3

Complex Numbers, Complex Maps, and Trigonometry

Introduction

Complex numbers—numbers of the form $a+bi$ where a and b are real numbers and $i^2 = -1$—emerged in mathematics as algebraic objects. First, they were useful in solving cubic equations. Later, it was found that every polynomial of degree n with real coefficients has exactly n roots in the complex numbers. This amazing fact—that the complex numbers are all you need to solve any polynomial with real coefficients—is known as the fundamental theorem of algebra. We won't prove the fundamental theorem in this chapter, because we want to go in a different direction. But you can find readable accounts in [2, 8].

"Exactly n roots" if you count multiple roots more than once. The sixth-degree equation $(x-3)^4(x+1)^2 = 0$ has six roots: 3, 3, 3, 3, −1, and −1.

Of course, you've now introduced a new kind of number into the mix. If you need complex numbers to solve polynomials with real coefficients, you probably need some other kind of number to solve polynomials with complex coefficients, right? Amazingly, no! Even if you allow the coefficients of a polynomial to be complex numbers, you still have exactly n roots for a polynomial of degree n, and those roots can all be found in the complex number system. Later in the chapter, we'll use the fundamental theorem of algebra to prove this generalized result.

In spite of its name, every proof of the fundamental theorem must use results that are strictly outside algebra.

Often in mathematics, the significance of new findings is not immediately clear. Complex numbers helped to solve many problems in algebra. But adding a geometric interpretation to these numbers allowed for new insights into other fields of study as well. In particular, the study of dynamics, chaos, iterated function systems, and fractals all involve functions on complex numbers.

Just as we wrote $f \in \mathbb{R}[x]$ in Chapter 2 to mean f is a polynomial in x with real coefficients, we'll write $f \in \mathbb{C}[x]$ to mean f is a polynomial in x with complex coefficients.

Complex numbers sit underneath many topics in school mathematics—from the very oldest topics in algebra to the newest visitors to school mathematics like fractals. Knowing your way around complex numbers helps you make sense of many topics and helps connect different ideas. In this chapter, we provide a short tour of complex numbers and some of their applications to algebra, geometry, analysis, and even arithmetic. We'll

- look at them as algebraic tools for solving equations,
- find a geometric representation that connects to trigonometry and geometry,
- look at functions of complex numbers and some famous fractal curves, and
- use complex numbers to generate and prove trigonometric identities.

The last bullet will involve an important family of polynomials, so we'll return to the themes of Chapter 2.

3.1 Complex Numbers

Complex numbers are numbers of the form $a+bi$ where a and b are real numbers and $i^2 = -1$. Most texts introduce the symbol i as a way to solve quadratic equations with no real roots. The first example is invariably

$$x^2 + 1 = 0.$$

This masks the history a bit. In fact, the "imaginary unit" i and complex numbers first came about in the solution to *cubic* equations.

Cardan's formula is to cubics what the quadratic formula is to quadratics: a way to find the roots for any cubic exactly, if the roots exist. Here's how it works: Suppose you have this cubic:

$$y^3 + by^2 + cy + d = 0.$$

A simple variable substitution (see problem 88 on page 87 of Chapter 2), $y = x - (b/3)$, allows you to get rid of the degree-2 term, so now you have

A proof of Cardan's formula is developed in problems 88–90 on page 87 of Chapter 2.

$$x^3 + px + q = 0.$$

Solutions to this equation are given by:

Actually, there are *three* choices for each cube root; we're choosing one of them here.

$$x = \sqrt[3]{\frac{-q + \sqrt{\frac{27q^2+4p^3}{27}}}{2}} + \sqrt[3]{\frac{-q - \sqrt{\frac{27q^2+4p^3}{27}}}{2}}.$$

Example: To solve $x^3 + x - 2 = 0$, note that $p = 1$ and $q = -2$. So Cardan's formula gives:

$$x = \sqrt[3]{\frac{2 + \sqrt{\frac{27(-2)^2+4}{27}}}{2}} + \sqrt[3]{\frac{2 - \sqrt{\frac{27(-2)^2+4}{27}}}{2}}.$$

Amazingly, this simplifies to 1! (Check it with a CAS.)

There are also two complex roots, which you can find by choosing different cube roots. Or, after finding that 1 is a root, you can divide the cubic by $x - 1$ to get

Since $x = 1$ is a root of the equation, $x - 1$ is a factor of the polynomial.

$$x^3 + x - 2 = (x - 1)(x^2 + x + 2) = 0.$$

You can then solve the remaining quadratic to find the other roots.

In applying Cardan's formula to $x^3 - 15x - 4 = 0$, Bombelli found this as a root:

$$\begin{aligned} x &= \sqrt[3]{\frac{4 + \sqrt{\frac{27(-4)^2+4(-15)^3}{27}}}{2}} + \sqrt[3]{\frac{2 - \sqrt{\frac{27(-4)^2+4(-15)^3}{27}}}{2}} \\ &= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}. \end{aligned}$$

Now, Bombelli knew that this polynomial had nice roots—in fact, 4 is one of the roots—so the square roots of negative numbers were puzzling. He was able to complete calculations like this by pretending that things like $\sqrt{-121}$ exist, but he disapproved of such methods.

Notice that, if we calculate formally,

$$\begin{aligned} (2 + \sqrt{-1})^3 &= 2^3 + 3 \cdot 2^2 \cdot \sqrt{-1} + 3 \cdot 2 \cdot (\sqrt{-1})^2 + (\sqrt{-1})^3 \\ &= 8 + 12\sqrt{-1} - 6 - \sqrt{-1} \\ &= 2 + 11\sqrt{-1} \\ &= 2 + \sqrt{-121}. \end{aligned}$$

In other words,

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}.$$

Well, $2 + \sqrt{-1}$ is *one* of the cube roots of $\sqrt[3]{2 + \sqrt{-121}}$.

Similarly,

$$\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}.$$

If you make these substitutions into Cardan's solution, you get

$$x = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4.$$

As is always the case in mathematics, ideas created by one mathematician are refined and generalized by others. Euler was the first one to introduce the symbol i for $\sqrt{-1}$, and it took time for the algebra of the complex number system to be understood.

Just to get some perspective: Cardan lived from 1501–1575. The first rigorous proof of the Fundamental Theorem of Algebra—the fact that a polynomial of degree $n \geq 1$ (with real or complex coefficients) has exactly n roots in the complex numbers—was published by Gauss in 1799. It was more than 200 years after Bombelli worked with $\sqrt{-121}$ while computing the roots of a cubic.

Actually, Cardan should not be given much credit here. Scipio Ferro first discovered a general solution to cubics, but he preferred to keep it a secret shared by only his students. 40 years later, Tartaglia independently came up with the same method. He showed his method to Cardan who, despite his promise to secrecy, published the result.

Facts and Notation

We all agree that $\sqrt{9}$ means 3. So, the two square roots of 9 are $\sqrt{9}$ and $-\sqrt{9}$. This is just convention, part of the definition of $\sqrt{}$.

Similarly, we agree that $\sqrt{-1} = i$, so that the two square roots of -1 are i and $-i$. And, one more convention: $\sqrt{-121} = 11i$, so the two square roots of

-121 are $11i$ and $-11i$. Similarly, $\sqrt{-3} = i\sqrt{3}$, and, if $c > 0$, $\sqrt{-c} = i\sqrt{c}$ (note: $\sqrt{c}$ is a real number). Just for typographical reasons, $\sqrt{-16}$ is written as $4i$, while $\sqrt{-17}$ is written as $i\sqrt{17}$.

