Math 305

Advanced Algebra and Trigonometry!

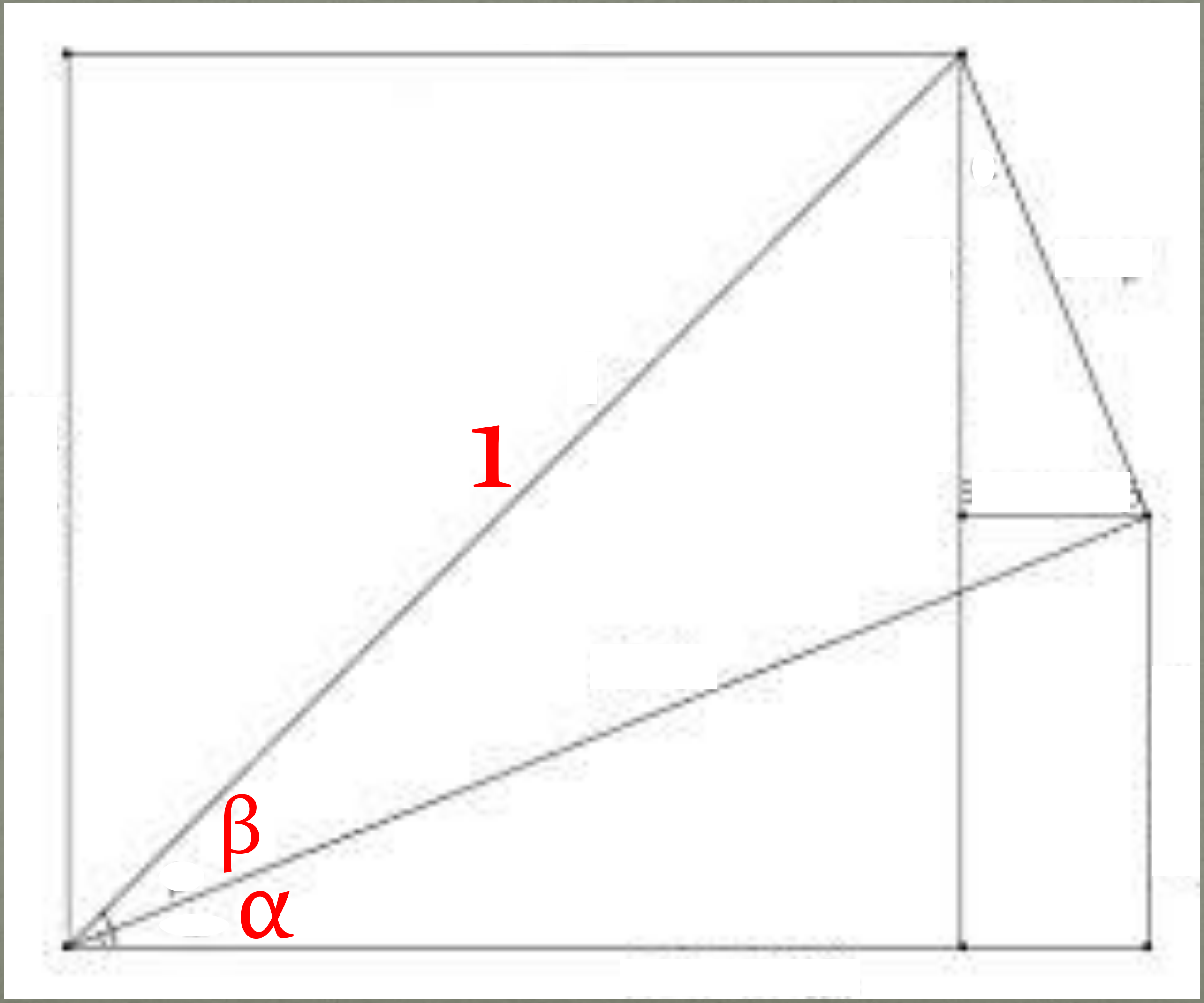
Goodbye Circles!

time to get hyperbolic?!

