2

Form and Function: The Algebra of Polynomials

Introduction

In Chapter 1, our focus was on fitting polynomials to tables. In this chapter, we take polynomials themselves as the objects of study.

We're in a curious position with what we know about polynomials as a result of Chapter 1. We know that if we write a polynomial f as a combination of combinatorial polynomials, $\binom{x}{k}$, we have a simple interpretation of the numerical coefficients: Theorem 9 on page 37 of Chapter 1 says that if f is a polynomial of degree m, we can express f this way:

$$f(x) = \sum_{k=0}^{m} \Delta^k(f)(0)\binom{x}{k}.$$

In case it's been a while since you've looked at Chapter 1, if f is a function, $\Delta(f)$ is the function defined by $x \mapsto f(x+1) - f(x)$. The numbers $\Delta^k(f)(0)$ come from the "0th" row of the difference table. See page 22 of Chapter 1 for a refresher for all this.

So, if

$$f(x) = 4\binom{x}{3} - 13\binom{x}{2} + 5\binom{x}{1} - 6\binom{x}{0},$$

we know exactly where 4, −13, 5, and 6 come from:

$$\begin{aligned} -6 &= \Delta^0(f)(0) = f(0) \\ 5 &= \Delta^1(f)(0) \\ -13 &= \Delta^2(f)(0) \\ 4 &= \Delta^3(f)(0) \end{aligned}$$

Perhaps it's more interesting that you can go the other way, too: you can construct f from the 0th row of its difference table.

and the difference table for f starts out:

n	$f(n)$	$\Delta(f)(n)$	$\Delta^2(f)(n)$	$\Delta^3(f)(n)$	$\Delta^4(f)(n)$	$\Delta^5(f)(n)$	$\Delta^6(f)(n)$	…
0	−6	5	−13	4	0	0	0	…
1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

This is the same f as the one above, just written in a different form.

But what if you write a polynomial in the "normal" form? Say

$$f(x) = \frac{2}{3}x^3 - \frac{17}{2}x^2 + \frac{77}{6}x - 6.$$

What interpretation can you give to the numbers

$$\frac{2}{3}, \quad \frac{17}{2}, \quad \frac{77}{2}, \quad \text{and} \ -6?$$

In this chapter, we'll answer this question in two ways: More precisely,

- We'll relate the coefficients of a polynomial to the values of the function (and the values of its derivatives).
- We'll see how the derivative arises in a completely algebraic way, with no recourse to calculus.
- We'll also see how the coefficients are related to the zeros of the function—generalizing the "sum and product of the roots" topics from algebra.

To be precise, functions have zeros and equations have roots, but we'll use the terms "zero" and "root" interchangeably.

More generally, we'll look into how changing the form of a polynomial expression gives you insight into the underlying function.

2.1 Polynomials

Let's be a little more precise about what we mean by the word "polynomial." The word is used in two closely related ways in school mathematics:

- **Polynomial *Functions*:** When you think about the "letter" in a polynomial as a *variable*, you are thinking of the polynomial as a machine that takes inputs to outputs. It generates a table, has a graph, and has all the other attributes of real-valued functions of a real variable. This is the point of view underneath graphing and tabulation (spreadsheet) technology. Using this point of view, two polynomial functions are equal if they represent the *same function*: They produce the same output for any input—they always tabulate the same and they have the same graph.
- **Polynomial *Forms*:** When you think of a polynomial as a formal expression, the "letter" is an *indeterminate*. This is the "algebra 2" perspective that you use when you factor, add, multiply, combine like terms, and so on. It's the point of view underneath CAS technology. Looked at as forms, two polynomials are equal if, when written in "standard" form, they have the same degree and the same coefficients. Of course, you may have to use some algebra to get them into the same standard form.

The mathematician Herman Weyl put it this way:

This is from an essay that Weyl wrote in 1931, entitled "Topology and Abstract Algebra as Two Roads of Mathematical Comprehension."

> "The system of real numbers is like a Janus head with two oppositely directed faces. In one respect it is the domain of + and × and their inverses, and in another it is a continuous manifold, and the two are continuously related. One is the algebraic and the other is the topological face of numbers. The idea that the argument x is a variable that traverses continuously

> its values is foreign to algebra; it is just an indeterminate, an empty symbol that binds the coefficients of the polynomial into a uniform expression that makes it easier to remember the rules for addition and multiplication. 0 is the polynomial all of whose coefficients are 0 (not the polynomial that takes on the value 0 for all values of the variable x).”

For our purposes, form and function represent two different ways to *think about* polynomials.