Every complex number can be written as $a+bi$ with a and b real. A complex number written as $a+bi$ with a and b real is said to be in *standard form.*

If you know how to manipulate expressions like $3+4x$, you can add, subtract, and multiply complex numbers. Do the calculations as you usually would, and then whenever you have i^2, substitute -1.

$$(3+4i)+(2-i) = 5+3i$$
$$(3+4i)-(2-i) = 1+5i$$
$$(3+4i)(2-i) = 6+5i-4i^2 = 6+5i+4 = 10+5i$$

Proof If $a_1 + b_1 i = a_2 + b_2 i$ and $b_1 \neq b_2$, then i would equal

$$\frac{a_2 - a_1}{b_1 - b_2},$$

a nice real number. That's silly (why?). So, $b_1 = b_2$. This implies that $a_1 = a_2$ (why?).

If a complex number is written in standard form as $a+bi$ with a and b real numbers, a is called the *real part* and b is called the *imaginary part.* Two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$. That is, for two complex numbers to be equal, the real parts must be equal and the imaginary parts must be equal.

If $z = a+bi$ is a complex number, then $\overline{z} = a-bi$ is known as the *conjugate* of z.

$$\begin{array}{ll} \text{If } z = 2+3i, & \text{then } \overline{z} = 2-3i; \\ \text{if } z = -1+\pi i, & \text{then } \overline{z} = -1-\pi i; \\ \text{if } z = 2-7i, & \text{then } \overline{z} = 2+7i. \end{array}$$

Dividing complex numbers is not as straightforward as adding or multiplying. A number like

$$\frac{2-3i}{1+i}$$

is still a complex number, but to work with it, we want it to look like $a+bi$ with a and b real. Even if you separate the fraction above into two parts like this:

$$\frac{2}{1+i} - \frac{3i}{1+i},$$

you still don't know what is the "real part" and what is the "imaginary part."

What you want to do is get rid of the "imaginary part" in the denominator. You can do this by noticing a handy fact: When you multiply a complex number by its conjugate, you always get a real number. Check it out:

$$(2+3i)(2-3i) = 4-9i^2 = 4+9 = 13$$
$$(1-i)(1+i) = 1-i^2 = 1+1 = 2$$
$$(\sqrt{2}+\sqrt{3}i)(\sqrt{2}-\sqrt{3}i) = 3-3i^2 = 2+3 = 5$$

This is a lot like "rationalizing the denominator." Instead of getting rid of square root signs, we're getting rid of *i*s.

So if you want to find

$$\frac{2-3i}{1+i},$$

you can multiply by $1 - i/1 - i$:

$$\frac{2-3i}{1+i} \cdot \frac{1-i}{1-i} = \frac{-1-5i}{2} = -\frac{1}{2} - \frac{5}{2}i.$$

This is in the $a + bi$ form, so it's a complex number we can deal with more easily.

Problems

1. Find one real root for each cubic using Cardan's formula.

(a) $x^3 - 18x - 35 = 0$ **(b)** $x^3 + 24x - 56 = 0$

2. Suppose $z = 1 - 4i$ and $w = 5 + 2i$. Do each calculation.

(a) $z + w$ **(b)** $z - w$ **(c)** zw

(d) z^2 **(e)** w^2 **(f)** $(z + w)^2$

3. (a) What is the conjugate of a real number? For example, if $z = -2$, what is $\overline{z}$? Explain your answer.

(b) What is the conjugate of a "pure imaginary" number? For example, if $z = 3i$, what is $\overline{z}$? Explain your answer.

(c) What is the conjugate of $\overline{z}$? Explain your answer.

(d) Characterize all complex numbers z so that $\overline{z} = z$.

(e) Find a complex number z so that $\overline{z} = z^2$.

4. Suppose $z = 2 - 3i$ and $w = 5 - i$. Do each calculation. Look for patterns.

(a) $z + \overline{z}$ **(b)** $z - \overline{z}$ **(c)** $z\overline{z}$

(d) $w + \overline{w}$ **(e)** $w - \overline{w}$ **(f)** $w\overline{w}$

(g) $\frac{z}{w}$ **(h)** $\frac{w}{z}$ **(i)** $\frac{z}{i}$

What is $\overline{i}$?

5. (a) State a conjecture about the product of a complex number and its conjugate: $z\overline{z}$.

(b) Prove your conjecture.

Hint: Let $z = a + bi$. What is $\overline{z}$? Now find the product and sum.

6. (a) State a conjecture about the sum of a complex number and its conjugate: $z + \overline{z}$.

(b) Prove your conjecture.

7. Prove the following theorem:

Theorem 1. *If z and w are complex numbers, then*

(a) $\overline{z + w} = \overline{z} + \overline{w}$, *and*

(b) $\overline{zw} = \overline{z}\,\overline{w}$

Say it like this: "The conjugate of the sum is the sum of the conjugates" and "the conjugate of the product is the product of the conjugates."

8. If $z = a + bi$ and $w = c + di$, express each in standard form (in terms of a, b, c, and d):

(a) $\frac{1}{z}$ **(b)** $\frac{w}{z}$

One way to go about this: Let $w = a + bi$. Find zw. Note that the real part must be 1 and the imaginary part must be 0. Another way: simplify $1/z$.

9. A **reciprocal** for z is a number w so that $zw = 1$. Try to find each reciprocal:

(a) $z = 1 + 2i$ **(b)** $z = 2 - i$ **(c)** $z = i$

10. Show that a quadratic equation with real coefficients and with one real root has, in fact, two real roots (that may possibly be the same).

Gaussian Integers are numbers of the form $a + bi$ where a and b are *integers*. For example, $27 - 15i$ is a Gaussian integer, but $3 + \frac{7}{2}i$ and $\sqrt{2} + i$ are not because $\frac{7}{2}$ and $\sqrt{2}$ are not integers. The following problems deal with Gaussian integers.

11. Is the sum of two Gaussian integers also a Gaussian integer? Explain how you know.

12. Is the product of two Gaussian integers also a Gaussian integer? Explain how you know.

13. Is the reciprocal of a Gaussian integer also a Gaussian integer? Explain how you know.

14. Note that in the Gaussian integers, you can factor some numbers that you usually think of as primes:

$$2 = (1 + i)(1 - i)$$
$$5 = (1 + 2i)(1 - 2i) = (2 + i)(2 - i)$$

(a) Which of these numbers are primes in the Gaussian integers and which are not: 3, 7, 11, 13, 17, 23, 29?

(b) Can you find a pattern? Which primes in the integers will still be primes in the Gaussian integers, and which will factor?

15. Pick five different Gaussian integers $a + bi$ where $a \neq b$. For each number you choose:

- Square your number to get a new number, $c + di$.
- Find $c^2 + d^2$. What kind of number is it?

16. Problem 15 suggests that squaring Gaussian integers is a way to generate Pythagorean triples. The conjecture is something like this: Suppose $a + bi$ is a Gaussian integer. If $(a + bi)^2 = c + di$, then $c^2 + d^2$ is a perfect square. In other words, c and d are the legs of some integer-sided right triangle.

(a) Prove this conjecture.

(b) Do you get all possible Pythagorean Triples this way?

Problem 16b is not so easy. See [12] for more details.

