1.2 Doing It with Differences

Let's switch perspective a little. The goal is still to find a simple polynomial function that agrees with a table, but, instead of working from "right to left" in these difference tables, let's work from "bottom to top." An example will help show what we mean. Here's a table (the one from problem 8), together with all the difference columns, until we get a constant:

When looking for an algorithm that solves a class of problems, it's a good idea to work out some numerical examples, concentrating on the rhythm of the calculations rather than on the numbers with which you work. This next discussion is a good example of that.

Input	Output	Δ	Δ^2	Δ^3
0	1	−2	14	12
1	−1	12	26	12
2	11	38	38	12
3	49	76	50	12
4	125	126	62	12
5	251	188	74	
6	439	262		
7	701			

n	$f(n)$	Δ	Δ^2	Δ^3
0	1	−2	14	12
1	−1	12	26	12
2	11	38	38	12
3	49	76	50	12
4	125	126	62	12
5	251	188	74	
6	439	262		
7	701			

The third difference is constant. Suppose we want to find a function f (a polynomial, say) that agrees with this table. Let's just see what we can get from the "up and over" feature of the table.

Notice that, by the way we built difference tables, *every* entry in the interior of the table is the sum of its "up and over." $49 = 11 + 38$, $62 = 50 + 12$, $262 = 188 + 74, \ldots$. So, we can take any entry in the $f(n)$ column, replace it by its up and over, replace these two numbers by *their* up and overs, and keep moving up the table toward the first row. Let's try it with $f(3)$:

$$\begin{aligned} f(3) &= 49 \\ &= 11 + 38 \\ &= (-1 + 12) + (12 + 26) = -1 + 2 \cdot 12 + 26 \\ &= (1 + -2) + 2 \cdot (-2 + 14) + 14 + 12 = 1 + 3 \cdot (-2) + 3 \cdot 14 + 12. \end{aligned}$$

Look again, with some emphasis added and details suppressed:

There are many ways to write $f(3)$ as a sum of numbers. Here, we're trying to follow the *form* of how you'd chase $f(3)$ up to the 0th row. Try it yourself, writing each step in terms of numbers in the table.

$$\begin{aligned} f(3) &= \mathbf{1} \cdot 49 \\ &= \mathbf{1} \cdot 11 + \mathbf{1} \cdot 38 \\ &= \mathbf{1} \cdot -1 + \mathbf{2} \cdot 12 + \mathbf{1} \cdot 26 \\ &= \mathbf{1} \cdot 1 + \mathbf{3} \cdot (-2) + \mathbf{3} \cdot 14 + \mathbf{1} \cdot 12. \end{aligned}$$

Could it be? It looks as if Pascal's triangle is rearing its head once again. Let's try it for $f(4)$.

$$
\begin{aligned}
f(4) &= \mathbf{1}\cdot 125\\
&= \mathbf{1}\cdot 49+\mathbf{1}\cdot 76\\
&= \mathbf{1}\cdot(11+38)+\mathbf{1}\cdot(38+38)\\
&= \mathbf{1}\cdot 11+\mathbf{2}\cdot 38+\mathbf{1}\cdot 38\\
&= \mathbf{1}\cdot(-1+12)+\mathbf{2}\cdot(12+26)+\mathbf{1}\cdot(26+12)\\
&= \mathbf{1}\cdot -1+\mathbf{3}\cdot 12+\mathbf{3}\cdot 26+\mathbf{1}\cdot 12\\
&= \mathbf{1}\cdot(1+-2)+\mathbf{3}\cdot(-2+14)+\mathbf{3}\cdot(14+12)+\mathbf{1}\cdot(12+0)\\
&= \mathbf{1}\cdot 1+\mathbf{4}\cdot(-2)+\mathbf{6}\cdot 14+\mathbf{4}\cdot 12+\mathbf{1}\cdot 0.
\end{aligned}
$$

In order to make this work, we had to add more columns to the difference table:

It *is* true that $\Delta^4 = 0$ for this table, right?

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	−2	14	12	0	0
1	−1	12	26	12	0	0
2	11	38	38	12	0	0
3	49	76	50	12	0	
4	125	126	62	12		
5	251	188	74			
6	439	262				
7	701					

Facts and Notation

For our current discussion about Pascal's triangle, we need three things: some notation for its entries, an algorithm for generating the triangle, and an explicit formula for the entries in terms of factorials. To that end:

Pascal's triangle has many wonderful properties and is connected to a host of mathematical phenomena. For more details and a complete development, see, for example, [18].

There are several different ways to get the numbers in Pascal's triangle. It's by no means obvious that all the ways produce the same numbers, but they do. We'll list several ways here. The proofs that all these methods yield the same results can be carried out by induction or by other means (see [15], for example). If you have time, try proving some of the equivalences for yourself. We'll use $\binom{n}{k}$ to stand for the entry in the nth row and kth position in the triangle. Depending on your previous experiences, this notation might conjure up any one of the following ways to think about Pascal's triangle. Think of *that* way as a definition and think about the rest of the ways as theorems.

