2.36 of [11].

Theorem A.14. *Let E be a vector bundle over M as above. Let $1 \leq p < \infty$, let $k \geq 0$ be an integer, and let $n = \dim M$. Then if the integer $r \geq 0$ is such that*

$$\frac{1}{p} < \frac{k-r}{n},$$

then there is a natural continuous embedding

$$L^p_k(E) \hookrightarrow C^r(E),$$

where $C^r(E)$ is the Banach space of r-times continuously differentiable sections of E.

A.3 Elliptic Regularity

Elliptic regularity comes in several parts. First we have the statement about regularity of solutions to elliptic PDEs. The following theorem is 1.4.1 of [7].

Theorem A.15. *Let M be a compact Riemannian manifold, E, E' vector bundles over M with metrics on the fibers, and $Q \in PDO(E, E')$ an elliptic operator of order k.*

Suppose $Q(u) = v$ holds weakly, with $u \in L^1(E)$ and $v \in L^1(E')$, i.e. that

$$\int_M \langle \varphi, v \rangle \operatorname{vol}_g = \int_M \langle Q^*(\varphi), u \rangle \operatorname{vol}_g$$

for all $\varphi \in C^\infty(E')$, and where Q^* is the formal adjoint of Q.

Then if $v \in L^p_l(E')$, we have that $u \in L^p_{k+l}(E)$, and there is a constant $C > 0$ independent of u, v such that

$$\|u\|_{L^p_{k+l}} \leq C(\|u\|_{L^1} + \|v\|_{L^p_l}).$$

Remark. This makes precise the statement about $Q(u) = v$ implying that u admits k more derivatives than v.

An important corollary of theorem A.15 is the following.

Corollary A.16. *Let M and $Q \in PDO(E, E')$ be as above. Then for each integer $l \geq 0$, Q induces a continuous linear map*

$$Q : L^p_{k+l}(E) \to L^p_l(E'),$$

and the kernel $\ker Q \subset L^p_{k+l}(E)$ is finite-dimensional and contained in the image of the embedding $C^\infty(E) \hookrightarrow L^p_{k+l}(E)$.

Finally we have the condition for $Q(u) = v$ to have a solution in u. Again this is a consequence of A.15. The following theorem is 1.5.3 in [7].

Theorem A.17. *Let M and $Q \in PDO(E, E')$ be as above. Then the adjoint operator Q^* is also elliptic of order k, and if $v \in L^p_l(E')$ then there exists $u \in L^p_{k+l}$ such that $Q(u) = v$ if and only if $v \perp_{L^2} \ker Q^*$. Moreover if we demand $u \perp_{L^2} \ker Q$ then u is unique.*

In particular, Q induces a continuous linear isomorphism

$$\begin{tikzcd}
(\ker Q)^\perp \arrow[r, "\simeq"] \arrow[d, hook] & (\ker Q^*)^\perp \arrow[d, hook] \\
L^p_{k+l}(E) & L^p_l(E')
\end{tikzcd}$$

between the finite codimension subspaces $(\ker Q)^\perp$ and $(\ker Q^*)^\perp$.

In view of this last theorem, we introduce the notion of the Green's operator of an elliptic differential operator.

Definition A.18. Let M and $Q \in PDO(E, E')$ be as above. Then for any given integer $l \geq 0$, the *Green's operator of Q* is the continuous linear map

$$G_Q : L^p_l \to L^p_{k+l}$$

which on $(\ker Q^*)^\perp$ inverts $Q : (\ker Q)^\perp \xrightarrow{\simeq} (\ker Q^*)^\perp$ and is zero on $\ker Q^*$.

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