- When you think of forms, you are thinking about algebraic calculations and transformations with polynomials, forgetting about the meaning of the variable, and establishing *identities*. An identity is established by showing that each side can be transformed into the same “normal form” using the rules of algebra.
- When you think about functions, you worry about tables and graphs, notions like continuity, slopes of tangent lines (rates of change), the possible numerical *values* you can obtain, and, what’s especially important in this chapter, the properties of the function that you can figure out by substituting numbers and expressions for the variable.

Let’s adopt the convention that “normal” means that you write the polynomial as a sum of monomials whose powers go from largest to smallest, like $5x^4 - 3x^3 + 3x^2 - 2$. Your CAS may use a different convention, but it should be easy to translate its normal form to this one.

Form and function come together in the *theory of equations*. Sometimes, when you change the form of a polynomial expression, you gain insight into properties of the associated function. Writing $f(x) = x^2 - 1$ as $f(x) = (x+1)(x-1)$ tells you what inputs produce 0 as an output.

It’s certainly true that polynomials that are equal as forms are equal as functions. Indeed, if you have a polynomial identity, you have infinitely many numerical identities, because you can substitute numbers for the variable. So, since $x^2 - 1 = (x-1)(x+1)$ *as polynomials*, we know that

- $5^2 - 1 = 4 \cdot 6$,
- $8^2 - 1 = 7 \cdot 9$,
- $900^2 - 1 = 899 \cdot 901$,
- $3 - 1 = (\sqrt{3} - 1)(\sqrt{3} + 1)$,

and so on. In fact, you can substitute *expressions* (other polynomials, for example) for x and obtain other identities:

- $(a+2)^2 - 1 = (a+1)(a+3)$,
- $(3x^2 - 4x + 4)^2 - 1 = (3x^2 - 4x + 3)(3x^2 - 4x + 5)$,
- $(x^8 - 1)^2 - 1 = (x^8 - 2)x^8$.

Let’s call this the “form implies function” property of polynomials.

Some people describe this “true under any substitution” property as the “universal property” of identities.

Because we are working over $\mathbb{R}$, the converse is also true: If two polynomials define the same function, then they are equal as polynomials, and we’ll prove that in section 2.2. But this isn’t the case over every number system.

“Function implies form” (if we are working over $\mathbb{R}$).

The distinctions between polynomial forms and functions tend to be ignored in school mathematics. Getting the distinctions straight has made it easier to emphasize the right things at the right times with students, even if they are never asked to “learn” the precise distinctions for a test. For example, students often wonder about the importance of the quadratic formula, given the existence of calculators that will approximate roots of equations to any reasonable accuracy.

If you care about the *value* of your answer, numerical approximation may be good enough. If you care about its *form*, approximations will not do the trick.

Well, the quadratic formula tells us about the *algebraic* character of the roots. The numerical routines in the calculator give us information about the *values* of the roots. When looking for the numbers that make $x^2 - x + 1$ equal to 0, 10-place accuracy is probably all you'd ever need. But if you want to use the roots of this equation to generate and prove properties of Fibonacci numbers (see problems 28–30), the *algebraic* properties of the roots are what you care about.

Example: Here are two algebraic expressions:

$$f(x) = \frac{x^3 - 3x^2 - 4x + 12}{x - 2} \qquad \text{and} \qquad g(x) = x^2 - x - 6.$$

Are they equal? It depends on your frame of mind:

In abstract algebra, people say that "$f = g$ in the field of rational expressions over $\mathbb{R}$."

- If you look at them as formal expressions, they *are* equal because "equal" in this context means you can transform one into the other using legal moves:

$$\begin{aligned}\frac{x^3 - 3x^2 - 4x + 12}{x - 2} &= \frac{(x-2)(x+2)(x-3)}{x-2} \\ &= (x+2)(x-3) \\ &= x^2 - x - 6.\end{aligned}$$

People sometimes say that "$f = g$ except at 2." The graph of f has a hole above 2, but the graph of g is a continuous curve.

- If you look at them as functions, they are *not* equal because they have different domains: $g(2) = -4$, but $f(2)$ is undefined.

In abstract algebra, what we are calling a "system" is known as a *ring*.

Ways to think about it

Let's look at this a little more abstractly. We have two systems, polynomial forms in one variable with real coefficients (usually denoted by $\mathbb{R}[x]$), and polynomial *functions* on $\mathbb{R}$ that come from these polynomial forms (let's denote the system of these functions by $\mathbb{R}\langle x\rangle$).

