Math E305
Advanced Algebra and Trigonometry!

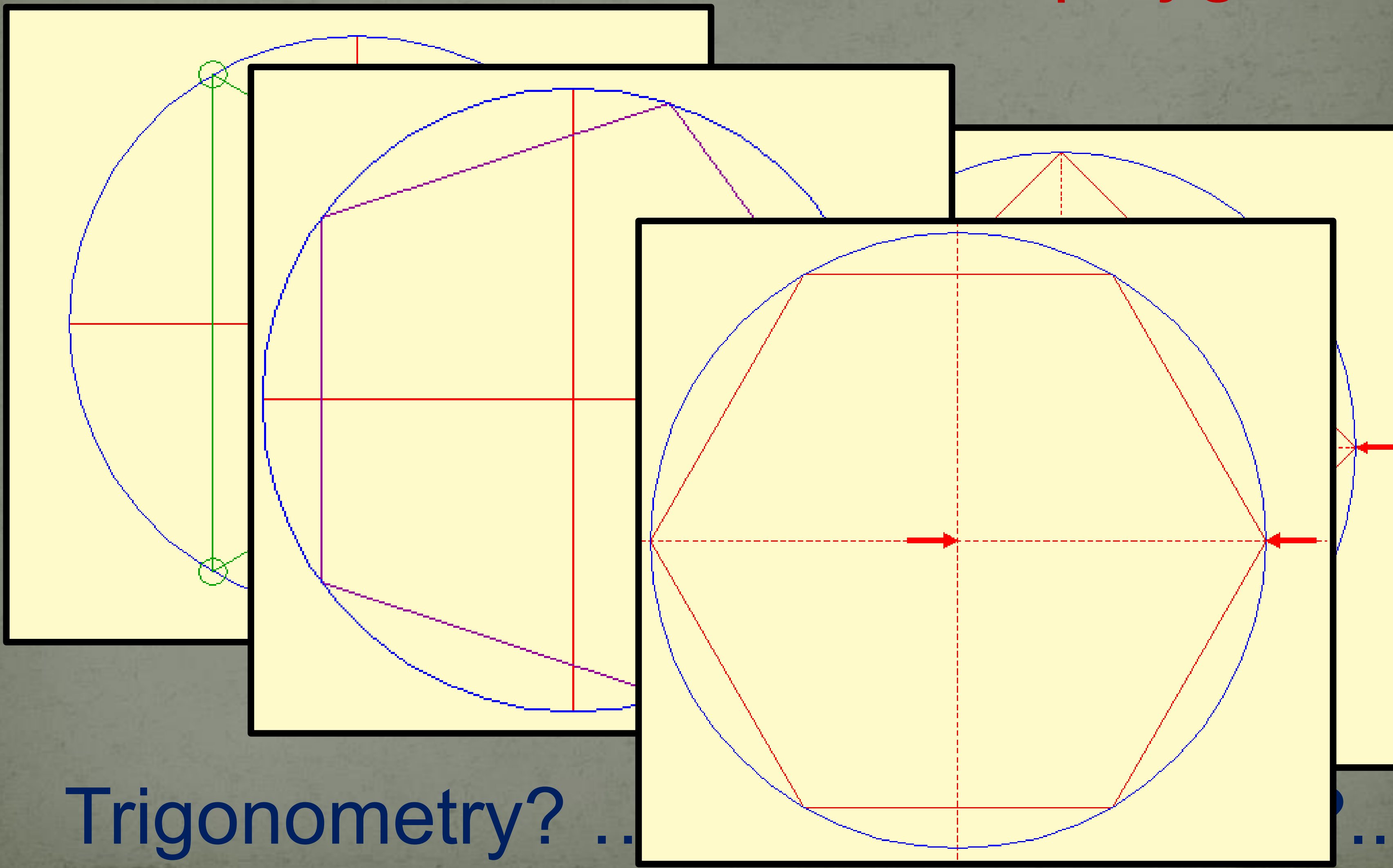
Time to get Complex
…two dimensional number lines!

Tenth Class – Friday, July 11th

- POTD
  - *First… the hard way!*
- Back to the Future!
  - *well, post Cardano, that is!*
- But what's this strange new number?
  - *time to tour the complex world*
  - *…but from a different perspective!*
- Matrices and Trigonometry?!
  - *connecting a number of ideas!*
  - *back to the POTD?!*

POTD – Time for Circles!

Take a look at these inscribed polygons…

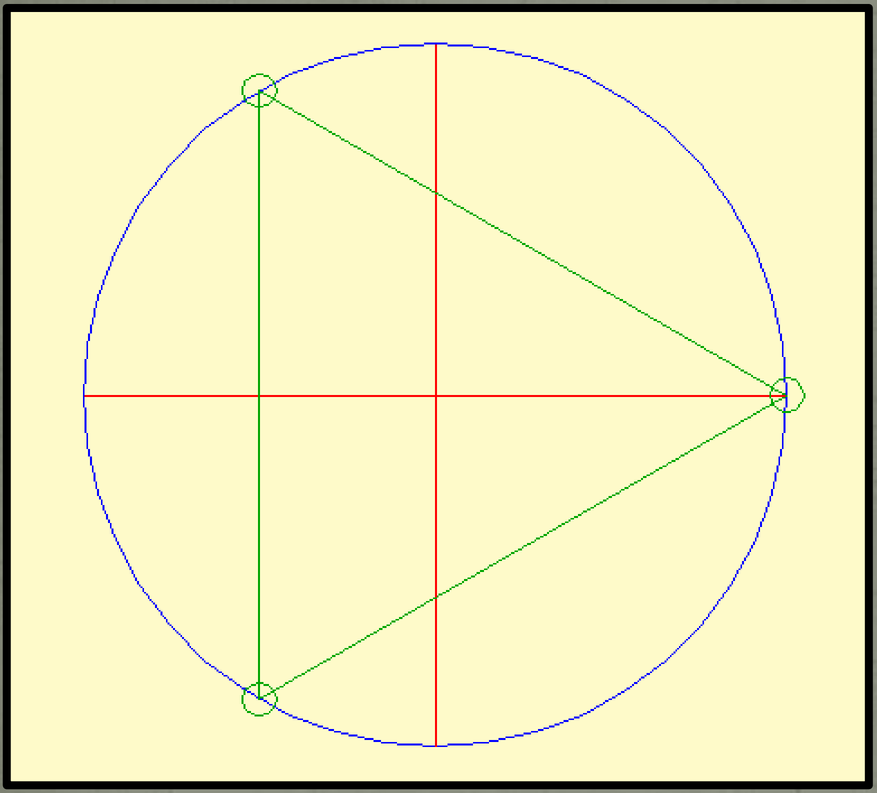


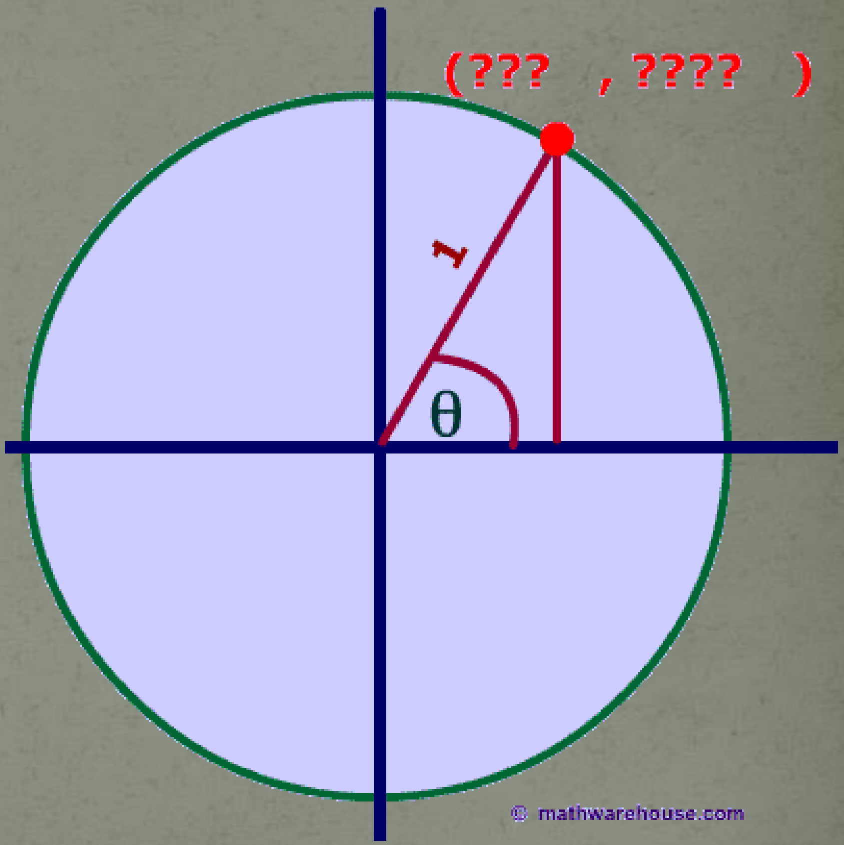
Trigonometry? ..

