Homework 10 Solutions

1) Following up on the first Puzzle of the Day involving the formulas for cosine and sine of angle sums, now come up with a formula for the tangent of an angle sum, i.e. what does Tan(A+B) equal in terms of Tan(A) and Tan(B)? To do this we know that Tan(X) is just Sin(X)/Cos(X), and given that we have formulas for Sin(A+B) and Cos(A+B), then you can get started with Tan(A+B) by using these existing formulas.

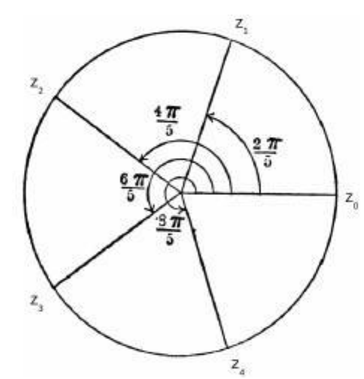
$$\mathrm{Tan}(A+B) = \frac{\mathrm{Sin}(A+B)}{\mathrm{Cos}(A+B)} = \frac{\mathrm{Sin\,A\,Cos\,B + Cos\,A\,Sin\,B}}{\mathrm{Cos\,A\,Cos\,B - Sin\,A\,Sin\,B}}$$

Now divide the top and bottom by Cos A Cos B and you get…

$$= \frac{(\mathrm{Sin}\,A\,\mathrm{Cos}\,B + \mathrm{Cos}\,A\,\mathrm{Sin}\,B)/\,\mathrm{Cos}\,A\,\mathrm{Cos}\,B}{(\mathrm{Cos}\,A\,\mathrm{Cos}\,B - \mathrm{Sin}\,A\,\mathrm{Sin}\,B)/\,\mathrm{Cos}\,A\,\mathrm{Cos}\,B}$$

$$= \frac{\left(\frac{\mathrm{Sin\,A}}{\mathrm{Cos\,A}} + \frac{\mathrm{Sin\,B}}{\mathrm{Cos\,B}}\right)}{\left(1 - \left(\frac{\mathrm{Sin\,A}}{\mathrm{Cos\,A}}\right)\left(\frac{\mathrm{Sin\,B}}{\mathrm{Cos\,B}}\right)\right)} = \frac{\mathrm{Tan\,A + Tan\,B}}{1 - \mathrm{Tan\,A\,Tan\,B}}$$

2) Start by considering Z_1 = e^(2Pi i / 5). What is Z_1^5? Let Z_2 = Z_1^2. What is Z_2^5? What the other "fifth roots of 1"? Write down all five "fifth roots of unity" (i.e. fifth roots of 1) in terms of powers of Z_1 and note that if you plotted these five complex numbers on an Argand diagram then these would just be the five vertices of a regular pentagon inscribed in the unit circle, which you can see in the diagram below (in the picture note that "Z_0" just equals 1, and that it is the same thing as Z_5 = Z_1^5)



Following the suggestions in the problem we find that

$$Z_1 = e^{2\pi i/5}$$
$$Z_2 = Z_1^2 = e^{4\pi i/5}$$
$$Z_3 = Z_1^3 = e^{6\pi i/5}$$
$$Z_4 = Z_1^4 = e^{8\pi i/5}$$
$$Z_5 = Z_1^5 = e^{10\pi i/5} = e^{2\pi i} = 1$$

Note, it was not enough to say that $Z_2 = Z_1^2, Z_3 = Z_1^3$, etc, because this doesn't actually show me that you can compute the values.

3) Nothing to write down/turn in for this problem.

4)…now the Puzzle of the Day said to consider the product of the four diagonal lengths, which in the picture are equivalent to the distances from Z_1 to Z_0, from Z_2 to Z_0, from Z_3 to Z_0, and from Z_4 to Z_0, respectively… we know that $Z_1 - Z_0$ (which is really just $Z_1 - 1$) is the complex number whose modulus is the same as the distance from Z_1 to Z_0. Similarly, the other three distances we want to consider are the moduli of $Z_2 - 1$, and $Z_3 - 1$, and $Z_4 - 1$ respectively. So now we just need to multiply these four complex numbers together, and find the modulus of the resulting product to get the answer in this particular case.

Backing up, try using this approach to find the answer for the (simpler!) case of a square inscribed in the unit circle - in this case the fourth roots of unity are at i, -1, -i, and 1.

Relatively simple… we find the product of $(i – 1)((-1) – 1)((-i) – 1) = (-2)(-i^2 – i + i +1) = -4$. Now the modulus of -4 is just 4 (and 4 is what we expected to get given our findings that the product of the diagonal lengths is N for an N-gon).

Remember that the modulus is the generalization of the concept of an absolute value to complex numbers, and when you take the modulus of an old fashioned real number, then this just acts like the old fashioned absolute value. As was pointed out in class at one point, the absolute value is just like the modulus/length function in that it just measures the distance of a number from the origin (which is just "0" on an (old fashioned) number line).

