# Homework 8 Solutions

1) *Start with* $X^2 - 14X + 40 = 0$. *This time use the transformation* $X = Z + 7$, *substituting this in to the equation. Solve the resulting equation in Z and then find the solutions to the original* $X^2 - 14X + 40 = 0$ *equation by using the* $X = Z + 7$ *relationship.*

*To see what's going on visually, graph both the* $X^2 - 14X + 40$ *quadratic, and the quadratic in Z that you found on the same graph. Aha - you should see (or already know) that the* $X = Z + 7$ *transformation just "shifts" the* $X^2 - 14X + 40$ *curve to the left by 7.*

So we transform $X^2 - 14X + 40 = 0$ into an equation involving Z using the substitution given:

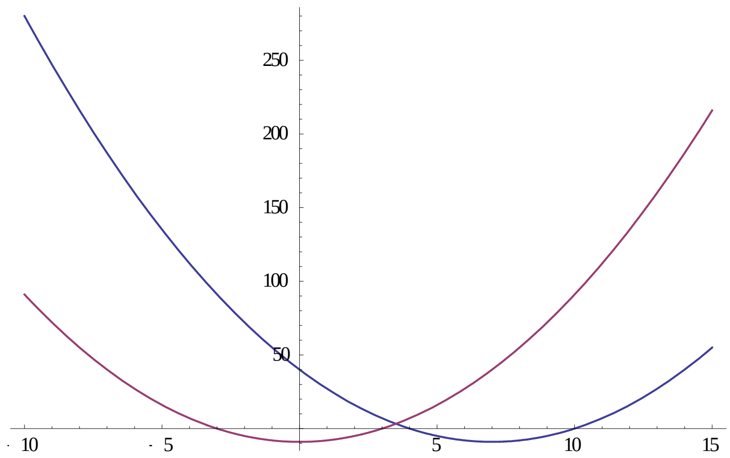
$$(Z+7)^2 - 14(Z+7) + 40 = 0$$

which is $Z^2 + 14Z + 49 - 14Z - 98 + 40 = 0$

simplifying to $Z^2 - 9 = 0$

with solutions $Z = 3$ and $Z = -3$. Thus given that $X = Z + 7$ then this means that $X = 3 + 7 = 10$ and $X = (-3) + 7 = 4$ are solutions to the original equation (which they are).

Graphically we see the original (blue) parabola simply shifted 7 to the left (to the red parabola) – which is the graphical equivalent to substituting Z+7 for X. The new red parabola ($Z^2 - 9$) is symmetric around the y axis, which is the equivalent graphically of a quadratic with a pair of roots that are inverses of each other (e.g. +/- 3 in this case).



*2) …now what transformation would you have to use to solve $X^2 + BX + C = 0$ in a similar fashion? Try using this transformation, solving the resulting quadratic, and then "undo" the transformation the same way you did in the previous problems to find the solutions to the original $X^2 + BX + C = 0$ equation and compare your solution to the one you'd get using the usual quadratic formula!*

Try $X = Z - B/2$ which "gets rid" of the second term as follows…

$$\left(Z - \frac{B}{2}\right)^2 + B\left(Z - \frac{B}{2}\right) + C = 0$$

solving: $Z^2 - \frac{B^2}{4} + C = 0$

so that $Z = \pm\frac{\sqrt{B^2-4C}}{2}$

and so we end up with $X = Z - \frac{B}{2} = \frac{\pm\sqrt{B^2-4C}}{2} - \frac{B}{2} = \frac{-B\pm\sqrt{B^2-4C}}{2}$

which looks awfully familiar (just missing the A coefficient given that we worked with the equation
$X^2 + BX + C = 0$ which had $A = 1$.

**Bonus:**

Okay, so if A doesn't equal 1, then we are working with the more general equation $AX^2 + BX + C = 0$

Now we'll try the substitution $X = Z - B/2A$ and we'll get the following:

$AX^2 + BX + C = 0$ becomes $A\left(Z - \frac{B}{2A}\right)^2 + B\left(Z - \frac{B}{2A}\right) + C = 0$

and this simplifies to $AZ^2 - \frac{B^2}{4A} + C = 0$

so that $AZ^2 = \frac{B^2}{4A} - C$ which we can solve easily enough for Z getting

$$Z = \pm\sqrt{\frac{B^2 - 4AC}{4A^2}} = \pm\frac{\sqrt{B^2 - 4AC}}{2A}$$

and then substituting back using $X = Z - B/2A$ we get… drum roll please…

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

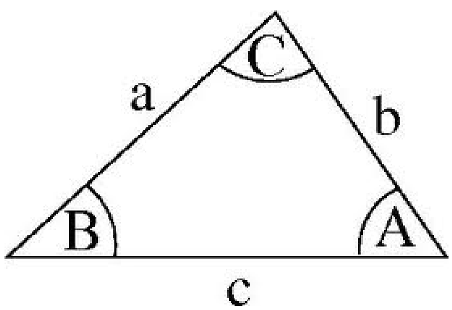
yea!

