Math S305

Advanced Algebra and Trigonometry!

Sequences

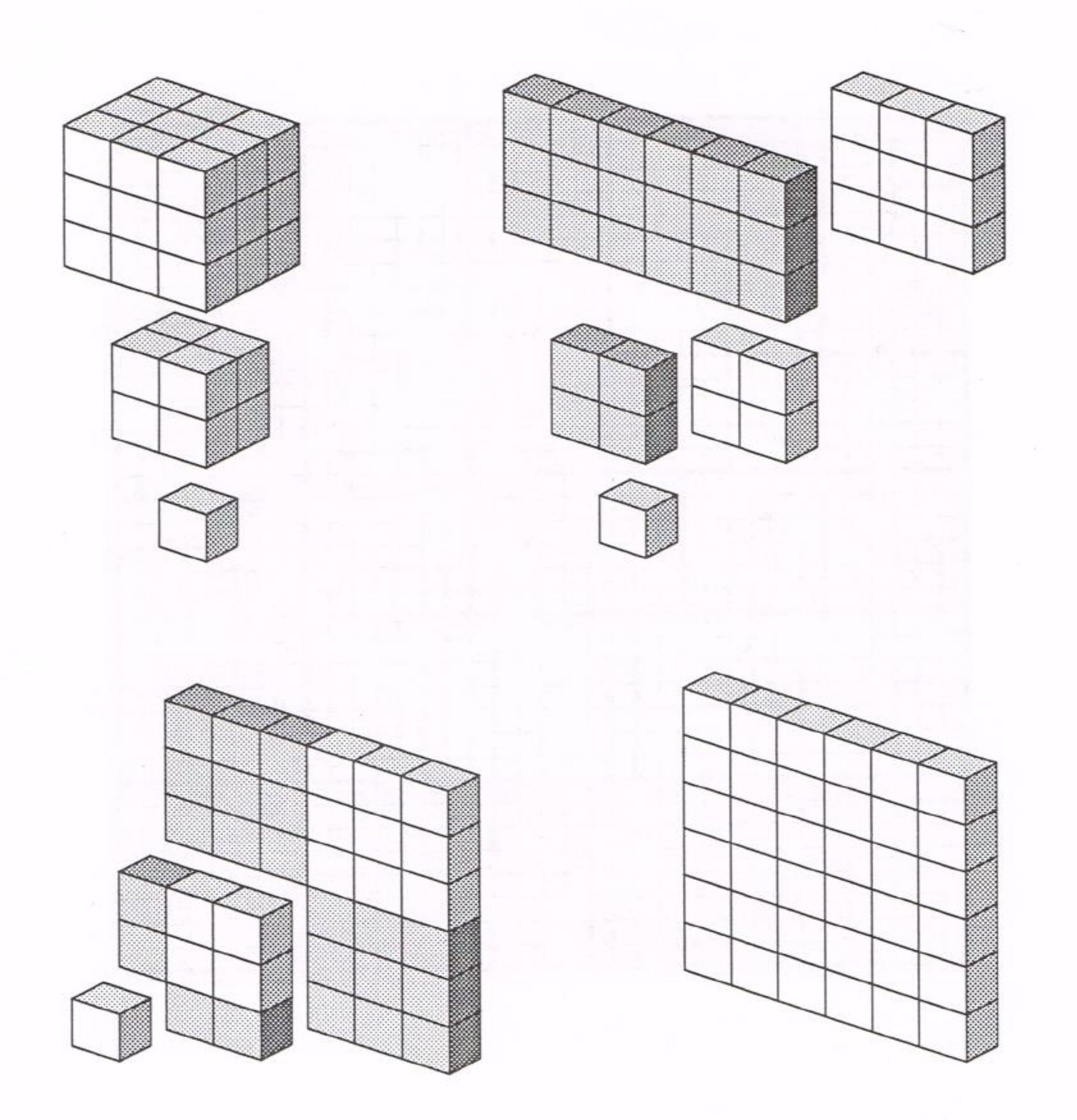
continued!

