Math E305

Advanced Algebra and Trigonometry!

Fitting polynomials

…"forwards" and "sdrawkcab" part Two

Seventh Class – Monday, July 7th

- Back to Creating Polynomials
  - *studying a polynomial from differences*
  - *two different approaches…*
- Combinatorial Polynomials?!
  - *and a precursor to Taylor's Theorem…*
- Prove It!
  - *a do it yourself summary of basic results!*

Start back with the fence problem… - an annoying homework problem?!

	Dimension			
Fences	1	2	3	4
0	1	1	1	1
1	2	2	2	2
2	3	4	4	4
3	4	7	8	8
4	5	11	15	16
5	6	16	26	31
6	7	22	42	57

recall our work in 3D first…

Fences	Regions in 3rd dim.	Δ	Δ^2	Δ^3
0	1	1	1	1
1	2	2	2	1
2	4	4	3	1
3	8	7	4	1
4	15	11	5	1
5	26	16	6	1
6	42	22	7	1

3D fence problem continued…

Fences	Regions in 3rd dim.	Δ	Δ equals
0	1	1	0 + 1
1	2	2	1 + 1
2	4	4	3 + 1
3	8	7	6 + 1
4	15	11	10 + 1
5	26	16	15 + 1
6	42	22	21 + 1

3D fence problem continued…

Fences	Regions in 3rd dim.	Δ	Δ equals
0	1	1	$\frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1$
1	2	2	$\frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1$
2	4	4	$\frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1$
3	8	7	$\frac{1}{2} \cdot 3^2 + \frac{1}{2} \cdot 3 + 1$
4	15	11	$\frac{1}{2} \cdot 4^2 + \frac{1}{2} \cdot 4 + 1$
5	26	16	$\frac{1}{2} \cdot 5^2 + \frac{1}{2} \cdot 5 + 1$
6	42	22	$\frac{1}{2} \cdot 6^2 + \frac{1}{2} \cdot 6 + 1$

3D fence problem continued…

Fences	Regions in 3rd dim.	Δ	Δ equals
0	1	1	$\frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1$
1	2	2	$\frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1$
2	4	4	$\frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1$
3	8		

so, for example, $f(3) = 8 = 1 + 1 + 2 + 4$

$$= 1 + \frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1$$

$$+ \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1$$

$$+ \frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1$$

3D fence problem continued…

so, for example, $f(3) = 8 = 1 + 1 + 2 + 4$

$$= 1 + \tfrac{1}{2} \cdot 0^2 + \tfrac{1}{2} \cdot 0 + 1$$

$$+ \tfrac{1}{2} \cdot 1^2 + \tfrac{1}{2} \cdot 1 + 1$$

$$+ \tfrac{1}{2} \cdot 2^2 + \tfrac{1}{2} \cdot 2 + 1$$

$$= 1 + \tfrac{1}{2}(0^2 + 1^2 + 2^2) + \tfrac{1}{2}(0 + 1 + 2) + (1 + 1 + 1)$$

and in general…

$$f(n) = 1 + \tfrac{1}{2}(0^2 + 1^2 + \ldots + (n-1)^2) + \tfrac{1}{2}(0 + 1 + \ldots + (n-1)) + n \cdot 1$$

3D fence problem continued…

$$f(n) = 1 + \tfrac{1}{2}\,(0^2 + 1^2 + \ldots + (n-1)^2\,) + \tfrac{1}{2}\,(0 + 1 + \ldots + (n-1)) + n \cdot 1$$

and now using the formula for the sum of the first n squares equaling n(n+1)(2n+1)/6 we get…

(remembering to sub in "(n-1)" for "n" in this formula!)

$$f(n) = 1 + \tfrac{1}{2}\,[(n-1)n(2(n-1)+1)/6] + \tfrac{1}{2}\,[n(n-1)/2] + n$$

which simplifies as…

$$f(n) = [n^3 + 5n + 6]/6$$

which works – i.e. gives the values we want – phew!

So now how about the fourth dimension?!

