

Math E305

Advanced Algebra and Trigonometry!

Fitting polynomials

...*“backwards”* and
“forwards”

Sixth Class – Thursday, July 3rd

- POTD
 - *let's fit some curves!*
 - *we are all worker bees!*
 - *...an old friend?!*
- Back to Creating Polynomials
 - *studying a polynomial from differences*
 - *two different approaches...*
- Combinatorial Polynomials?!
 - *and the precursor to Taylor's Theorem...*

Getting ready for today's POTD!

How many points does it take to describe a unique polynomial...

How many points does it take to describe a line?

Suppose you have two points,
 $(0, 0)$ and $(1, 1)$...

Let's fit some lines...

Let's find some quadratics!

And now... the POTD!

Suppose someone has a 10^{th} degree polynomial in mind $P(x)$...

and that all of its coefficients are positive integers

(or at least non-negative – some could be zero).

You can ask for the polynomial to be evaluated at any number of input values of your choice...

And now... the POTD!

Suppose someone has a 10^{th} degree polynomial in mind $P(x)$...

and that all of its coefficients are positive integers

(or at least non-negative – some could be zero).

What's the fewest points that it will take to determine what $P(x)$ equals?

hint... a lot fewer than you think!

Let's finish off some old business!

So can you find a number X with a similarly recursive pattern...?

Find X so that $X^n = X^{n-2} + X^{n-1}$!

Aha! $\varphi = \frac{1+\sqrt{5}}{2}$

and phi's (somewhat negative!) sidekick...

$$\psi = \frac{1-\sqrt{5}}{2}$$

Now put them to work!

Big idea, yes the following two sequences obey the recursive relation:

$$(\varphi^n)_{n \in \mathbb{N}} \quad (\psi^n)_{n \in \mathbb{N}}$$

But now think about what happens to a sequence defined by their sum...

$$s_n = \varphi^n + \psi^n$$

Now put them to work!

In fact any sequence of the form

$$s_n = A \varphi^n + B \psi^n$$

with constants A and B will fulfill
the recursive constraint...!

Could we find A and B so that the sequence

$s_n = A \varphi^n + B \psi^n$ begins with

1 1 2 3 5 ...

The Honeycomb Puzzle!

How many ways can you get from here to there?



Pascal's Triangle

The coefficients in Pascal's Triangle are connected with the Binomial Theorem... why?

$$(x+y)^0 = \mathbf{1} \quad \text{-----} \quad \text{0th row}$$

$$(x+y)^1 = \mathbf{1}x + \mathbf{1}y \quad \text{-----} \quad \text{1st row}$$

$$(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2 \quad \text{-----} \quad \text{2nd row}$$

$$(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3 \quad \text{-----} \quad \text{3rd row}$$

$$(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4 \quad \text{-----} \quad \text{4th row}$$

$$(x+y)^5 = \mathbf{1}x^5 + \mathbf{5}x^4y + \mathbf{10}x^3y^2 + \mathbf{10}x^2y^3 + \mathbf{5}xy^4 + \mathbf{1}y^5 \quad \text{-----} \quad \text{5th row}$$

Consider $(a + b)^3$ as steps down the
honeycomb...

a to the left

b to the right



$$(a + b)^3$$

$$= (a+b)(a+b)(a+b)$$

$$= aaa + aab + aba + baa + abb + bab + bba + bbb$$

$$= 1 a^3 + 3 a^2 b + 3 a b^2 + 1 b^3$$

And now the full-bore binomial theorem!

$$(a + b)^n$$

$$= (a+b) (a+b) \cdots (a+b) \quad (n \text{ times})$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \\ \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

$$= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

So if we examine the Binomial Theorem...

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^0 = \mathbf{1} \quad \text{_____} \quad \text{0th row}$$

$$(x+y)^1 = \mathbf{1}x + \mathbf{1}y \quad \text{_____} \quad \text{1st row}$$

$$(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2 \quad \text{_____} \quad \text{2nd row}$$

$$(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^2 \quad \text{_____} \quad \text{3rd row}$$

one ends up with $\binom{n}{k}$ calculated as

“n choose k” – the number of ways of selecting k objects out of n objects.

Remember the fence problem?

Four Dimensional Space...

Five Dimensional Space...!!



The fence problem...

Fences	Dimension			
	1	2	3	4
0	1	1	1	1
1	2	2	2	2
2	3	4	4	4
3	4	7	8	8
4	5	11	15	16
5	6	16	26	31
6	7	22	42	57

The fence problem continued...

Fences	Regions			
	in 3 rd dim.	Δ	Δ^2	Δ^3
0	1	1	1	1
1	2	2	2	1
2	4	4	3	1
3	8	7	4	1
4	15	11	5	1
5	26	16	6	1
6	42	22	7	1

The fence problem continued...

Fences	Regions in 3 rd dim.	Δ	Δ equals
0	1	1	0 + 1
1	2	2	1 + 1
2	4	4	3 + 1
3	8	7	6 + 1
4	15	11	10 + 1
5	26	16	15 + 1
6	42	22	21 + 1

The fence problem continued...

Fences	Regions in 3 rd dim.	Δ	Δ equals
0	1	1	$\frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1$
1	2	2	$\frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1$
2	4	4	$\frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1$
3	8	7	$\frac{1}{2} \cdot 3^2 + \frac{1}{2} \cdot 3 + 1$
4	15	11	$\frac{1}{2} \cdot 4^2 + \frac{1}{2} \cdot 4 + 1$
5	26	16	$\frac{1}{2} \cdot 5^2 + \frac{1}{2} \cdot 5 + 1$
6	42	22	$\frac{1}{2} \cdot 6^2 + \frac{1}{2} \cdot 6 + 1$

The fence problem continued...