Fifth Class – Wednesday July 2nd

- POTD
  - *A chocolate game?!*
- Final thoughts about sequences
  - *…what about Fibonacci?*
  - *…what about other recursive sequences?*
- And now… something different?!
  - *Singapore Math!*

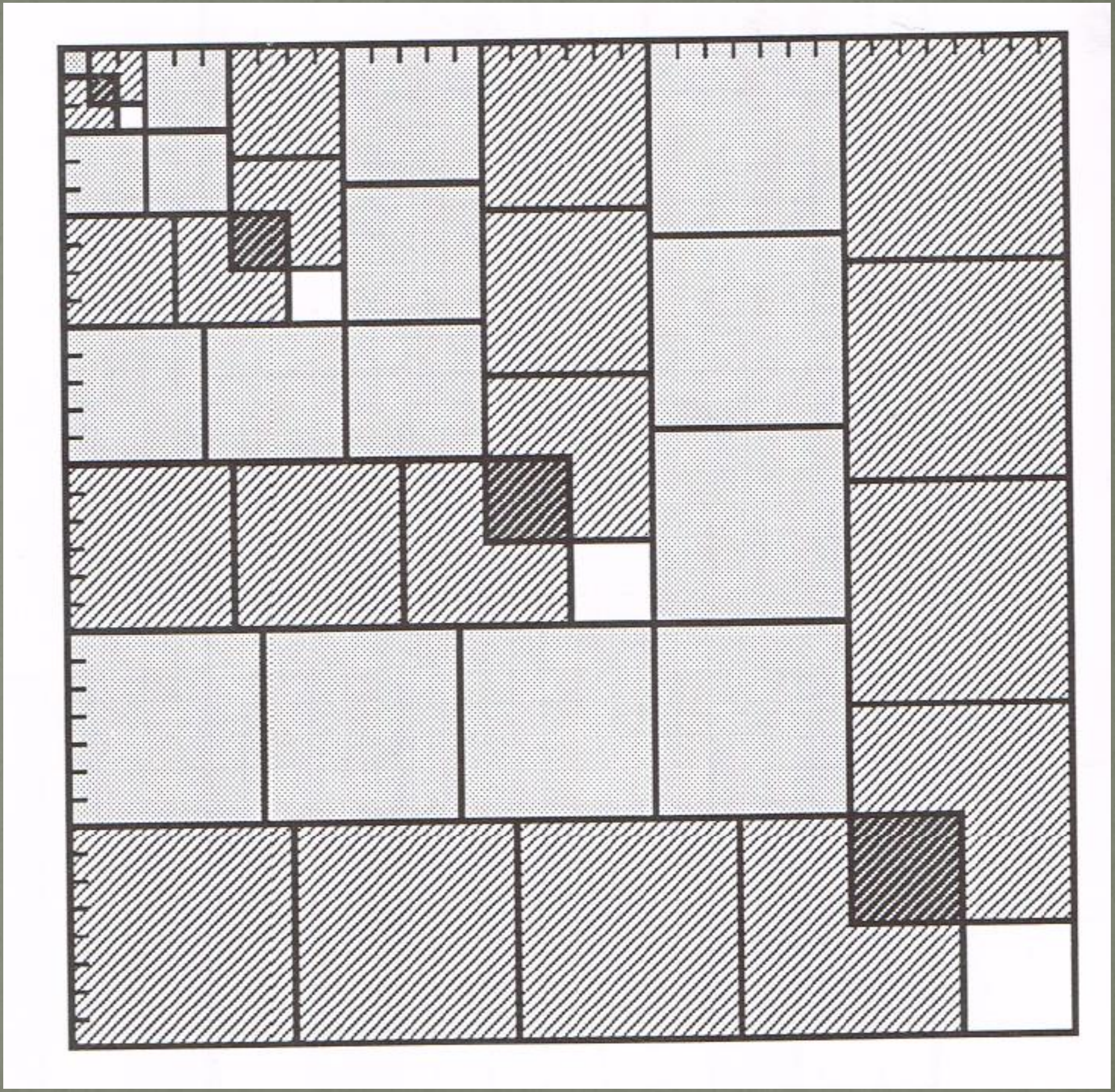
POTD… !