A lot of you had a tough time figuring out what this question was asking. You were supposed to do it for the square inscribed in a circle. If you did it (the hard way) for the pentagon, I gave you credit, but I urge you to make sure you know what the problem is looking for.

5) Back to the fifth roots of unity, Z_1, Z_2, Z_3, Z_4, and Z_5. Since we know that (Z_i)^5 = 1 for all the fifth roots of unity then this means they are the five roots of the polynomial $X^5 - 1$. Given, as we noted in class, the fact that $X^5 - 1$ factors as $(X - 1)(X^4 + X^3 + X^2 + X + 1)$, then the four roots of $(X^4 + X^3 + X^2 + X + 1)$ must be Z_1, Z_2, Z_3, and Z_4, (since $Z_5 = 1$ is the root of the factor (X-1), so the roots of the other factor, $(X^4 + X^3 + X^2 + X + 1)$ must be the other four numbers).

Now we're actually interested in the complex numbers $Z_1 - 1$, $Z_2 - 1$, $Z_3 - 1$, and $Z_4 - 1$, not Z_1, Z_2, Z_3, and Z_4. So here's the trick for the whole Puzzle of the Day. Find a polynomial whose roots are $Z_1 - 1$, $Z_2 - 1$, $Z_3 - 1$, and $Z_4 - 1$ (instead of Z_1, Z_2, Z_3, and Z_4). To do this remember that we know that Z_1, Z_2, Z_3, and Z_4 are the four roots of $(X^4 + X^3 + X^2 + X + 1)$. So try subbing in "X+1" for "X" in this same fourth degree polynomial and write down the result (no need to expand it out at this point)... will this work?

There's very little to actually do for this problem except to follow the hint. Instead of the polynomial $X^4 + X^3 + X^2 + X + 1$ we consider the following polynomial …

$$(X+1)^4 + (X+1)^3 + (X+1)^2 + (X+1) + 1$$

That was all you needed to do, but…

(and for those of you interested in what this looks like expanded out, one can check using Mathematica that you get:

Expand[(X + 1)^4 + (X + 1)^3 + (X + 1)^2 + (X + 1) + 1]

$5 + 10 X + 10 X^2 + 5 X^3 + X^4$

Now note that if you let $X = Z_1 - 1 = e^{(2Pi\ i)/5} - 1$, for instance, then you'll just get

$$((Z_1 - 1) +1)^4 + ((Z_1 - 1) +1)^3 + ((Z_1 - 1) +1)^2 + ((Z_1 - 1) +1) + 1,$$

which simplifies to just

$$(Z_1)^4 + (Z_1)^3 + (Z_1)^2 + (Z_1) + 1$$

which is just 0, given that we know that Z_1 is a root of the polynomial $X^4 + X^3 + X^2 + X + 1$

and in fact we can check this in Mathematica using the expanded out form of the polynomial from above, i.e. the polynomial $5 + 10 X + 10 X^2 + 5 X^3 + X^4$

Now we compute in Mathematica as follows:

P[X_] := Expand[(X + 1)^4 + (X + 1)^3 + (X + 1)^2 + (X + 1) + 1]

P[E^((2 Pi I)/5) – 1]

and we get

1 + E^(-((2 I \[Pi])/5)) + E^((2 I \[Pi])/5) + E^(-((4 I \[Pi])/5)) + E^((4 I \[Pi])/5)

which we'll see equals 0 by the last problem on the set, or we can just calculate

N[P[E^((2 Pi I)/5) – 1], 10]

(where "N[*** , 10]" is the command that gives 10 decimal places of accuracy for whatever is input as ***) and we get 0.*10^-100 + 0.*10^-100 I (the equivalent of "0" in Mathematica-speak!)

6) Now if you actually expanded out the polynomial you just found in the last question, what would its constant term be equal to? (no need to find all the terms, just figure out what the constant term would equal)

Well, we already just saw that the constant term is 5 from the Mathematica output, but this is also easy enough to see by hand given that the expansion of anything of the form $(X + 1)^n$ will have a constant term equal to 1, and there are 4 such terms in the polynomial expression

$$(X+1)^4 + (X+1)^3 + (X+1)^2 + (X+1) + 1$$

plus the "1" at the end.

7) And time for the "Aha!" - if you have an Nth degree polynomial that starts $X^N + \ldots$ *and it has roots* $R_1, R_2, R_3, \ldots, R_N$*, then by the work we'd done with polynomials earlier in the semester, the polynomial must factor as* $(X - R_1)(X - R_2) \ldots (X - R_N)$*. If you expand this polynomial, multiplying out all the terms, what would its constant term equal? (important note - given that we just want the modulus (absolute value in this case) of the root's product, then it doesn't matter whether the result is positive or negative)*

The constant term of the polynomial $(X - R_1)(X - R_2) \ldots (X - R_N)$ is equal to $(+/–)R_1 R_2 R_3 \ldots R_N$, i.e. just the product of the roots of the polynomial. Try this out with a simple example to convince yourself, for instance look at $(X – 3)(X – 4)$ which has roots 3 and 4. If you expand out $(X – 3)(X – 4)$ you get $X^2 – 7X + 12$, and yes the constant term 12 is just the product of the roots.