- *Pascal's triangle is a recursively generated number pattern.*

$$\binom{0}{0}=1$$
$$\binom{1}{0}=1 \quad \binom{1}{1}=1$$
$$\binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$$
$$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$$
$$\binom{4}{0}=1 \quad \binom{4}{1}=4 \quad \binom{4}{2}=6 \quad \binom{4}{3}=4 \quad \binom{4}{4}=1$$
$$\binom{5}{0}=1 \quad \binom{5}{1}=5 \quad \binom{5}{2}=10 \quad \binom{5}{3}=10 \quad \binom{5}{4}=5 \quad \binom{5}{5}=1$$

Notice that the numbering starts with 0. So, for example,

$$\binom{6}{0}=1 \quad \text{and} \quad \binom{6}{1}=6.$$

The rule for *generating* the triangle in this way can be stated in words:

This "two-term recurrence" (every number is the sum of the two above it) is satisfied by many mathematical phenomena.

Each row starts and ends with 1, and any interior element is the sum of the two above it

or in symbols:

$$\binom{n}{k}=\begin{cases} 1 & \text{if } k=0 \text{ or if } k=n \text{ (each row starts and ends with 1), and} \\ \binom{n-1}{k-1}+\binom{n-1}{k} & \text{if } 0<k<n \text{ (an interior element is the sum of the two above it).} \end{cases}$$

- *The entries in Pascal's triangle count subsets.* Suppose you have a set of five elements, say $\{A, B, C, D, E\}$. How many three-element subsets are there? There are $\binom{5}{3}=10$. Here they are:

$$\{A,B,C\}, \{A,B,D\}, \{A,B,E\}, \{A,C,D\}, \{A,C,E\}, \{A,D,E\},$$
$$\{B,C,D\}, \{B,C,E\}, \{B,D,E\}, \{C,D,E\}.$$

This is called the binomial theorem, and it is proved in [15].

- *The entries in Pascal's triangle are the coefficients in $(a+b)^n$.* The entries in the nth row are the coefficients in the expansion of $(a+b)^n$. More precisely,

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{n-2}a^2b^{n-2} \\ &\quad +\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n \\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.\end{aligned}$$

- *The entries in Pascal's triangle are quotients of factorials.* There's an explicit formula for $\binom{n}{k}$ in terms of factorials:

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}.$$

So, for example,

$$\begin{aligned}\binom{12}{5} &= \frac{12!}{5!7!}\\ &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1\times 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\\ &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1}\\ &= 792.\end{aligned}$$

Making this factorial formula hold for the cases $k = 0$ and $k = n$ is one of the reasons for defining 0! to be 1.

- *The entries in Pascal's triangle are rational expressions.* Sometimes, it's useful to do the cancellations in the factorial expression and write $\binom{n}{k}$ as a product of factors:

$$\begin{aligned}\binom{n}{k} &= \frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)\ (n-k)\cdots 1}{k!\,(n-k)!}\\ &= \frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{k!}.\end{aligned}$$

One advantage of this expression over all the others is that n can be any number here. In a sense, this expression extends the formula for entries in Pascal's triangle from integers to real (or even complex) numbers. This will be important in section 1.3.

The symbol $\binom{\pi}{3}$ doesn't make sense in the context of any of the other methods. But here, it's just

$$\frac{\pi(\pi-1)(\pi-2)}{6}.$$

So, it looks as if we have a general method for calculating $f(n)$ for all the inputs in our table on page 11. Let's see if we can turn this into an explicit (polynomial) formula for f. First of all, we can ignore the Δ^e columns for $e > 3$, because they are all 0. So far we have:

$$\begin{aligned}f(3) &= 49 &= \mathbf{1}\cdot 1+\mathbf{3}\cdot(-2)+\ \mathbf{3}\cdot 14+\ \mathbf{1}\cdot 12,\\ f(4) &= 125 &= \mathbf{1}\cdot 1+\mathbf{4}\cdot(-2)+\ \mathbf{6}\cdot 14+\ \mathbf{4}\cdot 12,\\ f(5) &= 251 &= \mathbf{1}\cdot 1+\mathbf{5}\cdot(-2)+\mathbf{10}\cdot 14+\mathbf{10}\cdot 12,\\ f(6) &= 439 &= \mathbf{1}\cdot 1+\mathbf{6}\cdot(-2)+\mathbf{15}\cdot 14+\mathbf{20}\cdot 12,\\ f(7) &= 701 &= \mathbf{1}\cdot 1+\mathbf{7}\cdot(-2)+\mathbf{21}\cdot 14+\mathbf{35}\cdot 12.\end{aligned}$$

Many discoveries in algebra are made through a careful and mechanical analysis of a calculation, concentrating on the *form* of the calculation rather than on its *value*. Notice that the equations start with $f(3)$ and not $f(0)$. Why?

And, if we determined further outputs for f by "pulling out" from the constant difference of 12, the same method would work for $f(8)$, $f(9)$, and $f(10)$: If you snake back through the difference table, winding your way to the top row by replacing each number by its "up and over," you'll get a combination of the elements in the first row, and the coefficients will be precisely the entries in the "input" row of Pascal's triangle. So, it looks as if we could calculate $f(n)$ by the rule

$$f(n) = \binom{n}{0}\cdot 1+\binom{n}{1}\cdot(-2)+\binom{n}{2}\cdot 14+\binom{n}{3}\cdot 12.$$

By "pulling out from the constant difference of 12," we mean adding a few more 12s in the Δ^3 column and then filling the table in, right to left, using the "up and over" rule.