These are *systems* rather than *sets*, because each of them allows for addition and multiplication: You can add and multiply polynomial forms and you can add and multiply polynomial functions:

- The way you add and multiply forms is, well, *formally*: You add expressions term by term, and you multiply expressions using the general distributive law, collecting like terms at the end.
- The way you add and multiply functions has nothing to do with the form used to represent the functions; the operations are carried out *functionally*—that is, by telling what the sum and product do to an input. And what they do is defined by two equations:

$$(f + g)(x) = f(x) + g(x), \quad \text{and} \quad (fg)(x) = f(x) \cdot g(x).$$

(*continued*)

Technically, two functions are equal if they have the same domain and produce the same output for each input in that domain. Since polynomial functions all have the same domain (all of $\mathbb{R}$), we don't need to mention it in the definition of addition and multiplication.

> These are subtle equations. In the first one, for example, the "+" means different things on both sides of the equation—on the left, it's the sum of two *functions*, and on the right, it's the sum of two *outputs* (numbers in our case). The equation defines addition of functions in terms of addition of outputs.

Anyway, back to the abstractions: What we have is a correspondence between $\mathbb{R}[x]$ and $\mathbb{R}\langle x\rangle$, a function of sorts, that associates each polynomial form with a polynomial function. If we wanted to be fancy, we could name this correspondence, say, Λ and write

$$\Lambda : \mathbb{R}[x] \to \mathbb{R}\langle x\rangle$$

where Λ is defined by

$$\Lambda(f) = (x \mapsto f(x)).$$

This says that Λ of a polynomial form f is the function that assigns a number to what you get when you substitute the number for x in the polynomial.

Most of what has been discussed in this section can be stated in terms of Λ:

1. Λ is *well defined*. That means that a polynomial form can give rise to only one polynomial function ("form implies function").
2. Λ is *one-to-one*. That means that two different polynomial forms can't give rise to the same polynomial function ("function implies form").
3. Λ *preserves the structure* of $\mathbb{R}[x]$ and $\mathbb{R}\langle x\rangle$. This means that if you add two forms and look at the function they define, you get the same function as adding the functions defined by each form ("Λ of a sum" is the same as "the sum of the Λs"). Mathematicians like to illustrate this with what they call a "commutative diagram"—we'll spare you the details. And the same holds for multiplication.

More details about the Λ correspondence can be found in [2].

Facts and Notation

- For the rest of the book, when we say "polynomial," we'll usually mean *polynomial form*.
- When we want to think about a polynomial as a *function*, we'll usually say that—"let f be a polynomial function," for example.
- When we say a property of polynomials is *algebraic* or is "part of algebra" we mean that it can be derived using formal calculations of the kind you find on a CAS, thinking of polynomial forms.
- When we say a property of polynomials is *analytic* or is "part of analysis" we usually mean that it depends on thinking of polynomials as functions.

A sentence like "let f be a polynomial" usually means we are thinking of forms. If we write "$f(x)$," it's usually a context clue that we're thinking of functions. None of these conventions is hard and fast.

We'll be a little sloppy about this "one variable" restriction, because sometimes we'll want to replace "x" by "$x + a$" or even "$x + y$." In Chapter 3, we'll allow complex coefficients.

- Both flavors of polynomials will mean "polynomials in *one* variable with *real* coefficients." This includes things like

$$3x^2 - \frac{5}{2}x + 1, \quad \sqrt{3}y + y^5, \quad \text{and} \quad \frac{5(x-4)^4 + 7x^2 - 1}{\pi},$$

but not things like

Unless, of course, $x + 2$ is a factor of $5(x - 4)^4 + 7x^2 - 1$. Is it?

$$3y^2 - \frac{5}{2}x + 1, \quad \sqrt{3y} + y^5, \quad \text{and} \quad \frac{5(x-4)^4 + 7x^2 - 1}{x+2}.$$

Some of the problems are just fun. As you do them, ask yourself "am I thinking *form* or *function* or *both*?"

The next problem set provides some examples for the two ways to think about polynomials. Not all the problems are needed in what follows, but some of them are previews of ideas that will be fully developed later in the chapter. We'll refer back to these as we go on, so if you want to skip around in this set, you can do the appropriate problems as they are cited later.

Problems

1. Express each result in normal form:

$f(x)$	$f(2)$	$f(a)$	$f(a+2)$
$6x^2 - x - 2$			
$x^2 + x - 1$			
$x^2 + x + 1$			

2. Write each answer as a polynomial in x with coefficients that are polynomials in a:

$f(x)$	$f(x+a)$	$f(x) + f(a)$	$f(x-a)$	$f(x) - f(a)$	$f(ax)$	$f(a)f(x)$
$6x^2 - x - 2$						
$x^2 + x - 1$						
$x^2 + x + 1$						

3. For each function f, find

$f(x)$	The numbers α such that $f(\alpha) = 0$	The numbers α such that $f(\alpha) = 3$	The smallest and largest values of $f(x)$
$6x^2 - x - 2$			
$x^2 + x - 1$			
$x^2 + x + 1$			

4. Suppose

$$f(x, y) = (x + y + 1)^5 - x^5 - y^5 - 1.$$

(a) Show that

$$f(x, y) = 5(y + 1)(x + 1)(x + y)(x^2 + xy + y^2 + x + y + 1).$$

(b) Write f as a polynomial in x with coefficients that are polynomials in y.