This is just the beginning. The Gaussian integers are just one example of algebraic systems that sit inside $\mathbb{C}$. These subsystems can be used to solve all kinds of problems from arithmetic and geometry. Reference [12] expands on the previous problem set and develops some general methods for generating other useful triples of numbers similar to Pythagorean triples.

Take it Further

On page 101 we mentioned the fact that the fundamental theorem of algebra guaranteed that every polynomial with *complex* coefficients had all its roots in $\mathbb{C}$. The next problem set helps you prove that if every polynomial with real coefficients has its roots in $\mathbb{C}$, so does every polynomial with complex coefficients.

17. Prove Lemma 1:

Lemma 1. *Suppose f is a polynomial with complex coefficients. Define $\overline{f}$ to be the polynomial you get by replacing each coefficient in f by its conjugate. Then:*

(a) $\overline{f+g} = \overline{f} + \overline{g}$. *(b)* $\overline{fg} = \overline{f}\,\overline{g}$.
(c) $\overline{f} = f \Leftrightarrow f(x) \in \mathbb{R}[x]$. *(d)* $f\overline{f} \in \mathbb{R}[x]$.
(e) $\overline{f(z)} = \overline{f}(\overline{z})$.

18. Assume that every polynomial with real coefficients and degree at least 1 has at least one root in $\mathbb{C}$. Use the Factor Theorem to show that a polynomial with real coefficients and degree n has n roots (if you count multiple roots as distinct) in $\mathbb{C}$.

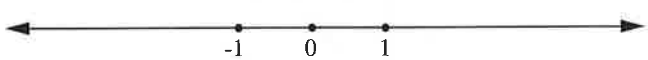
Sometimes, the statement "Every polynomial with real coefficients and degree at least 1 has at least one root in $\mathbb{C}$" is taken *as* the statement of the fundamental theorem of algebra.

19. Suppose $f \in \mathbb{C}[x]$. If a complex number z is a root of f, show that $\overline{z}$ is a root of $\overline{f}$.

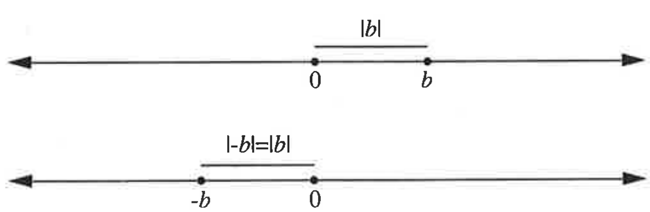
20. Suppose $f(x) \in \mathbb{C}[x]$. Then by Lemma 1, if we define $g(x) = f(x)\overline{f}(x)$, $g(x) \in \mathbb{R}[x]$. Show that if $g(z) = 0$, either $f(z) = 0$ or $f(\overline{z}) = 0$. Hence conclude that if every polynomial with real coefficients and degree at least 1 has a root in $\mathbb{C}$, then every polynomial with *complex* coefficients and degree at least 1 has a root in $\mathbb{C}$.

3.2 The Complex Plane

We want a way to represent complex numbers geometrically. Real numbers can be pictured as all the points along the number line.



The absolute value of a number b, written $|b|$, is the distance of b from the point 0. So, $|b| = |-b|$.

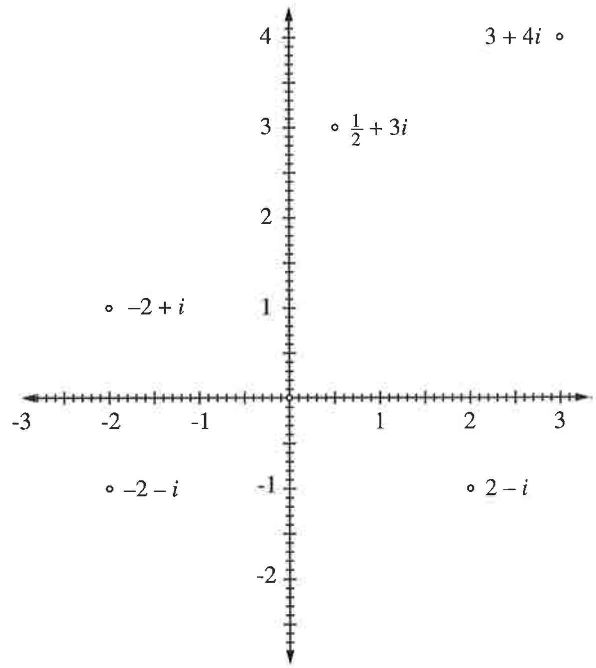


$|b|$ is the length of a segment

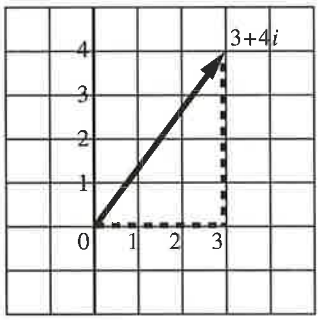
It took a couple of centuries after complex numbers were introduced into algebra before mathematicians (Gauss, Argand, and Wessel) hit upon this geometric representation. It may seem natural today, but it was quite a breakthrough in the history of the subject.

The designation of which is the real axis and which is imaginary is arbitrary. But it's important that everyone does the same thing.

To think about complex numbers, you need to think about *two* real numbers—the real and imaginary parts. One way to represent this situation is to use *two* number lines. The x-y plane below shows two real lines intersecting at the origin. The x-axis becomes the "real axis," where you graph the a (real part) of $a + bi$. The y-axis becomes the "imaginary axis," where you graph the b (imaginary part) of $a + bi$.



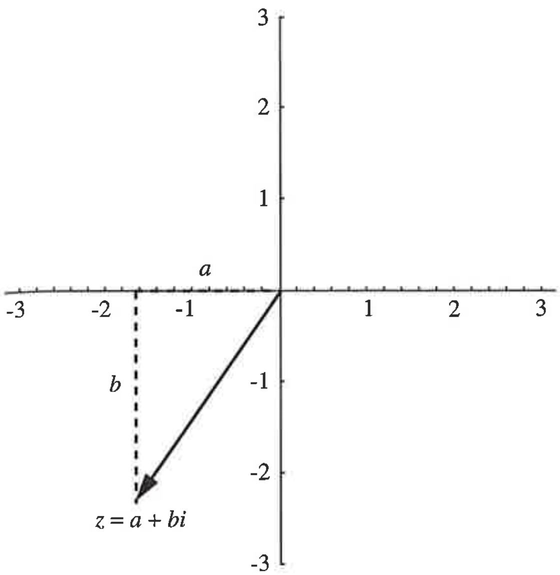
This representation is called the *complex plane*, and with it we can create an absolute value function for complex numbers. It measures the distance from a point $a + bi$ to the origin.

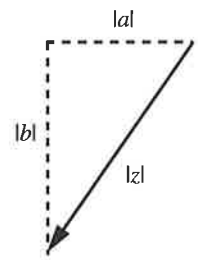


The symbol $|z|$ is usually read as the "absolute value of z" or the "length of z." In older books, it was sometimes called the "modulus of z."

Denote the length of the vector shown by $|z|$. By the Pythagorean Theorem, $|z|^2 = 3^2 + 4^2 = 25$, so $|z| = 5$. For a general complex number, this means $|z|^2 = a^2 + b^2$, so

$$|z| = \sqrt{a^2 + b^2}.$$





Now we have a picture of complex numbers as points in the plane. Notice that since every number $a + bi$ corresponds to a unique ordered pair (a, b) (and vice-versa), there is a one-to-one correspondence between points in the plane and complex numbers. Can we describe geometrically what addition and multiplication look like?