	Dimension			
Fences	1	2	3	4
0	1	1	1	1
1	2	2	2	2
2	3	4	4	4
3	4	7	8	8
4	5	11	15	16
5	6	16	26	31
6	7	22	42	57

We reversed the table…

Fences	4 Dim. Regions	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	1	1	1	1	0
1	2	2	2	2	1	0
2	4	4	4	3	1	0
3	8	8	7	4	1	0
4	16	15	11	5	1	0
5	31	26	16	6	1	0
6	57	42	22	7	1	0

And remembered what we'd found for the first differences…

Fences	4 Dim. Regions	Δ
0	1	1
1	2	2
2	4	4
3	8	8
4	16	15
5	31	26
6	57	42

$\Delta(n) = [n^3 + 5n + 6]/6$

so what, for example, does f(3) equal?

Now how about that other approach we looked at…

input	output	Δ	Δ^2	Δ^3	Δ^4
0	1	-2	14	12	0
1	-1	12	26	12	0
2	11	38	38	12	0
3	49	76	50	12	0
4	125	126	62	12	
5	251	188	74		
6	439	262			
7	701				

Binomial Coefficients!

these come in handy now!

Row						
0	$\binom{0}{0}=1$					
1	$\binom{1}{0}=1$	$\binom{1}{1}=1$				
2	$\binom{2}{0}=1$	$\binom{2}{1}=2$	$\binom{2}{2}=1$			
3	$\binom{3}{0}=1$	$\binom{3}{1}=3$	$\binom{3}{2}=3$	$\binom{3}{3}=1$		
4	$\binom{4}{0}=1$	$\binom{4}{1}=4$	$\binom{4}{2}=6$	$\binom{4}{3}=4$	$\binom{4}{4}=1$	
5	$\binom{5}{0}=1$	$\binom{5}{1}=5$	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{5}=1$

Formulas for Binomial Coefficients

so now consider the binomial coefficient $\binom{n}{k}$ *…*

we can calculate this as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

or, more simply(?!), just as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}$$

Formulas for Binomial Coefficients

Using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

what is $\binom{4}{3}$ *?* *what is* $\binom{\pi}{3}$ *?!*

now consider using this instead:

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-(k-1))}{k!}$$

this is called a "combinatorial polynomial"
- it generalizes binomial coefficients
to non-integer values, like π!

and note(!)

$$\binom{x}{0} = 1$$

Back to our polynomial…

using

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}$$

We end up with

$f(n) = \binom{n}{0} \cdot 1 + \binom{n}{1} \cdot (-2) + \binom{n}{2} \cdot 14 + \binom{n}{3} \cdot 12$

and so… does this work?!

and go ahead and write x instead of n!

Yes it does!

If $f(x) = \binom{x}{0} \cdot 1 + \binom{x}{1} \cdot (-2) + \binom{x}{2} \cdot 14 + \binom{x}{3} \cdot 12$

then given that $\binom{x}{0} = 1$, $\binom{x}{1} = \frac{x}{1} = x$,

$\binom{x}{2} = \frac{x(x-1)}{2 \cdot 1}$, and $\binom{x}{3} = \frac{x(x-1)(x-2)}{3 \cdot 2 \cdot 1}$

then we can sub these in and work out that…

$$f(x) = 1 \cdot 1 + x \cdot (-2) + \frac{x(x-1)}{2} \cdot 14 + \frac{x(x-1)(x-2)}{3 \cdot 2 \cdot 1} \cdot 12$$

which just simplifies to…

$$f(x) = 1 - 5x + x^2 + 2x^3$$

which gives outputs matching our mystery function!

So then for the 4D problem…

Fences	4 Dim. Regions	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	1	1	1	1	0
1	2	2	2	2	1	0
2	4	4	4	3	1	0
3	8	8	7	4	1	0
4	16	15	11	5	1	0
5	31	26	16	6	1	0
6	57	42	22	7	1	0

And now we end up with the Ultimate Polynomial Fitting Formula!

Given a polynomial f(n),
if we calculate a difference table,
i.e. use the Δ operators on f,
and compute their values at 0…

then $f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k(f)(0)$

Woah!

how about $f(x) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k(f)(0)$

So now how about the fourth dimension?!

now it's easy!