What do you make of this?!



POTD… !

How about this?!



Now another POTD !

A simple yet tasty game!

My boys are fighting over a chocolate bar… and use a game to see who gets it

Who should go first?!

Sequences – descriptions

What about something this simple?

.9 .99 .999 .9999 .99999 …

Looks suspiciously like it's heading towards…

Okay, so define a new term!

Convergence!

A sequence $\{a_n\}_{n\in\mathbb{N}}$ *converges to L if…*

$$\forall\epsilon > 0 \quad \exists B \in \mathbb{N} \quad s.t. \forall n \geq B \quad |a_n - L| < \epsilon$$

Closed formulas – not always!

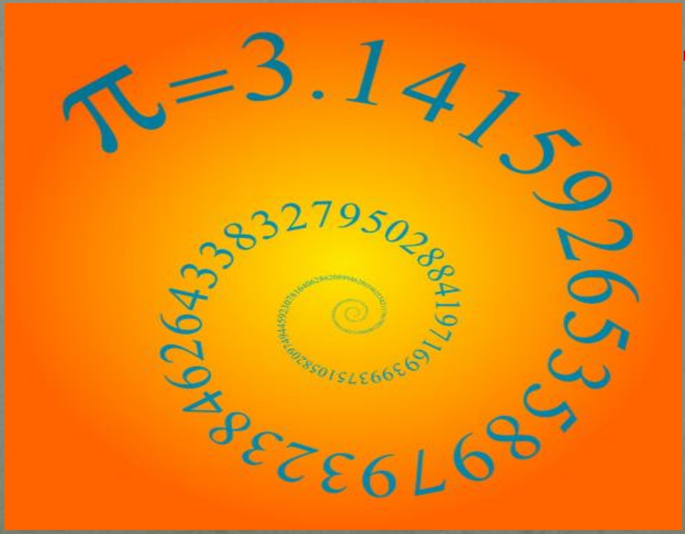
Here's a pretty simple sequence…

$(a_k)_{k\in\mathbb{N}}$ $\quad a_k = 2k - 1$

What about P_n = nth prime number

Back to P_k = first k digits of π

3 3.1 3.14



15 …

What is

$P_{10000000000000}$

$= P_{100\ \text{trillion}}$?

Major classes of sequences

Geometric sequences…

constant ratio between consecutive terms

Arithmetic sequences…

constant difference between consecutive terms

Can we sum them?

…Series!

What about finite sequences? …can't we sum them at least?

Geometric sequences…

starting with 1, with ratio r…

Arithmetic sequences…

starting with 0, with difference d…

And now vary the starting point…

Off to Infinity! Summing series

consider an arithmetic sequence...

still just starting with 0, with difference d…

take a look at the formula!

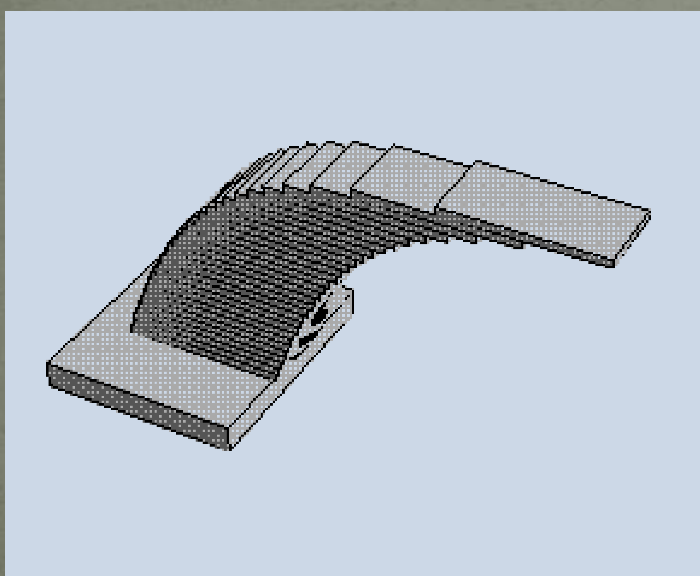
Now consider a geometric sequence...