This certainly works if n is 3, 4, 5, 6, or 7. Before we worry about the details, let's see if this will help us get a general formula for f. Since we're looking for a polynomial, let's get rid of the $\binom{n}{k}$ expressions and replace them with their algebraic equivalents:

$$f(n) = 1 \cdot 1 + n \cdot (-2) + \frac{n(n-1)}{2} \cdot 14 + \frac{n(n-1)(n-2)}{6} \cdot 12.$$

Oh, look at this. A little algebra (by hand or with a CAS) can turn this into a nice cubic:

$$f(n) = 2n^3 + n^2 - 5n + 1.$$

You can check that this function agrees with the table.

Ways to think about it

There are many loose ends here. For one, the formula that gave us the nice cubic:

$$f(n) = \binom{n}{0} \cdot 1 + \binom{n}{1} \cdot (-2) + \binom{n}{2} \cdot 14 + \binom{n}{3} \cdot 12 \qquad (*)$$

doesn't work if n is 0, 1, or 2, because, for example, we have no meaning for $\binom{2}{3}$. In fact, chasing the "up and over" property for 0, 1, and 2 up to the top row produces what look like different equations, "truncated" versions of $(*)$:

$$\begin{aligned} f(0) &= \binom{0}{0} \cdot 1, \\ f(1) &= \binom{1}{0} \cdot 1 + \binom{1}{1} \cdot (-2), \\ f(2) &= \binom{2}{0} \cdot 1 + \binom{2}{1} \cdot (-2) + \binom{2}{2} \cdot 14. \end{aligned}$$

But formula $(*)$ says that these equations should be

$$\begin{aligned} f(0) &= \binom{0}{0} \cdot 1 + \binom{0}{1} \cdot (-2) + \binom{0}{2} \cdot 14 + \binom{0}{3} \cdot 12, \\ f(1) &= \binom{1}{0} \cdot 1 + \binom{1}{1} \cdot (-2) + \binom{1}{2} \cdot 14 + \binom{1}{3} \cdot 12, \\ f(1) &= \binom{2}{0} \cdot 1 + \binom{2}{1} \cdot (-2) + \binom{2}{2} \cdot 14 + \binom{2}{3} \cdot 12. \end{aligned}$$

Ah, but if we look at $\binom{n}{3}$ as a rational expression in n (as in the last bullet on page 13), $\binom{2}{3}$, for example, and more generally $\binom{n}{k}$ with $k > n$, make sense and are, in fact, 0:

$$\binom{2}{3} = \frac{2(2-1)(2-2)}{3!} = 0.$$

Let's do that from now on: Whenever we see $\binom{n}{k} = 0$ with $k > n$, we'll think of the rational expression form, and replace it by 0.

So, from now on, $\binom{3}{5} = 0$, $\binom{5}{21} = 0$, $\binom{6}{7} = 0$, and $\binom{0}{1} = 0$. This even makes sense using the "subset" interpretation of $\binom{2}{3}$: How many 3-element subsets does a 2-element set have? None. In some investigations, it's even useful to extend Pascal's triangle to negative numbered rows.

Problems

10. Recall this table from page 11:

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	−2	14	12	0	0
1	−1	12	26	12	0	0
2	11	38	38	12	0	0
3	49	76	50	12	0	
4	125	126	62	12		
5	251	188	74			
6	439	262				
7	701					

(a) Show that

$$\begin{aligned} f(5) &= \mathbf{1}\cdot 1 + \mathbf{5}\cdot(-2) + \mathbf{10}\cdot 14 + \mathbf{10}\cdot 12 + \mathbf{5}\cdot 0 + \mathbf{1}\cdot 0 \\ &= \mathbf{1}\cdot 1 + \mathbf{5}\cdot(-2) + \mathbf{10}\cdot 14 + \mathbf{10}\cdot 12. \end{aligned}$$

(b) Show similar results for $f(6)$ and $f(7)$ too.

It might be handy to pull out a copy of Pascal's triangle for reference.

11. Use the method of this section to find a polynomial that agrees with each table.

(a)

Input	Output
0	1
1	5
2	25
3	79
4	185
5	361
6	625
7	995

(b)

Input	Output
0	−1
1	0
2	−5
3	−22
4	−57
5	−116
6	−205
7	−330

(c)

Input	Output
0	−1
1	0
2	7
3	26
4	63
5	124
6	215
7	342

(d)

Input	Output
0	−1
1	4
2	43
3	194
4	583
5	1384
6	2819
7	5158

(e)

Input	Output
0	2
1	9
2	26
3	53
4	90
5	137
6	194
7	261

(f)

Input	Output
0	1
1	6
2	63
3	364
4	1365
5	3906
6	9331
7	19608

12. If a table has 12 inputs, for $n = 0, \ldots, 11$, how many Δ columns can be created?

For fun, fill in the rest of the table and check your answer.

13. Find a polynomial function that agrees with this table:

Input	Output	Δ	Δ^2	Δ^3
0	3	-4	5	2
1				2
2				
3				
4				
5				

Is there more than one possible answer to this? Is there more than one polynomial function of a given degree that agrees with the table?

14. Find a polynomial function that agrees with this table:

Input	Output	Δ	Δ^2	Δ^3
0	a	b	c	d
1				d
2				
3				
4				
5				

Is there more than one possible answer to this?