	Dimension			
Fences	**1**	**2**	**3**	**4**
0	1	1	1	1
1	2	2	2	2
2	3	4	4	4
3	4	7	8	8
4	5	11	15	16
5	6	16	26	31
6	7	22	42	57

Try it out for Sums of Squares!

input	output	Δ	Δ^2	Δ^3	Δ^4
0	0	1	3	2	0
1	1	4	5	2	0
2	5	9	7	2	0
3	14	16	9	2	
4	30	25	11		
5	55	36			
6	91				

What about Sums of Cubes?!

input	output	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0	1	7	12	6	0
1	1	8	19	18	6	0
2	9	27	37	24	6	
3	36	64	61	30		
4	100	125	91			
5	225	216				
6	441					

Working with Polynomials –

First, although we've been working with polynomials we should probably distinguish between polynomial functions (algebra 1) and polynomial forms (algebra 2)…

What?!

consider $f(x) = \sum_{k=0}^{n} a_k\, x^n$

Working with Polynomials – the classic theorems…

First let's think about polynomial degree

what is it?

If f and g are polynomials, then what is:

deg(f + f) = deg(2f)

deg(f g)

deg(f + g)

What are polynomials of degree 0?

What is deg(0)?

Working with Polynomials – the classic theorems…

Let’s begin with the classic Long Division theorem…

what is it?

If f and g are polynomials, with $f \neq 0$, then there exist polynomials q and r such that $g = fq + r$,

where the degree of r is less than the degree of f

Prove it!

Working with Polynomials – the classic theorems…

You can prove the Long Division theorem using induction(!)

If f and g are polynomials, with $f \neq 0$, there exist polynomials q and r

such that $g = fq + r$

where the degree of r is less than the degree of f

suppose $f = a_n x^n + \ldots + a_1 x + a_0$

and $g = b_m x^m + \ldots + b_1 x + b_0$

If f and g are polynomials, with $f \neq 0$, there exist polynomials q and r

such that $g = fq + r$

where the degree of r is less than the degree of f

then the induction step goes as follows…

assume the theorem is true for every polynomial h of degree less than m

(the degree of g)… (that $h = f\,q' + r'$)

okay, so now consider

$$h(x) = g(x) - (b_m / a_n)\, f(x)\, x^{m-n}$$

so if

$$h(x) = g(x) - (b_m/a_n)\, f(x)\, x^{m-n}$$

$$h(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0$$
$$- (b_m/a_n)\,[a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0\,]\, x^{m-n}$$

$$= b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0$$
$$- (b_m/a_n)\,[a_n x^n]\, x^{m-n}$$
$$- (b_m/a_n)\,[a_{n-1} x^{n-1}]\, x^{m-n}$$
$$- \ldots - (b_m/a_n)\,[a_0]\, x^{m-n}$$

Aha!

$$h(x) = [b_{m-1} - (b_m/a_n)\, a_{n-1}]\, x^{m-1} + \ldots + b_0$$

…then by the induction hypothesis

$$h(x) = s(x)\, f(x) + r(x)$$

for some polynomials s(x), and r(x), whose degree is less than the degree of f(x)…

but so $h(x) = g(x) - (b_m/a_m)\, f(x)\, x^{m-n}$

$$= s(x)\, f(x) + r(x)$$

Aha! solve for g(x)…

$$g(x) = [(b_m/a_m)\, x^{m-n} + s(x)]\, f(x) + r(x)$$

Now it's your turn!

Consider the remainder theorem…

so what is it?

If f is a polynomial, then the remainder when f(x) is divided by (x – a) is f(a)

Prove it!

hint – use the long division theorem with f(x) and (x-a)

a few more corollaries!

The factor theorem…

If f is a polynomial,

then (x-a) is a factor of f(x)

if and only if f(a) = 0

Prove it!

The number of roots theorem…

A polynomial of degree n can have at most n roots

Prove it!

and a few more corollaries!

Values determining polynomials…

If two polynomials of degree at most n agree for more than n inputs, then they are exactly the same polynomial

Prove it!

Values determining polynomials part 2

A polynomial of degree n is uniquely determined by any n+1 of its values

Prove it!