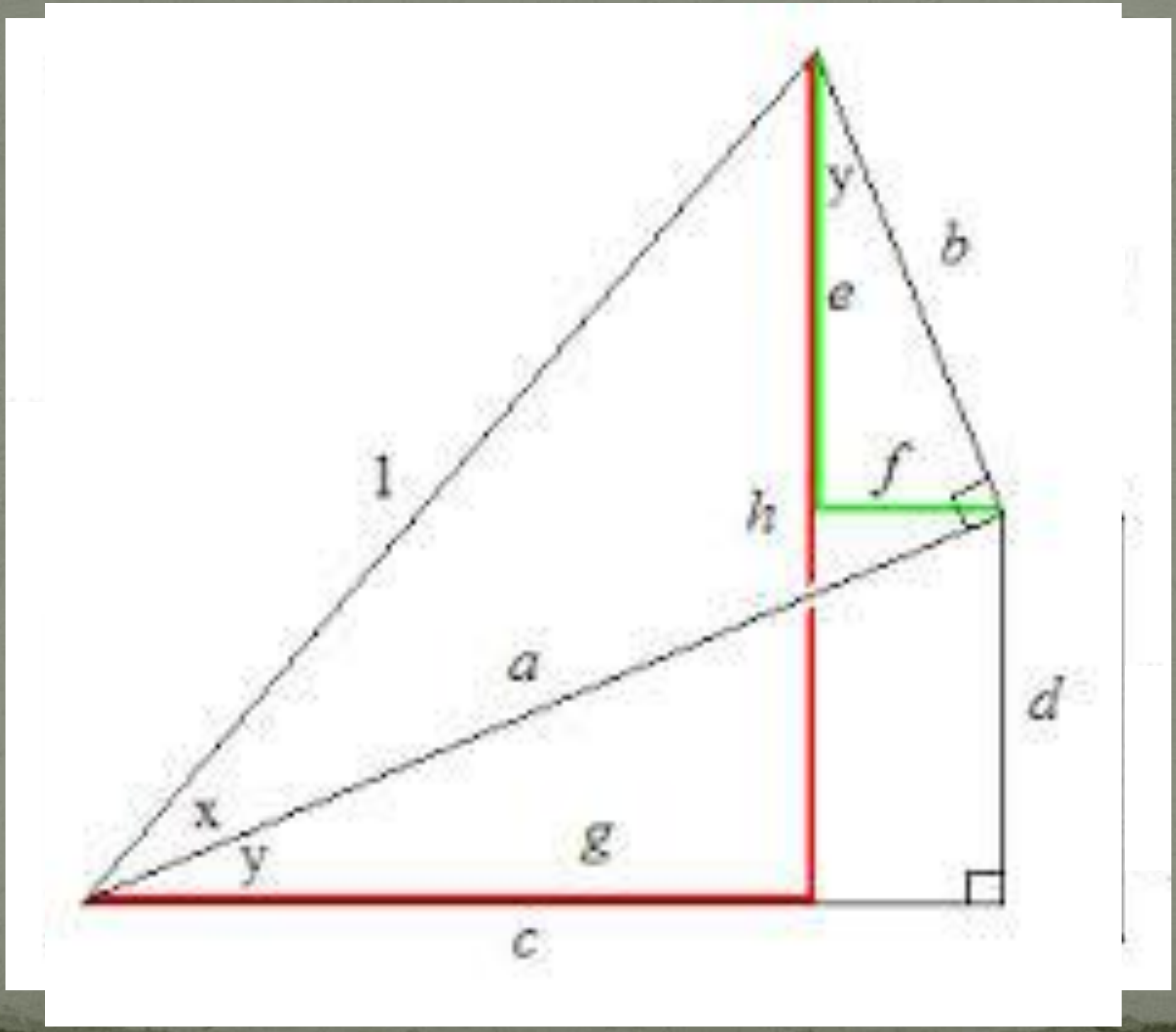
Twelfth Class – Tuesday, July 15th

- POTD
  - *Starting and ending the class with…!*
- Finishing off circular trigonometry
  - *tying together various approaches*
- …and finally that diagonal puzzle!
  - *pretty straightforward at this point!*
- And our last new topic – is there time for one last new foray?
  - *a few new shapes…*
  - *hyperbolic trigonometry!*
  - *…along with some new formulas!*

