Math S305
Summer 2014
Solutions for Problem Set 1

Homework 1 Solutions

3) Using a calculator, multiply 11202361 by 54700343 by multiplying the polynomials 1120t + 2361 and 5470t + 343 and evaluating the product at an appropriate value for t.

(1120t + 2361)(5470t + 343) = $6126400t^2$ + 384160t + 12914670t + 809823 = $6126400t^2$ + 13298830t + 809823. Evaluate this at t = 10,000 to get 612,772,989,109,823. Many of you worked out the full solution, either by hand or with Mathematica, which is excellent. You only needed to come up with the fact that t needs to be 10,000 to get credit.

4) Next, it's time to dive into some more generating functions(!) First, remember that a generating function is a "formal" expression - equivalent to an "infinite polynomial" $A + B X + C X^2 + D X^3 + E X^4 + \dots$ (all the generating functions we saw today started with A = 1) meaning that it might or might not make sense when viewed as a polynomial function(!)

(a) As a first quick reminder of what we worked on in class, what is 1 / (1 - x) equal to as an "infinite" polynomial? (using the idea that if A times B = 1, then we can say that 1 / A is "formally" equivalent to B)

$$\begin{array}{rl} (1 - x)(1 + x + x^2 + \dots + x^n) = & 1 + x + x^2 + \dots + x^n \\ & \underline{\quad - x - x^2 - \dots - x^n - x^{n+1}} \\ & 1 - x^{n+1} \end{array}$$

This should equal 1 since this would technically never stop at a final term involving a power of x – all the terms except for the constant term "1" would all be canceled by the – x term.

We know that since $(1 - x)(1 + x + x^2 + \dots) = 1$, then $1/(1-x) = (1 + x + x^2 + \dots)$.

(b) Thinking about this result, then what (formally) is the infinite polynomial that "equals" 1 / (1 + x) ? The reason I keep writing "formally" is that we don't know if these infinite polynomials "make sense" in the way we're used to regular finite polynomial functions - it's not as if we can just substitute any value for x and expect to get a finite result - we're just working with these infinite polynomials using the same rules we have for regular/finite polynomials - but we're not treating them as if they're the same as polynomial functions.

In this case, we need to alternate between positive and negative terms in the infinite polynomial so that the terms will line up and cancel each other out. So, we know $(1 - x + x^2 - \dots) = 1/(1 + x)$. For example,

$$\begin{array}{rl} (1 + x)(1 - x + x^2 - \dots + x^n) = & 1 - x + x^2 - \dots + x^n \\ & \underline{\quad + x - x^2 + \dots - x^n + x^{n+1}} \\ & 1 + x^{n+1} \end{array}$$

Another helpful way to think of this is to recognize that $1/(1 + x) = 1/(1 – (-x))$. Replacing x with –x in the sequence $1 + x + x^2 + \ldots$ will give us a positive coefficient for all the even powers and a negative coefficient for all the odd powers.

5)(a) Now, formally applying rules of addition of polynomials, term by term, what is the infinite polynomial that is "equal" to 1 / (1 - x) + 1 / (1 + x) ?

$1/(1 – x) + 1/(1 + x) = (1 + x + x^2 + \ldots) + (1 – x + x^2 – \ldots) = 2 + 2 x^2 + 2 x^4 + \ldots$

(b) Thinking about this in another way, the infinite polynomial 1 + x^2 + x^4 + x^6 + ... equals 1 / P(x) for what polynomial P(x)?

$1/(1 – x) + 1/(1 + x) = 2/(1 – x^2)$. So, $1 + x^2 + x^4 + \ldots = 1/(1 – x^2)$

(c) And generalizing this result, the infinite polynomial 1 + x^3 + x^6 + x^9 + ... equals 1 / R(x) for what polynomial R(x)?

Generalizing this result, we have $1 + x^3 + x^6 + \ldots = 1/(1 – x^3) = 1/(1 – x)(1 + x + x^2)$. Multiplying this out gives us $(1 – x^3)(1 + x^3 + x^6 +\ldots) =$ $(1 + x^3 + x^6 +\ldots)(- x^3 - x^6 – x^9\ldots) = 1$, confirming our generalization.

(d) and, some last "food for thought" - in our next class we'll look at the infinite polynomial F(x) = 1 + x + 2 x^2 + 3 x^3 + 5 x^4 + 8 x^5 + 13 x^6 + ...
...why did I label this infinite polynomial "F(x)"?!

Because the coefficients are the Fibonacci sequence!

6) Finally, here's a challenging problem involving more coin counting, using the generating functions that you just investigated, as an extension of what we did in class last Tuesday... good luck!

How many ways are there to make 40 cents out of pennies, nickels, dimes, and quarters? You can almost just work this out by hand, but here's a much more sophisticated way to go about this type of problem...

First, there is only one way to represent any particular as the sum of pennies (i.e. using pennies there is only one way to make 6 cents - use 6 pennies!). Now we can use a generating function to encode this information as follows:

$P(x) = 1 + x + x^2 + x^3 + \ldots$ which formally equals $1/(1 - x)$

Here the coefficients of P(x) are all equal to 1, representing the 1 way that each amount can be represented (e.g. the coefficient of x^6 is 1, representing the 1 way 6 cents can be represented using pennies).

Next, here's a generating function for the number of ways C cents can be made using just nickels:

$N(x) = 1 + x^5 + x^{10} + x^{15} + x^{20} + \ldots$ which formally equals $1 / (1 - x^5)$

Why is this? ...there's only 1 way to make any multiple of 5 cents, and there are 0 ways of making any other amount, so the coefficients are all either equal to 1 (for the powers of x equal to multiples of 5), or 0 (for all the other powers of x).

So, here's the cool thing - look at P(x) times $N(x) = 1 + x + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + 3x^{11} + \ldots$

which formally equals $1/(1-x)$ times $1/(1-x^5)$ which is just $1 / [(1-x)(1-x^5)]$

Now, take a look - given that the coefficient of x^{10} is 3, then this means that there are 3 ways of representing 10 cents with just pennies and nickels: (i) 10 pennies, or (ii) 1 nickel and 5 pennies, or (iii) just 2 nickels.

So now write down a way to use this approach to figure out the number of ways N cents can be represented using pennies, nickels, dimes, and quarters - I'll show you how to use Mathematica to actually calculate the result for any value of N cents, but for now don't worry about calculating any actual numerical answers.

Pennies can be represented by the generating function $1/(1 – x)$; Nickels by $1/(1 – x^5)$; Dimes by $1/(1 – x^{10})$; Quarters by $(1 – x^{25})$. So the number of ways N cents can be represented using these four coins can be found from the coefficients of the generating function given by their product $1/((1 – x)(1 – x^5)(1 – x^{10})(1 – x^{25}))$. So to find the number of ways to make 40 cents, we need to find the coefficient of x^{40}, which happens to be 31.

As a side note, many of you noticed that there are obvious places to truncate the polynomials in order to computer the coefficient of x^{40}. For pennies, nickels, and dimes, we need to take the polynomials out to x^{40} in order to be sure that we will “catch” all the terms that multiply out to x^{40}. There is no x^{40} term in the polynomial for quarters, and there is no reason to include the x^{50} term, so we can truncate this polynomial at x^{25}. That makes the Mathematica calculations much faster, but it means we have to be aware that the coefficients of the terms after x^{40} will be too low because there are missing terms. [You didn’t need to make this observation to get credit].