(c) Write f as a polynomial in y with coefficients that are polynomials in x.

(d) For what numbers a is $f(x, a)$ identically 0?

(e) For what numbers a is $f(a, x)$ identically 0?

5. Suppose $f(x, y)$ and $g(x, y)$ are two polynomials in x and y, and we have the polynomial identity

$$f(x, y) = g(x, y).$$

Suppose also that $a_1, a_2, a_3, \ldots$ is a sequence of numbers. If, in f and g, we replace each power of y, y^k, with a_k, we get two polynomials in one variable, x. Show that these, too, are identically equal.

6. Let's let $\deg(f)$ denote the degree of the polynomial f (assume f is a polynomial in one variable x). If f and g are polynomials, find a relationship among

Try some concrete examples to find a pattern.

(a) $\deg(fg)$, $\deg(f)$, and $\deg(g)$.

(b) $\deg(2g)$ and $\deg(g)$.

(c) $\deg(f + g)$, $\deg(f)$, and $\deg(g)$.

7. What are the polynomials of degree 0?

8. What about the number 0? The convention is to call the degree of 0 "$-\infty$." Explain why this is a good idea.

Hint: Look at your solution to problem 6a.

9. Show that a polynomial of degree n can have at most n linear factors (that is, factors of the form $ax + b$).

10. Let f be any polynomial function. What is the domain of f?

11. If f and g are any functions, $f \circ g$ is the function defined by

$$(f \circ g)(x) = f\big(g(x)\big).$$

(a) Find $h \circ k$ if $h(x) = x^3 - 1$ and $k(x) = x^2 + 1$.

(b) Show that, if f and g are polynomial functions, so is $f \circ g$.

How is the polynomial $f \circ g$ obtained from the polynomials f and g?

(c) Show that, in general, $f \circ g \neq g \circ f$.

(d) Find two polynomials f and g so that $f \circ g = g \circ f$.

(e) True or false? $\deg(f \circ g) = \deg(g \circ f)$.

(f) If $r(x) = 2x^2 - 1$ and $s(x) = 5x - 20x^3 + 16x^5$, show that $r \circ s = s \circ r$.

In Chapter 3, we'll see how to construct "commuting" polynomials like r and s. It's connected to trigonometry.

Here Δ is the usual difference operator, a function in its own right. This problem shows that, if you're careful, you can do "algebra" with powers of Δ, thinking of $+$ and $\circ$ as the $+$ and $\times$ of first-year algebra.

12. Show that

$$(\Delta + \Delta^2) \circ (\Delta^3 + \Delta^4) = \Delta^4 + 2\Delta^5 + \Delta^6.$$

We'll use the shorthand "the graph of f" to mean the graph of the equation $y = f(x)$: the set of all points in the Cartesian plane of the form $(x, f(x))$, as x runs over the real numbers.

13. If f and g are polynomial functions,

(a) How is the the graph of $y = f(x) + g(x)$ related to the graphs of f and g?

(b) How is the the graph of $y = 2f(x)$ related to the graph of f?

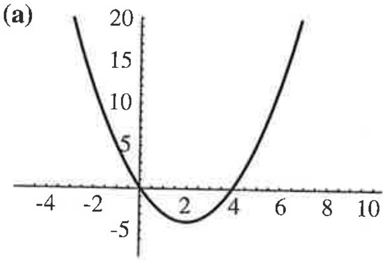
(c) How is the the graph of $y = f(x) \cdot g(x)$ related to the graphs of f and g?

(d) How is the the graph of $y = f \circ g(x)$ related to the graphs of f and g?

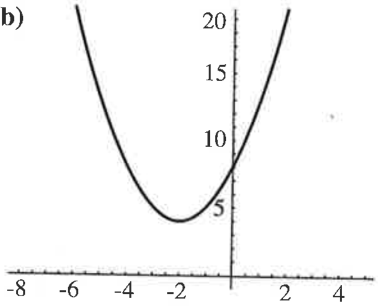
Assume the graph doesn't do anything strange outside the shown window.