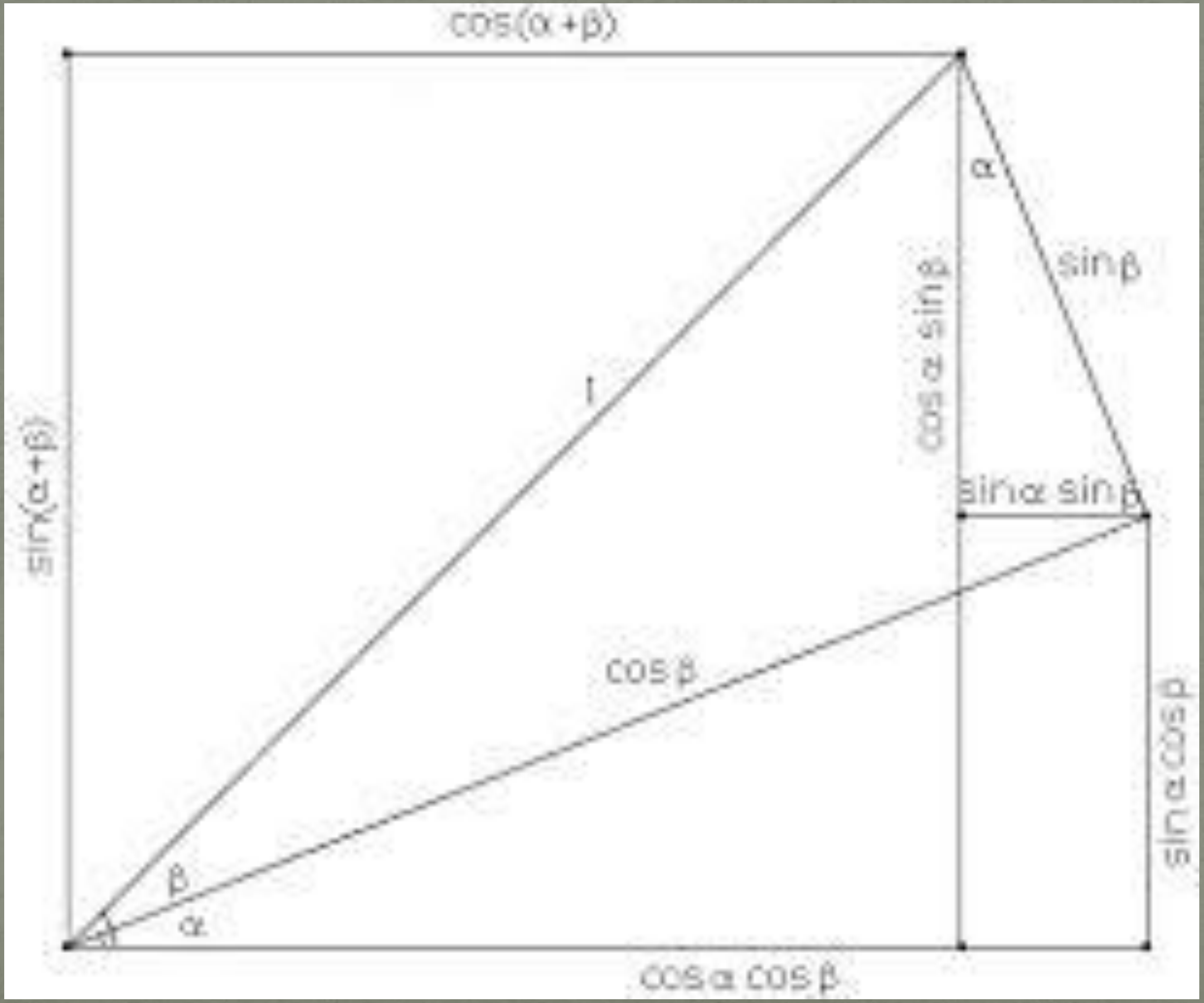
POTD *The proof is in the picture!*



POTD Follow the alphabet!



POTD You can see it all from here!



POTD – and now the famous…!

Now if $e^{ix} = \cos(x) + i \sin(x)$

then what is $e^{i\pi}$*?*

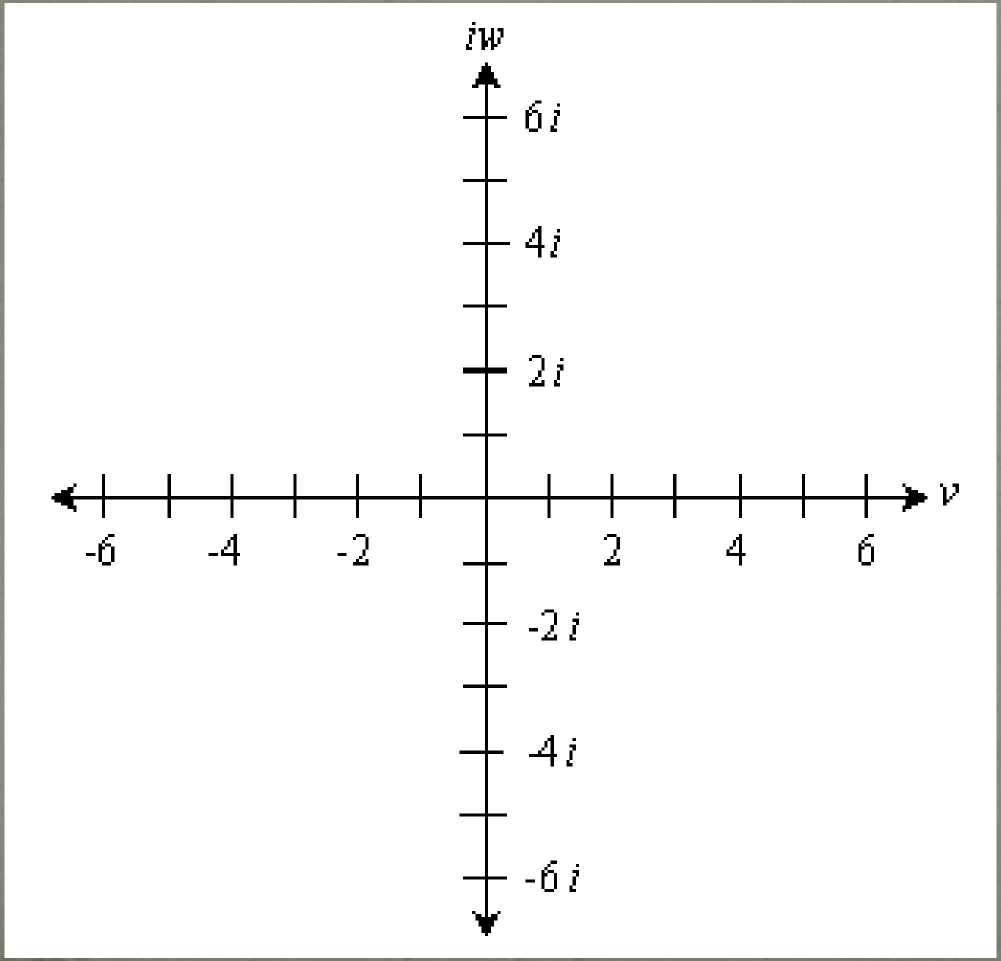
and while we're here…

okay, now write i *as* $e^{i\theta}$

…so what is i^i?

Back to our Complex Number Line!

The "complete" number line!

