Math E305

Advanced Algebra and Trigonometry!

From Cardano…

…to Argand!

Ninth Class – Wednesday, July 9th

- Back to the Quadratic Formula
  - *but wait, let's back it up a bit!*
- An off to Cardano!
  - *we begin with a transformation…*
  - *…then let's head for the Cubic Formula!*
- But what's this strange new number?
  - *time to tour the complex world*
  - *…but from a different perspective!*
- Matrices and Trigonometry?!
  - *connecting a number of ideas!*

POTD – a simple/complex one!

A couple quick questions for those of you who feel comfortable with i...

If i is the square root of -1, then what is the square root of i?

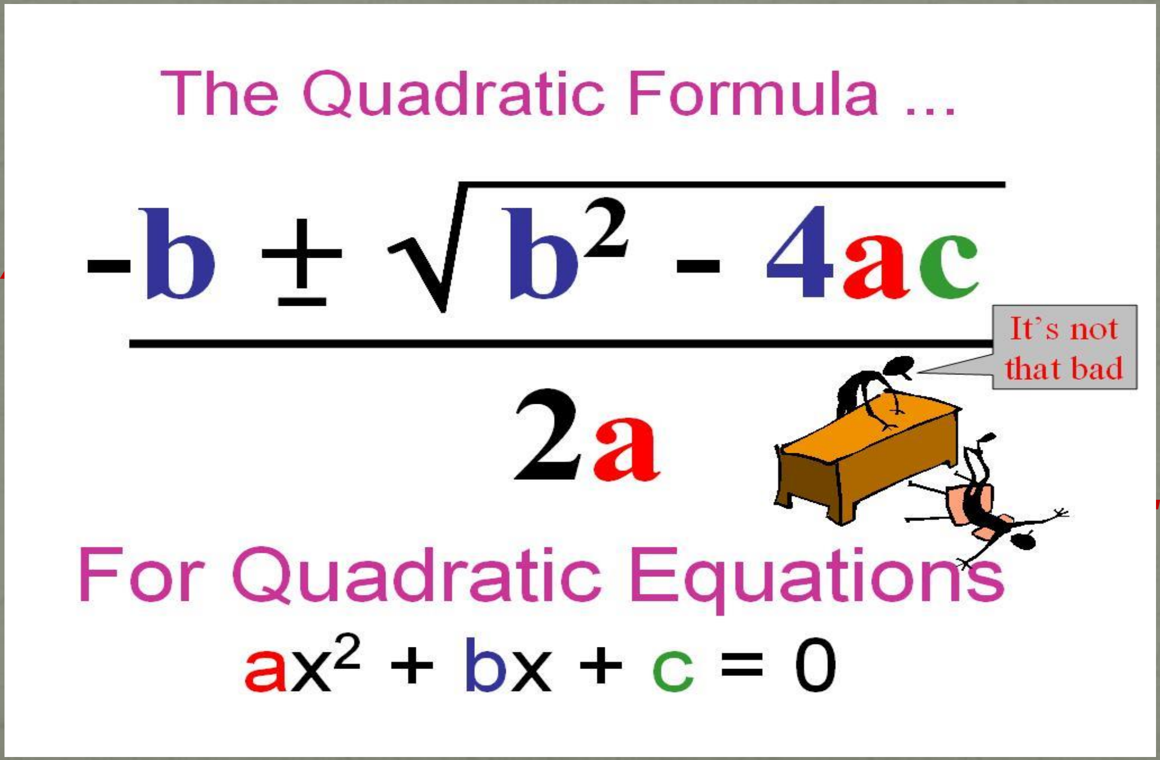
…and taking it up a notch, what is i^i?!

and now time for a little tour!

The Quadratic Formula…

why "quadratic"?

doesn't that mean "four"?



two

…

Time for a few questions!

Solve $x^2 = 64$…

Solve $(w - 3)^2 = 64$

Solve $4(m + 3)^2 - 64 = 0$

Solve $16(z + 2)^2 + 64 = 0$

Keep going!

