Homework 6 Solutions

Binomial Theorem Chapter Exercises 1, 2, 3, 4, 6

1. Compute the powers: 101^0, 101^1, 101^2, 101^3, 101^4, 101^5.

   $101^0 = 1$
   $101^1 = 101$
   $101^2 = (100 + 1)(100 + 1) = 10{,}000 + 2\cdot 100 + 1 = 10{,}000 + 200 + 1 = 10{,}201$
   $101^3 = (10{,}000 + 200 + 1)(100 + 1) = 1{,}000{,}000 + 3\cdot 10{,}000 + 3\cdot 100 + 1 =$ $1{,}000{,}000 + 30{,}000 + 300 + 1 = 1, 030{,}301$
   $101^4 = (1{,}000{,}000 + 30{,}000 + 300 + 1)(100 + 1) = 100{,}000{,}000 + 4\cdot 1{,}000{,}000 + 6\cdot 10{,}000 + 4\cdot 100 + 1 = 104{,}060{,}401$
   $101^5 = (100{,}000{,}000 + 4{,}000{,}000 + 60{,}000 + 400 + 1)(100 + 1) =$ $10{,}510{,}100{,}501$

   Notice that the terms of Pascal's triangle form the numbers in the expansion. So the next power would be 1,061,520,150,601.

2. Without a calculator, compute 201^6.

   $201^6 = (200 + 1)^6 = 200^6 + 6\cdot 200^5 + 15\cdot 200^4 + 20\cdot 200^3 + 15\cdot 200^2 + 6\cdot 200 + 1$ $= 64{,}000{,}000{,}000{,}000 + 6\cdot 320{,}000{,}000{,}000 + 15\cdot 1{,}600{,}000{,}000 + 20\cdot 8{,}000{,}000 + 15\cdot 40{,}000 + 6\cdot 200 + 1 = 64{,}000{,}000{,}000{,}000 + 1{,}920{,}000{,}000{,}000 + 24{,}000{,}000{,}000 + 160{,}000{,}000 + 600{,}000 + 1{,}200 + 1 = 65{,}944{,}160{,}601{,}201$.

3.
   a. Without doing the arithmetic, explain why $64 – 6\cdot 32 + 15\cdot 16 - 20\cdot 8 + 15\cdot 4 - 6\cdot 2 +1$ must equal 1.

      We see that the above expression is equivalent to the following:
      $(-2)^6 + 6\cdot(-2)^5 + 15\cdot(-2)^4 + 20 \cdot(-2)^3 + 15\cdot(-2)^2 + 6\cdot(-2) + 1 = (-2 + 1)^6 = (-1)^6 = 1$.

   b. What can be deduced from examining $(3 – 1)^6$?

      We see that this can be expanded using the binomial theorem.
      $(3)^6 + 6\cdot(3)^5(-1) + 15\cdot(3)^4(-1)^2 + 20 \cdot(3)^3(-1)^3 + 15\cdot(3)^2(-1)^4 + 6\cdot(3)(-1)^5 + (-1)^6 = 2^6 = 64$

6. Suppose we wished to expand the product $(x + y + z)^7$.
   a. How many times will the term y^7 appear?

      One time

   b. How many times will the term xz^6 appear?

      Seven times

c. How many times will the term $x^2y^3z^2$ appear?

210 times

$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7!}{2!3!2!}$ = 210 because you choose the two x's out of the 7 trinomials, then choose 3 y's from the remaining trinomials, and finally the 2 z's from the remaining trinomials.

d. Write a formula for the number of times a general term $x^ay^bz^c$ will appear.

Using the rational from part c, the general term is as follows:

$\binom{7}{a}\binom{7-a}{b}\binom{7-a-b}{c} = \frac{7!}{a!b!c!}$, where a + b + c = 7.

e. State the general "trinomial theorem:" $(x + y + z)^n =$

Sum the terms

$$\frac{7!}{a!b!c!}x^ay^bz^c,$$

where the exponents, a, b, c, range over all possible nonnegative integer combinations that sum to 7, i.e.

$$\sum_{a,b,c} \frac{7!}{a!\,b!\,c!} x^a y^b z^c$$

again where a, b, and c range over all combinations of nonnegative integers summing to n.

Mathematical Connections Problems 11a, b, c and 13

11a) First, find the differences in the following table and using them to compute the polynomial.

n	f(n)	Δ(n)	Δ^2(n)	Δ^3(n)
0	1	4	16	18
1	5	20	34	18
2	25	54	52	18
3	79	106	70	18
4	185	176	88	18
5	361	264	106	
6	625	370		

7	995			

Using the method from the reading, we see that

$$f(n)=\binom{n}{0}\cdot 1+\binom{n}{1}\cdot 4+\binom{n}{2}\cdot 16+\binom{n}{3}\cdot 18$$
$$=1\cdot 1+n\cdot 4+\frac{n(n-1)}{2}\cdot 16+\frac{n(n-1)(n-2)}{6}\cdot 18$$
$$=1+4n+8n(n-1)+3n(n^2-3n+2)$$
$$=1+4n+8n^2-8n+3n^3-9n^2+6n$$
$$=3n^3-n^2+2n+1$$

Note that you can skip all this working out by hand algebraic work by just writing this down in *Mathematica* as
F[n_]: = 1Binomial[*n*, 0] + 4Binomial[*n*, 1] + 16Binomial[*n*, 2] + 18Binomial[*n*, 3]

Which you can then simply using the command
Simplify[F[n]]

to get to the same polynomial.
In fact, this really was the main point for giving you *Mathematica* to work with, so that you could avoid all the slow (and not very useful) computations by hand for just such problems – getting more practice (and more annoyance!) with basic algebraic expression manipulation is not the point of this problem, after all! (and not the focus for the course either!)