Fences	Regions in 3 rd dim.	Δ	Δ equals
0	1	1	$\frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1$
1	2	2	$\frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1$
2	4	4	$\frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1$
3	8		

so, for example, $f(3) = 8 = 1 + 1 + 2 + 4$

$$= 1 + \frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1$$

$$+ \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1$$

$$+ \frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1$$

The fence problem continued...

so, for example, $f(3) = 8 = 1 + 1 + 2 + 4$

$$\begin{aligned} &= 1 + \frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 0 + 1 \\ &\quad + \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1 + 1 \\ &\quad + \frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2 + 1 \end{aligned}$$

$$= 1 + \frac{1}{2} (0^2 + 1^2 + 2^2) + \frac{1}{2} (0 + 1 + 2) + (1 + 1 + 1)$$

and in general...

$$f(n) = 1 + \frac{1}{2} (0^2 + 1^2 + \dots + (n-1)^2) + \frac{1}{2} (0 + 1 + \dots + (n-1)) + n \cdot 1$$

The fence problem continued...

$$f(n) = 1 + \frac{1}{2} (0^2 + 1^2 + \dots + (n-1)^2) + \frac{1}{2} (0 + 1 + \dots + (n-1)) + n \cdot 1$$

and now using the formula for the sum of the first n squares equaling $n(n+1)(2n+1)/6$ we get...

(remembering to sub in “(n-1)” for “n” in this formula!)

$$f(n) = 1 + \frac{1}{2} [(n-1)n(2(n-1)+1)/6] + \frac{1}{2} [n(n-1)/2] + n$$

which simplifies as...

$$f(n) = [n^3 + 5n + 6]/6$$

which works – i.e. gives the values we want – phew!

Then we went to the fourth
dimensions...! *and figured it out!*

Fences	Dimension			
	1	2	3	4
0	1	1	1	1
1	2	2	2	2
2	3	4	4	4
3	4	7	8	8
4	5	11	15	16
5	6	16	26	31
6	7	22	42	57

We reversed the table...

4 Dim.

Fences	Regions	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	1	1	1	1	0
1	2	2	2	2	1	0
2	4	4	4	3	1	0
3	8	8	7	4	1	0
4	16	15	11	5	1	0
5	31	26	16	6	1	0
6	57	42	22	7	1	0

And remembered we'd already found formula for the first differences...

	4 Dim.	
Fences	Regions	Δ
0	1	1
1	2	2
2	4	4
3	8	8
4	16	15
5	31	26
6	57	42

$$\Delta(n) = [n^3 + 5n + 6]/6$$

so what, for example, does $f(3)$ equal?

Now let's try doing this backwards!

input	output	Δ	Δ^2	Δ^3	Δ^4
0	1	-2	14	12	0
1	-1	12	26	12	0
2	11	38	38	12	0
3	49	76	50	12	0
4	125	126	62	12	
5	251	188	74		
6	439	262			
7	701				

Binomial Coefficients!

Hey! We just saw those!

Row

0										$\binom{0}{0} = 1$				
1									$\binom{1}{0} = 1$	$\binom{1}{1} = 1$				
2									$\binom{2}{0} = 1$	$\binom{2}{1} = 2$	$\binom{2}{2} = 1$			
3									$\binom{3}{0} = 1$	$\binom{3}{1} = 3$	$\binom{3}{2} = 3$	$\binom{3}{3} = 1$		
4									$\binom{4}{0} = 1$	$\binom{4}{1} = 4$	$\binom{4}{2} = 6$	$\binom{4}{3} = 4$	$\binom{4}{4} = 1$	
5									$\binom{5}{0} = 1$	$\binom{5}{1} = 5$	$\binom{5}{2} = 10$	$\binom{5}{3} = 10$	$\binom{5}{4} = 5$	$\binom{5}{5} = 1$

Formulas for Binomial Coefficients

so now consider the

binomial coefficient $\binom{n}{k}$...

we can calculate this as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

or, more simply(?!), just as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Formulas for Binomial Coefficients

Using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

what is $\binom{4}{3}$? *what is $\binom{\pi}{3}$?!*

now consider using this instead:

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

this is called a “combinatorial polynomial”
- it generalizes binomial coefficients
to non-integer values, like π !

Back to our polynomial...

using

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

We end up with

$$f(n) = \binom{n}{0} \cdot 1 + \binom{n}{1} \cdot (-2) + \binom{n}{2} \cdot 14 + \binom{n}{3} \cdot 12$$

and so... does this work?!

Yes it does!

$$\text{If } f(n) = \binom{n}{0} \cdot 1 + \binom{n}{1} \cdot (-2) + \binom{n}{2} \cdot 14 + \binom{n}{3} \cdot 12$$

then given that $\binom{n}{0} = 1$, $\binom{n}{1} = \frac{n}{1} = n$,

$$\binom{n}{2} = \frac{n(n-1)}{2 \cdot 1}, \text{ and } \binom{n}{3} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1}$$

then we can sub these in and work out that...

$$f(n) = 1 \cdot 1 + n \cdot (-2) + \frac{n(n-1)}{2} \cdot 14 + \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} \cdot 12$$

which just simplifies to...

$$f(n) = 1 - 5n + n^2 + 2n^3$$

which gives outputs matching our mystery function!