Solve $(w - 1)^2 = 49$

Solve $(x + 3)^2 = 11$

Solve $z^2 + 6z + 9 = 17$

Aha! *Try to rewrite equations in the form*

$(x + A)^2 = B$

What about solve $z^2 + 6z + 9 = 9$*?*

Now – Algebra → Geometry!

Singapore Math?!

interpret $(x + 3)^2$ *as an area…*

	x	3
x	x^2	3x
3	3x	9

so… what do you need to put in for ◌
to make a perfect square for…

$x^2 + 12x +$ ◌ $\qquad x^2 - 8x +$ ◌

Getting to a formula?!

So now what about $x^2 + bx + ◌$ *?...*

	x	?
x	x^2	
?		??

what about…

$x^2 + \sqrt{5}x + ◌$ $\quad$ $x^2 + \frac{1}{2}x + ◌$

A few more...

Solve $x^2 + 4x = 21$

	x	2
x	x^2	2x
2	2x	??

Solve $x^2 - 6x = 27$

A couple twists…

Now solve $3x^2 – 4x + 1 = 0$

There are a couple of ways to go…

try multiplying through by 3

What about $x^2 – 3x + 2 = 20$

middle coefficient odd…

try multiplying through by 4!

And here we go…!

First solve $3x^2 + 7x + 4 = 0$

And now let’s do it all!

Solve $ax^2 + bx + c = 0$

try going ahead and drawing out a box to help you through all the steps!

But curiously the story of complex numbers actually starts here…

Solve $ax^3 + bx^2 + cx + d = 0$

first idea – can you reduce the number of coefficients by 1 so that you've got to work with three instead?

Solve $x^3 + Ax^2 + Bx + C = 0$

cool idea – can you work with this equation to make it so that you're dealing with just two coefficients instead of three?

Try solving $x^2 - 4x - 5 = 0$

Time for a shift! try subbing $w = x-2$

But curiously the story of complex numbers actually starts here…

Solve $x^3 + Ax^2 + Bx + C = 0$

Now, simplify by subbing $z - A/3$ in for x

So now solve $z^3 = Dz + E$

Girolamo Cardano (1501 – 1576)
knew this trick…
but needed to do more
to solve this new cubic…

actually Cardano gets more credit than he should!

Solving the cubic…!

Compare $(r+s)^3$ *to* $r^3 + s^3$

Given this relation…

$$(r+s)^3 = 3rs\,(r+s) + (r^3 + s^3)$$

now try to solve $z^3 = 6z + 9$

aha – if you can find an r and an s
so that $3rs = 6$ *and* $r^3 + s^3 = 9$
then you're done!

Solving the cubic… one last step!

A final coefficient trick…

Instead of trying to solve

$$z^3 = Dz + E$$

first rewrite it as

$$z^3 = 3Pz + 2Q$$

aha – if you can find an r and an s
so that rs = P and $r^3 + s^3 = 2Q$ then…

z just equals r + s

Solving $z^3 = 3Pz + 2Q$... the last step

If we rewrite $rs = P$ *and* $r^3 + s^3 = 2Q$

then $s = P/r$ *…and* $r^3 + (P/r)^3 = 2Q$

then multiplying through by r^3*…*

$$(r^3)^2 + P^3 = (2Q)\, r^3$$

and now write $M = r^3$ *so that…*

$$M^2 - (2Q)\, M + P^3 = 0$$

and s^3 *satisfies the same quadratic…*

now solve it and find z = r + s!

And here we go…!

Now, using Cardano's formula…

Rafael Bombelli

(1526 – 1572)

solved

$z^3 = 15z + 4$

and got…

$$z = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

But what does it mean? ...!

$$z = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

Bombelli noted that if one were simply to calculate formally…

what is $2+\sqrt{-1}$ *cubed?*

and likewise what is $2-\sqrt{-1}$ *cubed?*

Time for some new numbers!!

