Math S305
Summer 2014
Solutions for Problem Set 5

Homework 5 Solutions

1) First, revisit the whole issue of finding formulas for sequences defined recursively. Here's a reminder of the basic ideas we used for the Fibonacci sequence - Finding formulas for recursively defined sequences. Now it's your turn to go through all of the steps yourself (including solving a relatively simple system of linear equations in two unknowns).
So, suppose there's a sequence starting $a_0 = 1$, $a_1 = 4$, and that for the rest of the sequence a_n is found by taking the average of the last two terms (e.g. $a_2 = (a_0 + a_1)/2 = (1 + 4)/2 = 2.5$, etc.). Now use the approach we used to find a formula for the nth term of the Fibonacci sequence to find a formula for the nth term of this new sequence.
Note that this is a pretty involved process(!), so that even though this is just one question, it will require several steps to solve. At the end of your work, please check your formula by writing down the first five terms of the sequence as they're given by the recursive formula $a_0 = 1$, $a_1 = 4$, and $a_n = (a_{n-1} + a_{n-2})/2$) and then by the formula you've just found (good luck!)

Note, many of you tried to start with the closed form for the Fibonacci sequence and build from there. In fact, the solution to this problem has nothing whatsoever to do with phi or psi. Instead, you needed to reuse the *process* we used to find the closed form for the Fibonacci sequence.

Start with $a_0 = 1$, $a_1 = 4$, and $a_n = (a_{n-1} + a_{n-2})/2$. We want to see if we can represent this sequence as a geometric sequence. To do this, we know that $1, r, r^2, r^3, \ldots$ must also satisfy the formula for a_n. This means that $r^n = (r^{n-1} + r^{n-2})/2 = r^{n-2}(r + 1)/2$, or that $2r^2 = r + 1$. This means we can solve the quadratic, $2r^2 - r - 1 = 0$, to determine the value(s) of r. Using the quadratic formula, we get $r = -½$ and $r = 1$.

Now we want to find values for A and B so that our sequence, $a_n = A(-½)^n + B(1)^n$ or $a_n = A(-½)^n + B$. Using our first three values for a_0, a_1, and a_2, we have a system to solve.

$$A + B = 1$$

$$\frac{-A}{2} + B = 4$$

$$\frac{A}{4} + B = 2.5$$

We quickly see that $B = 1 - A$, so $-3A/2 = 3$, or $A = -2$. So $B = 3$. Hence, our sequence can be represented as $a_n = -2(-½)^n + 3$.

Using this we find the following to check our sequence:

$$a_0 = 1 = -2 + 3 = -2\left(\frac{-1}{2}\right)^0 + 3$$

$$a_1 = 4 = 1 + 3 = -2\left(\frac{-1}{2}\right) + 3 = -2\left(\frac{-1}{2}\right)^1 + 3$$

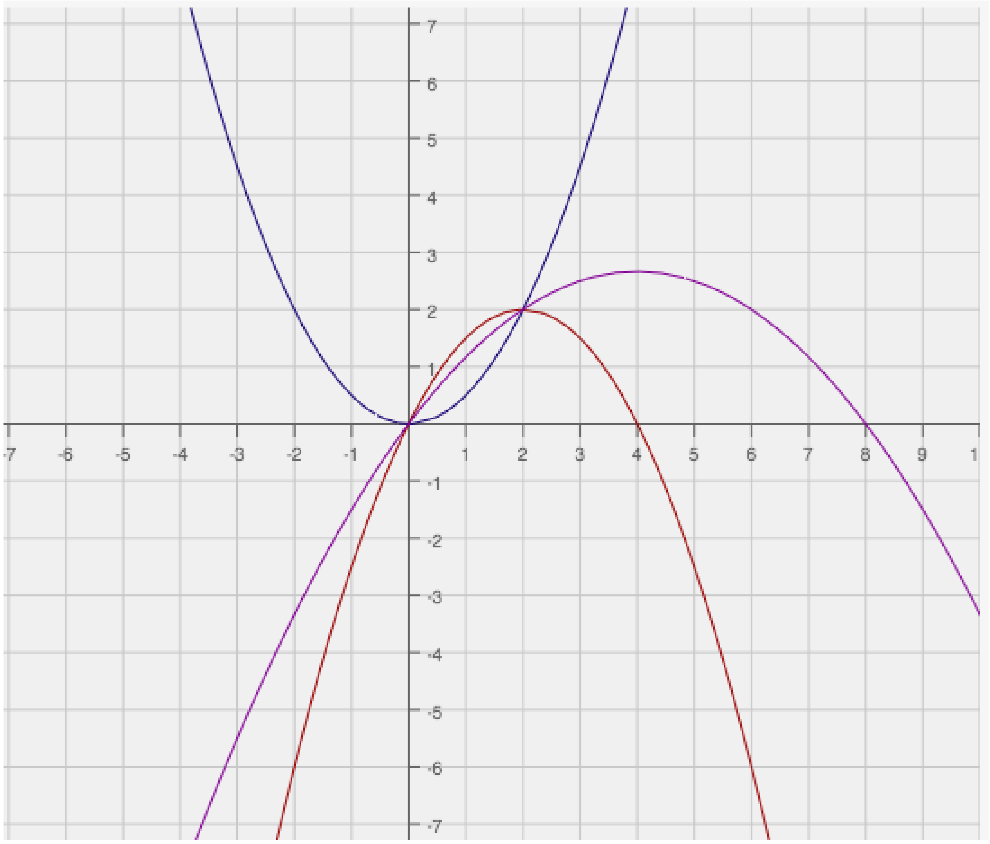
$$a_2 = \frac{(1+4)}{2} = \frac{5}{2} = 2.5 = \frac{-1}{2} + 3 = -2\left(\frac{1}{4}\right) + 3 = -2\left(\frac{-1}{2}\right)^2 + 3$$