14. Name at least two polynomial functions that could have each picture as its graph.

(a)



(b)



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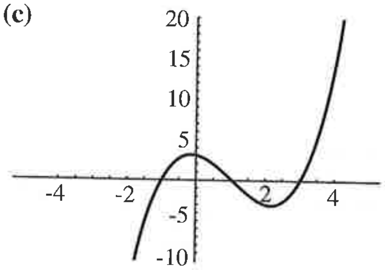

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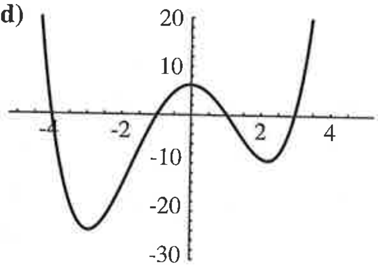


15. Show that the square of an odd number is always one more than a multiple of 4.

If you project the graph of f onto the y-axis, you'll cover the axis completely.

16. Suppose f is a polynomial function with the property that every real number is an output of f. What can you say about $\deg(f)$? Proof?

17. Show that a polynomial (with real coefficients) of odd degree has a real zero.

i.e., one-to-one

18. What degrees are possible for polynomials f that have the property that whenever $f(a) = f(b)$, $a = b$?

19. If n is a positive integer, what is the normal form for this product?

$$(x-1)(1 + x + x^2 + \cdots + x^{n-1}).$$

Hint: Use problem 19 with $x = 2$.

20. If you earn one cent today and you double your daily rate every day, how many days will it take to earn more than $1000?

21. Suppose a number r, written in base 10, is a string of 100 "8"s. What is the highest power of 10 that is a factor of $9r + 8$?

22. Let $f(x) = 1 + x + x^2 + x^3 + x^4$. Show, without expanding, that all the coefficients of $f(x+1)$ except the leading one are divisible by 5.

Hint: Use problem 19 and the Binomial theorem.

23. If n is a positive integer, simplify

$$(x+1)(1 - x + x^2 - x^3 \pm \cdots \pm x^{n-1}).$$

Remember, "simplify" means "put in normal form." In this case, just expand the product and write the the terms in descending order.

24. If n is a positive integer, simplify

$$(x+1)(x^2+1)(x^4+1)(x^8+1)\cdots(x^{2^n}+1).$$

25. Factor over $\mathbb{R}$:

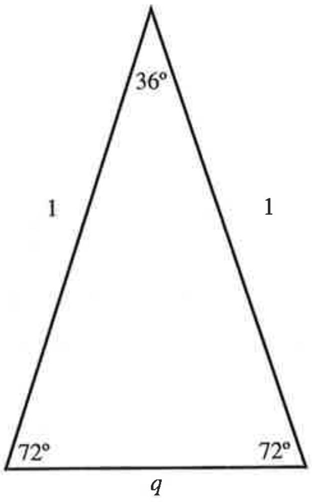
$$x^4 + x^2 + 1.$$

26. Prove the *quadratic formula*: The roots to $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

27. There's something special about a 72° angle: There's only one isosceles triangle (up to similarity) whose base angle is twice the vertex angle. It is the "72-72-36 triangle."

For geometry teachers: One consequence of this problem is that the side of a regular decagon inscribed in a circle can be constructed with straight-edge and compass. So the regular decagon, 20-gon, and $10n$-gon are all constructible with these tools. So is the pentagon (connect every other vertex of the decagon). How long is the side of the pentagon if the circle's radius is taken to be 1?



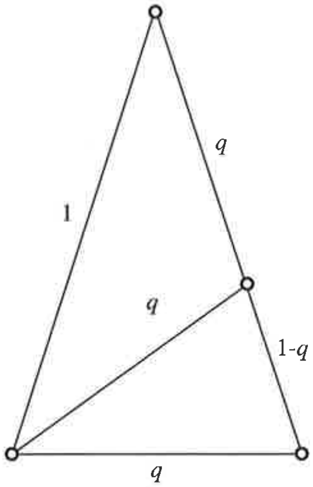
Suppose the equal sides of the triangle have length 1, and let q stand for the length of the base.

(a) Bisect one of the base angles of the triangle.

(b) Show that the small triangle is similar to the whole triangle.

(c) Use (b) to show that

$$\frac{1}{q} = \frac{q}{1-q}.$$



(d) Show that $q = 2\cos 72^\circ$.

(e) Solve for q and for $\cos 72^\circ$.

28. The Fibonacci numbers are the outputs of the function defined by the rule

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1, \\ f(n-1) + f(n-2) & \text{if } n > 1. \end{cases}$$

These numbers have many wonderful properties (see [11] for some of them). The sequence

$$\{f(0), f(1), f(2), f(3), f(4), f(5), f(6), \dots\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$$

is neither geometric (constant ratio between terms) nor arithmetic (constant difference between terms). But it still exhibits some regularity. Write out a few terms of the sequence of ratios

$$\left\{\frac{f(1)}{f(0)}, \frac{f(2)}{f(1)}, \frac{f(3)}{f(2)}, \frac{f(4)}{f(3)}, \frac{f(5)}{f(4)} \dots\right\}.$$

If you write these fractions as decimals, they seem to converge. What's a good approximation to the limit? What's the exact value of the limit?