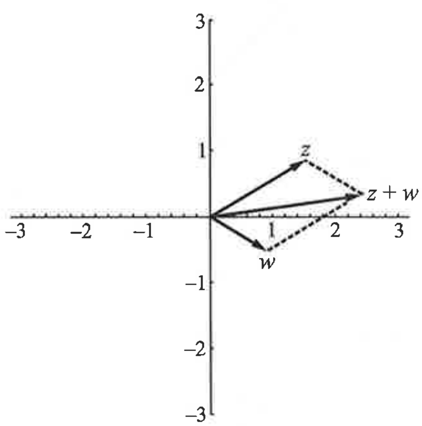
Just as there is a one-to-one correspondence between points on the number line and real numbers.

Well, if you add two complex numbers, you add the real parts and add the imaginary parts.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

This looks just like vector addition:

Proving this is the object of problem 21 on page 111.

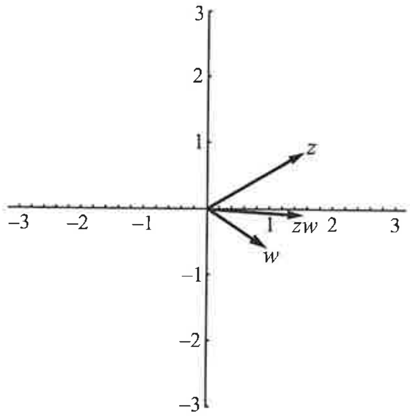


The vector $z + w$ is the diagonal of the parallelogram described by adjacent sides z and w.

Multiplication is trickier. Suppose $w = a + bi$ and $z = c + di$. We know that

$$wz = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

But is there a way to locate zw on the complex plane if you know z and w?

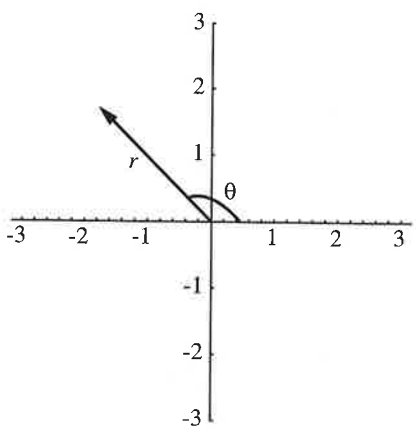


How do you get wz from w and z?

Nothing seems to jump out from this picture, or from the algebra. In fact, there is a relationship, and it's a simple one at that.

Notice that the radius of z is $|z|$. The argument is sometimes denoted by $\arg(z)$. So, $r = |z|$ and $\theta = \arg(z)$.

For multiplication of two complex numbers, it helps to think about characterizing the numbers in what's called **polar form**, rather than the standard $a + bi$ form. If z is a complex number, instead of saying how far over and up you go to get to z, you could describe how far z is from the origin (its *radius*) and what angle you have to turn through from the positive x-axis to get to z (it's *argument*). Hence, we can locate a complex number by an ordered pair (r, θ), where r is the length of the vector and θ is the angle it makes with the x-axis:



Locating z by a radius and an argument

Using this way to describe complex numbers, you can experiment with several examples, comparing the length and argument of zw to the lengths and arguments of z and w. What emerges is a pretty amazing conjecture. Let's stop here and work on some problems that bring together what we have and that investigate this conjecture.

Problems

21. Suppose $z = a + bi$ and $w = c + di$. Show that 0, z, w, and $z + w$ are the vertices of a parallelogram.

How can you show that a quadrilateral is a parallelogram?

22. For each number below, find

$$|z|,\ |\overline{z}|,\ |iz|,\ \text{and } |-z|.$$

(a) $z = 5 + 12i$ (b) $z = -\sqrt{2} + \sqrt{2}i$ (c) $z = -\frac{7}{5} - \frac{24}{5}i$

23. Let $z = a + bi$ be any complex number. Show that $|z| = |\overline{z}| = |iz| = |-z|$.

24. Describe geometrically *all* the points with the same absolute value as z.

25. For each pair of numbers, find $|z|$, $|w|$, and $|zw|$.
(a) $z = 3 + 4i, \quad w = \sqrt{3} + i$ (b) $z = \sqrt{5} + 2i, \quad w = -i$

26. If $z \in \mathbb{C}$, show that $-z$ is symmetric to z with respect to the origin.

27. If $z \in \mathbb{C}$, show that $\overline{z}$ is symmetric to z with respect to the real axis.

28. Let $z = a + bi$ and $w = c + di$. Show that
(a) $|z||w| = |zw|$.
(b) Use this to show that $|z^n| = |z|^n$ for any nonnegative integer n.

29. Suppose z is a complex number such that $z^n = 1$ for some nonnegative integer n. Show that $|z| = 1$.

30. Let $z = a + bi$ and $w = c + di$. Use geometry to show that
(a) $|z + w| \le |z| + |w|$ (b) $|z + w| \ge |z| - |w|$
Under what conditions are these inequalities actual equalities?

The results of this problem are sometimes called the "triangle inequalities." Why?

31. For each number, find $|z|$ and $z\overline{z}$.
(a) $z = -3 - 4i$ (b) $z = \sqrt{3} - i$
(c) $z = 5$ (d) $z = -2i$

32. Let $z = a + bi$. Show that $|z| = \sqrt{z\overline{z}}$.

Problems 33–37 are experiments to help see the geometry behind the multiplication of complex numbers. Feel free to experiment on your own.

33. What happens to complex numbers on the circle of radius 5 (centered at the origin) when you multiply them by 2? How about when you multiply them by $\sqrt{2} - \sqrt{2}i$? As you would expect, multiplying by 2 is not the same as multiplying by $\sqrt{2} - \sqrt{2}i$. What differences do you notice?

34. (a) Choose five points on the circle of radius 1 and graph them.
(b) Multiply the points by 2 and graph the results. Describe how multiplying by 2 affects the location of the points.
(c) Multiply the points by $\sqrt{2} - \sqrt{2}i$ and graph the results. Describe how multiplying by $\sqrt{2} - \sqrt{2}i$ affects the location of the points.

35. Choose five points on the circle of radius 3, and repeat steps (b) and (c) of problem 34. Make a general conjecture about how multiplying by 2 and

multiplying by $\sqrt{2} - \sqrt{2}i$ affect the locations of points in the complex plane.

You may want to choose some points on the circle of radius 1, as you did in problem 34, to form conjectures.

36. Describe (as completely as you can) how multiplying by z affects the locations of points in the complex plane.

(a) $z = \frac{1}{4}$ **(b)** $z = -1$ **(c)** $z = 3$ **(d)** $z = -5$

(e) $z = i$ **(f)** $z = -i$ **(g)** $z = -3i$ **(h)** $z = 1 + i$

(i) $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ **(j)** $z = \sqrt{3} - i$

37. In general, describe how multiplying by a complex number $z = x + yi$ affects the locations of points in the complex plane. Be as specific as possible.