?...

POTD – *The Unit Circle*!

In fact, circles are the way to go for trig…

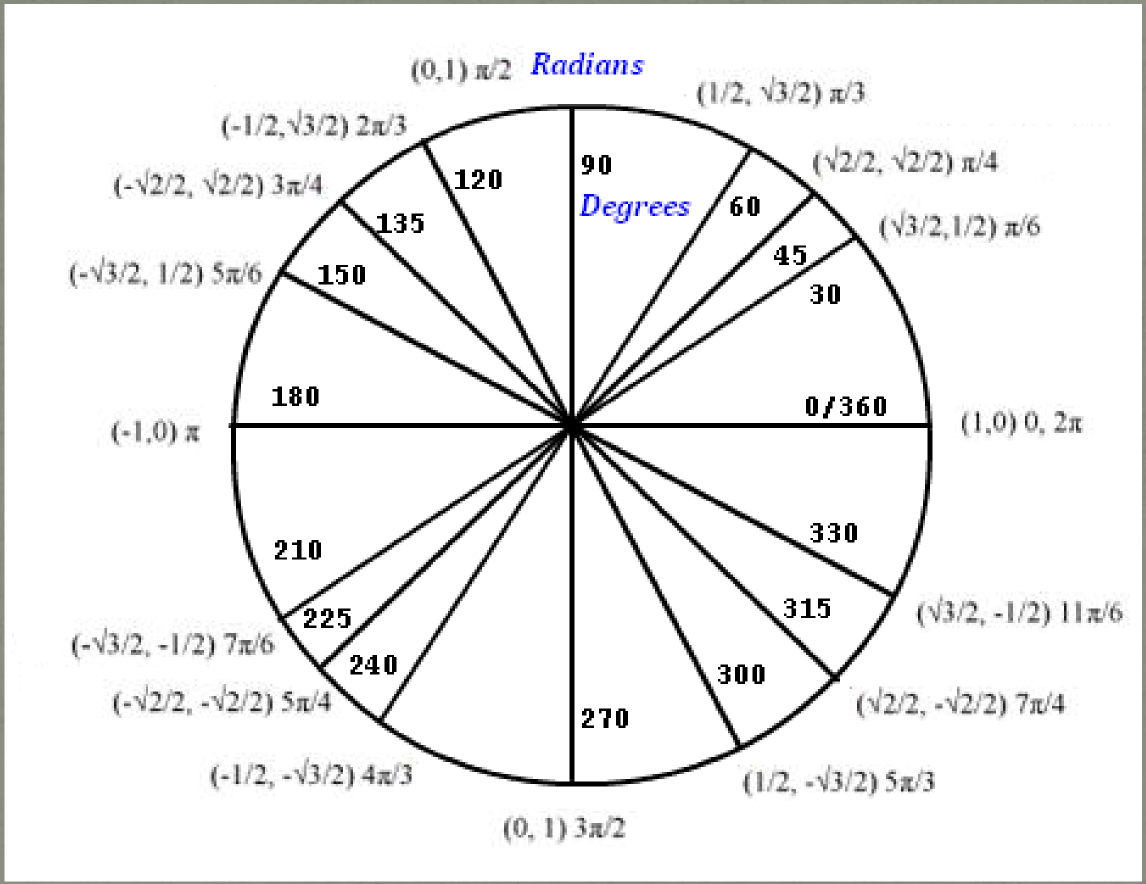




Trigonometry? …or Circle-ometry?!...

The Unit Circle in all its glory!

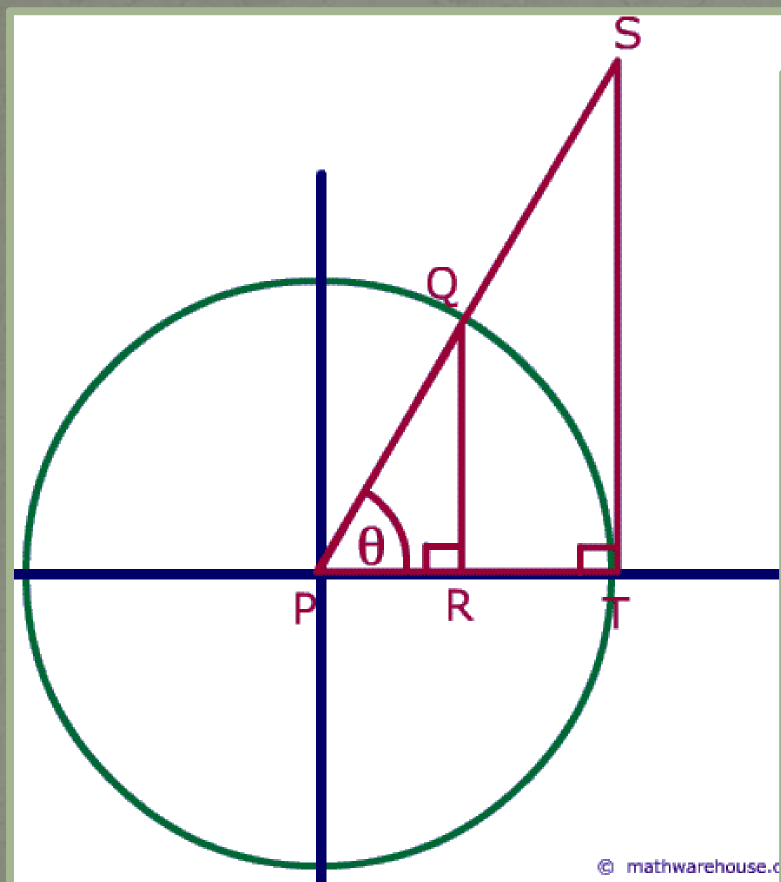
There are a number of “easy” values…

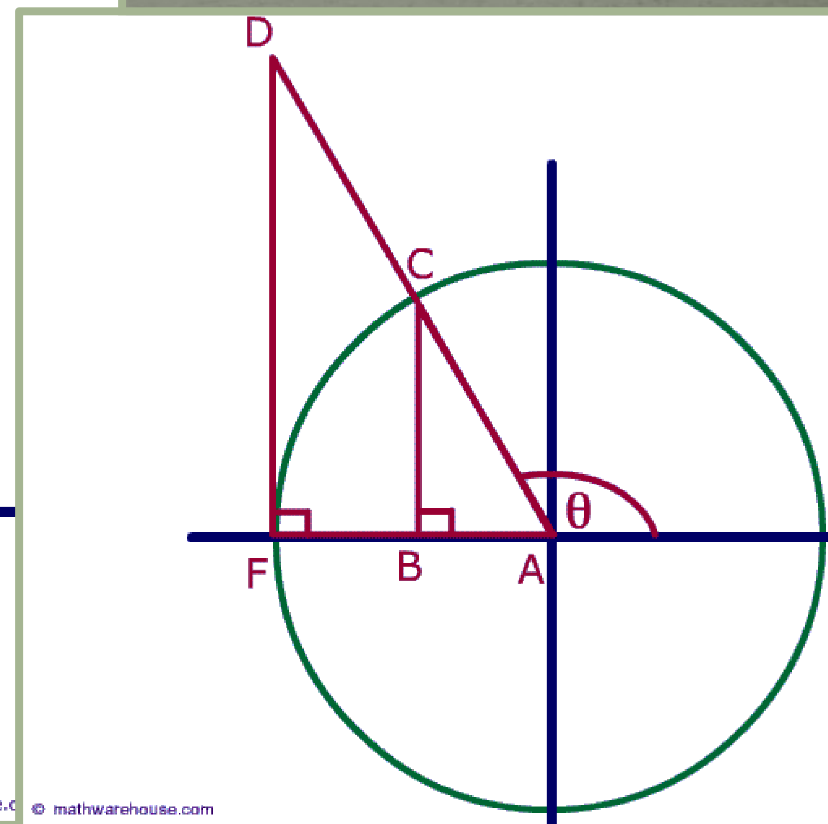


Trigonometry? …or Circle-ometry?!...