$$a_3 = \frac{(4+2.5)}{2} = \frac{6.5}{2} = 3.25 = \frac{1}{4} + 3 = -2\left(\frac{-1}{8}\right) + 3 = -2\left(\frac{-1}{2}\right)^3 + 3$$

$$a_4 = \frac{(2.5+3.25)}{2} = \frac{5.75}{2} = 2.875 = \frac{-1}{8} + 3 = -2\left(\frac{1}{16}\right) + 3 = -2\left(\frac{-1}{2}\right)^4 + 3$$

3) Next, do a bit more work on fitting polynomials to points the way we worked on in our last class - suppose you are given the two points (0,0) and (2,2). Working with the polynomial A X^2 + B X + C, plot a quick graph of the two points and then try sketching at least three parabolas (quadratic functions in X) that go through the two points (try sketching them before you do any actual calculations of what the polynomials formulas would have to be - they don't have to be super accurate - just try to imagine what the quadratics might look like!). Next, find an equation that gives *all the possible* quadratics that go through the two points (your resulting equation will be a quadratic expression in X, but should have an "extra variable" in it that can take on different values - the so-called "parameter" that we talked about in class).

Given two point (0, 0) and (2, 2). First, here are three possible quadratics going through those two points.



Now, to find the equation for all possible quadratics, we can set up a system of equations:

$$0 = A \cdot 0^2 + B \cdot 0 + C$$

$$2 = A \cdot 2^2 + B \cdot 2 + C$$

From the first equation, we see that c = 0. Then, using the second equation we find

$$2 = 4A + 2B$$

$$2 - 4A = 2B$$

$$1 - 2A = B$$

Thus, any quadratic with the form y = A X^2 + (1 – 2A)X goes through the points (0, 0) and (2, 2).

4) Next find two cubic polynomials (A X^3 + B X^2 + C X + D) that go through the same two points (note - no need to sketch them for this problem). Can you then find a cubic polynomial that starts with the first term 10 X^3... that goes through these two points? Can you find a cubic starting with 1000 X^3 instead?

First, as long as D = 0 then the cubic polynomial will go through (0,0). So now you can simply play with the other coefficients to make sure the cubic goes through (2,2). In fact in general this just means (subbing in X = 2), that.

$$A \cdot 2^3 + B \cdot 2^2 + C \cdot 2 + D = 2$$

So, as long as D = 0 and 8A + 4B + 2C = 2, then the cubic will go through (0,0) and (2,2). One example might be to let B = C = 0. Then A = 0.25 works and the cubic polynomial is simply 0.25 X^3. Another example might be to let B = C = 1. Then A = -0.5 and the cubic polynomial is -0.5 X^3 + X^2 + X.

As for a cubic that starts with 10 X^3, this simply requires finding values for B and C so that 8A + 4B + 2C = 2, so that with A = 10, then this comes down to 80 + 4B + 2C = 2. Keeping things as simple as possible, let B = 0, and then solving for C yields C = -39, so the cubic 10 X^3 - 39 X works.

Similarly, we can work out that 1000 X^3 - 3999 X works too.

There are actually quite a few different solutions (well, infinitely many) that work for the last two parts, these are simply some of the easiest.

5) Find a 100th degree polynomial that goes through these same two points (i.e. starts A X^100 + B X^99 +... where A is not equal to 0) - this is actually quite easy to do if you think about keeping the polynomial "as simple" as possible.