*3) ...head back to Cardano's formula and use what we all discovered in class together to find one real root for the following cubic: $x^3 - 18x - 35 = 0$.*

Change the equation to $x^3 = 18x + 35$ and try to find r and s so that $3rs = 18$ and $r^3 + s^3 = 35$. By inspection we see that $r = 3$ and $s = 2$ works ($3*3*2 = 18$ and $3^3 + 2^3 = 27 + 8 = 35$). From class we saw that this gave us the solution r + s, which in this case gives us a solution $x = 5$.

*5) Next, here's a short foray based on our work a few classes ago with the Law of Sines and the Law of Cosines. You might not know it but Heron's formula giving the area of a triangle, can be proved using the Law of Cosines, and this problem will guide you through this proof (and if you're not familiar with Heron's formula, you'll find out about that too!) Quick background - Heron's formula gives the area of any triangle just in terms of its side lengths (i.e. no mention of angles at all).*

*First off, suppose you have a triangle (not necessarily right) with sides a, b, and c, and angles A, B, and C:*




*(i) Using the Law of Cosines, first write down what $a^2$ equals in terms of b, c, and Cos A.*

$$a^2 = b^2 + c^2 - 2bc(\text{Cos } A)$$

*(ii) Next, using the previous result, solve for Cos A in terms of a, b, and c.*

$$a^2 = b^2 + c^2 - 2bcCosA$$

$$a^2 - b^2 - c^2 = -2bcCosA$$

$$\frac{a^2 - b^2 - c^2}{-2bc} = CosA$$

*(iii) Next, recall the basic trig result $(\text{Cos } A)^2 + (\text{Sin } A)^2 = 1$, and now using the previous result, find Sin A in terms of a, b and c*

$$(CosA)^2 + (SinA)^2 = 1$$

$$\left(\frac{a^2 - b^2 - c^2}{-2bc}\right)^2 + (SinA)^2 = 1$$

$$\frac{\left(a^2 - b^2 - c^2\right)^2}{4b^2c^2} + (SinA)^2 = 1$$

$$(SinA)^2 = 1 - \frac{\left(a^2 - b^2 - c^2\right)^2}{4b^2c^2}$$

$$(SinA)^2 = \frac{4b^2c^2}{4b^2c^2} - \frac{a^4 - 2a^2b^2 - 2a^2c^2 + b^4 + 2b^2c^2 + c^4}{4b^2c^2}$$

$$(SinA)^2 = \frac{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}{4b^2c^2}$$

$$SinA = \frac{\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}}{2bc}$$

*(iv) Next, write down the area of the above triangle in terms of b, c, and Sin A.*

One can work out that the height is equal to b Sin(A), and so the area is just 1/2 base times height, or ½ b c (Sin A)

*(v) Aha! Given that you found out what Sin A equals using just a, b, and c (in part (iii)) now write down what the area of the triangle equals just in terms of its side lengths, a, b, and c.*

By using the formula above we have

$$A = \frac{\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}}{4}$$

*Yea! This is Heron's formula, just in a completely unrecognizable form given that everything is/should be completely messy at the moment…Heron's formula says the area of a triangle with side lengths a, b, and c, equals the square root of s(s-a)(s-b)(s-c) where s is the so-called "semi-perimeter" which is half of the true perimeter (a+b+c).*

***Bonus*** *- If you'd like to test out your Mathematica simplification skills… see if what you got in part (v) really is equivalent to Heron's formula.*

Here you can just expand out Heron's formula given above and check it against your answer using the following *Mathematica* command:

s = (a + b + c)/2; Expand [s (s - a) (s - b) (s - c)]

and this will give you:

$$-\frac{a^4}{16}+\frac{a^2b^2}{8}-\frac{b^4}{16}+\frac{a^2c^2}{8}+\frac{b^2c^2}{8}-\frac{c^4}{16}$$