What about the other functions?

Can find others using the circle as well…

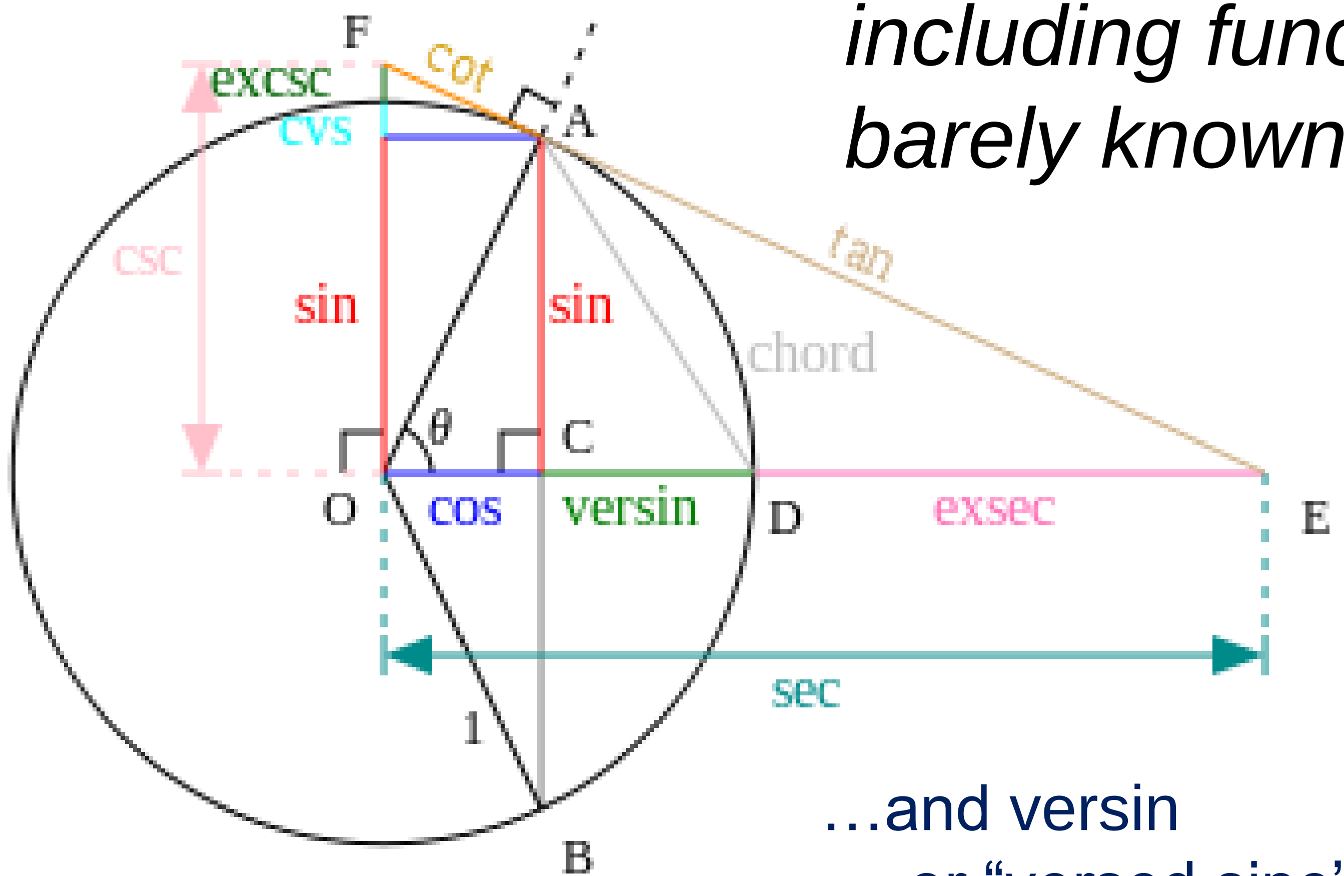




Find Tan X!...

…need to be aware
of signs too (not *sines*!)

And even more?!



including functions barely known today!

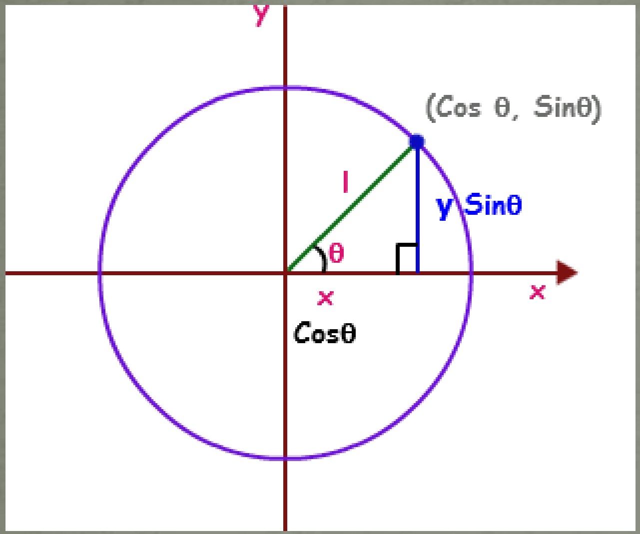
…and versin
or "versed sine"
which equals 1 – cos
in (much!) older textbooks

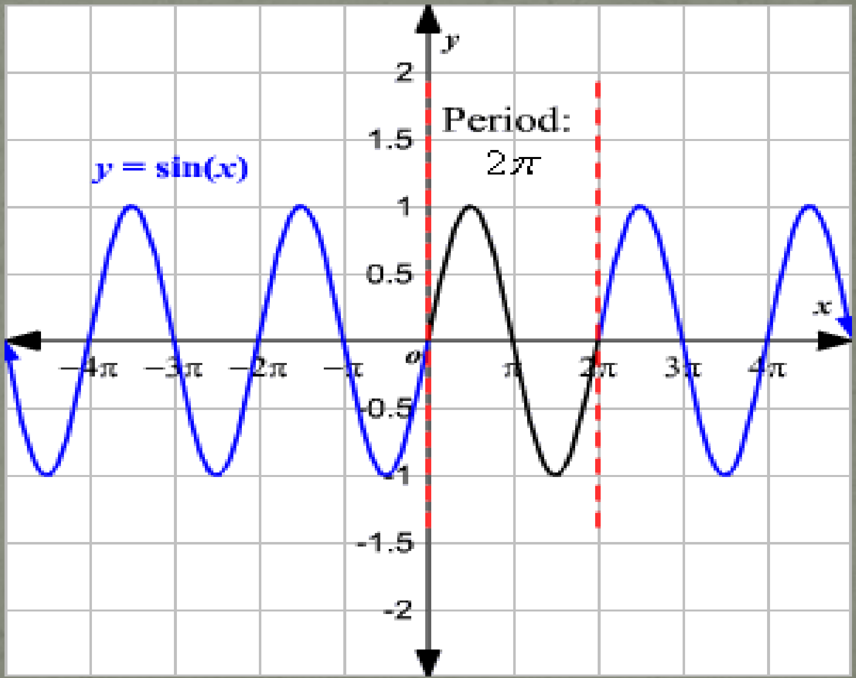
exsec = secant - 1

once important in surveying…!

Circular motion – Periodic functions!