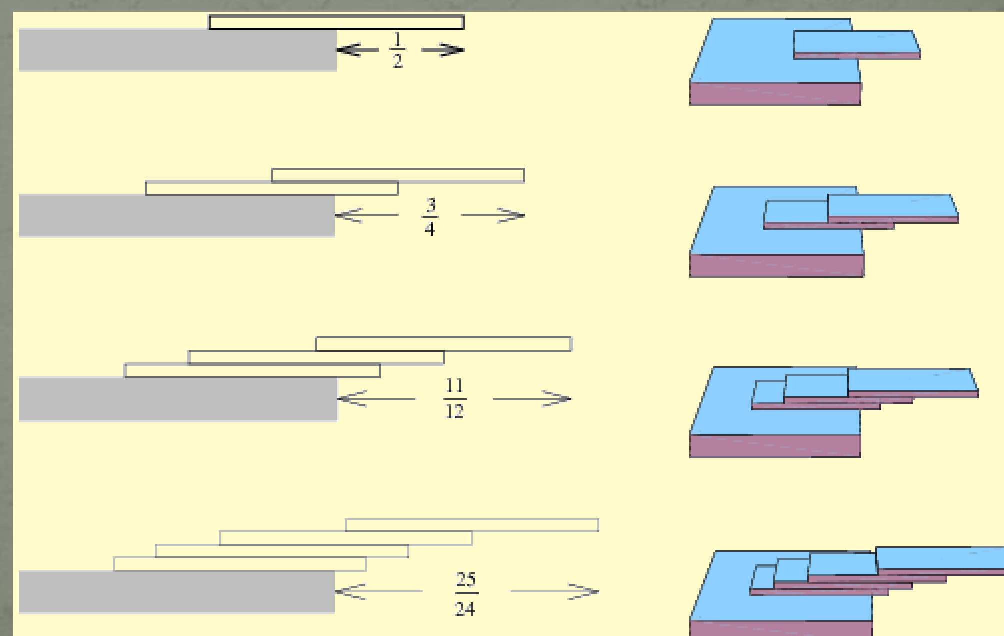
starting with 1, with ratio r…

Off to Infinity! and more!

*...does every **series** sum to a finite number if its **sequence** of terms converges to 0?*

Say hello to the ***HARMONIC SERIES*** !





The tipping point – hard to tell!

But if you tamper with
the Harmonic Series ever so slightly…

What about a "depleted"
HARMONIC SERIES?

Toss out all the terms with
a 9 in the denominator…

S = 1/1 + 1/2 + … + 1/8 + 1/10 + … + 1/18
+ 1/20 + … + 1/88 + 1/100 + 1/101 + …

The Comparison Test et al.

The first several convergence tests are relatively self-explanatory…

If a_n is a sequence such that $\sum_{n=1}^{\infty} a_n$ exists, and $0 < b_n < a_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ exists too.

can do a bit more by considering absolute values of the terms, and also just need the comparison to eventually happen for large enough n…

The Ratio Test

essentially compares a sequence to a geometric series, which converges if the ratio is < 1…

If a_n is a sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is < 1, then $\sum_{n=1}^{\infty} a_n$ exists too *(i.e. the series has a finite sum).*

oops – we need to fix this every so slightly – see the hole in this test? Think negative thoughts (instead of "thinking positive thoughts"!)

And now for something different…

Suppose $a_1 = 1$ $a_2 = 1$

and $a_3 = a_1 + a_2$

…and then $a_n = a_{n-1} + a_{n-2}$

RECURSIVE FORMULAS!

The Fibonacci sequence is an example of a linear recursive sequence…

Is it arithmetic, or geometric?

The Fibonacci Sequence!

