Homework 4 Solutions

1. *And the classic summing problem (this shows up in some form in practically every text on the subject!) You hold a tennis ball 4 feet above the floor and drop it. After each bounce, it bounces back up to a height that's 3/4ths of the height of the previous bounce (i.e. after the first bounce it bounces back up to 3 feet, and after the second bounce it bounces back up to 2 1/4 feet (= 9/4 feet). How far has the ball "traveled" by the time it hits the floor for the Nth time?*

When the ball hits the floor the first time, it has traveled 4 feet. When it hits the floor a second time, the ball has traveled 4 + 3 + 3 = 10 feet. The third time, 10 + 2.25 + 2.25 = 14.5 feet. We see that this is a pattern. That is, when the ball has hit the floor the n^{th} time, to determine how far it has traveled, we see that it has traveled $2\left(4 + 4\left(\frac{3}{4}\right) + 4\left(\frac{3}{4}\right)^2 + \cdots + 4\left(\frac{3}{4}\right)^{n-1}\right) - 4$ feet (where we subtract a "4" as the ball just travels down before the first bounce, not up and down as it does before every subsequent bounce).

We can rewrite this as $8\left(\left(\frac{3}{4}\right)^0 + \left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-1}\right) - 4$.

Note that this is a *finite* sum. Many people tried to use the formula for an infinite geometric series, but that will give you the total travel distance only if you let the bouncing process go on forever, as opposed to just n times. To compute a finite sum, we get:

$$8\left(\frac{1-\left(\frac{3}{4}\right)^n}{1-\left(\frac{3}{4}\right)}\right) - 4$$
$$= 8\left(\frac{1-\left(\frac{3}{4}\right)^n}{\frac{1}{4}}\right) - 4$$
$$= 32\left(1-\left(\frac{3}{4}\right)^n\right) - 4 = 28 - 32\left(\frac{3}{4}\right)^n$$

2) In class we came up with formulas for sums of finite arithmetic sequences, along with formulas for sums of geometric series (both finite and infinite). Now try your hand at using these formulas to find the sums (if they exist!) of the following series (each series is either arithmetic or geometric). To begin with please review these by writing down the formulas for 1) the sum of the first N terms of an arithmetic sequence, 2) the sum of the first N terms of a geometric sequence, and 2a) the sum of an infinite geometric sequence (assuming that there is a finite sum - i.e. that R, the ratio of terms is less than 1, if all the terms are positive, otherwise that the absolute value of R is less than 1).

(a) 70 + 84 + 98 + 112 + ... + 280 + 294
(b) 90 + 60 + 40 + ...
(c) 1 + 1.1 + 1.21 + 1.331 + ... + 2.357947691

Finite Arithmetic: $S_n = (n/2)(2a_1 + (n - 1)d) = (n/2)(a_1 + a_n)$
Finite Geometric: $S_n = a_1((1 - r^n)/(1 - r))$
Infinite Geometric: $S = a_1(1/(1 - r))$

a. Finite arithmetic: $n = 17$, $d = 14$, $a_1 = 70$; $S_{17} = 3094$
b. Infinite geometric: $r = 2/3$, $a_1 = 90$; $S = 270$
c. Finite geometric: $n = 10$, $r = 1.1$, $a_1 = 1$; $S_{10} = 15.9374246$

3) Let $\{a_n\}$ be the sequence given explicitly by $a_n = (n2 - n) / 2$.

(a) First find terms $a1$, a_2, a_5, and a_{10}
(b) Next show that $a_{n+1} = (n2 + n) / 2$ and then find a formula for $a_{n+1} - a_n$
(c) Finally, use this to come up with recursive description for the same sequence (i.e. a starting point a_1 along with a formula for finding a_{n+1} in terms of a_n)

Given $a_n = (n^2 - n)/2$

a. $a_1 = 0$; $a_2 = 1$, $a_5 = 10$; $a_{10} = 45$
b. $a_{n+1} = \frac{(n+1)^2 - (n+1)}{2} = \frac{n^2 + 2n + 1 - n - 1}{2} = \frac{n^2 + n}{2}$, so we know

$$a_{n+1} - a_n = \frac{n^2 + n}{2} - \frac{n^2 - n}{2} = \frac{n^2 + n - n^2 + n}{2} = \frac{2n}{2} = n$$

c. Using the second equation above and solving for a_{n+1}, we see that if we define $a_1 = 0$, then $a_{n+1} = a_n + n$.