Introducing trig functions in this way connects neatly with periodic functions

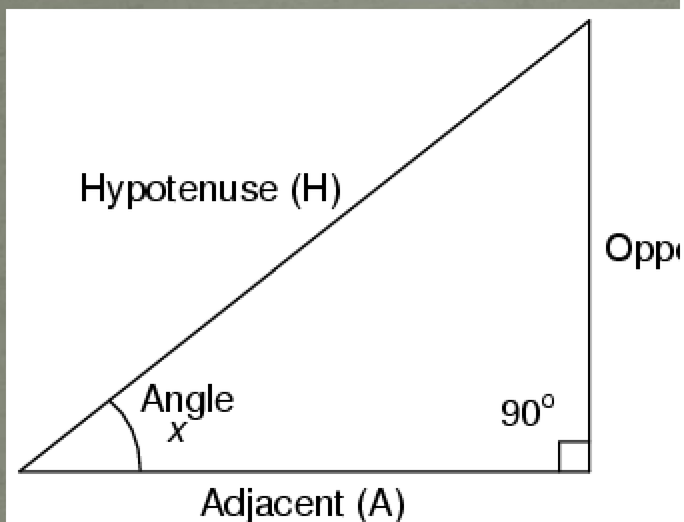


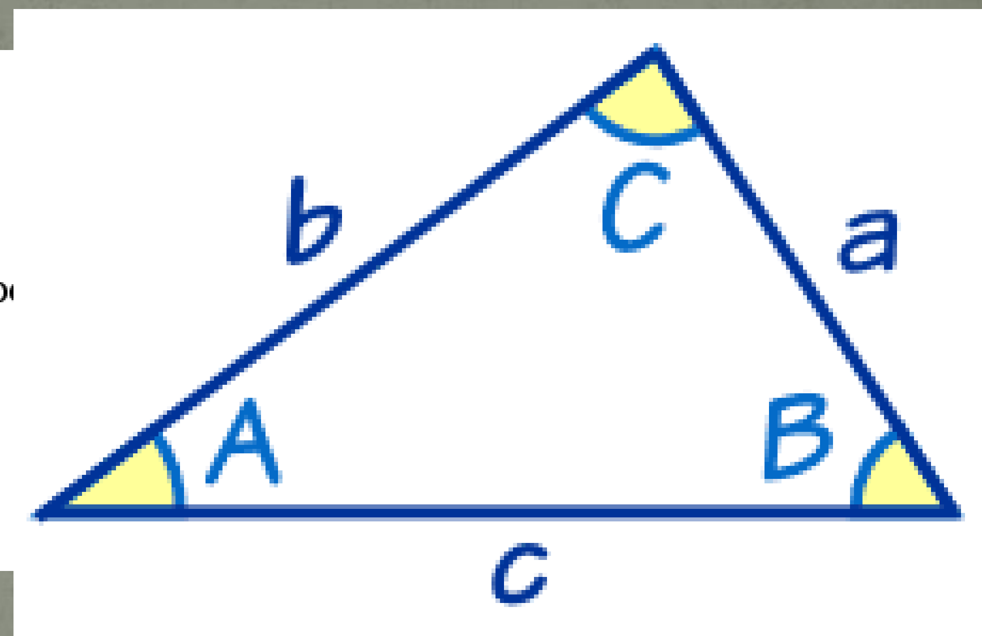


Circles become waves…

Back to Triangles… right?!

Of course triangle measurement is at the heart of things…

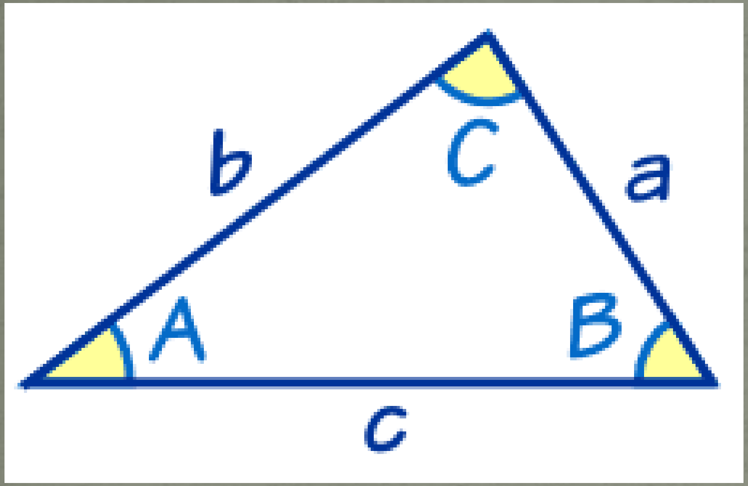




and what if it's not "right"?…

The Laws…!

Law of Sines…



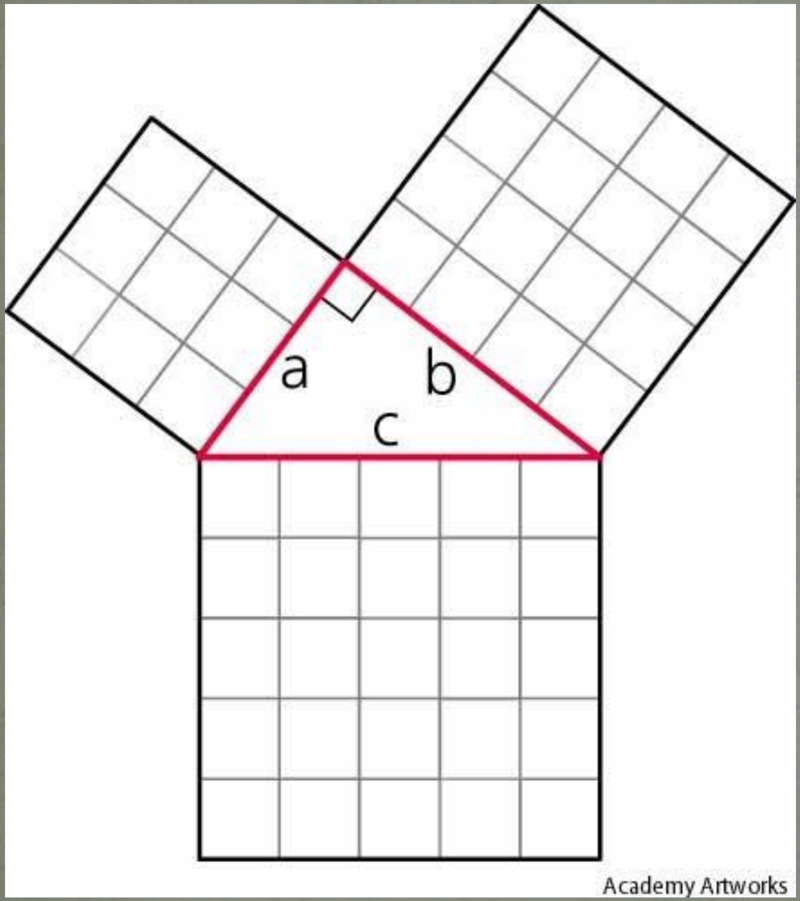
a/(Sin A)
= b/(Sin B)
= c/(Sin C)

Law of Cosines…

generalizes the Pythagorean Theorem!

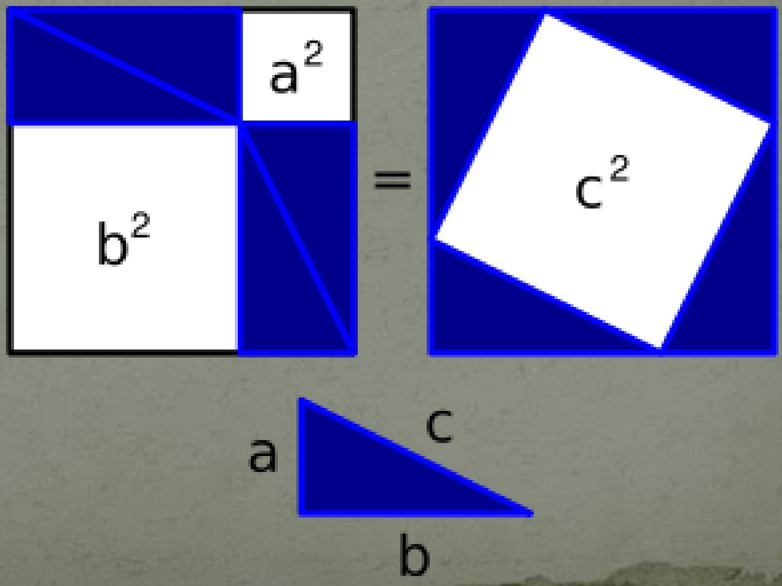
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2r$$

