Math S305
Summer 2014
Solutions for Problem Set 2

Homework 2 Solutions

1) Using the fact we found in class that 1/(1 - a X) = 1 + a X + a^2 X^2 + a^3 X^3 + ... (formally, that is!), come up with a fraction that equals 1 + 0.2 + 0.04 + 0.008 + 0.0016 + 0.00032... If you think about this closely, it's a little bit weird to think that everything sums up by overlapping so beautifully/seamlessly as to end up with a number that actually ends after just two decimal places - how is that possible? (must be magic!!) For example, somewhere in this sum is the term (this is the 20th power of 2, starting 20 places to the right of the decimal point):

+ 0.00000000000001048576 +...

so how do those seven digits (1048576) turn out to exactly match all the digits around it so that they essentially disappear in the final result?!

Let a = (1/10) and X = 2. Then 1/(1 – (1/10)*2)) = 1 + 0.2 + 0.04 + 0.008 + ... What's interesting about this, is that 1/(1 – (1/10)*2)) = 1/(1 – 2/10) = 1/(4/5) = 5/4 = 1.25. It's amazing that all the individual terms perfectly "mesh" so that they all sum up to exactly 1.250000000... but that's part of the magic of infinite sums!

2) The Fibonacci Sequence is just one type of recursively generated sequence (we'll look at these a bit more in the next class), meaning a sequence in which each new term is derived by creating a combination of the previous several terms. Suppose you start off a sequence that begins 1 2 6 16 44 120... where each new term is the *twice* the sum of the two previous terms (as opposed to the "regular" Fibonacci sequence which just sums the two previous terms) - and note that the sequence starts "1 2" instead of "1 1"

(a) First off, write out three more terms in this sequence.

The sequence is 1, 2, 6, 16, 44, 120, 328, 896, 2448 because 328 = (120 + 44)*2, 896 = (328 + 120)*2, and 2248 = (896 + 328)*2.

(b) Next come up with a polynomial (in the form 1/P(X)) which is equivalent to the generating function for this new "modified" Fibonacci sequence, i.e. you need to find the polynomial P(X) such that 1/P(X) = 1 + 2 X + 6 X^2 + 16 X^3 + 44 X^4 + 120 X^5 + ...

The polynomial P(x) = 1 – 2x - $2x^2$ works here (as you can check using the command "Series[1/(1-2x-2x^2), {x,0,10}]" in Mathematica).

3) Now using your result in the last question find a fraction that equals the sum 1 + .2 + .06 + .016 + .0044 + .00120 + ... where each term comes from the sequence in the last problem, but shifted one decimal place to the right each time - similar to the type of result we saw for the Fibonacci sequence. (Do all sequences that are defined recursively share this same type of

summing up property? you don't need to answer this - just food for thought!)

> Find the fraction equaling the sum 1 + 0.2 + 0.06 + 0.016 + 0.0044 + 0.00120 + ...
> And so here we can play the same “game” we played in class when we “summed up” Fibonacci sequence (when we summed up 0.1 + 0.01 + 0.002 + 0.0003 + ...) Here since we just found out that 1/(1-2x-2x^2) equals $1 + 2x + 6x^2 + 16x^3 + 44x^4 + \dots$
> Then we can see that 1 + 0.2 + 0.06 + 0.016 + 0.0044 + 0.00120 + ... equals $1 + 2x + 6x^2 + 16x^3 + 44x^4 + \dots$ with 0.1 substituted in for x.
> This we can calculate simply using the equivalent 1/(1-2x-2x^2) form, subbing in x =0.1 (or just 1/10 to get the answer in fraction form): 50/39

4) And a last (puzzle!) setting us up for some work in class at some point - try to find the polynomial that is equivalent to the generating function for the squares:

$$1 + 4X + 9X^2 + 16X^3 + 25X^4 + 36X^5 + \dots$$

We know $\frac{1}{(1-x)^3} = 1+3x+6x^2+10x^3+15x^4+21x^5+\dots$. We can also notice that the coefficients, when added to the next consecutive one add to perfect squares, e.g. 1 + 3 = 4, 3 + 6 = 9, 6 + 10 = 16. Therefore,

$$\frac{1}{(1-x)^3} + \frac{x}{(1-x)^3} =$$

$$\left(1+3x+6x^2+10x^3+15x^4+21x^5+\dots\right)+\left(x+3x^2+6x^3+10x^4+15x^5+21x^6+\dots\right) =$$

$$1+4x+9x^2+16x^3+25x^4+36x^5+\dots$$

5) Pushing along with the "Triple Fibonacci Sequence" we tinkered around with in class: 1, 1, 1, 3, 5, 9, 17, 31, ... where each term was the sum of the last ***three*** terms. We almost found a generating function for this sequence using 1/(1 - X - X^2 - X^3), but unfortunately that produced 1, 1, 2, 4, 7, 13, 24, 44, ... instead - starting with 1, 1, 2, instead of just 1, 1, 1. Now try your hand at "fixing" this - here's the idea - any sequence with the same "sum three terms to get the next term" rule will have a generating function of the form P(X) / (1 - X - X^2 - X^3), where P(X) is a polynomial (why? because the "1 - X - X^2 - X^3" is necessary for the "sum three terms to get the next term" rule to show up in the sequence).

We know that $\frac{1}{1-x-x^2-x^3} = 1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + \cdots$, giving us the correct coefficients for the first two terms. But we want the coefficient of x^2 to be 1 also, so let’s try multiplying the 1/P(x) by $-x^2$ and adding this to 1/P(x) to get rid of that coefficient. We get:

$$\frac{1-x^2}{1-x-x^2-x^3} = (1 + x + 2x^2 + 4x^3 + 7x^4 + \cdots) - (x^2 + x^3 + 2x^4 + 4x^5 + 7x^6 + \cdots)$$

$= 1 + x + 2x^2 - x^2 + 4x^3 - x^3 + 7x^4 - 2x^4 + \cdots =$
$1 + x + x^2 + 3x^3 + 5x^4 + \cdots$

So not only does this trick fix the x^2 term, it fixes everything after that, so this is our generating function!