And now on to ℂ!

we could just work with $\sqrt{-1}$
from a symbolic perspective…

actually we should really work with u, an unknown value that has the property that $u^2 = -1$…

from a purely formal algebraic perspective, then what is (1+ u) plus (2 – 3u)?

from a purely formal algebraic perspective, then what is (1+ u) times (2 – 3u)?

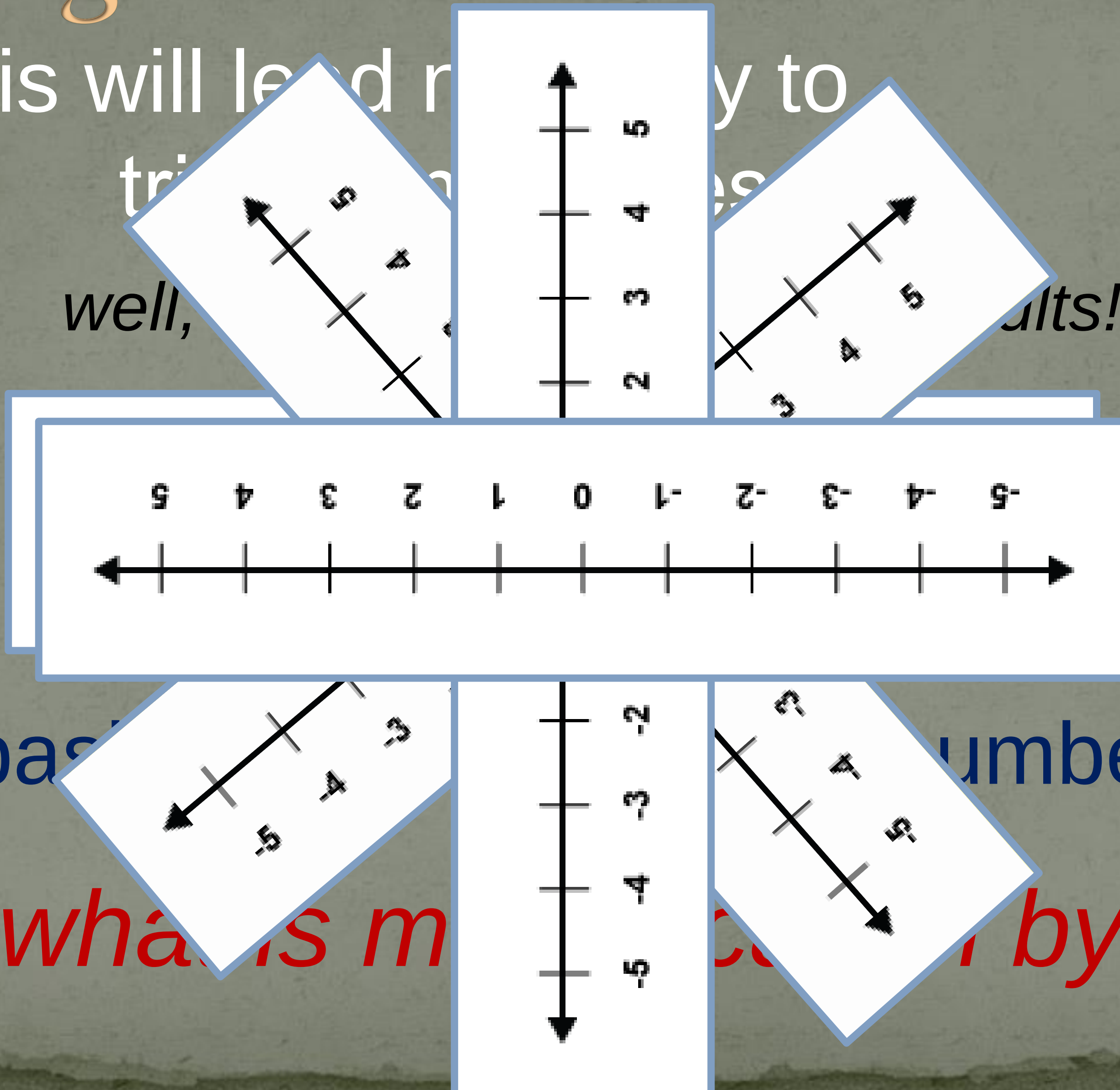
But let's do something different!