Law of Cosines

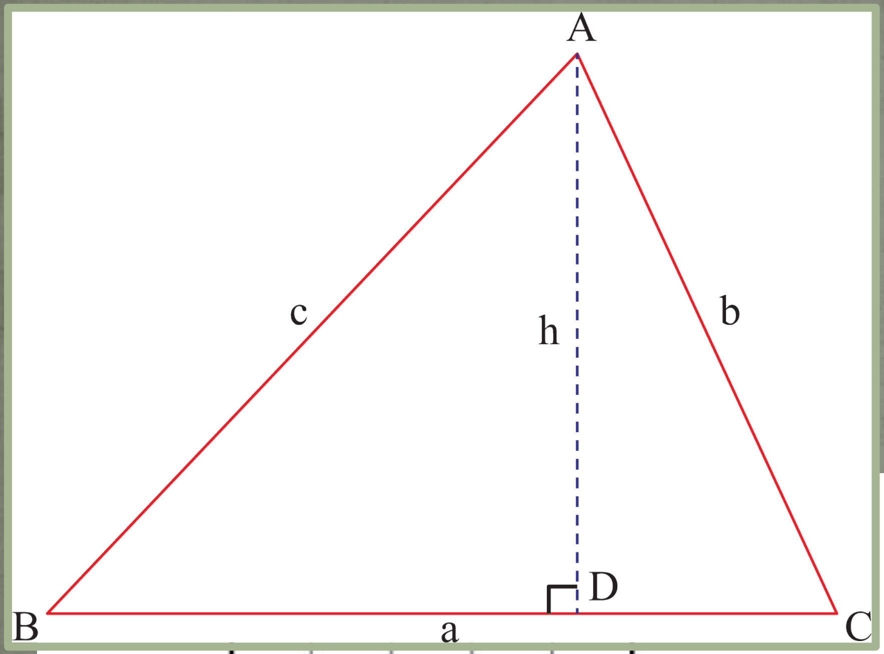


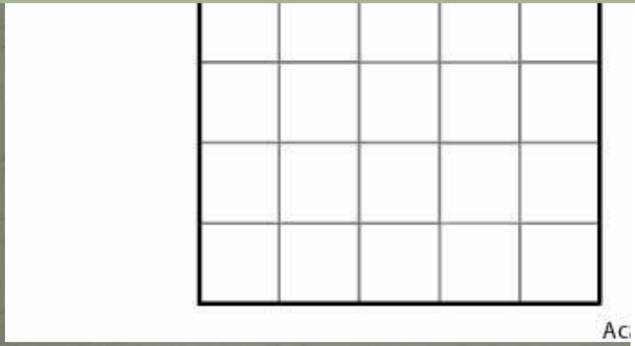
Law of Cosines…

generalizes the Pythagorean Theorem!