15. Prove Theorem 1 below:

Theorem 1. *Suppose we have a table with inputs* $0, \ldots, m$*:*

Input	*Output*	Δ	Δ^2	Δ^3	$\ldots$	Δ^m
0	a_0	a_1	a_2	a_3	$\ldots$	a_m
1						
2						
3						
4						
5						
6						
$\vdots$						
m						

If f is a function that agrees with the table, then for $0 \le n \le m$:

$$f(n) = \sum_{k=0}^{m} a_k \binom{n}{k}.$$

1.3 Finding a Formula: Combinatorial Polynomials

Theorem 1 is the basis for a simple technique, called *Newton's Difference Formula*, to find a polynomial—of smallest degree, even—that agrees with any finite input-output table (whose inputs are consecutive integers starting with 0). To do this, we introduce a new collection of polynomials that "extend" the binomial coefficients.

For example, is there a polynomial function g that has the property that when you take $g(n)$ (n an integer), you get $\binom{n}{3}$? The answer is deceptively simple. To think about $\binom{n}{3}$, look at the rational expression:

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

So, let

$$g(x) = \frac{x(x-1)(x-2)}{6}.$$

Then g is certainly a polynomial function, so you can take g of any real (or even complex) number. In fact, expanding the fraction, we get $g(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x$ so g is given by a cubic polynomial with rational coefficients. And looking at g in its factored form, one sees that $g(n) = \binom{n}{3}$ for any integer n. So, we have "extended" the binomial coefficient $\binom{n}{3}$ by a cubic polynomial.

More generally, we can define a sequence of polynomials that extends all the binomial coefficients.

We're using 3 here just for the sake of example. Any other positive integer would do.

In[15], we often use another device to define functions, especially when we don't care about the function's name. Instead of writing "$f(x) = x^2 + 1$", we often write "$x \mapsto x^2 + 1$" and say, "x maps to $x^2 + 1$."

Extension in this sense means "extend the function to values not in its domain." Until now, $n \mapsto \binom{n}{3}$ had the nonnegative integers as a domain (remember that $\binom{n}{3} = 0$ if $n \in \{0, 1, 2\}$). The function g agrees with $n \mapsto \binom{n}{3}$ for nonnegative integers but is defined on all of $\mathbb{R}$.

Definition Suppose k is a nonnegative integer. The kth *combinatorial polynomial*, $\binom{x}{k}$, is defined by the rule

$$\binom{x}{k} = \frac{x(x-1)(x-2)(x-3)\cdots(x-k+1)}{k!}$$

By convention, we take $\binom{x}{0}$ to be 1.

> *Ways to think about it*
>
> There's a suggestive piece of notation here. We use "$\binom{x}{k}$" to stand for the kth combinatorial polynomial. It extends the binomial coefficients $\binom{n}{k}$ to $\mathbb{R}$. You can think "when I replace x by n in $\binom{x}{k}$, I get $\binom{n}{k}$." But that's a *theorem*, made transparent by the notation.

These were generated with a CAS.

Here are a few of the combinatorial polynomials:

k	$\binom{x}{k}$
0	1
1	x
2	$\frac{-x+x^2}{2}$
3	$\frac{2x-3x^2+x^3}{6}$
4	$\frac{-6x+11x^2-6x^3+x^4}{24}$
5	$\frac{24x-50x^2+35x^3-10x^4+x^5}{120}$
6	$\frac{-120x+274x^2-225x^3+85x^4-15x^5+x^6}{720}$
7	$\frac{720x-1764x^2+1624x^3-735x^4+175x^5-21x^6+x^7}{5040}$

Before we investigate the properties of the various $\binom{x}{k}$, let's state one of the main results of this chapter:

Theorem 2. (Newton's Difference Formula.) *Suppose we have a table whose inputs are the integers between 0 and m:*

Input	*Output*	Δ	Δ^2	Δ^3	…	Δ^m
0	a_0	a_1	a_2	a_3	…	a_m
1						
2						
3						
4						
5						
6						
⋮						
m						

A polynomial function that agrees with the table is

$$f(x)=\sum_{k=0}^{m} a_k \binom{x}{k}.$$

Furthermore, f has degree at most m.

Proof The fact that f has degree at most m comes from the explicit formula for $\binom{x}{k}$:

$$\binom{x}{k} = \frac{x(x-1)(x-2)(x-3)\cdots(x-k+1)}{k!}.$$

There are k factors on top, so this has degree k. Since f is a sum of constant multiples of the $\binom{x}{k}$, its degree is no greater than the largest degree of any summand, that is, of $\binom{x}{m}$.

To see that the f given in the theorem does what it's supposed to do (that is, agrees with the table), use Theorem 1 and the fact that, if n is substituted for x in $\binom{x}{k}$, the result is $\binom{n}{k}$. ■

So, now fitting a polynomial to a table becomes a completely mechanical process.

Example: Here's the table from problem 11f on page 15.

The polynomial resulting from Newton's difference formula is quite simple in this case. That's because we wanted a simple example. Usually, the answers aren't so nice. But the method always works.

Input	Output
0	1
1	6
2	63
3	364
4	1365
5	3906
6	9331
7	19608

Let's complete its difference table:

We could *really* complete the difference table, adding the columns of 0s, but that wouldn't change the resulting function, right?