An example of a 100th degree polynomial can follow the same pattern as the first cubic function above. That is, we know the constant term will be equal to 0, and we can choose to make every other coefficient equal to 0 as well, with the exception of the leading coefficient. Then we have 2 = 2^{100} A, or a = $\frac{1}{2}^{99}$. Thus, the function could be y = $(\frac{1}{2}^{99})x^{100}$.

Pay particular attention to this solution, because this is a problem where you can easily go down the rabbit hole. You don't want to do that on an exam. That said, I saw a wide variety of clever, simple solutions to this problem. They may not have been (arguably) *the easiest*, but they weren't so tough to come up with.

6) Now add another point, (1, -1) to the mix so that you've got three points to deal with. How many quadratic functions go through these three points - find them (or it!) And one other quick, curious side note - we "know" that it takes three points to define a quadratic - what if the points had been (0,0), (1,1) and (2,2) instead of (1,-1) - is there a quadratic function that goes through these three points?

Find quadratics through (0, 0), (1, -1), and (2, 2). How many can you find?
We will only be able to find one quadratic that passes through these three points. That is, if we write up the equations we get by systematically subbing in X = 0, then 1, then 2, we end up getting the following:

$$a \times 0^2 + b \times 0 + c = 0$$

$$a \times 1^2 + b \times 1 + c = -1$$

$$a \times 2^2 + b \times 2 + c = 2$$

This is equivalent to the following:

$$c = 0$$
$$a + b = -1$$
$$4a + 2b = 2$$

Now, using the equations involving a and b, we find that there is one unique solution: $a = 2$, $b = -3$. Thus, there is one unique quadratic passing through these three points, namely $2x^2 - 3x$.

On the other hand, when we throw in the point (1,1) we get (using our result from problem 3) that $1 = A(1) + (1 - 2A)(1) = A + 1 - 2A$ or $A = 0$. In other words, this is not a quadratic, but a linear function.

An important note, many of you claimed that this meant that the *only* polynomial that goes through the points (0,0), (1,1) and (2,2) is the linear function y = x. *This is not true*. It is easy to fit a cubic polynomial (or any higher power) to these three points. We've merely shown that it's not possible to fit a *quadratic* to these three points.

7) Here's a bit of a twist on the Alexa polynomial puzzle (remember that we found the 10th degree polynomial that Alexa had created by just knowing two points it went through - asking what the value of P(1) was followed by P(100) (or in fact P(any power of 10 larger than P(1))). Okay... this just plays off of our base 10 system, but is it necessary to use a power of 10 to do this? What if I told you that P(X) is a fourth degree polynomial (i.e A X^4 + B X^3 + C X^2 + D X + E), where A, B, C, D, and E are each positive integers. If you know P(1) = 9, then telling you what P(10) equals will give away the coefficients just as we saw in class (since knowing P(1) = 9 tells you that all five coefficients are single digit numbers). What if instead of telling you what P(10) equals, I tell you that P(16) = 74,529. Can you figure out what the polynomial is from just this information? (hint - think binary! and you'll probably find the following base conversion website http://korn19.ch/coding/base_converter.php useful!)

Since P(x) is a 4th degree polynomial, we can write it as $P(x) = ax^4 + bx^3 + cx^2 + dx + e$. We also are given that $P(1) = a + b + c + d + e = 9$ and $P(16) = a(16)^4 + b(16)^3 + c(16)^2 + d(16) + e = 74{,}529$.

If we think in binary terms, we know that $16 = 2^4 = 10000_2$. Converting P(16) to binary as well, we see that $74{,}529 = 10010001100100001_2$. Thinking about this number in terms of "place value," and considering that the coefficients of the polynomial will be shifted by 10000 each time when written in binary, we see that each term in the binary form of 74, 529 are as follows: 1 0010 0011 0010 0001. We see that $1 = 1$, $2 = 0010_2$, $3 = 0011_2$, and $1 = 0001_2$. Hence, a = 1, b = 2, c = 3, d = 2, and e = 1.

Checking this with our original points we have $1 + 2 + 3 + 2 + 1 = 9$ and $1(16)^4 + 2(16)^3 + 3(16)^2 + 4(16) + 1 = 74{,}529$. Hence, our polynomial is $y = x^4 + 2x^3 + 3x^2 + 2x + 1$.

> Many of you actually chose to convert 74,529 to base 16 rather than base 2, which actually gives a simpler result. In base 16, this is 12,321, and we can just read off the coefficients (though we would have had to do some conversion back to base 10 if any of the coefficients had been larger than 9 base 10).

bonus - what other input values could be used to figure out the polynomial? ...what strategy would you use to do this?

> In fact, you can follow this same approach using different bases, so that you could also find the polynomial as long as you learn the value $P(B^n)$ where B is any positive integer with n chosen so that B^n is greater than the sum of the coefficients (in this case just greater than 9 since P(1) = 9)