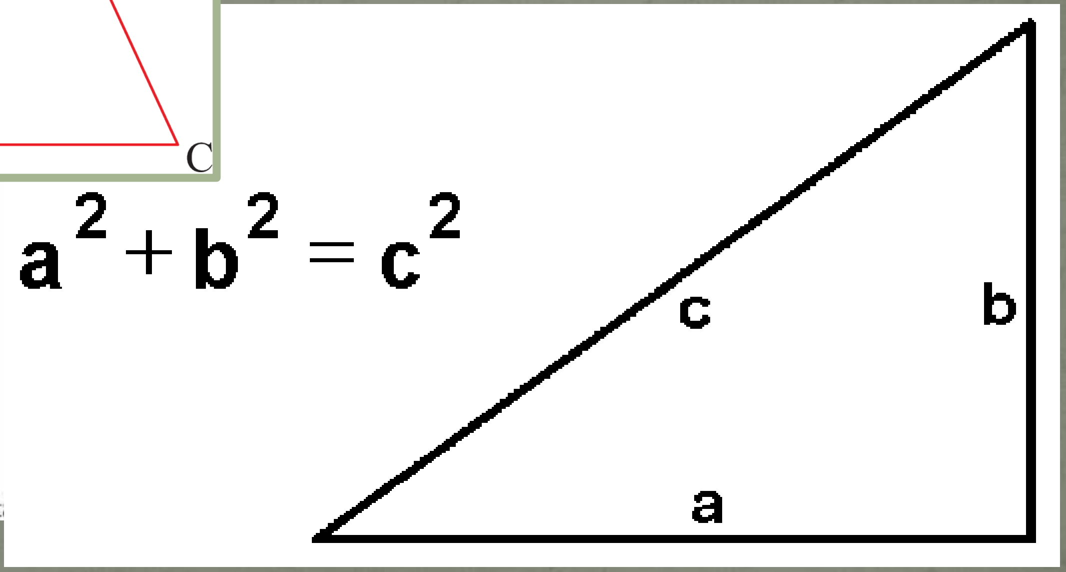


Law of Cosines



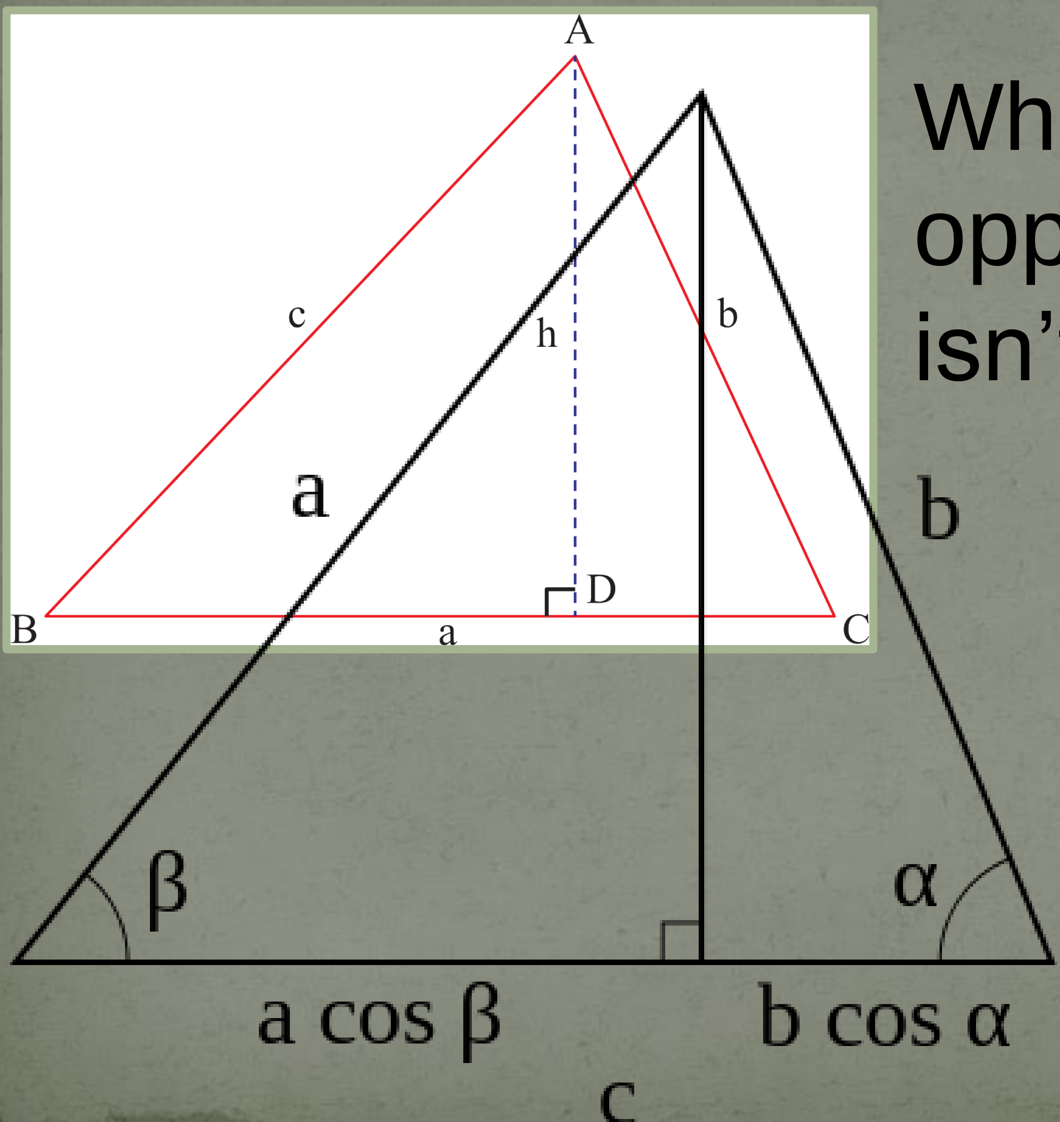


What if the angle opposite side c isn't a right angle?…

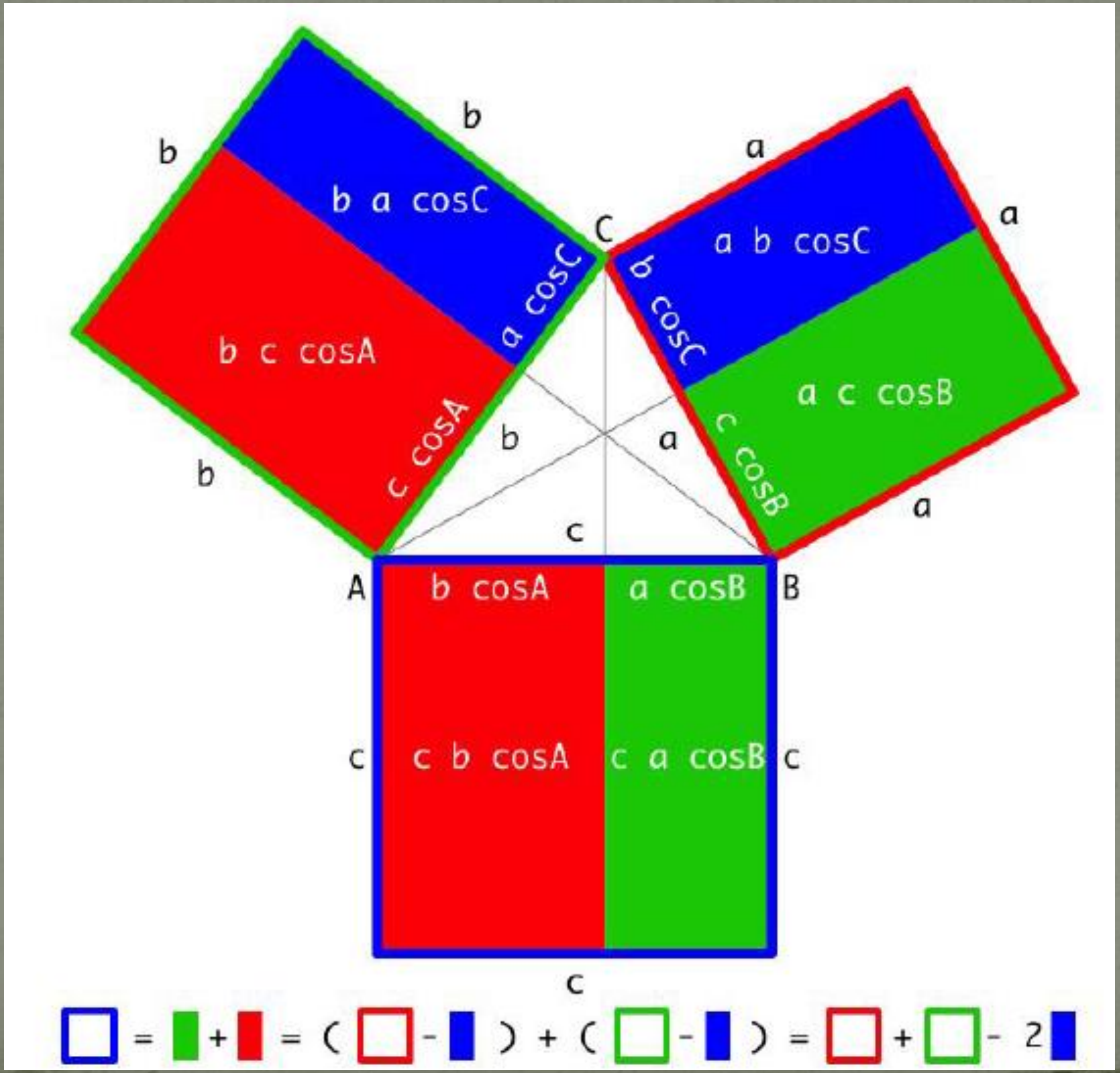


Law of Cosines

What if the angle opposite side c isn't a right angle?…

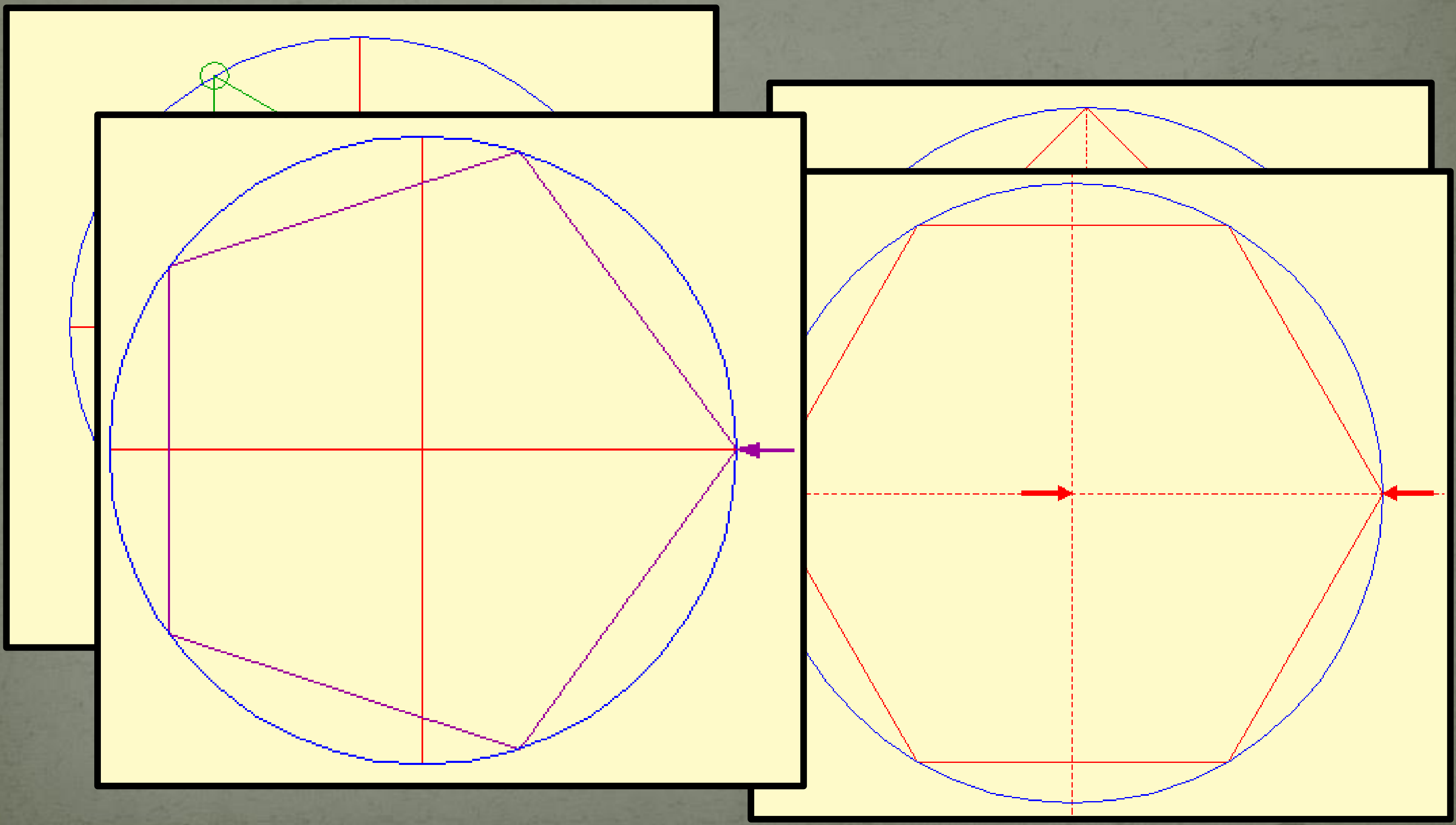


Law of Cosines – picture proof!



Now armed… back to the POTD!

Diagonal products!



And now, back at it…!

Using Cardano's formula…

Rafael Bombelli

(1526 – 1572)

solved

$$z^3 = 15z + 4$$

and got…

$$z = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

But what does it mean? …!

$$z = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Bombelli noted that if one were simply to calculate formally…

what is $2 + \sqrt{-1}$ *cubed?*

and likewise what is $2 - \sqrt{-1}$ *cubed?*

Time for some new numbers!!

And now on to $\mathbb{C}$!

we could just work with $\sqrt{-1}$ from a symbolic perspective…

actually we should really work with u, an unknown value that has the property that $u^2 = -1$…

from a purely formal algebraic perspective, then what is (1+ u) plus (2 – 3u)?

from a purely formal algebraic perspective, then what is (1+ u) times (2 – 3u)?