Input	Output	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	5	52	192	264	120
1	6	57	244	456	384	120
2	63	301	700	840	504	120
3	364	1001	1540	1344	624	
4	1365	2541	2884	1968		
5	3906	5425	4852			
6	9331	10277				
7	19608					

So, with no further fuss, here's a (degree 5) polynomial function that agrees with the table:

$$\begin{aligned} f(x) &= 1 \cdot \binom{x}{0} + 5 \cdot \binom{x}{1} + 52 \cdot \binom{x}{2} + 192 \cdot \binom{x}{3} + 264 \cdot \binom{x}{4} + 120 \cdot \binom{x}{5} \\ &= 1 + 5x + 52\frac{x(x-1)}{2} + 192\frac{x(x-1)(x-2)}{6} \\ &\quad + 264\frac{x(x-1)(x-2)(x-3)}{24} \\ &\quad + 120\frac{x(x-1)(x-2)(x-3)(x-4)}{120} \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

The last simplification was done with a little help from a CAS.

Notice that $f(x)$ can also be written as

$$\frac{x^6 - 1}{x - 1}.$$

Problems

These tables are from problem 11 on page 15. A CAS will help greatly here.

16. Use the result of Theorem 2 to find a polynomial that agrees with each table.

(a)

Input	Output
0	1
1	5
2	25
3	10
4	79
5	185
6	361
7	995

(b)

Input	Output
0	−1
1	0
2	−5
3	−22
4	−57
5	−116
6	−205
7	−330

(c)

Input	Output
0	−1
1	0
2	7
3	26
4	63
5	124
6	215
7	342

(d)

Input	Output
0	−1
1	4
2	43
3	194
4	583
5	1384
6	2819
7	5158

(e)

Input	Output
0	2
1	9
2	26
3	53
4	90
5	137
6	194
7	261

(f)

Input	Output
0	1
1	6
2	63
3	364
4	1365
5	3906
6	9331
7	19608

17. Fran hands you a table and asks you to find a function that agrees with it.

Input	Output
0	−5
1	−6
2	7
3	166
4	843
5	2770

Find a function f that will agree with Fran's table. To save you some work, here's the completed difference table:

Input	Output	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	−5	−1	14	132	240	120
1	−6	13	146	372	360	
2	7	1591	518	732		
3	166	677	1250			
4	843	1927				
5	2770					

18. Fran forgot one input in the table in problem 17; the real table is:

Input	Output
0	−5
1	−6
2	7
3	166
4	843
5	2770
6	4999

Does your function f from problem 17 still work?

- Find a function g that agrees with this new table.
- Let $h(x) = f(x) - g(x)$. Tabulate h between 0 and 7.

"Tabulate *h*" means "make a table for *h*."

19. Graph the functions $x \mapsto \binom{x}{k}$ for $k = \{0, \dots, 10\}$ between $x = -1$ and $x = 10$.

When you are done, put them on the same axes. Pretty, no?

Problems 20 and 21 look at the uniqueness of the function produced by Newton's difference formula.

Use anything you need from high school algebra to make the argument.

20. **Algebra Review.** Suppose that f and g are two polynomial functions of degree at most m. Show that if f and g agree at $m + 1$ inputs, they are identical polynomials.

21. Show that the function you get using Theorem 2 is the unique polynomial function of smallest degree that agrees with a given table.

1.4 Making It Formal: The Δ Operator

In this section, we'll refine our notion of Δ in a way that will give us a useful tool for the rest of this book. We'll also shift perspective a bit. Up until now, we had a table and we wanted to fit it with a function. Now we imagine starting with the function and we see what happens if we "Δ it."

So far, we've been using Δ to stand for an operation we perform on tables (on columns of numbers, really). There's another way to think about it. You can think of Δ as an operation on *functions*. For example, suppose $f(x) = x^2 + 1$. The table and its first differences look like this:

n	$f(n)$	Δ
0	1	1
1	2	3
2	5	5
3	10	7
4	17	9
5	26	11
6	37	

The Δ column defines a function in its own right. In fact, it's not hard to see that it can be taken to be $x \mapsto 2x + 1$. So, you can think of Δ as a process that transforms one function ($x \mapsto x^2 + 1$) to another one ($x \mapsto 2x + 1$). We could write it this way:

$$\Delta(x \mapsto x^2 + 1) = x \mapsto 2x + 1.$$

Example: Make a table for the function $x \mapsto x^3 + 3x$ and look at its first differences.

x	$x \mapsto x^3 + 3x$	Δ
0	0	4
1	4	10
2	14	22
3	36	40
4	76	64
5	140	94
6	234	

So, what's a formula for $\Delta(x \mapsto x^3+3x)$? You can poke around to find one (or, use the method of the previous section). But you could also reason like this:

This is the same thing as asking for a formula that fits the Δ column in the table.

First of all, name the thing you are "differencing": Let $f(x) = x^3+3x$. Then you want a formula for $\Delta(f)$. Use the algebraic habit of "form not value" and look at how you'd get various values of $\Delta(f)$.

It takes some getting used to to see "$\Delta(f)$" and think "a function." If it makes it easier, replace $\Delta(f)$ by some letter like g for awhile.

$$\begin{aligned}
\Delta(f)(0) &= f(1) - f(0)\\
\Delta(f)(1) &= f(2) - f(1)\\
\Delta(f)(2) &= f(3) - f(2)\\
\Delta(f)(3) &= f(4) - f(3)\\
\Delta(f)(4) &= f(5) - f(4)
\end{aligned}$$

Ah. There we go. We can do it once and for all and be done with it:

When we look at the table, we think of x as an integer. But x could be anything in the domain of f.

$$\begin{aligned}
\Delta(f)(x) &= f(x+1) - f(x)\\
&= \big((x+1)^3 + 3(x+1)\big) - (x^3+3x)\\
&= 3x^2 + 3x + 4.
\end{aligned}$$

So,

$$\Delta(x \mapsto x^3+3x) = x \mapsto 3x^2+3x+4.$$

If you check out this formula against the Δ column, you'll see an exact match.