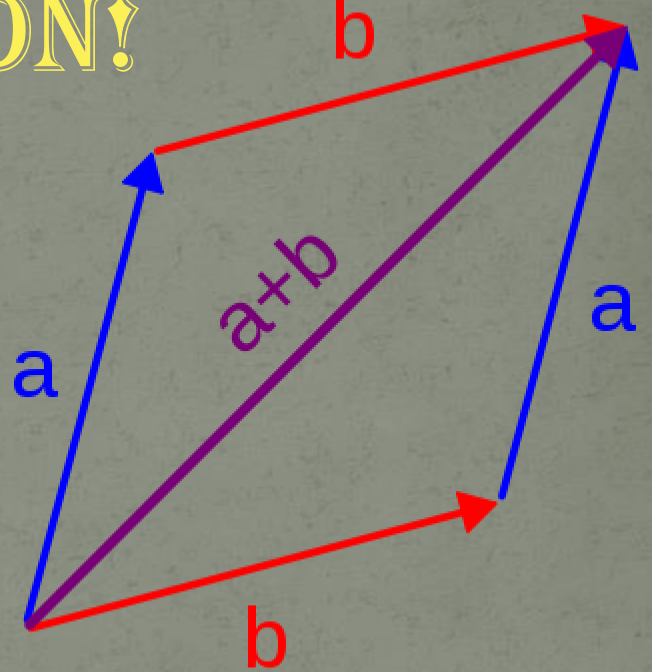


where all the action happens!

First, the easy stuff!

Now that you've been doing the hard stuff with complex multiplication, sit back and take a look at…

Complex Addition!



Algebraically, what is 1 + 3i plus 3 + i?

*Then, visually, we can think of addition using vectors… **the parallelogram rule!***

And picking back up multiplication…

so multiplying by **1** is equivalent to

the matrix transformation $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and **i** is equivalent to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

and multiplying by a + bi is equivalent to the matrix transformation…

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Multiplying complex numbers …as matrices!

We then tried $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

and we got $\begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix}$

which is equivalent to…

$(a + bi)\,(c + di)$

$= (ac - bd) + i(ad + bc)$

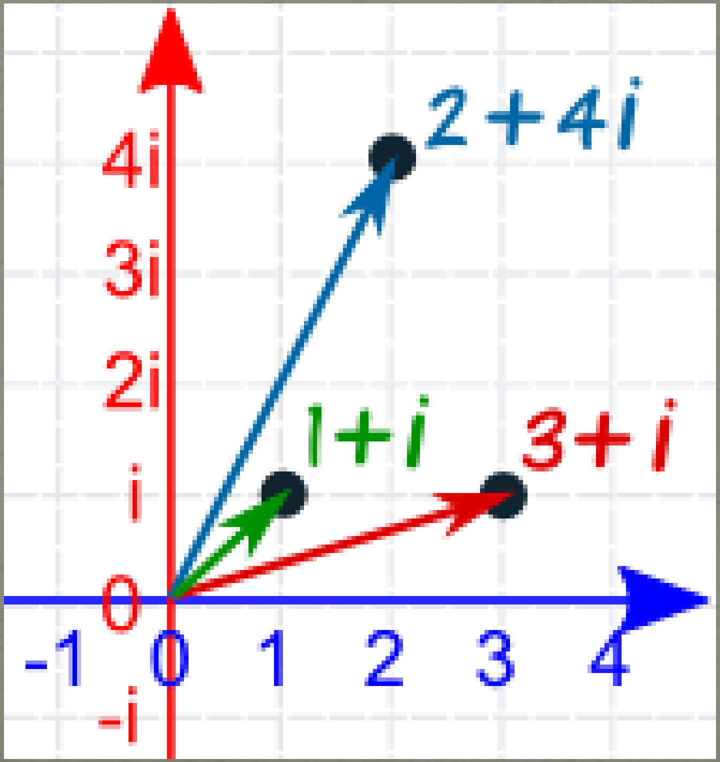
Let's get visual - geometrically...

Going back to our complex number line... *Argand!*

what does the product

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

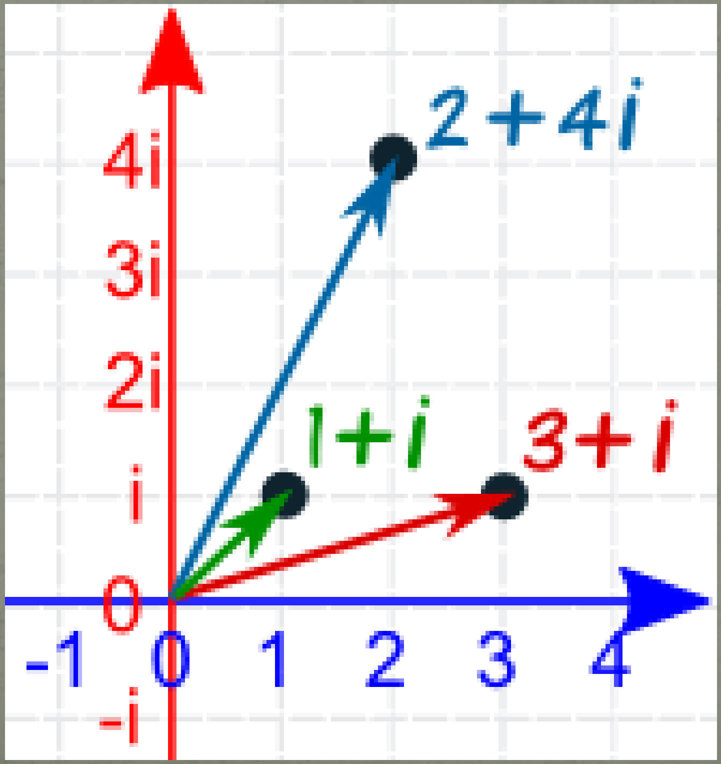
correspond to?