Hint: $\beta = \frac{1-\sqrt{5}}{2}$ is also a root of this polynomial.

29. Find a polynomial with integer coefficients for which

$$\alpha = \frac{1+\sqrt{5}}{2}$$

is a root.

Tabulate it between 1 and 20, for example.

Hint: If $\alpha = \frac{1+\sqrt{5}}{2}$, $\alpha^2 = \alpha + 1$.

30. Experiment with this function:

$$g(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Show that g produces the Fibonacci numbers as outputs.

These are special cases of the "sum of roots" and the "product of roots" formulas.

31. Show that, if α and β are roots of the quadratic equation $x^2 + bx + c = 0$, then

(a) $\alpha + \beta = -b$, **(b)** $\alpha\beta = c$.

32. Show that

(a) $(x-r)(x-s) = x^2 - (r+s)x + rs$,

(b) $(x-r)(x-s)(x-t) = x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst$.

33. Generalize the result of problem 32 to an identity that expands

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n).$$

This identity will be useful in section 2.5.

34. Show that

$$(r+s)^3 - 3rs(r+s) = r^3 + s^3.$$

35. Consider the polynomials:

$$\{x^2-1, x^3-1, x^4-1, x^5-1, \dots, x^n-1, \dots\}.$$

(a) Investigate how these polynomials factor over the integers.

(b) Make a table:

n	number of factors of x^n-1
1	
2	
3	
4	
5	
6	
⋮	
20	

(c) For what values of n does it seem that x^n-1 factors into *exactly* two factors?

(d) Find a function that agrees with the above table.

Beware: the simplest function may not be polynomial; you might want to describe its rule in words.

2.2 The Basic Theorems

In Chapter 1, we stated several facts from algebra that may or may not be familiar to you. The purpose of this section is to sketch proofs of these facts. The facts themselves will be useful throughout the rest of this book. If you already know proofs, see if your proofs are the same as ours.

Theorem 1. (Long division) *If f and g are polynomials, with $g \neq 0$, there exist polynomials q and r such that the degree of r is strictly less than the degree of g and*

$$f = gq + r.$$

We call this "long division" because the proof of the theorem is based on the usual algorithm for dividing one polynomial into another. Consider this example:

$$
\begin{array}{rr}
 & 3x+\tfrac{5}{2} \\
\cline{2-2}
2x^2-x-1 & \big)\; 6x^3+2x^2+x-1 \\
 & 6x^3-3x^2-3x \quad\;\; \\
\cline{2-2}
 & 5x^2+4x-1 \\
 & 5x^2-\tfrac{5}{2}x-\tfrac{5}{2} \\
\cline{2-2}
 & \tfrac{13}{2}x+\tfrac{3}{2}
\end{array}
$$

You first find a trial quotient, namely $3x$ ("how many times does $2x^2$ go into $6x^3$?"), multiply the divisor by the quotient, subtract (killing the leading term of the "dividend"), and repeat, until the degree of the divisor is larger than the degree of the dividend. Now, to make this all precise might seem like more trouble than it's worth, but, in case you'd like to see the gory details, here's a "formal" proof:

The case $m < n$ is like a fact that confuses many students in arithmetic: The remainder when 37 is divided by 72 is 37 itself (and the quotient is 0).

Proof Suppose

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0.$$

We'll use induction on m, the degree of g. If $m < n$, the theorem is trivially true because

$$g = 0 \cdot f + g \quad \text{and} \quad \deg(g) < \deg(f).$$

So, suppose $m \geq n$ and the theorem is true for every polynomial h of degree less than m. Then let

$$h(x) = g(x) - \frac{b_m}{a_n} f(x)x^{m-n}.$$

Try to see how this really is just a formalization of what you do when you do long division.

If you write this all out, you see that this construction kills off the leading term in g so that $\deg(h) < m$. By the induction hypothesis, there are polynomials $s(x)$ and $r(x)$ so that

$$h(x) = s(x)f(x) + r(x) \quad \text{and} \quad \deg(r) < \deg(f).$$

The fact that you can do division with remainder for polynomials (that's the basic conclusion in Theorem 1) forms the basis for the structural similarity between polynomials and integers. It's the reason you can talk about factorizations, greatest common divisors, and the rest of arithmetic in both systems.