Let's go Geometric!

this will lead n… y to

tri… es

well, … ults!

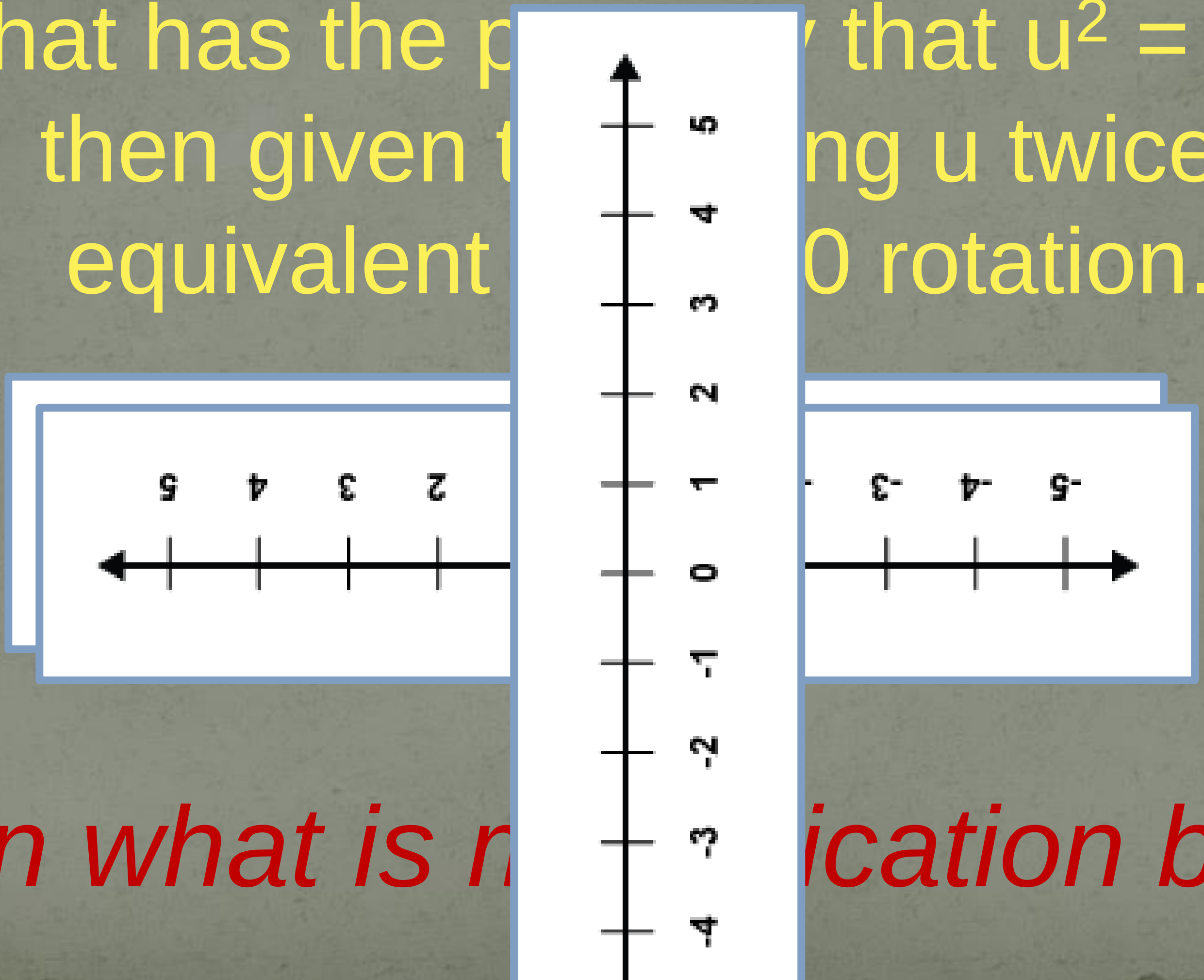


The bas… umber line…

so wha… is m… c… by -1?