Let's get visual - geometrically…

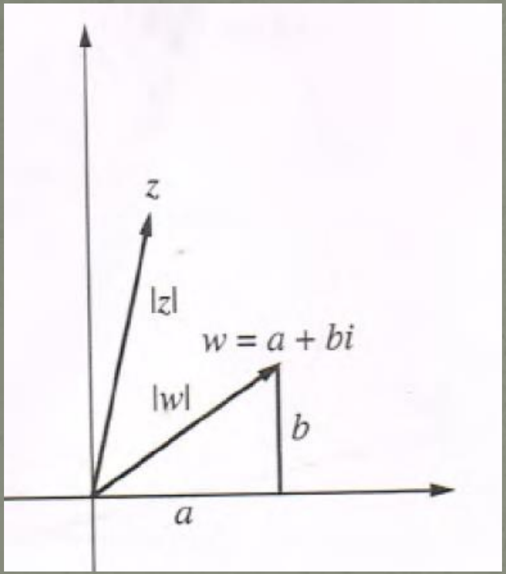
Going back to our complex number line… *Argand!*

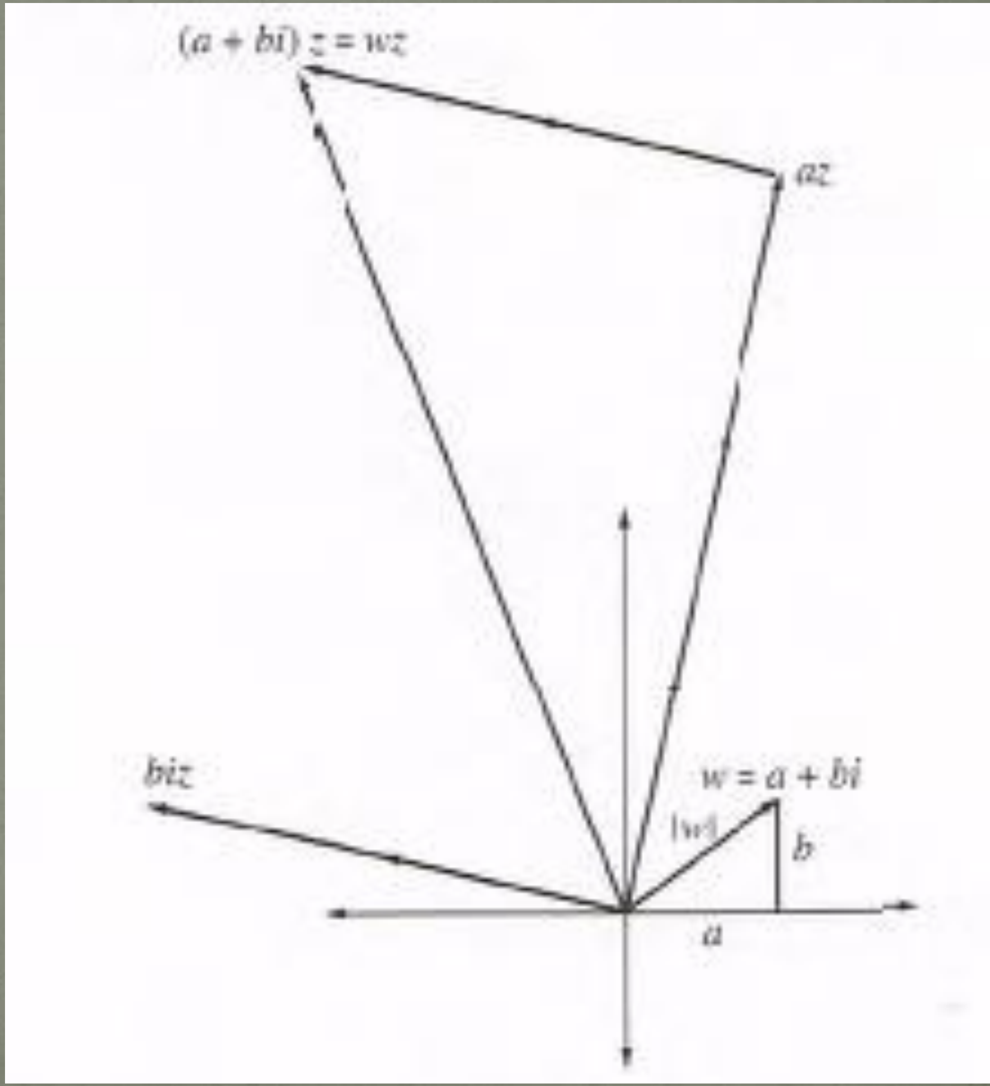
If $z = a+bi$ *and* $w = c + di$ *then how does* wz *compare to* w *and* z*?*



Do these diagrams help?