But then

$$g(x) - \frac{b_m}{a_n} f(x)x^{m-n} = s(x)f(x) + r(x)$$

so

$$g(x) = \left(\frac{b_m}{a_n}x^{m-n} + s(x)\right) f(x) + r(x).$$

If we let

$$q(x) = \left(\frac{b_m}{a_n}x^{m-n} + s(x)\right)$$

Note that the $r(x)$ is the same whether the dividend is h or g.

we have

$$g(x) = q(x)f(x) + r(x) \quad \text{and} \quad \deg(r) < \deg(f)$$

as advertised. ■

Theorem 2. (The remainder theorem) *If f is a polynomial, then the remainder when $f(x)$ is divided by $x - a$ is $f(a)$.*

We're assuming here that a is a number.

Proof Apply Theorem 1 to f and $x - a$. It says that

$$f(x) = q(x)(x - a) + r(x) \quad \text{and} \quad \deg(r) < \deg(x - a) = 1.$$

Well, the only polynomials of degree < 1 are the constants, so $r(x)$ is a *number*, call it r. So,

$$f(x) = q(x)(x - a) + r.$$

Now substitute a for x; you get

$$f(a) = q(a)(a - a) + r = q(a) \cdot 0 + r.$$

So, $r = f(a)$. ■

Theorems 1 and 2 are the basic results we need. Several useful corollaries follow:

Corollary 1. (The factor theorem) *If f is a polynomial, $(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$.*

The factor theorem is the classic example of form and function coming together: "$(x - a)$ is a factor of $f(x)$" is a statement about polynomial algebra, and "$f(a) = 0$" talks about f as a function.

The proof is an exercise (problem 36).

Corollary 2. (The number of roots theorem) *A polynomial f of degree n can have at most n zeros.*

Proof Every root produces a linear factor, and every linear factor increases the degree of the product of factors by 1. You can make this precise using an argument similar to that in the proof of Theorem 10 on page 43 of Chapter 1. ■

Note that this proof breaks down if $f = 0$ (that is, if $\deg(f) = -\infty$), so a variation on this corollary is

Corollary 2. (Alternate version) *A polynomial f with more roots than its degree is identically 0.*

Corollary 3. (The function implies form theorem) *If two polynomials of degree at most n agree for more than n inputs, they are exactly the same polynomial.*

This is the converse of the "form implies function" property described on page 55.

Proof If f and g agree at $m > n$ numbers $\alpha_1, \dots, \alpha_m$, then $f - g$, a polynomial of degree at most n, vanishes at m numbers, so it is identically 0. If $f - g = 0$, $f = g$. ■

Corollary 4. *A polynomial of degree n is completely determined by any $n + 1$ of its values.*

Proof Two polynomials of degree n that agree at these $n+1$ values have to be identical by Corollary 3.

Example: Suppose we are looking for a simple (polynomial) formula that lets you calculate the sum of the first n squares. That is, we want a polynomial $f(x)$ so that

$$f(n) = 0^2 + 1^2 + \cdots + (n-1)^2.$$

The corresponding polynomial for the sum of the *first* powers is $\frac{x(x-1)}{2} = \binom{x}{2}$ because $0+1+2+\cdots+n-1 = \frac{n(n-1)}{2}$.

Imagine we had such a polynomial. Could we determine its degree? Well (in the spirit of Chapter 1), on positive integers, f would tabulate like this:

n	$f(n) = 0^2 + 1^2 + \cdots (n-1)^2$	$\Delta(f)(n)$
1	0	1
2	1	4
3	5	9
4	14	16
5	30	25
6	55	36
7	91	

Remember the hockey stick property?

Look at the Δ column. It looks like perfect squares. In fact, if n is any positive integer,

$$\begin{aligned}\Delta(f)(n) &= f(n+1) \qquad - \qquad f(n)\\ &= \left(0^2 + 1^2 + \cdots n^2\right) - \left(0^2 + 1^2 + \cdots + (n-1)^2\right)\\ &= n^2.\end{aligned}$$

n	$f(n)$	$\Delta(f)(n)$
1	0	1
2	1	4
3	5	9
4	14	16
5	30	25
6	55	36
7	91	

And, if $\Delta(f)(x) = x^2$ and $f(1) = 0$, the hockey stick property of Chapter 1 implies that

$$f(n) = 0 + 1 + 4 + 9 + \cdots + (n-1)^2$$

and that's what we want. So, we're on the hunt for a polynomial f so that $f(1) = 0$ and $\Delta(f)(x) = x^2$. But, by problem 73 on page 37 of Chapter 1, this implies that f must have degree 3. We know that a cubic is determined by four values (Corollary 4), so fit a cubic to this table:

"Fit a cubic" using the methods of Chapter 1, for example.