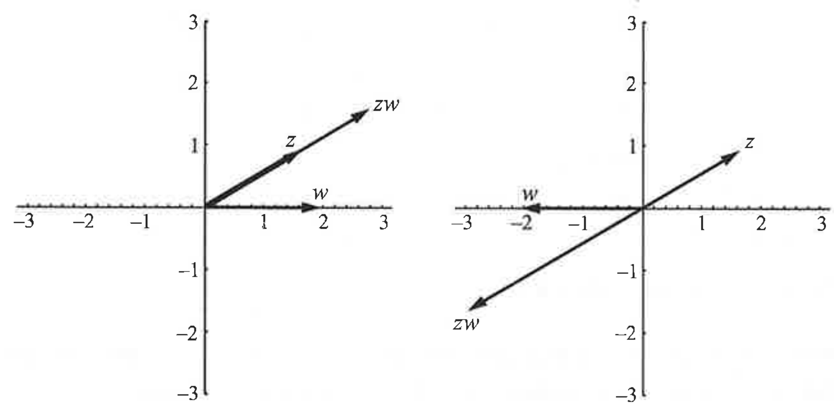
In these problems, $w = a + bi$ and $z = c + di$.

Problems 38–40 help you think about special cases of the problem of explaining the geometry behind the multiplication of complex numbers.

38. Suppose w is a real number, then $w = a + 0i$. So

$$wz = a(c + di) = (ac) + (ad)i.$$

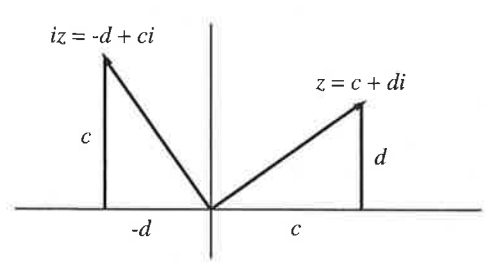
Show that multiplying z by w "stretches" z a factor of $|a|$. (If $a < 0$, you have a stretch by $|a|$ and a rotation by 180°.)



$w = a$. On the left, $a > 0$; on the right, $a < 0$.

In this picture, z is in the first quadrant. You should check the result for other quadrants.

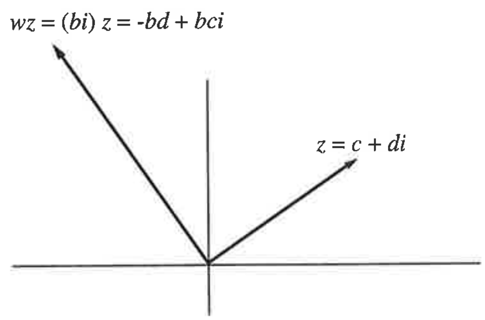
39. Suppose next that $w = i$. Show that multiplication by i rotates 90° counterclockwise.



iz is obtained by rotating z 90° counterclockwise

Hint: Algebraically, $wz = i(c + di) = -d + ci$. Show that the slope of the vector from the origin to $c + di$ is the negative reciprocal of the slope of the vector from the origin to $-d + ci$, so they are perpendicular. A little more work (analyzing cases) shows which way wz points and that wz is obtained from z by a counterclockwise rotation of 90°.

40. More generally, if w is "pure imaginary" (that is $w = bi$), show that wz is obtained from z by rotating 90° counterclockwise and then stretching by a factor of $|b|$.



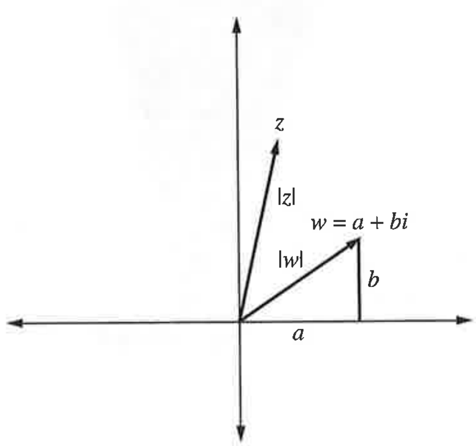
wz: rotate z 90° and stretch by $|b|$.

3.3 The Geometry behind Multiplying

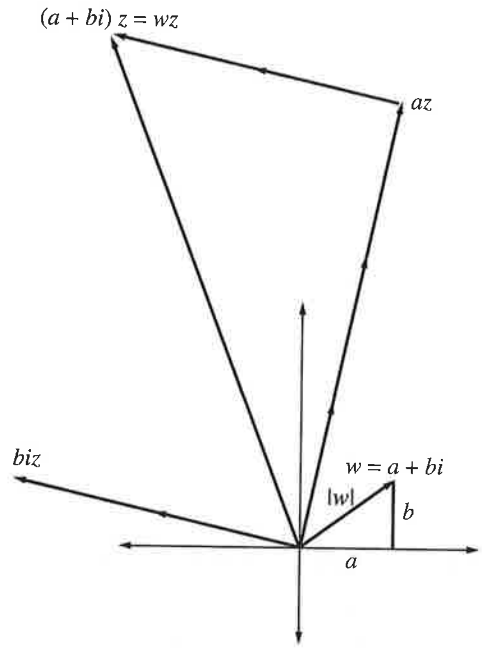
It seems from the previous experiments that the length of zw is the product of the lengths of z and w and the argument of zw is the *sum* of the arguments of z and w. And, in fact, it's true. But getting there usually involves developing and using trig identities. In [15], we tried to circumvent the complications by getting at the essential mathematics beneath the complicated trig, and we succeeded in a way. But then, in the summer of 2002, a group of teachers at the Park City Mathematics Institute in Utah discovered a very simple and elegant way to see what's going on. Let's look at what they found.

Added in proof: The "PCMI" method does eppear elsewhere, for example in [17].

Suppose that $w = a + bi$ is some arbitrary complex number, and we want to see the effect of multiplying by w. The PCMI teachers reasoned like this:

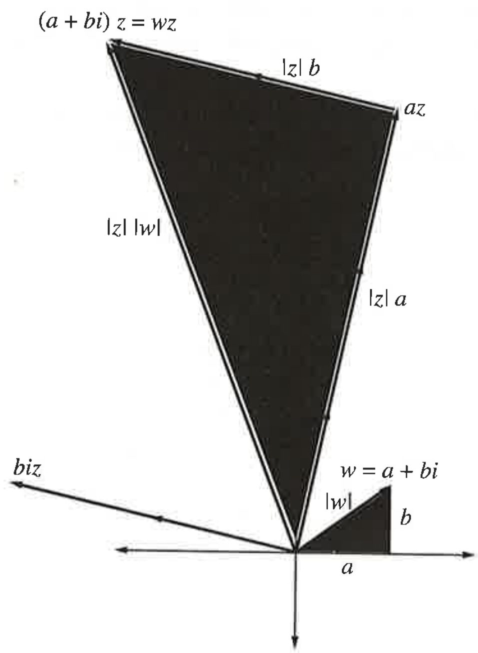


- Multiplying z by a scales it by a factor of $|a|$ (problem 38 from the previous section),
- Multiplying z by bi scales it by a factor of $|b|$ and rotates it 90° (problem 40 from the previous section), and
- $wz = (a + bi)z$ is the sum of az and $(bi)z$ (the distributive property in $\mathbb{C}$).



$$wz = az + biz$$

Now, the black triangles in the figure below are similar (SAS).



And the ratio of the sidelengths is $|z|$ (all the sides of the large one are $|z|$ times the corresponding sides of the small one). This implies that the hypotenuse of the large one has length $|z||w|$. But the length of this hypotenuse is just the absolute value of zw, so we have

$$|zw| = |z||w|.$$

An algebraic proof of this result is suggested in problem 28 on page 111.