1 1 2 3 5 8 13 21 34 55 89…

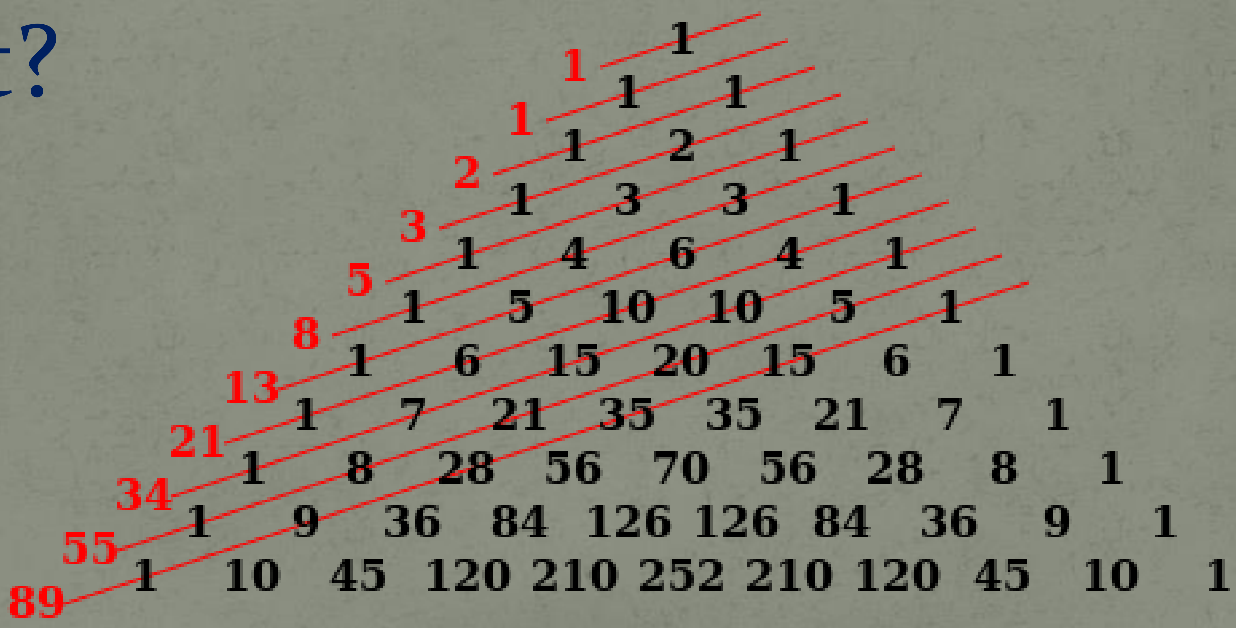
Given that each term in the Fibonacci sequence is defined using a linear recursive sequence… *then to find the n^{th} term don't you have to compute all the ones before it?*

So does that mean there's no closed formula?

Fibonacci Sequence – no formula?

1 1 2 3 5 8 13 21 34 55 89…

But there are some obvious patterns, right?



The infamous Fibonacci-Pascal connection!

Time for an amazing number!

Can you find a number X with a similarly recursive pattern…?

Find X so that $X^n = X^{n-2} + X^{n-1}$!

If you find such an X then any sequence of the form $a_k = X^k$ will follow the recursive constraint…

In fact any sequence of the form $a_k = C X^k$ (where C is any constant) will work just as well!

Time for an amazing number!

So can you find a number X with a similarly recursive pattern…?

Find *X* so that $X^n = X^{n-2} + X^{n-1}$!

Aha! $\varphi = \frac{1+\sqrt{5}}{2}$

and phi's (somewhat negative!) sidekick…

$$\psi = \frac{1-\sqrt{5}}{2}$$

Now put them to work!

Big idea, yes the following two sequences obey the recursive relation:

$$(\varphi^n)_{n\in\mathbb{N}} \qquad (\psi^n)_{n\in\mathbb{N}}$$

But now think about what happens to a sequence defined by their sum…

$$s_n = \varphi^n + \psi^n$$

Now put them to work!

In fact any sequence of the form

$$s_n = A\,\varphi^n + B\,\psi^n$$

with constants A and B will fulfill the recursive constraint…!

Could we find A and B so that the sequence $s_n = A\,\varphi^n + B\,\psi^n$ *begins with*

1 1 2 3 5 …