From *Mathematical Connections…*





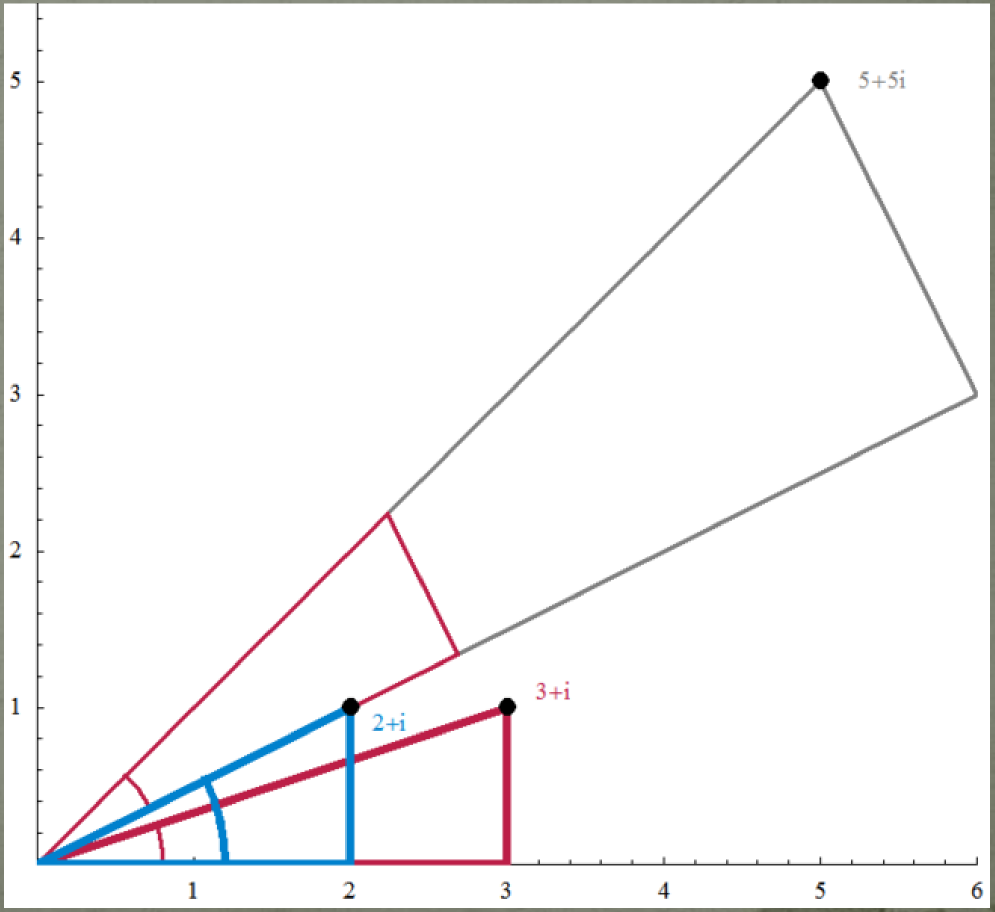
similar triangles?!

Back to matrix transformations…

So is it possible to recognize multiplication by 2 + i,

i.e. the transformation by $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

as a *rotation* followed by a *scaling*?

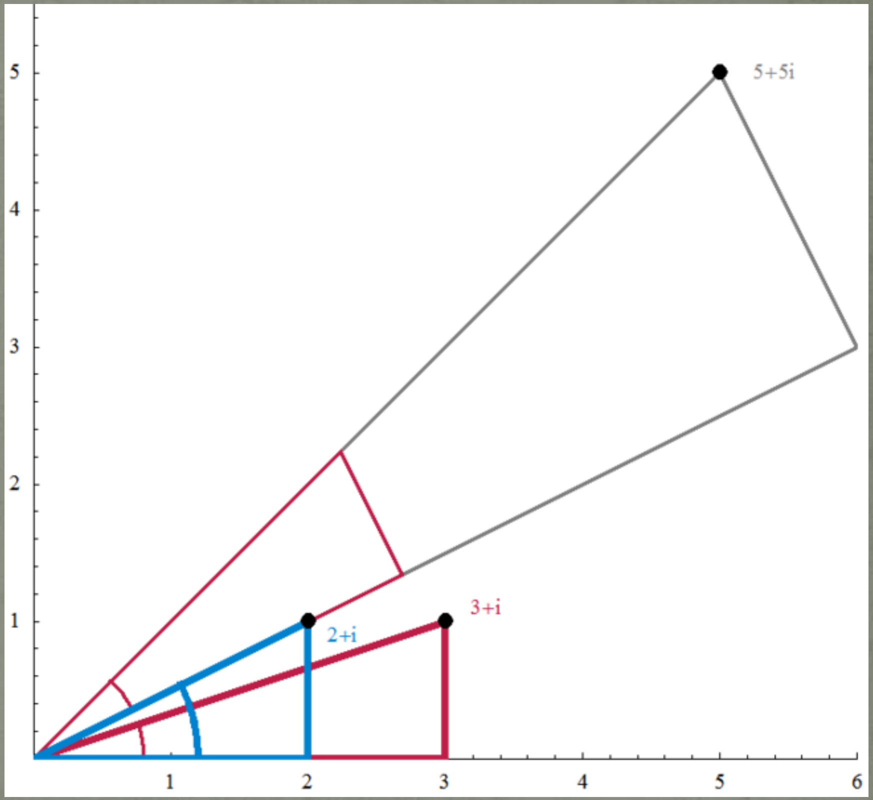


Deconstructing a transformation…

What if we "take out" the scaling factor…

how much does the transformation

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$ *scale by?*



Now rewrite

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$ as the product of two matrices

Deconstructing a transformation…

In general any complex number multiplication transformation

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$ can be rewritten as

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

where s *equals?*

…and what's special about c and d?