As for the angles, notice that the argument of zw is the sum of the argument of z plus the black angle at the origin in the large triangle. And the black angle at the origin in the large triangle is the same as the black angle at the origin in the small triangle, because they are corresponding angles in similar triangles. But the black angle at the origin in the small triangle is nothing other than the argument of w. So, we have

Remember, "the argument of a complex number" means the angle you get by rotating the real axis into the complex number in a counter-clockwise direction.

$$\text{argument}\,(zw) = \text{argument}\,(z) + \text{argument}\,(w).$$

So, we have the following theorem:

Theorem 2. *If z and w are complex numbers, then the length of zw is the product of the lengths of z and w and the argument of zw is the sum of the arguments of z and w. In short,*

$$|zw| = |z||w| \quad \textit{and} \quad \arg(zw) = \arg(z) + \arg(w).$$

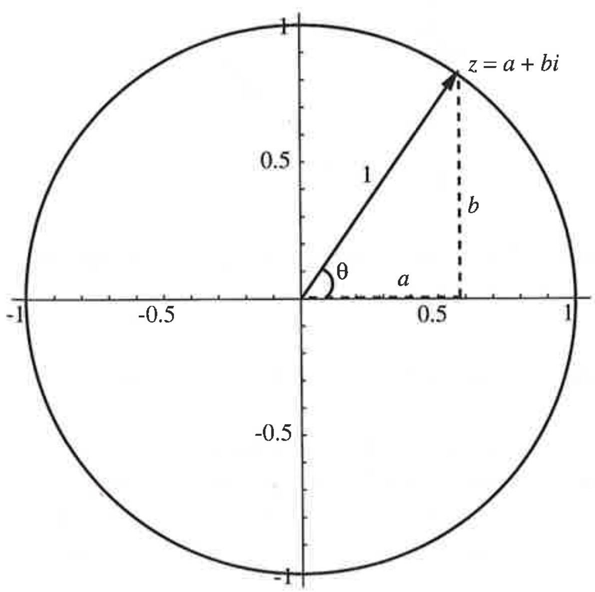
Some people say, "The length of the product is the product of the lengths, and the angle of the product is the sum of the angles."

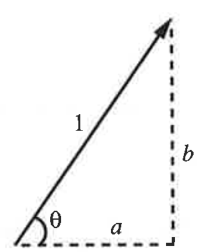
Complex Numbers and Trigonometry

Theorem 2 can be used as a tool to simplify many results in trigonometry. To see how, let's review the connections between trigonometric functions and complex numbers.

More detailed treatments of these connections can be found in chapter 3 of [15] or in any good trigonometry book.

First, we'll focus just on complex numbers with absolute value 1; that is, the points on the unit circle. In the first quadrant, you have right triangles with





hypotenuse 1. So the lengths of the sides of the triangle (the a and b values) can be found directly from the angle at the origin:

$$a = \frac{a}{1} = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}},$$

$$b = \frac{b}{1} = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}.$$

We're assuming you know a bit of trig in what follows. This will not be a complete course, starting from scratch. Most of the details are in [15].

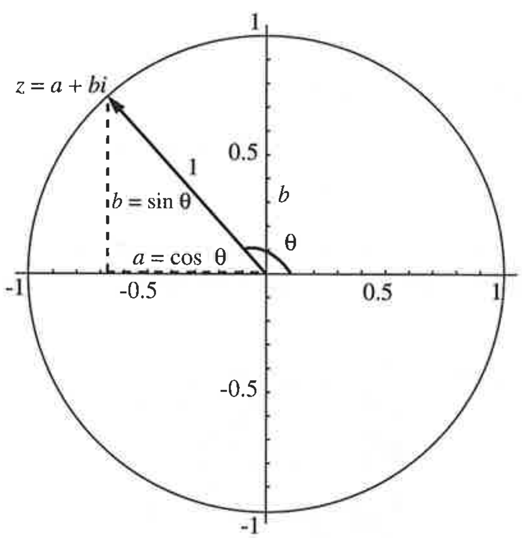
Of course, with these triangle-based definitions, sine and cosine don't make sense for angles greater than 90°. But we can *extend* the definitions of sine and cosine for these other angles based on the unit circle: Suppose $z = a + bi$ is a complex number on the unit circle, and z makes an angle of θ with the x-axis. We *define*:

$$\cos\theta = a: \text{ the } x\text{-coordinate of } z,$$

$$\sin\theta = b: \text{ the } y\text{-coordinate of } z.$$

So, we can write a and b in terms of the angle θ:

$$z = a + bi = \cos\theta + i\sin\theta.$$



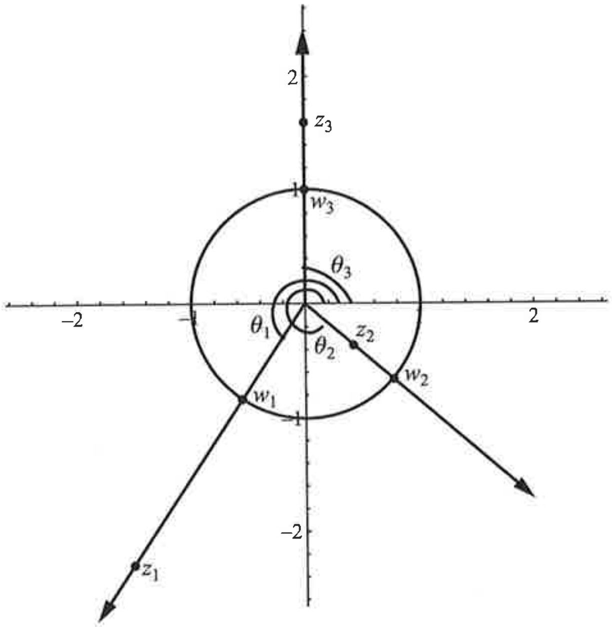
Now we have a trigonometric description for every point on the unit circle:

$$z = \cos\theta + i\sin\theta$$

where θ is the angle z makes with the positive x-axis. What about points not on the unit circle?

Each z corresponds to exactly one w on the unit circle.

Well, every point *not* on the unit circle can be mapped to a point with the same angle θ that *is* on the unit circle. To picture this geometrically, draw a ray from the origin through the point in question. That ray will cross the unit circle in exactly one point, and every point on that ray forms the same angle with the x-axis.



Algebraically, we can show that $w = z/|z|$ is on the unit circle. Suppose $z = a + bi$ is a nonzero complex number and $w = z/|z|$:

$$w = \frac{a+bi}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i,$$

$$|w| = \sqrt{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2}$$

$$= \sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1.$$

Since $|w| = 1$, w is on the unit circle. So we can write

$$w = \cos\theta + i\sin\theta.$$

Since $z/|z| = w$, $z = |z|w$:

$$z = |z|w = |z|(\cos\theta + i\sin\theta).$$

Ways to think about it

There's another way to think about this result that saves some of the algebraic drudgery. It uses the *multiplicativity* of absolute value that was established in Theorem 2 (and in problem 28 on page 111). Let z be any complex number and let w be the complex number on the unit circle that corresponds to z. Then $w = cz$ for some real $c > 0$. Then we can argue like this:

(continued)

$$
\begin{aligned}
1 &= |w| \\
&= |cz| = |c||z| \\
&= c|z|.
\end{aligned}
$$

So,

$$c = \frac{1}{|z|} \quad \text{and} \quad w = \frac{1}{|z|}z = \frac{z}{|z|}.$$

Summarizing, we have

Theorem 3. *Suppose z is a complex number and $\theta = \arg(z)$. Then*

$$z = |z|(\cos\theta + i\sin\theta).$$

Problems

Problems 41–43 are not just for "practice." There are patterns, and thinking about them gives you a feeling for the theorems.