Feel free to use *Mathematica* whenever you get to such a point – i.e. facing a fair amount of algebraic manipulation – and just jot down the commands you've used (and the output) for your homework – that's fine (and, again, expressly hoped for!)

11c) First, find the differences in the following table and using them to compute the polynomial.

n	f(n)	Δ(n)	$Δ^2$(n)	$Δ^3$(n)
0	-1	1	6	6
1	0	7	12	6
2	7	19	18	6
3	26	37	24	6
4	63	61	30	6
5	124	91	36	
6	215	127		

7	342			

And one last time, the method from the reading gives us the desired polynomial as follows:

$$f(n) = \binom{n}{0}\cdot(-1) + \binom{n}{1}\cdot 1 + \binom{n}{2}\cdot 6 + \binom{n}{3}\cdot 6$$
$$= 1\cdot(-1) + n + \frac{n(n-1)}{2}\cdot 6 + \frac{n(n-1)(n-2)}{6}\cdot 6$$
$$= -1 + n + 3n(n-1) + n(n^2 - 3n + 2)$$
$$= -1 + n + 3n^2 - 3n + n^3 - 3n^2 + 2n$$
$$= n^3 - 1$$

And, once more, we've provided the algebraic details for this problem, but you could have just skipped all this using *Mathematica* instead.

13) Using the partially completed table and assuming that the third difference remains constant, we can determine the polynomial as follows:

$$f(n) = \binom{n}{0}\cdot 3 + \binom{n}{1}\cdot(-4) + \binom{n}{2}\cdot 5 + \binom{n}{3}\cdot 2$$
$$= 1\cdot 3 + n\cdot(-4) + \frac{n(n-1)}{2}\cdot 5 + \frac{n(n-1)(n-2)}{6}\cdot 2$$
$$= 3 - 4n + \frac{5}{2}n(n-1) + \frac{1}{3}n(n^2 - 3n + 2)$$
$$= 3 - 4n + \frac{5}{2}n^2 - \frac{5}{2}n + \frac{1}{3}n^3 - n^2 + \frac{2}{3}n$$
$$= \frac{1}{3}n^3 + \frac{3}{2}n^2 - \frac{35}{6}n + 3$$

Again, doing this using commands in *Mathematica* is the preferred method!
If we're told that this is supposed to be a cubic polynomial, then yes, this is the only possible cubic polynomial (essentially playing off of the idea that knowing four values of any cubic uniquely identifies the cubic).

On the other hand there are an infinite number of polynomials of higher degree that would work given this same difference table. For instance if we continue to fill out the rest of the table using the "up and over" rule we get:

Input	Output	Δ	Δ^2	Δ^3	Δ^4
0	3	−4	5	2	0
1	−1	1	7	2	
2	0	8	9		
3	8	17			
4	25				
5					

Here we also filled in the next column, the $\Delta^4(n)$ column. Now note that this information doesn't determine the value of f(5), as we could essentially pick any value for f(5) and have it agree with the information that exists in this difference table. For example, suppose we pick f(5) = 97. We can then fill out the rest of the table as follows:

Input	Output	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	3	−4	5	2	0	44
1	−1	1	7	2	44	
2	0	8	9	46		
3	8	17	55			
4	25	72				
5	**97**					

And then we could get a new polynomial:

$$f(n)=\binom{n}{0}\cdot 3+\binom{n}{1}\cdot(-4)+\binom{n}{2}\cdot 5+\binom{n}{3}\cdot 2+\binom{n}{4}\cdot 0+\binom{n}{5}\cdot 44$$
$$=1\cdot 3+n\cdot(-4)+\frac{n(n-1)}{2}\cdot 5+\frac{n(n-1)(n-2)}{6}\cdot 2+\frac{n(n-1)(n-2)(n-3)(n-4)}{6}\cdot 44$$
$$=\frac{11}{30}n^5-\frac{11}{3}n^4+\frac{79}{6}n^3-\frac{101}{6}n^2+\frac{89}{30}n+3$$

And clearly we would get new polynomials by choosing different values for f(5) in this way.

Problem 37 (Corollary 2): A polynomial with more roots than its degree is the 0 polynomial.

Assume f(x) is a polynomial with roots $r_1,\ldots,r_m$, where m >n. The using the remainder theorem, we know that
$f(x) = q_1(x)(x-r_1)$ where $\deg(q_1) = n-1$
$= q_2(x)(x-r_2)(x-r_1)$ where $\deg(q_2) = n-2$
$= \cdots$
$= q_n(x)(x-r_n)\ldots(x-r_2)(x-r_1)$ where $\deg(q_n) = n-n = 0$
$= q_{n+1}(x)(x-r_{n+1})(x-r_n)\ldots(x-r_2)(x-r_1)$ where $\deg(q_{n+1}) = -1$

Uh oh, this is a problem. The only possible polynomial we can have with degree less than 0 is the 0 polynomial, which means $q_{n+1} = 0$, which means the entire polynomial $f(x) = 0$.