Rotation transformations…

What does the matrix of a rotation transformation look like?

First, where does a rotation of an angle of θ send the basis vectors?

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

Back to -1 …

thinking back to multiplication by -1…

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} -a \\ -b \end{bmatrix} = a\begin{bmatrix} -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

So a rotation by θ...

now we need cos(θ) and sin(θ) !

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$? = a\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} + b\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

So back to multiplying by complex numbers as matrix transformations…

Suppose z = a + bi, so multiplication by z is equivalent to a transformation by

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

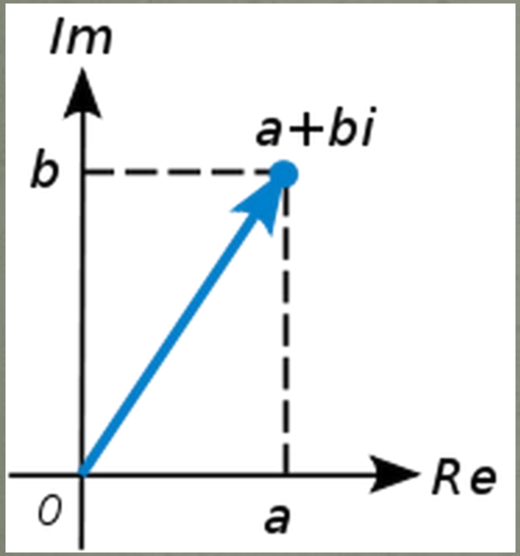
which can be rewritten as

$$\begin{bmatrix} |z| & 0 \\ 0 & |z| \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Aha! Arguments and Moduli!

Following up on scaling and rotation transformations, let's connect back to complex numbers…

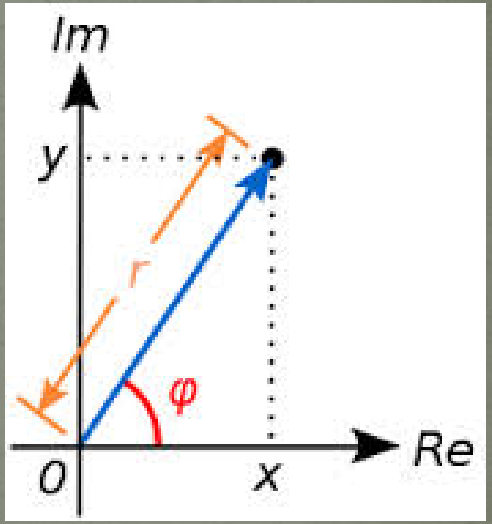
Instead of viewing complex numbers as pairs, a combination of real and imaginary numbers



let's use "polar coordinates"!

arg(z) = ?

|z| = ?



What are "length one" complex numbers?

Remember that amazing formula…?!

If z is a complex number,

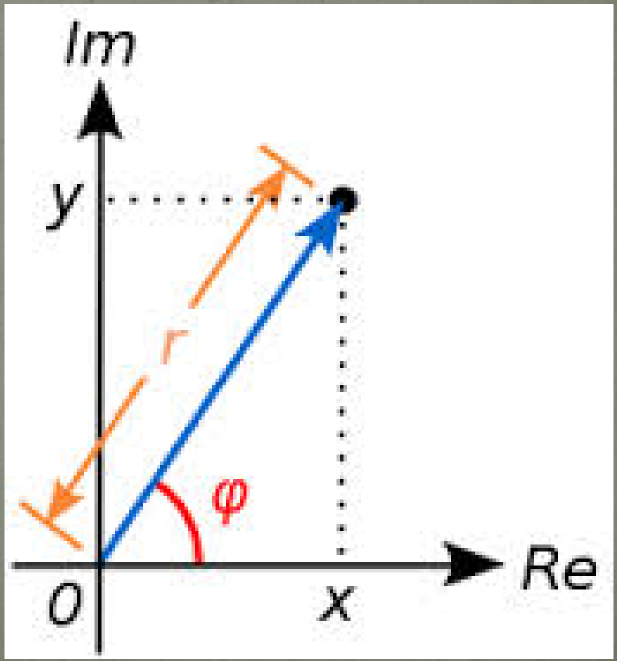
then what is $z / |z|$?

…a "length one" complex number

Recall that

$e^{i\theta} = \cos\theta + i \sin\theta$

So now rewrite z!



Now let's cycle back!!

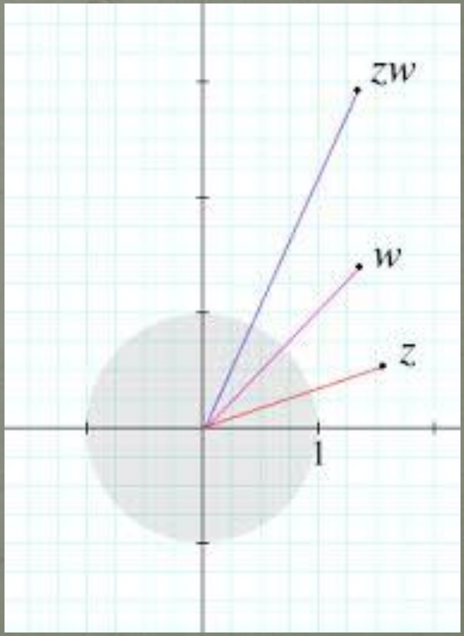
So if z is a complex number, then z can be written as $r\,e^{i\theta} = |z|\,e^{i\,arg(z)}$

where r is a real number

…now what happens if you multiply two complex numbers w and z?