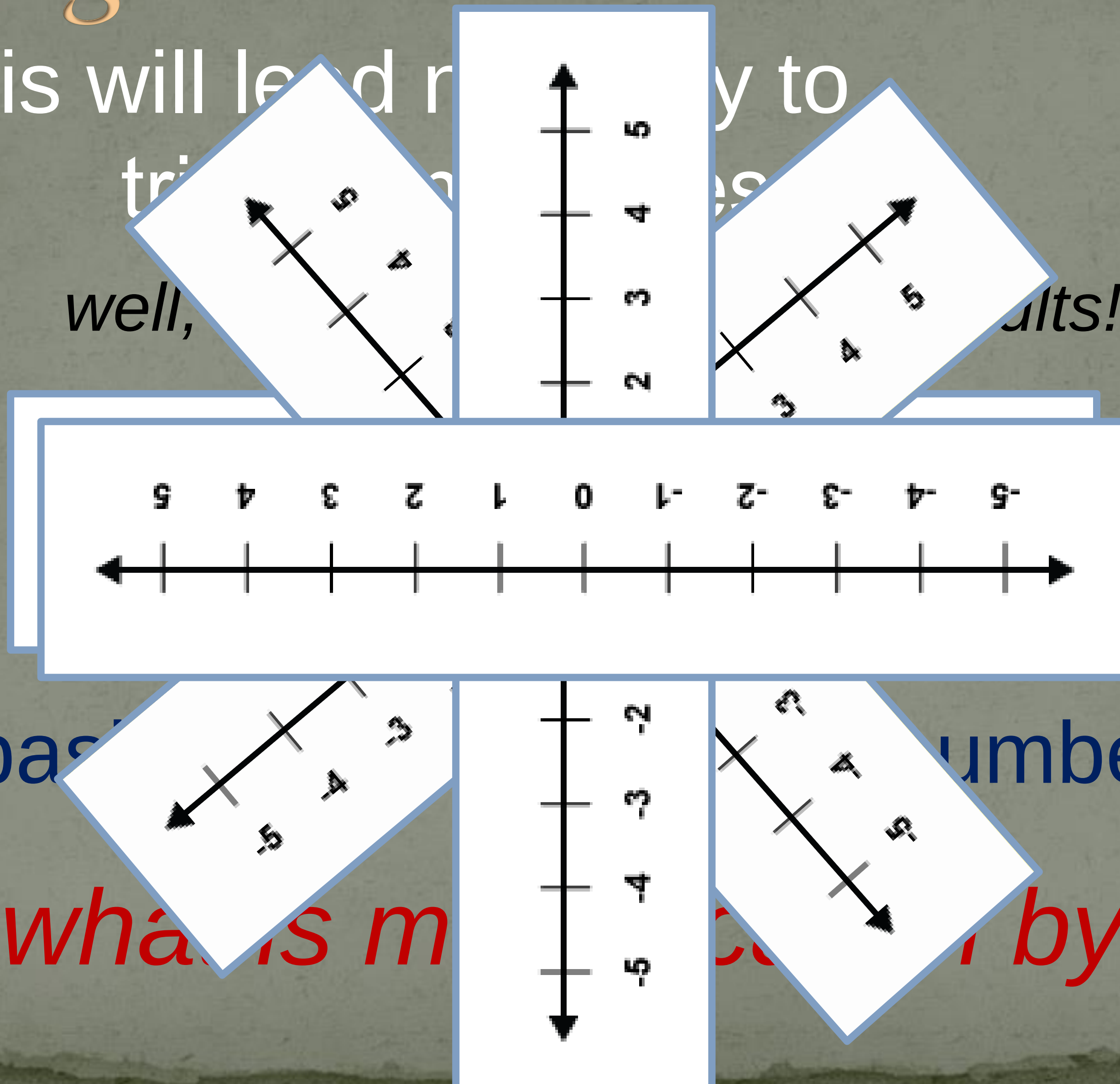
But let's do something different!

Let’s go Geometric!

this will lead n… y to

tri… es

well, … ults!

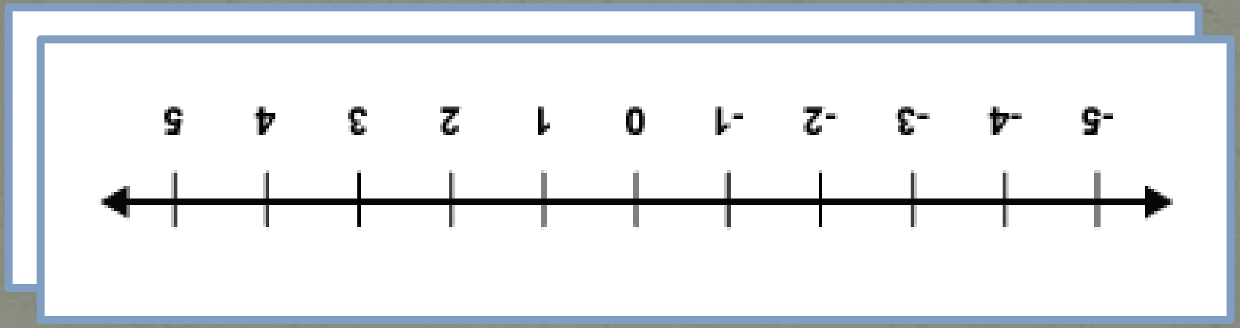


The bas… umber line…

so wha… is m… by -1?

Finding “u”!

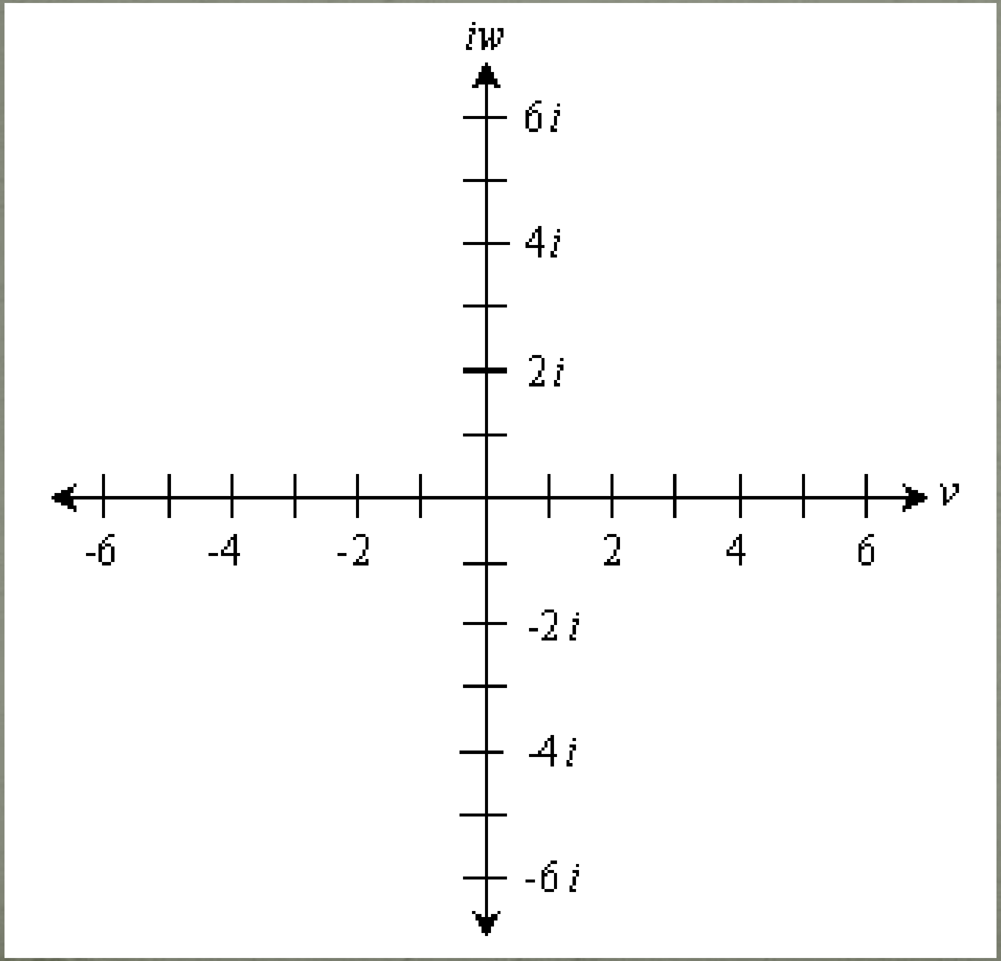
so now if we want something “u” that has the property that $u^2 = -1$, then given that doing u twice is equivalent to a 180 rotation...



then what is multiplication by u?

A New Number Line!

