Math S305
Summer 2014
Solutions for Problem Set 3

Homework 3 Solutions

First off, some problems from *Mathematical Connections*.

3c)) The second difference is constant for this table, so this can be represented using a quadratic equation.

Input	Output	Δ	Δ^2
0	3	1	3
1	4	4	3
2	8	7	3
3	15	10	3
4	25	13	
5	38		

We see that the first difference can be represented by the linear equation Δ(n) = 3n + 1. Using the "hockey stick" property in the reading, we can find a general solution for our table by considering output values of the table at different inputs. For example, f(2) = f(0) + Δ(0) + Δ(1) = 3 + 1 + 4 = 8; f(3) = f(0) + Δ(0) + Δ(1) + Δ(2) = 3 + 1 + 4 + 7 = 15; and f(4) = f(0) + Δ(0) + Δ(1) + Δ(2) + Δ(3) = 3 + 1 + 4 + 7 + 10 = 25. Looking for a pattern in each of these we notice the following:

f(2) = f(0) + Δ(0) + Δ(1) = 3 + [3(0) + 1] + [3(1) + 1]
f(3) = f(0) + Δ(0) + Δ(1) + Δ(2) = 3 + [3(0) + 1] + [3(1) + 1] + [3(2) + 1]
f(4) = f(0) + Δ(0) + Δ(1) + Δ(2) + Δ(3) = 3 + [3(0) + 1] + [3(1) + 1] + [3(2) + 1] + [3(3) + 1]

In general, this means that

$$f(n) = 3 + 3\left(0 + 1 + 2 + \ldots + (n-1)\right) + n(1) =$$
$$= 3 + 3\left(\frac{(n-1)n}{2}\right) + n = 3 + \frac{3}{2}\left(n^2 - n\right) + n =$$
$$= 3 + \frac{3}{2}n^2 - \frac{3}{2}n + n = \frac{3}{2}n^2 - \frac{1}{2}n + 3.$$

4) Since the first difference is constant, we know we can fill in the table as follow:

Input	Output	Δ
0	c	d
1	c + d	d
2	c + 2d	d
3	c + 3d	d
4	c + 4d	d

5	c +5d	

This is a linear function that can be represented by y = d(n) + c.

6) Since the second difference is constant, we know the first difference can be represented by the linear equation Δ(n) = 2n + 4. Using this, we can fill in the Δ column of the table.

Input	Output	Δ	$Δ^2$
0	3	4	2
1		6	2
2		8	2
3		10	2
4		12	
5			

Now, we can use the property that f(n) = f(0) + Δ(0) + Δ(1) + … + Δ(n-1) used in problem 3c to fill in the output column of the table.

Input	Output	Δ	$Δ^2$
0	3	4	2
1	7	6	2
2	13	8	2
3	21	10	2
4	31	12	
5	43		

We can then simplify f(n) = f(0) + Δ(0) + Δ(1) + … + Δ(n-1) as follows:

$$f(n) = 3 + [2(0)+4] + [2(1)+4] + \ldots + [2(n-1)+4]$$
$$= 3 + 2\left(0 + 1 + \ldots + (n-1)\right) + 4n$$
$$= 3 + 2\left(\frac{(n-1)n}{2}\right) + 4n$$
$$= 3 + 2\left(\frac{n^2 - n}{2}\right) + 4n$$
$$= 3 + n^2 - n + 4n$$
$$= n^2 + 3n + 3$$

Hence, the table can be represented by $y = n^2 + 3n + 3$.

7) Since the second difference is constant, we know the first difference can be represented by the equation Δ(n) = cn + b. Using this, we can fill in the Δ column of the table.

Input	Output	Δ	Δ^2
0	a	b	c
1		b + c	c
2		b + 2c	c
3		b + 3c	c
4		b + 4c	
5			

Now, we can use the property that f(n) = f(0) + Δ(0) + Δ(1) + … + Δ(n-1) used in problem 3c to fill in the output column of the table.

Input	Output	Δ	Δ^2
0	**a**	b	c
1	**a + b**	b + c	c
2	a + b + b + c = **a + 2b + c**	b + 2c	c
3	a + 2b + c + b + 2c = **a + 3b + 3c**	b + 3c	c
4	a + 3b + 3c + b + 3c = **a + 4b + 6c**	b + 4c	
5	a + 4b + 6c + b + 4c = **a + 5b + 10c**		

We can then simplify f(n) = f(0) + Δ(0) + Δ(1) + … + Δ(n-1) as follows:

$$
\begin{aligned}
f(n) &= a + [c(0)+b] + [c(1)+b] + \ldots + [c(n-1)+b] \\
&= a + c\left(0+1+\ldots+(n-1)\right) + bn \\
&= a + c\left(\frac{(n-1)n}{2}\right) + bn \\
&= a + c\left(\frac{n^2-n}{2}\right) + bn \\
&= a + \frac{c}{2}n^2 - \frac{c}{2}n + bn \\
&= a + \frac{c}{2}n^2 + \left(\frac{2b-c}{2}\right)n
\end{aligned}
$$

Theorem: For any table in which the first output, at input = 0, is equal to a, the first difference at input = 0 is equal to b, and second difference is determined by the constant, c, the output can be determined by the formula

$$f(n) = \frac{c}{2}n^2 + \left(\frac{2b - c}{2}\right)n + a.$$

8) Let's start with the table of differences:

n	F(n)	Δ(n)	Δ^2(n)	Δ^3(n)
0	1	-2	14	12
1	-1	12	26	12
2	11	38	38	12
3	49	76	50	
4	125	126		
5	251			
6	439			
7	701			

(Note, we could keep going on the difference, but we have enough information here.)

Since the third difference is a constant, this is enough to tell us that the first difference will be a quadratic function, so let's use what we did in the last problem to find Δ(n). In this case, we have a = -2, b = 14, and c = 12, so:

$$\Delta(n) = \frac{12}{2}n^2 + \frac{2(14) - 12}{2}n - 2$$

$$= 6n^2 + 8n - 2$$

Now, we can use this along with the "hockey stick" to compute f(n):

$$f(n) = 1 + \sum_{i=0}^{n-1} 6i^2 + 8i - 2$$

$$= 1 + 6\sum_{i=0}^{n-1} i^2 + 8\sum_{i=0}^{n-i} i - 2n$$

$$= 1 + 6\left(\frac{n(n-1)(2n-1)}{6}\right) + 8\left(\frac{n(n-1)}{2}\right) - 2n$$

After working out all the algebra here, we end up with: $f(n) = 2n^3 + n^2 - 5n + 1$, and indeed, checking this for the values we have suggests that it works.

2) Sequence puzzles!

a. 4, 9, 25, 49, 121, **169** (squares of prime numbers)
b. 1, 2, 6, 30, 210, 2310, **30030** (multiply previous term by the next consecutive prime)

c. 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, **885** (alternate adding and multiplying by consecutive integers)
d. 1, 2, 3, 2, 1, 2, 3, 4, 2, 1, 2, 3, 4, 3, 2, 3, 4, 5, 3, 2, **3** (number of letters in Roman numeral representation of n)