On the other hand, with three roots, such as $(X – 2)(X – 3)(X – 4) = X^3 – 9X^2 + 26X – 24$ then the constant term -24 is negative the product of the three roots. In general you'll get plus the product if there are an even number of roots, and minus the product in the case of an odd number of roots.

8) And so finally, for the general case, one could go through everything from above, but instead of doing it with the fifth roots of unity, one could work with the Nth roots of unity, where Z_1 *would equal e^(2Pi i / N), and* Z_2 *would equal* Z_1*^2,* $Z_3 = Z_1$*^3, etc. all the way up to* $Z_N = Z_1$*^N = 1.*

In this case, you'd just need to figure out the constant term of $(X+1)^{N-1} + \ldots + (X+1)^2 + (X+1) + 1$ *(what is this?) and see that that would equal the product of* $(Z_1 - 1)(Z_2 - 1)\ldots(Z_{N-1} - 1)$*, whose modulus (just absolute value in this case) is the same as the product of the lengths of the diagonals... done!*

And so now in the general case we have that all the Z_i are roots of the polynomial $X^N – 1$, and that factoring out the factor $(X – 1)$ as before, we're left with the polynomial

$$(X + 1)^{N-1} + (X + 1)^{N-2} + \cdots (X + 1) + 1$$

whose roots are the $Z_i - 1$ for the $N – 1$ roots as i takes values from 1 to $N – 1$.

The constant term will just be (N – 1) x 1 + 1 = N, and we know that this will equal plus or minus the product of the roots, i.e. so N is equal to $\pm(Z_1 - 1)(Z_2 - 1) \cdots (Z_{N-1} - 1)$

But we know that the modulus of this product is just the product of the lengths of the diagonals, that we're trying to calculate, and so, in conclusion, the modulus (or just absolute value in this case) of this product will always be equal to N, the number of sides of the inscribed polygon.

Bonus) To see one more result with all the machinery you just worked through, calculate the sum of the N Nth roots of unity, i.e. the sum of Z_1 = e^(2Pi i / N), plus Z_2 (which equals Z_1^2), plus Z_3 (which equals Z_1^3), etc. all the way up to Z_N = Z_1^N = 1. Here's the trick. Think about this sum S = $Z_1 + Z_2 + Z_3 + ... + Z_N$. Given what we know about each of the numbers, then S can also be written as Z_1 + Z_1^2 + Z_1^3 + ... + Z_1^N. Now multiply this whole sum by Z_1, and think about what you get as a result (remembering that Z_N = Z_1^N just equals 1) i.e. what is Z_1 times S? Now if Z_1 times S just equals S (as you should have just figured out!), then what does this imply that S equals (knowing that Z_1 doesn't equal 1)?

You can picture this geometrically on an Argand diagram - it's easy to see what the answer must be in the case where N is even. Draw out the case of N = 6 for yourself, drawing a vector from the origin to each 6th root of unity and think about how each vector sums with the vector directly opposite (it's a bit more mysterious geometrically what happens when N is odd!).

So we start with S = $Z_1 + Z_2 + \ldots + Z_N$, where $Z_1 = e^{2\pi i/N}$

Then we can also write that $S = Z_1 + Z_1^2 + Z_1^3 + \cdots Z_1^N$
and then if we calculate S times Z_1 we get $Z_1^2 + Z_1^3 + \cdots Z_1^N + Z_1^{N+1}$

but Z_1 to the N+1 power is just Z_1 as Z_1^N equals 1 so this whole thing equals

$$Z_1^2 + Z_1^3 + \cdots Z_1^N + Z_1$$

which is just equal to S again (just rearrange the last term).

Okay, so S times Z_1 equals S, or S $\times$ Z_1 = S, so that $S(Z_1) - S = 0$, or $S(Z_1 - 1) = 0$.

Given that ($Z_1 - 1$) is not equal to 0, then for the product S ($Z_1 - 1$) to equal 0, it must be the case that S equals 0 (if the product of two numbers is 0, then at least one of them must be 0).

Another way to think about this is that the N roots of unity are equally spaced around the unit circle. If their sum was nonzero, then the sum, a complex number would have an argument (we know that as the angle that the complex number makes relative to the x axis) which can't be possible given that the roots are symmetric around the origin (i.e. equally spaced all the way around). Therefore, by symmetry, the sum must be 0.

In the even cases (i.e. when N is even), each root of unity has a root of unity just opposite it (across the origin), so that they neatly pair off, summing to 0 each time.

Consider the case of N = 8, for example (where the roots in the picture are labeled W_N^1, W_N^2 W_N^3 etc. It's clear that adding W_N^1 to W_N^5, for example, will just equal 0, as they're opposite from each other (or, better put, W_N^5 is just equal to $-W_N^1$, and likewise W_N^6 is just equal to $-W_N^2$, etc.), thus there are four pairs of 8th roots of unity, each of which sum to 0, so the whole sum (of all 8 roots of unity) equals 0.