So, if we think of Δ as an operator that takes one function and produces another, we can make a definition:

Definition If f is any function defined on, say, the real numbers, $\Delta(f)$ is the new function defined by

$$\Delta(f)(x) = f(x+1) - f(x)$$

We realize that this is getting pretty notation-heavy. This is it for a while. Almost.

Ways to think about it

The operator Δ is an example of what computer scientists call a "higher-order" function: a function whose inputs and outputs are themselves functions. You may have met other higher-order functions in calculus: The *derivative* can be looked at as a machine that turns functions into functions. In linear algebra, taking the *transpose* can be thought of as a way to change linear transformations into other linear transformations.

What's confusing sometimes is when we write "$\Delta(f)(8)$." To parse this, think, "I apply Δ to f and get a function. I apply this new function to 8 (and get a number, probably)." We could add more parentheses to make

(*continued*)

Alternatively, if the world you care about is polynomials, you can think of Δ as an operation that assigns a polynomial to another polynomial, where you think of a polynomial as a formal expression.

the order explicit, writing something like

$$(\Delta(f))(8),$$

but this only adds notational clutter with no guarantee that it makes the ideas clearer.

Here's the last little bit of notation: $\Delta^2(f)(n)$ means $\Delta(\Delta(f))(n)$ etc. Let's let $\Delta^0(f)$ be f itself.

So, now we can write our difference tables in a way that makes all the column headings consistent: For the function $f(x) = 2x^3 + x^2 - 5x + 1$, we have (same table, new look):

n	$f(n)$	$\Delta(f)(n)$	$\Delta^2(f)(n)$	$\Delta^3(f)(n)$	$\Delta^4(f)(n)$	$\Delta^5(f)(n)$
0	1	−2	14	12	0	0
1	−1	12	26	12	0	0
2	11	38	38	12	0	0
3	49	76	50	12	0	
4	125	126	62	12		
5	251	188	74			
6	439	262				
7	701					

We'll see even more applications of this way of thinking about Δ as a higher-order function in the next section.

This section would be a huge waste of time if all it did was make column headings consistent. But it does more: It allows us to express any *row* of the difference table generically, for any function f:

n	$f(n)$	$\Delta(f)(n)$	$\Delta^2(f)(n)$	$\Delta^3(f)(n)$	$\Delta^4(f)(n)$	$\Delta^5(f)(n)$
0	$f(0)$	$\Delta(f)(0)$	$\Delta^2(f)(0)$	$\Delta^3(f)(0)$	$\Delta^4(f)(0)$	$\Delta^5(f)(0)$
1	$f(1)$	$\Delta(f)(1)$	$\Delta^2(f)(1)$	$\Delta^3(f)(1)$	$\Delta^4(f)(1)$	$\Delta^5(f)(1)$
2	$f(2)$	$\Delta(f)(2)$	$\Delta^2(f)(2)$	$\Delta^3(f)(2)$	$\Delta^4(f)(2)$	$\Delta^5(f)(2)$
3	$f(3)$	$\Delta(f)(3)$	$\Delta^2(f)(3)$	$\Delta^3(f)(3)$	$\Delta^4(f)(3)$	$\Delta^5(f)(3)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮

The beauty is that you don't even need to have a specific function f in mind to think about this.

Look carefully at this table, and chase $f(3)$ up to the top row using the up and over scheme. You should get:

$$f(3) = \binom{3}{0} f(0) + \binom{3}{1} \Delta(f)(0) + \binom{3}{2} \Delta^2(f)(0) + \binom{3}{3} \Delta^3(f)(0).$$

More generally, we could write:

$$\begin{aligned} f(n) &= \binom{n}{0} f(0) + \binom{n}{1}\Delta(f)(0) + \binom{n}{2}\Delta^2(f)(0) + \binom{n}{3}\Delta^3(f)(0) + \cdots \\ &\quad + \binom{n}{n}\Delta^n(f)(0) \\ &= \sum_{k=0}^{n} \binom{n}{k}\Delta^k(f)(0). \end{aligned}$$

This is a beautiful formula. Some of its features include:

- It tells you how to find the values of a function from the "zeroth" row of its difference table.
- It works for *any* function, not just polynomial functions.
- It works for any nonnegative integer n (with the proviso that $\binom{n}{k} = 0$ if $k > n$).

Example: Let's check out the formula for $f(x) = x^3 + 3x$. Using the definition

$$\Delta(f)(x) = f(x+1) - f(x)$$

we can derive the formulas for the successive differences:

$$\begin{array}{lclcr} f(x) & = & & & x^3 + 3x, \\ \Delta(f)(x)) & = & f(x+1) - f(x) & = & 3x^2 + 3x + 4, \\ \Delta^2(f)(x)) & = & \Delta(f)(x+1) - \Delta(f)(x) & = & 6x + 6, \\ \Delta^3(f)(x)) & = & \Delta^2(f)(x+1) - \Delta^2(f)(x) & = & 6, \\ \Delta^4(f)(x)) & = & \Delta^3(f)(x+1) - \Delta^3(f)(x) & = & 0. \end{array}$$

This allows easy construction of the table:

Notice how the columns go all the way down to the bottom. We have *formulas* now. If the original function is known, so are all its Δs.