4) We saw that the Harmonic Series doesn't converge (i.e. the series 1 + 1/2 + 1/3 + 1/4 + ... doesn't have a finite sum). I also mentioned that if you created a new series by getting rid of all the terms with 9's in them, then in fact what was left did converge (the sum has a finite value). Is this also the case if you start with the Harmonic Series and this time only keep the terms whose denominators end in 0? Thus you're left with the series 1/10 + 1/20 + 1/30 + ... + 1/100 + 1/110 + 1/120 + ... showing off the fact that "most" of the terms of the original Harmonic series are gone, so does this now converge - did we throw out enough terms?

Consider the fact that 1/10 + 1/20 + 1/30 + 1/40 + ... = (1/10)(1 + ½ + 1/3 + ¼ + ...). The sum of 1/10 + 1/20 + 1/30 + 1/40 + ... is one-tenth the sum of the original Harmonic series. Thus, we need to convince ourselves that 1/10 of an infinite sum is still infinite.

5) Explain why it's the case that if you start with the Harmonic Series and get rid of a finite number of terms in the series, that what's left will still not have a finite sum.

There are two ways to think about this informally. One is to realize that if we take away a finite number of terms, there will be a "last missing term". We can simply ignore all the terms up until we run out of missing terms, and then go right back to grouping terms until we get sums greater than ½. Yes, that will take a while, but the nice thing about infinity is that we have a lot of room to play. Another way to think about it is to realize that the terms we remove will sum up to something finite (they have to because there are a finite number of terms). Taking a finite number away from infinity still leaves us with infinity.

For those who are interested, here is a more formal proof:

> Consider the Harmonic Series with terms $a_n = 1/n$. We know that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, meaning that it does not have a finite value. But then it must also be the case that $\sum_{n=i}^{\infty} \frac{1}{n}$ also diverges where you consider the same series, but start the sum at n = i for any i > 1, instead of starting with 1/1 (for example consider 1/100 + 1/101 + 1/102 + ... starting at n = 100). Why? Well, all you've done is take the same sum but subtracted a finite value = the sum of the terms up to 1/i (in our example, the sum 1/1 + 1/2 + 1/3 + ... + 1/99), and an "infinite value" minus a finite value is still infinite (i.e. "∞ – M" is still "∞" for any finite number M).
>
> Now start with the harmonic series and get rid of a finite number of terms in the series. Since this is a finite set of terms, we can find the term in the set that's "furthest along" = has the highest index, and then consider the rest of the harmonic series starting just past this term, say starting at $a_m = 1/m$. We just said that $\sum_{n=i}^{\infty} \frac{1}{n}$ diverges for the sum starting at any point i > 1. Since we didn't necessarily remove all the terms up to $a_m = 1/m$, then what's left is at least equal to $\sum_{n=m}^{\infty} \frac{1}{n}$ which is already infinite (i.e. doesn't have a finite value), so "larger than infinity" means the sum is still "infinite"! (i.e. still doesn't have a finite value). Hence, removing a finite number of terms from the harmonic series will always result in a series that also does not have a finite sum.

7) And finally, thinking about defining sequences recursively instead of with closed formulas, figure out what sequence is described by the following "recipe"

$a_1 = 1$, $a_2 = 4$, $a_3 = 9$, and from then on $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$

Now try to show why this is the case (i.e. why the sequence will continue the way it looks like it is!) Hint - if three consecutive terms a_{n-3}, a_{n-2} and a_{n-1} are squares (and equal to $(n-3)^2$, $(n-2)^2$, and $(n-1)^2$, respectively) then what does that make a_n equal to? (feel free to use Mathematica/Wolfram Alpha to do the "dirty work" for you!!) This is essentially a proof by induction if you throw in the fact that the sequence starts with $a_1 = 1^2$, $a_2 = 2^2$, and $a_3 = 3^2$)

In any case, this once more shows that it's possible to describe the same sequence in a variety of ways - both "direct" (i.e. with closed formulas) and "indirect" (i.e. with a recursive formula)

First, let's compute a_n given that $a_{n-3} = (n-3)^2, a_{n-2} = (n-2)^2, a_{n-1} = (n-1)^2$. In this case,

$$a_n = 3(n-1)^2 - 3(n-2)^2 + (n-3)^2$$
$$= 3(n^2 - 2n + 1) - 3(n^2 - 4n + 4) + n^2 - 6n + 9$$
$$= n^2$$

This was enough to get credit for this problem. However, many of you tried to do a full induction proof and got rather confused, so let's look at that.

Step i) The initial cases given in the problem are enough to give us our base cases.

Step ii) The trick to doing this induction is that we either need to use strong induction (i.e. assume that $a_k = k^2$ for every k less than n, or we have to assume that the situation holds for at least the three terms before n. Once we do that, then the calculation we did above is enough to get the induction to work.

(Many of you did the calculation for n and then did an "induction step" by doing the calculation for n+1, but this really doesn't tell you anything.)