Finding “u”!

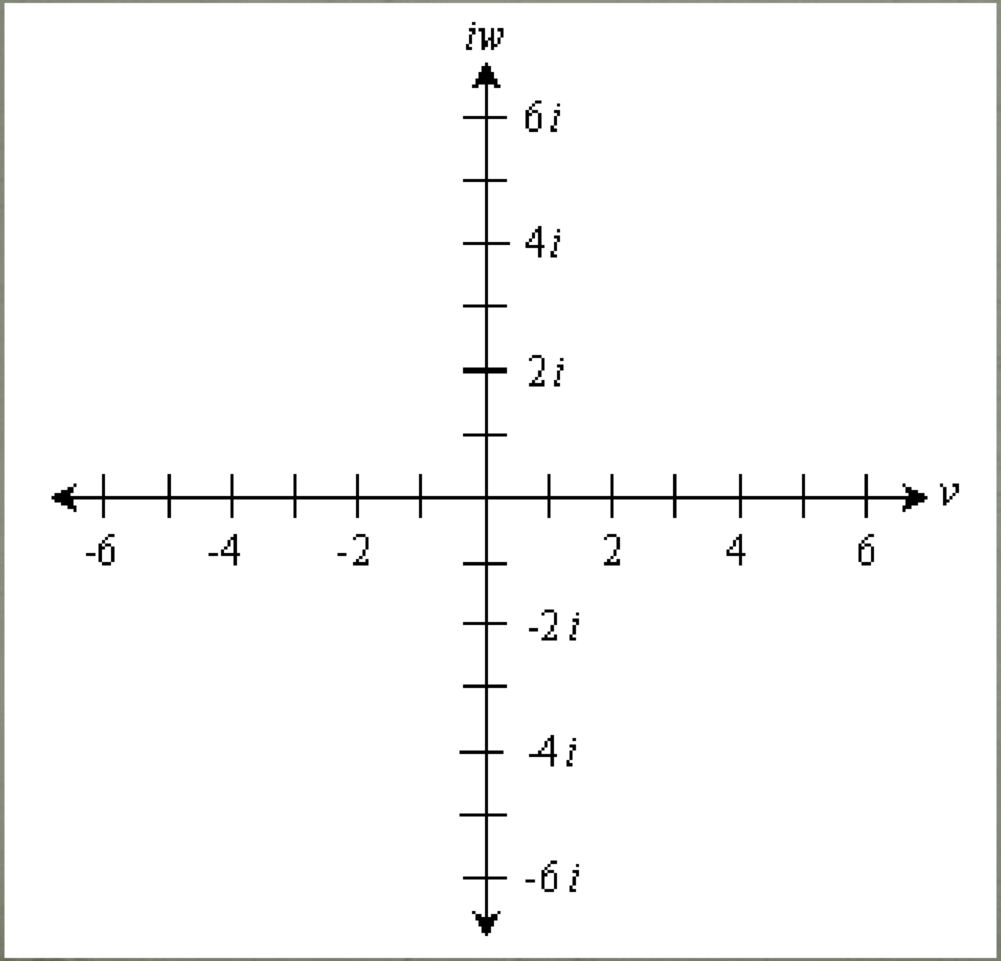
so now if we want something “u” that has the p … that $u^2 = -1$, then given t … ng u twice is equivalent … 0 rotation...



then what is m … ication by u?

A New Number Line!

The "complete" number line!



and of course we'll know "u" as "i" instead!