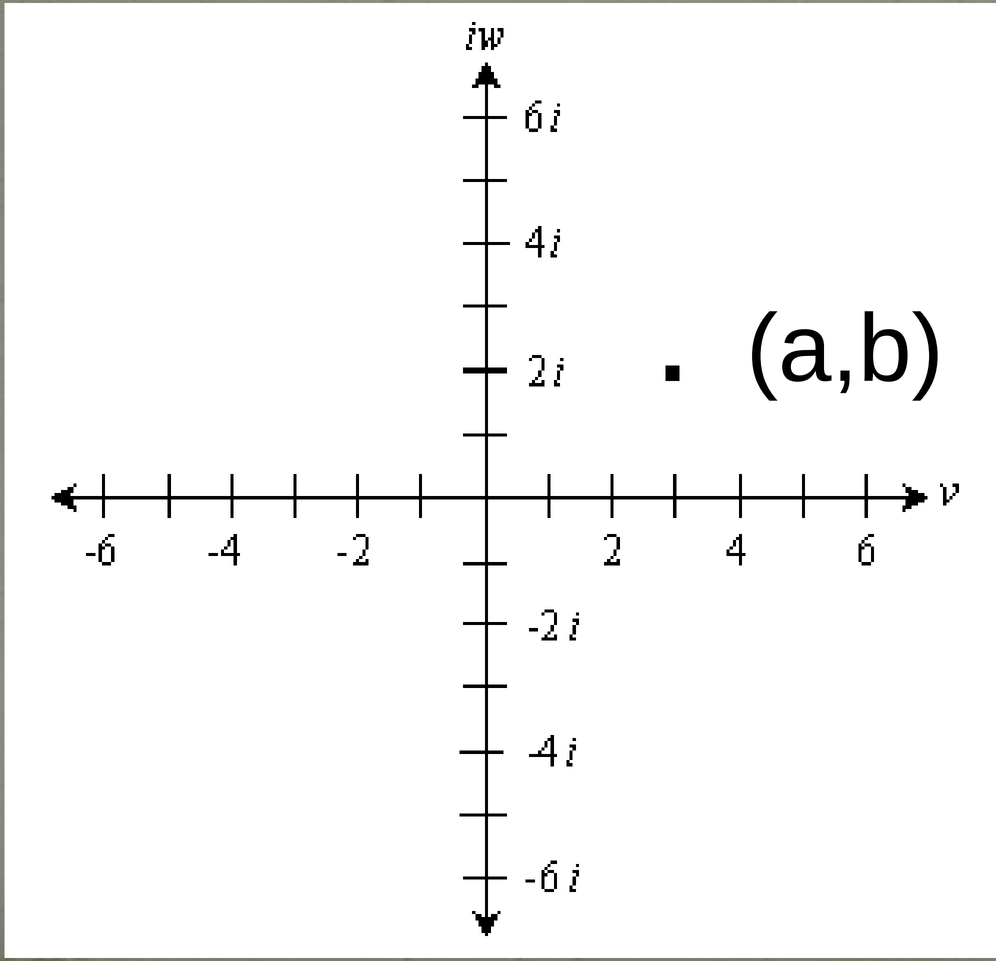
The "complete" number line!



and of course we'll know "u" as "i" instead!

Now time for some vectors!

Let's go at this a bit differently…



where we can think of a+bi as point (a,b)

or as (*vectors*!)

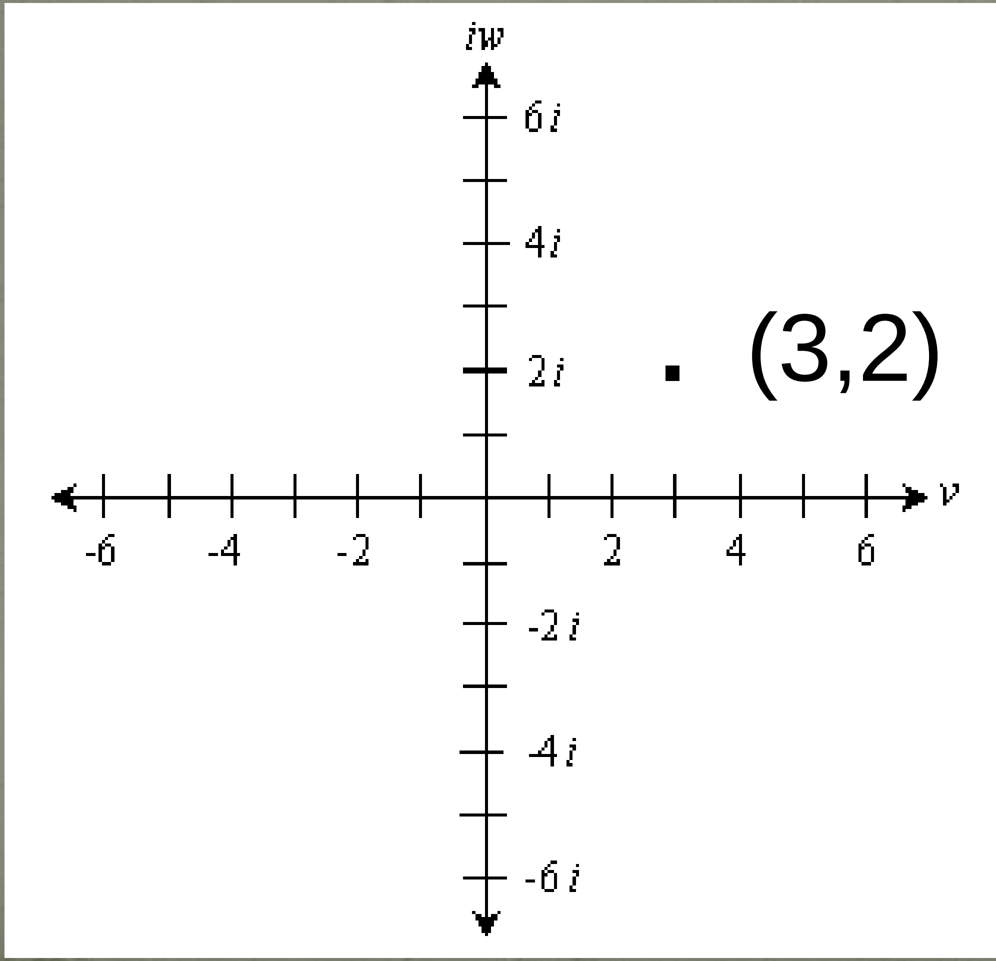
a(1,0) + b(0,1)

for example here (3,2) *is*

3(1,0) + 2(0,1)

Now time for some vectors!

but 3(1,0) + 2(0,1) looks a bit odd…



write (a,b) vertically instead…

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

now (3,2) *is*

$$3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Working with vectors…

so what is multiplication by -1 equivalent to?

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} -a \\ -b \end{bmatrix} = a\begin{bmatrix} -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Working with vectors…

so multiplication by -1 is equivalent to *a change in basis vectors*…

sending $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

to $\begin{bmatrix}-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-1\end{bmatrix}$

or $\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}-1\\0\end{bmatrix},\begin{bmatrix}0\\-1\end{bmatrix}$

Working with vectors…

time for some new notation!

instead of writing $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

we could write $\overset{a}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{b}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ *hmm…*

how about $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix}$

or just $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ *hello, matrices!*

Back to -1 …

thinking back to multiplication by -1…

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} -a \\ -b \end{bmatrix} = a\begin{bmatrix} -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

Matrix multiplication…

what should…

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \end{bmatrix}\right)$$ *do?*

Aha! A B … $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \mathrm{x} \\ y \end{bmatrix} = ?$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} \mathrm{x} \\ y \end{bmatrix}, \begin{bmatrix} \mathrm{w} \\ z \end{bmatrix}\right) = ?$$

so how about $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & w \\ y & z \end{bmatrix} = ?$

Finding "i"…!

so now…

if multiplication by -1 is equivalent to multiplication by

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

then find a matrix A such that

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

wow! a quadratic equation… in matrices!

Finding "i"...!

Aha!

so **1** is equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and **i** is equivalent to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

so then a + bi is equivalent to...?

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Multiplying complex numbers …as matrices!

Now try out $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

which is equivalent to…

$(a + bi)\ (c + di)$

$= (ac - bd) + i(ad + bc)$

Roots of Unity

Now if the number i has the property that $i^2 = -1$, then what is i^4?

…so i is a fourth root of one ("unity")

and now tell me about the roots/factorization of $x^4 - 1 = 0$

we know **i** is equivalent to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

How about this?!

Okay, you just found that multiplication by i is equivalent to the transformation by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then try finding a matrix A with

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \text{“}i\text{”}$$

wait, wasn't that the puzzle of the day?!