	$f(n)$	$\Delta(f)(n)$	$\Delta^2(f)(n)$	$\Delta^3(f)(n)$	$\Delta^4(f)(n)$
n	$n^3 + 3n$	$3n^2 + 3n + 4$	$6n + 6$	6	0
0	0	4	6	6	0
1	4	10	12	6	0
2	14	22	18	6	0
3	36	40	24	6	0
4	76	64	30	6	0
5	140	94	36	6	0
6	234	130	42	6	0
7	364	172	48	6	0

So, let's see:

$$\begin{aligned} f(7) &= \binom{7}{0}0 + \binom{7}{1}4 + \binom{7}{2}6 + \binom{7}{3}6 + \binom{7}{4}0 + \binom{7}{5}0 + \binom{7}{6}0 + \binom{7}{7}0 \\ &= 1\cdot 0 + 7\cdot 4 + 21\cdot 6 + 35\cdot 6 + 35\cdot 0 + 21\cdot 0 + 6\cdot 0 + 1\cdot 0 + \cdots \\ &= 364. \end{aligned}$$

Yup. It works. Notice how the Δ columns become 0 before the $\binom{7}{k}$ "run out." Now let's try it with a smaller number:

$$\begin{aligned} f(2) &= \binom{2}{0}0 + \binom{2}{1}4 + \binom{2}{2}6 + \binom{2}{3}6 + \binom{2}{4}0 + \binom{2}{5}5 + \binom{2}{6}0 + \cdots \\ &= 1\cdot 0 + 2\cdot 4 + 1\cdot 6 + 0\cdot 6 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 + \cdots \\ &= 14. \end{aligned}$$

Yup again, except this time the $\binom{2}{k}$ became 0 before the Δs ran out.

Finally, notice that for all $n > 2$ (to ensure that $\binom{n}{3} \neq 0$), if we look at our formula algebraically, we recover the formula that defines f:

$$\begin{aligned} f(n) &= \binom{n}{0}0 + \binom{n}{1}4 + \binom{n}{2}6 + \binom{n}{3}6 + \binom{n}{4}0 + \cdots \\ &= 1\cdot 0 + n\cdot 4 + \frac{n(n-1)}{2}\cdot 6 + \frac{n(n-1)(n-2)}{6}\cdot 6 \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{4!}\cdot 0 + \cdots \\ &= n^3 + 3n. \end{aligned}$$

Again, the proof of this formula is easier to talk about than to write down.

We should state our formula as a theorem and play with it a bit before we move on:

In terms of fitting a function to a table, the theorem says that any function that agrees with a table has to produce values at n that are given by this formula, where the "$\Delta^k(f)(0)$" are taken from the "zeroth" row of the table.

Theorem 3. *If f is any function defined (at least) on the nonnegative integers, then, for all integers $n \geq 0$,*

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k(f)(0).$$

Problems

22. Find a formula for the Δ of each function:

- **(a)** $x \mapsto 4x^2 + 1$
- **(b)** $x \mapsto 8x^2 + 2$
- **(c)** $x \mapsto 3x$
- **(d)** $x \mapsto 4x^2 + 3x + 1$
- **(e)** $x \mapsto 3x + 5$
- **(f)** $x \mapsto x^3 + 3x + 728$

23. Suppose $f(x) = x^4 + 5x^3 - 7x^2 + 3x - 1$.

"Our favorite formula" is the one in Theorem 3.

- **(a)** Find formulas for $\Delta^k(f)$ for $k = 1, 2, 3, 4,$ and 5.
- **(b)** Use part 23a and our favorite formula to find $f(9)$.

24. Here's a difference table:

n	$f(n)$	$\Delta(f)(n)$	$\Delta^2(f)(n)$	$\Delta^3(f)(n)$	$\Delta^4(f)(n)$
0	1	-2	4	6	8
1					8
2					8
3					8
4					8
5					8
6					8
7					8

Assuming $\Delta^4(f)(n) = 8$ for all n, find $f(12)$. Find a formula for $f(n)$.

25. What about functions that are not polynomial? For example, complete the table for $f(n) = 2^n$.

Input	Output	Δ	Δ^2	Δ^3	Δ^4	Δ^5	...
0	1						
1	2						
2	4						
3	8						
4	16						
5							
6							
7							
8							
9							
10							

Express $f(n)$ in terms of the numbers in the first row of the table. Does this lead to anything interesting?

Problems 26–28 let you find the differences of a combinatorial polynomial.

26. Show that if $k \geq 1$, we have a polynomial identity:

Hint: See problem 20 on page 22.

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$

27. Show that, for $k \geq 1$,

What happens if $k = 0$?

$$\Delta\left(\binom{x}{k+1}\right) = \binom{x}{k}$$

28. State and prove a theorem that gives you a formula for

$$\Delta^n \left(x \mapsto \binom{x}{m} \right)$$

for any nonnegative integers m and n.

Problems 29–31 explore the differences of a power of x.

29. Find formulas for the Δ of each function.

(a) $x \mapsto 1$

(b) $x \mapsto x$

(c) $x \mapsto x^2$

(d) $x \mapsto x^3$

(e) $x \mapsto x^4$

(f) $x \mapsto 5x^4 - 2x^3 + 3x^2 - x - 1$

Hint: Use the Binomial Theorem.