If you stumbled on an f so that $\Delta(f)(x) = x^2$ but $f(1) \neq 0$, you could subtract the number $f(1)$ from f and not ruin the fact that $\Delta(f)(x) = x^2$ (see problem 48 on page 34 of Chapter 1).

Input	Output
1	0
2	1
3	5
4	14

We get

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x = \frac{x(x-1)(2x-1)}{6}.$$

You can check that $\Delta(f)(x) = x^2$. Since $f(1) = 0$, we have the nice formula

$$0^2 + 1^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}.$$

Ways to think about it

We're being a little cavalier about the way we use words like "root" and "number." Even if we restrict the coefficients of our polynomials to be real numbers, the zeros of the corresponding functions may well turn out to be complex numbers. We'll deal explicitly with complex numbers in the next chapter, but for now, we have to allow them to lurk behind the scenes. For example, in problem 31 on page 62, the sum and product of the roots are what the problem claims they are (nice real numbers), even if the roots are complex.

In fact, in section 2.6, we'll make implicit use of a sort of converse to the "number of roots theorem" (Corollary 2), the celebrated *fundamental theorem of algebra* that says that every polynomial of degree n with real (or even complex) coefficients factors into exactly n linear factors in the complex numbers. The FTA implies that every polynomial equation of degree n has n roots in the complex numbers (if we count multiple roots as distinct). So, we're safe in assuming that a polynomial *has* roots, and we know how many it has, as long as we assume the FTA.

The FTA is not so easy to prove. All proofs of the theorem need a so-called "analytic step:" a result that's part of analysis and not strictly algebra. For example, the fact that a polynomial of odd degree has a real root (problem 17 on page 60) is such an analytic step, one that is often used in proofs of the FTA.

Problems

36. Prove Corollary 1.

37. Prove that a polynomial with more roots than its degree is identically 0 (the "alternate version" of Corollary 2).

38. Prove that two polynomials that agree infinitely often are identical.

39. Find the remainders when each polynomial is divided by $x - 3$:

(a) $x^3 - 1$ **(b)** $x - 1$
(c) $x^2 + x + 1$ **(d)** $x^4 + x^2 + 1$
(e) $(x^4 + x^2 + 1)(x - 1)$ **(f)** $x^4 + x^3 + x^2$

Hint: Is long division really necessary?

40. Suppose f and g are polynomials and that $f(2) = 5$ and $g(2) = -4$. Find the remainder when each polynomial is divided by $x - 2$:

(a) $3f + g$ **(b)** fg^2
(c) $(x^2 + x + 1)f(x) + g(x)$

Is the converse of problem 41 true? See [13] for more on this theme.

41. Suppose $f(S, C)$ is a polynomial in two variables that vanishes for all pairs of numbers (S, C) such that $S^2 + C^2 = 1$. Show that there is a polynomial $g(S, C)$ so that

$$f(S, C) = (S^2 + C^2 - 1)g(S, C).$$

Problems 42–44 outline a different proof of the remainder theorem.

This generalizes problem 19 on page 60.

42. Establish the identity

$$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + a^3x^{n-4} + \cdots + a^{n-2}x + a^{n-1}).$$

43. Suppose

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

Show that

$$f(x) - f(a) = (x - a)g(x)$$

for some polynomial g (in x and a), and write down an explicit expression for g.

44. Use problem 43 to prove the remainder theorem.

45. Find a formula for the sum of the first n cubes, starting with 0:

$$0^3 + 1^3 + 2^3 + \cdots + (n - 1)^3.$$

2.3 Coefficients and Values

We're writing our generic polynomial in "reverse" normal form, just because it makes some formatting easier later in this section.

Let's take up the question we asked on page 54: What if we write a polynomial, not as a linear combination of the $\binom{x}{k}$, but as a linear combination of the "standard basis" (the x^k)? Suppose

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_mx^m.$$

What can you say about the c_k? Are they related to anything like the "$\Delta^k(f)(0)$"? Well, yes, and the exact analogy might be a bit of a surprise. Here's the story.

One of the things you need to do when you calculate $\Delta(f)$ is to figure out $f(x+1)$. The question here is more general: what is a formula for $f(x+a)$?

When you teach about polynomial functions, you probably ask students, given a formula for $f(x)$, to find a formula for $f(-x)$, $f(x + 2)$, or maybe even $f(x + a)$.

Example: Suppose $f(x) = 1 - 4x + 5x^2 + 3x^4$. Then

$$\begin{aligned} f(x + a) &= 1 - 4(x + a) + 5(x + a)^2 + 3(x + a)^4 \\ &= 1 - 4(x + a) + 5(x^2 + 2xa + a^2) \\ &\quad + 3(x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4) \end{aligned}$$