Notice that in problems 41(l) and 41(m), we switched to radian measure. We'll gradually make the transition to using radians. Remember, radians measure distance around the unit circle, starting from $(1, 0)$, so that 2π radians is the same as $360°$. For a refresher, see chapter 3 of [15].

41. Plot z, w, and zw, and then verify Theorem 2 for the following pairs z and w:

(a) $z = 1 + i$, $w = i$ **(b)** $z = 2 + 2i$, $w = i$

(c) $z = -2 + 2i$, $w = 1 + i$ **(d)** $z = 5i$, $w = -1 + i\sqrt{3}$

(e) $z = -4$, $w = \dfrac{-1 + i\sqrt{3}}{2}$ **(f)** $z = -2 - 2i$, $w = -1 - i$

(g) $z = \dfrac{-1 + i\sqrt{3}}{2}$, $w = z$ **(h)** $z = \dfrac{-1 + i\sqrt{3}}{2}$, $w = z^2$

(i) $z = 1 + i$, $w = \cos 30° + i \sin 30°$

(j) $z = \dfrac{-1 + i\sqrt{3}}{2}$, $w = \overline{z}$

(k) $z = 2(\cos 30° + i \sin 30°)$, $w = \cos 120° + i \sin 120°$

(l) $z = 2\left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right)$, $w = \cos \dfrac{3\pi}{2} + i \sin \dfrac{3\pi}{2}$

(m) $z = 2\left(\cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}\right)$, $w = \cos \dfrac{\pi}{3} + i \sin \dfrac{\pi}{3}$

42. Verify Theorem 3 (and plot the numbers on the complex plane) for the following values of z:

(a) $1 + i$ **(b)** $2 + 2i$ **(c)** $2 - 2i$ **(d)** $5i$

(e) -3 **(f)** $-2 - 2i$ **(g)** $-1 + i\sqrt{3}$ **(h)** $\dfrac{-1 + i\sqrt{3}}{2}$

(i) $\dfrac{-1 - i\sqrt{3}}{2}$ **(j)** $3 + 4i$ **(k)** $3 - 4i$ **(l)** $-3 + 4i$

43. Calculate z, z^2, z^3, z^4, and z^5 and then plot each of these powers on the complex plane.

(a) $z = 1 + i$ **(b)** $z = \sqrt{2} + i\sqrt{2}$ **(c)** i

(d) $5i$

(e) -3

(f) $\dfrac{-1+i\sqrt{3}}{2}$

(g) $\dfrac{-1+i\sqrt{3}}{4}$

(h) $\dfrac{1+i\sqrt{3}}{2}$

(i) $\cos 20° + i \sin 20°$

(j) $\cos \dfrac{\pi}{6} + i \sin \dfrac{\pi}{6}$

(k) $\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}$

(l) $\cos \dfrac{2\pi}{5} + i \sin \dfrac{2\pi}{5}$

44. Show that if $z = \cos\theta + i\sin\theta$ is a complex number on the unit circle,

$$\frac{1}{z} = \bar{z}.$$

Is the converse of this result true?

45. Suppose $z = \cos\theta + i\sin\theta$, and $w = \bar{z} = \cos\theta - i\sin\theta$.

(a) Show that z and w are roots of $x^2 - 2ax + 1$, where $a = \cos\theta$.

(b) Find (in terms of a), a quadratic equation whose roots are z^2 and w^2.

46. Prove Theorem 4:

Theorem 4. *For any value of* θ, $\sin^2\theta + \cos^2\theta = 1$.

The result of Theorem 4 is sometimes called the Pythagorean Identity. Why? **Hint**: Use problem 44.

47. Show that if $z = \cos\alpha + i\sin\alpha$ and $w = \cos\beta + i\sin\beta$ are complex numbers on the unit circle,

$$zw = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

48. Prove Theorem 5:

The results of Theorem 5 are called the *addition formulas for sine and cosine*. **Hint**: Use problem 47.

Theorem 5. *For all values of* α *and* β,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\beta\cos\alpha.$$

49. Prove Theorem 6 (*DeMoivre's theorem*):

Theorem 6. *If* n *is a positive integer, then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

50. Use DeMoivre's theorem to solve the equation $x^6 - 1 = 0$.

For problems 49–52, plot the roots on the complex plane.

51. Use DeMoivre's theorem to solve the equation $x^8 - 1 = 0$.

52. Use DeMoivre's theorem to solve the equation $x^8 - 128 = 0$.

53. Show that the roots of $x^n - 1 = 0$ (n a positive integer) lie on the vertices of a regular n-gon on the complex plane.

54. If

$$z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$$

write z^{243} in the form $\cos\theta + i\sin\theta$.

3.4 Trigonometric Identities

And some don't . . .

Going back to the definitions uses the geometry of $\mathbb{C}$. Exploiting the multiplication rule uses its algebra.

What's the point to having students prove trigonometric identities? Sure, some are useful, and it's certainly true that some students get very good at the identity game. But isn't it just a bag of *ad-hoc* tricks?

There are two methods that bring some coherence (and even prettiness) to the topic of trigonometric identities. And the methods not only let students establish identities, they give them machines for generating their own. One is a special case of a bigger idea: *go back to the definitions*. Many teachers use a version of this technique, but it seems to be one of those "pick it up on the job" skills. The second seems to be fairly unknown in high school circles: *exploit the multiplication rule for complex numbers*. Let's first see how each works. Then we'll look at one special sequence of trig identities that shows up all over mathematics.

Go Back to the Definition

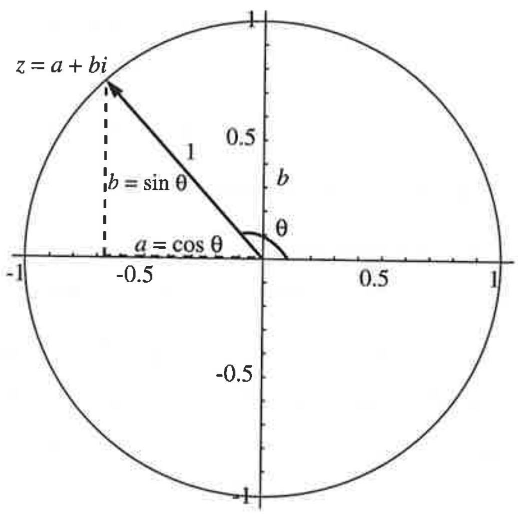
Remember that we extended definitions of sine and cosine from right triangles to the unit circle. $z = a + bi$ is a complex number on the unit circle, and z makes an angle of θ with the x-axis. We *define:*