30. Show that

$$\Delta(x \mapsto x^n)$$

is a polynomial of degree $n - 1$. What's the degree of $\Delta^2(x \mapsto x^n)$?

31. Show that, if $k > n$, $\Delta^k(x \mapsto x^n) = 0$.

Is this really an "infinite" sum?

32. Show that, if f is a polynomial function, f can be written as

$$f(x) = \sum_{k=0}^{\infty} \Delta^k(f)(0) \binom{x}{k}.$$

33. Let $f(x) = 5x^6 + 4x^5 + 3x^4 - 2x^3 + x^2 - x - 8$. For integers $r \geq 1$, let

$$f_r(x) = \sum_{k=0}^{r} \Delta^k(f)(0) \binom{x}{k}.$$

What would f_8 be?

(a) Tabulate each of the functions f, f_1, f_2, f_3, f_4, f_5, f_6, and f_7 between 0 and 10. How are the tables related?

(b) Sketch the graphs of f, f_1, f_2, f_3, f_4, f_5, f_6, and f_7. How are they related?

34. Same as problem 33 except $f(x) = 2^x$.

35. If f is any function and r is an integer, $r \geq 1$, let

$$f_r(x) = \sum_{k=0}^{r} \Delta^k(f)(0) \binom{x}{k}.$$

Show that $\Delta^k(f)(0) = \Delta^k(f_r)(0)$ if $0 \leq k \leq r$.

36. In what sense is the result of problem 32 true if $f(x) = 2^x$?

37. Theorem 2 allows you to write any function whose differences are eventually constant as a *linear combination* (that is, as a sum of constant multiples) of the $\binom{x}{k}$. Write each polynomial function as a linear combination of the $\binom{x}{k}$:

(a) $x \mapsto 1$ **(b)** $x \mapsto x$ **(c)** $x \mapsto x^2$
(d) $x \mapsto x^3$ **(e)** $x \mapsto x^4$ **(f)** $x \mapsto x^5$

Problems 38–41 look at differences for some nonpolynomial functions.

38. Suppose $f(n) = 3^n$.
- **(a)** Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
- **(b)** Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

39. Suppose $f(n) = a^n$.
- **(a)** Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
- **(b)** Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

40. Suppose $f(n) = (1+a)^n$.
- **(a)** Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
- **(b)** Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

41. Suppose $f(n) =$ the nth Fibonacci number. So, the table for f looks like this:

Any output (after 0) is the sum of the previous two outputs.

n	$f(n)$
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13

- **(a)** Describe the difference table for f. In particular, find a formula for $\Delta^k(f)(0)$ in terms of the f column.
- **(b)** Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

A closed form for this table is particularly difficult. Look for ways to relate the differences and the $f(n)$.

42. Suppose you have a mystery function g with the property that $\Delta(g)(x) = 5x - 2$. Find formulas for:

(a) $\Delta\big(x \mapsto 2\,g(x)\big)$ **(b)** $\Delta\big(x \mapsto 2\,g(x) + x^3\big)$
(c) $\Delta\big(x \mapsto 2\,g(x) + 7\big)$ **(d)** $\Delta\big(x \mapsto 2\,g(x) + 9\big)$

What would you need to know to find a formula for $g(x)$?

43. Suppose $f(x)$ is a polynomial. Show that f is a constant function if and only if $\Delta(f) = 0$.

44. Suppose $f(x)$ is a polynomial. Show that you can construct $\Delta(f)$ by the following process:

  (a) Write f as a sum of powers of x (that is, remove all parentheses and write f in "normal" form).

  (b) Replace each power x^k in f by $(x+1)^k - x^k$.

  (c) Simplify the result.

Problems 45 and 46 are not necessary for what follows. But they are fun.

45. Using the ideas from this section, write a convincing argument to show that, if f is a polynomial and $\Delta^3(f)$ is constant and nonzero, f has degree 3.

46. Using the ideas from this section, write a convincing argument to show that, if f is a polynomial and $\Delta^m(f)$ is constant and nonzero, f has degree m.

Historical Perspective

The expansion of a polynomial f in problem 32:

$$f(x) = \sum_{k=0}^{\infty} \Delta^k(f)(0)\binom{x}{k}$$

actually holds for a much wider class of functions than polynomials (although the sum is no longer finite for nonpolynomials). Some people call this expression the "Mahler expansion" for f, and they even call the polynomials $\binom{x}{k}$ the "Mahler polynomials." This has nothing to do with the composer Gustav Mahler; rather it is in honor of the German-born mathematician Kurt Mahler (1903–1988). In the 1960s, Mahler showed that this kind of expansion could be used to extend some important classical functions to a number system useful in number theory (and known as the "p-adic numbers").

1.5 Going the Other Way: Polynomials to Tables

So, if we assume (or even better, know) that the mth difference in a table is constant and nonzero, we know that there is a polynomial of degree m that agrees with the table. In fact, that polynomial can be expressed as:

$$\sum_{k=0}^{m} a_k \binom{x}{k}$$

where the a_k are the entries in the zeroth row of the difference table (before they all become 0). There's one open question: If you start with a polynomial of degree m, will its mth differences be constant?

The answer is yes. To see this, we'll need two things:

- We'll prove an important property of the Δ operator known as *linearity*. Linearity will allow us to take the Δ of a linear combination of polynomials by taking the Δ of each polynomial in the sum.