Suppose $z = r\,e^{i\theta}$
and $w = s\,e^{i\varphi}$

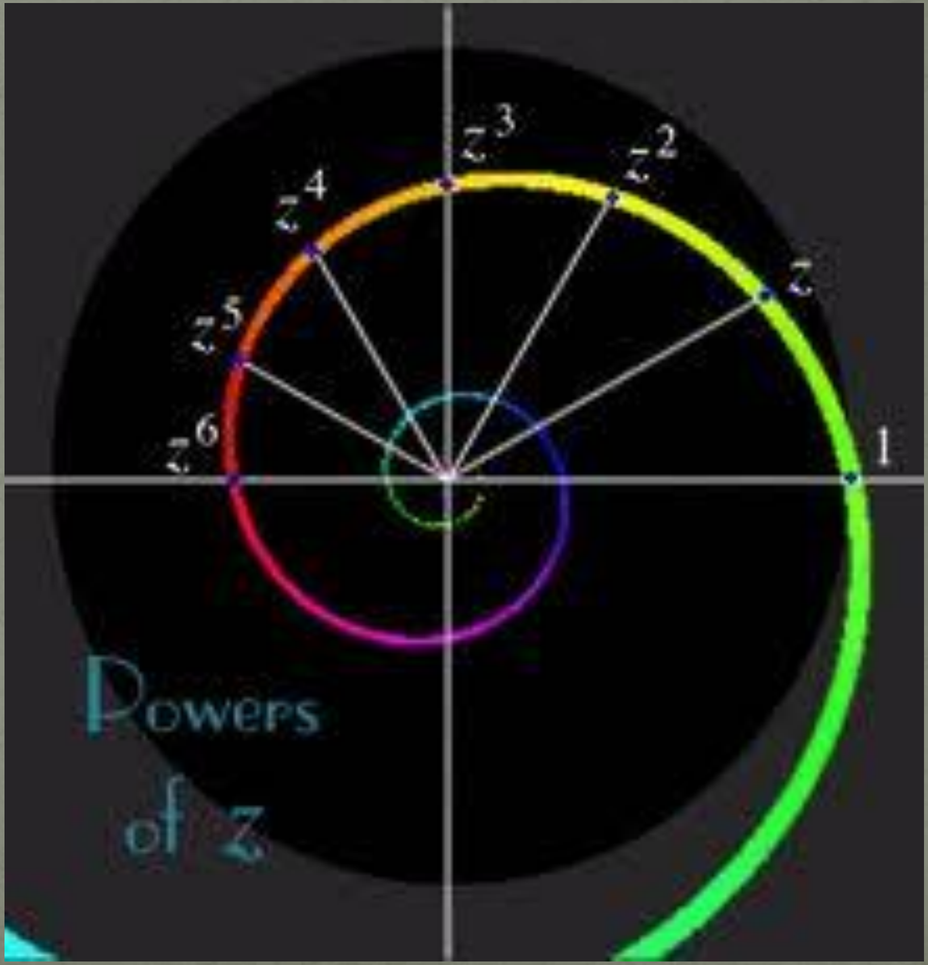
Well, what is
$r\,e^{i\theta}$ *times* $s\,e^{i\varphi}$?



Powers in the Complex Plane!

Suppose z has modulus less than 1…

what do the powers of z look like?



And one last matrix connection!!

Suppose z and w have absolute value 1

So what are z and w equal to?

As matrix transformations, then, multiplying by z and w is equivalent to…

and multiplying by z, and then by w, is equivalent to what kind of transformation?

Now multiply together the matrices representing multiplication by w and z…

Connections!

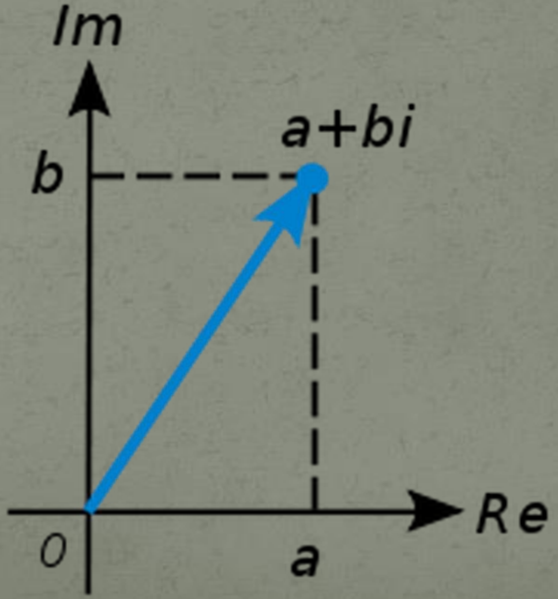
Algebra (complex numbers)...

a + bi

Matrices…

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Geometry...



and Trigonometry!...