$$\cos\theta = a: \text{ the } x\text{-coordinate of } z,$$

$$\sin\theta = b: \text{ the } y\text{-coordinate of } z.$$

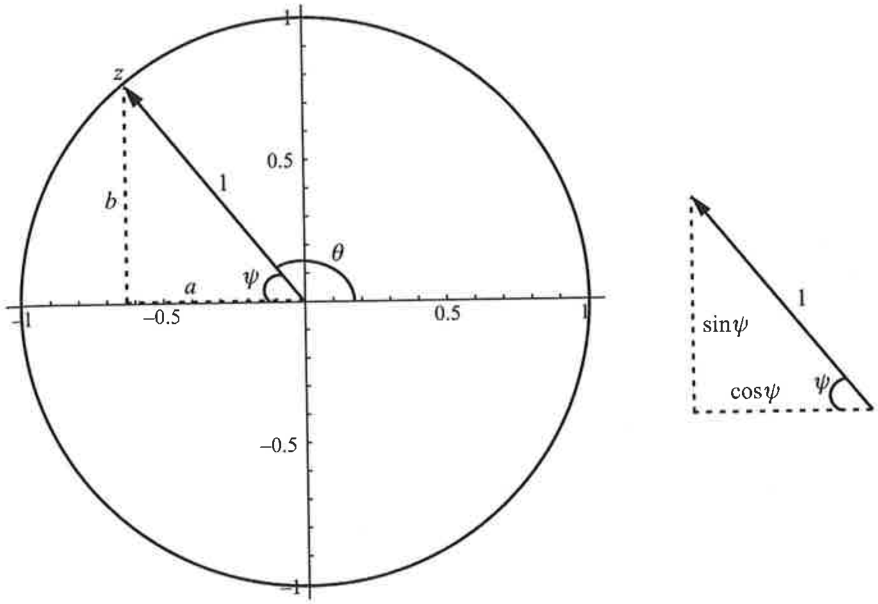
So, we can write a and b in terms of the angle θ:

$$z = a + bi = \cos\theta + i\sin\theta.$$



From this definition, we can form lots of trigonometric identities.

Example: Here is a picture that shows two ways to think about sine and cosine—on the unit circle, relating to the angle θ that a vector makes with the real axis, and in a right triangle, relating to an angle ψ that is less than $\pi/4$.



$a = \cos\theta$ is negative; $\cos\psi$ is positive.

In the picture, $\cos\theta = -\cos\psi$. (Since the point is in the second quadrant, a is negative, so $\cos\theta$ is negative. But for right triangles, sine and cosine are always between 0 and 1.) Also $\theta + \psi = \pi$. Putting these two things together, we get an identity:

So the lengths of the segments are the same, but the sign is different.

$$\cos\theta = -\cos(\pi - \theta).$$

Similarly, $\sin\theta = \sin\psi$, since they are the same length and the same sign. In this case, we have:

$$\sin\theta = \sin(\pi - \theta).$$

So one way to prove identities is to draw pictures of the unit circle, pull out the appropriate right triangle, and look for relationships in the angles, the sines, and the cosines.

Exploit the Multiplication Rule for Complex Numbers

If we stick to the unit circle, suppose we have two numbers:

$$z = \cos\theta + i\sin\theta \qquad w = \cos\psi + i\sin\psi$$

Our multiplication rule says that to make zw, simply rotate z by an angle of ψ. (There is no stretching because $|w| = 1$.)

$$zw = \cos(\theta + \psi) + i\sin(\theta + \psi)$$

If you did problem 48 on page 119, this will look very familiar.

But suppose we just perform the multiplication, algebra style:

$$\begin{aligned} zw &= (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= \cos\theta\cos\psi + i\cos\theta\sin\psi + i\sin\theta\cos\psi + i^2\sin\theta\sin\psi \\ &= (\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \sin\theta\cos\psi) \end{aligned}$$

If both of these expressions are equal to zw, then the real parts must be equal and the imaginary parts must be equal, giving us the celebrated *addition formulas for sine and cosine*:

$$\cos(\theta + \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$
$$\sin(\theta + \psi) = \cos\theta\sin\psi + \sin\theta\cos\psi$$

This is just the beginning. For example, here's one way to see that cosine is even and sine is odd. Let $z = \cos\theta + i\sin\theta$. Because z is on the unit circle,

$$\frac{1}{z} = \bar{z}.$$

Notice:

$$\begin{aligned} 1 &= (\cos\theta + i\sin\theta) \\ &\quad\times(\cos\theta - i\sin\theta) \\ &= \cos^2\theta + \sin^2\theta. \end{aligned}$$

So, $1 = z\bar{z}$, and the conjugate of z is its reciprocal. But by the multiplication rule,

$$(\cos\theta + i\sin\theta)\big(\cos(-\theta) + i\sin(-\theta)\big) = \cos 0 + i\sin 0 = 1.$$

So, $\cos(-\theta) + i\sin(-\theta)$ is also the reciprocal of z. But z has only one reciprocal, so

$$\cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta.$$

Comparing real and imaginary parts, we have:

$$\cos(-\theta) = \cos\theta \qquad \text{(cosine is an } \textit{even} \text{ function)}$$
$$\sin(-\theta) = -\sin\theta \qquad \text{(sine is an } \textit{odd} \text{ function)}$$

Ways to think about it

Notice how we are standing the development on its head. Most developments of the multiplication rule *use* the addition formulas. We used Theorem 2 on page 115 instead. So we're free to use the multiplication rule for complex numbers to prove the addition formulas. And this exhibits a common device in mathematics. Just as Cardan and Bombelli could solve cubics by "moving up" to the complex numbers and then moving back down again, our method establishes equality of functions defined in $\mathbb{R}$—the trigonometric functions—by doing calculations in $\mathbb{C}$.

Rumor has it that all trigonometric identities can be reduced to complex number calculations, and that this is the basis for *Mathematica*'s trig-identity checker.

Example: Suppose you want to *find* the "double angle formulas," for cosine and sine; that is, formulas for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. You could replace ψ by θ in the addition formulas, but one calculation with complex numbers will give you both formulas at once: Just multiply $\cos\theta + i\sin\theta$ by itself:

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\\ &= (\cos^2\theta - \sin^2\theta) + i(\sin\theta\cos\theta + \cos\theta\sin\theta)\\ &= (\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta.\end{aligned}$$

So,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

and

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

The formula for $\cos 2\theta$ is often re-written:

$$\cos 2\theta = \cos^2\theta - \sin^2 = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1.$$

Multiple Angle Formulas

What about $\cos 3\theta$? Or, more generally, $\cos n\theta$? Or $\sin n\theta$? It turns out that we can get these "trig identities" by combining some ideas from Chapter 2 with DeMoivre's theorem.

For extensions of this idea, see [14].

Let's start with $\cos 3\theta$. We can try to proceed inductively, expanding $\cos(3\theta)$ as $\cos(\theta + 2\theta)$:

$$\begin{aligned}\cos(3\theta) &= \cos(\theta + 2\theta)\\ &= \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta)\\ &= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta)\\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

So, $\cos(3\theta)$ is a polynomial (a cubic, in fact) in $\cos\theta$. Oh my. We have

$\cos 1\theta$	$\cos\theta$
$\cos 2\theta$	$-1 + 2(\cos\theta)^2$
$\cos 3\theta$	$-3(\cos\theta) + 4(\cos\theta)^3$

Will $\cos 4\theta$ be a fourth degree polynomial in $\cos\theta$?

We could keep going, but already the algebra is getting hefty. Before we invest in more hand calculation, let's use the CAS to generate some data. The question on the table is "Can $\cos(n\theta)$ be expressed as a polynomial (maybe of degree n) in $\cos\theta$?"