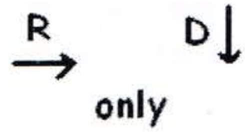
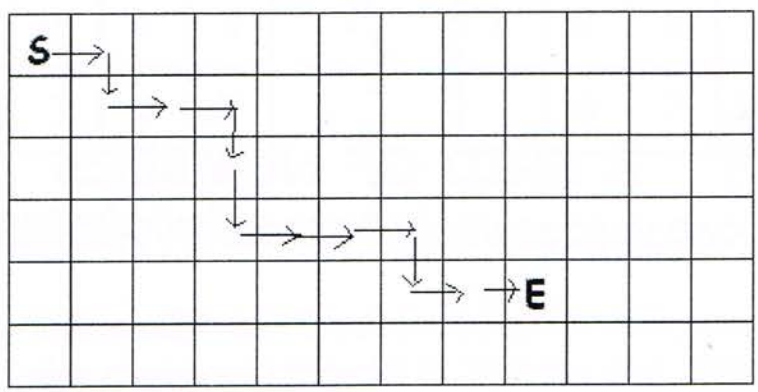
THE GRID

Here's a famous puzzle:

> *Starting at the top-left cell marked S and taking horizontal steps one place only to the right or vertical steps downwards only, how many different paths are to the location marked E?*





Play with this puzzle for a while before reading on. As you play, perhaps contemplate the following questions:

1. Given the location of the point E, is the grid shown in the diagram unnecessarily large?

2. Marking in different paths from S to E is awfully complicated. One could first count paths to different cells first, ones easier to handle, and look for patterns. For example, how many distinct paths are there from S to any cell on the top row? Write the answers in those cells. How many distinct paths take you to any cell in the leftmost column? To cells in the second row? Second column? Third row?

3. If you are willing to trust patterns, can you make a good guess as to the answer to the original puzzle?

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

There are two ways to approach this puzzle.

APPROACH NUMBER 1: FORMULAS

Every path from S to E can be described by a sequence of letters R and D. For example, the path given in the diagram can be described by the sequence:

RDRRDDRRRDRR

This sequence contains eight Rs and four Ds. Moreover, any sequence of eight Rs and four Ds corresponds to a path from S to E.

Exercise: Mark in on the diagram the paths given by DDRRRRRDRRRD and RRRRRRRRDDDD.

Thus ... the number of paths from S to E matches the number of ways to arrange twelve letters - eight Rs and four Ds. (That is, to label twelve slots with eight Rs and four Ds). The answer to the original puzzle is:

$$\frac{12!}{8!4!} = 495 \text{ paths.}$$

Exercise: How many paths are there from S to the bottom-right cell of the grid?

Exercise: Suppose the cell E is a steps to the right of S and b steps down from S. Show that the number of paths from S to E is given by:

$$\begin{pmatrix} a+b \\ a \quad b \end{pmatrix} = \frac{(a+b)!}{a!b!}$$

In fact, number the rows 0, 1, 2, ... (with the top row being the zero-th row) and we number the columns 0, 1, 2, ... (with the leftmost column being the zero-th column). The cell E in the original diagram thus has position row 4, column 8, and the number of paths to it is $\frac{12!}{4!8!}$. (Paths to this cell involve 4 Rs and 8 Ds.)

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

In general, numbering rows and columns this way, the cell row *a* and column *b* requires *a* Rs and *b* Ds to get to it and so the number of paths to it is:

$$\frac{(a+b)!}{a!b!}$$

Exercise: Is this formula still correct for the cells in the zero-th row? In the zero-th column? (Good thing we set $0!=1$.)

What value should we place in the cell labeled S - row zero, column zero? How many ways should we say that start at S and end at S?

APPROACH 2: PATTERNS

If you fill in the answers for the number of paths to each cell, the following grid of numbers appears:

S	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12
1	3	6	10	15	21	28	36	45	55		
1	4	10	20→	35	56	84	120	165			
1	5	15	35	70	126	210	330	495			
1	6	21	56	126	332	542					

(The exercise above suggests that the position labeled S should also be assigned the number 1.)

Exercise: Explain why the table is symmetrical about the southeast diagonal line.

We have the formula $\frac{(a+b)!}{a!b!}$ for the entry in the *a*-th row and *b*-th column (starting the counts at zero).

Have you noticed that each entry in an interior cell is the sum of two numbers - the number just above the cell and the number just to the left of the cell? This makes sense in terms of counting paths. Consider the circled

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

cell. To reach this cell one can either first reach the cell just above - there are 15 ways to do this - and then step down, or reach the cell just to its left - there are 20 ways to accomplish this - and then step right. This gives a total of 15 + 20 = 35 paths to the circled cell.

Exercise: Use this observation to fill in the remainder of the table.

This grid of numbers possesses a number of curious properties. For example, start at any "1" on the top row, head down any number of cells, and then turn right for one cell to create a stocking. The number in the toe of the stocking always equals the sum of the numbers in the leg of the stocking.

1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12
1	3	6	10	15	21	28	36	45	55		
1	4	10	20	35	56	84	120	165			
1	5	15	35	70	126	210	330	495			
1	6	21	56	126	332	542					

$$1+5+15+35+70 = 126$$

There is a horizontal version of this stocking property also.

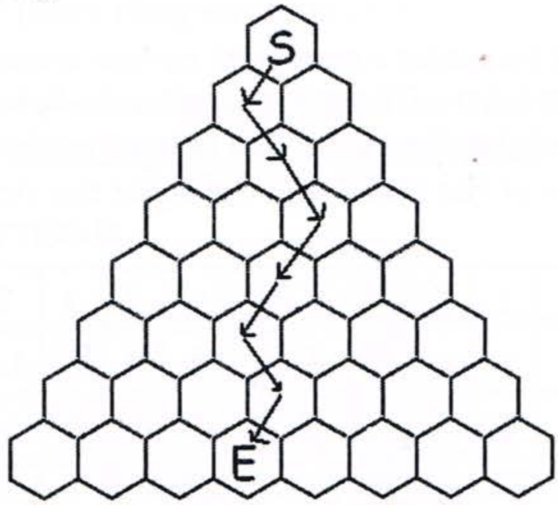
Exercise: Explain why the stocking property works.

<u>HINT</u>: Use the fact that the number in the toe is really the sum of two other numbers in the grid.

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

EXERCISE: *Here is a second path walking puzzle, this time we walk through a honeycomb of cells:*

a) Starting at the top cell marked S and moving only downwards to adjacent cells, how many different paths are there to the cell marked E. (Each path from S to E requires how many southeast right steps and how many southwest left steps?)



b) Fill in each cell of the honeycomb with a number indicating the number of paths to it from S.

c) Starting all counts with 0 (so that the cell marked E in the diagram lies in row 7 of the honeycomb, 3 places in from the left and 4 places in from the right), explain why the number of number of paths to a cell in row *n*, *k* places in from the left and $n-m$ places in from the right is $\frac{n!}{k!(n-k)!}$.

d) Explain why every number in the honeycomb is the sum of the two numbers above it.

e) Is this honeycomb problem different from the grid problem of the previous few pages? What number should go in the cell marked S?

The numbers appearing in the rectangular grid of cells, or equivalently, the honeycomb of cells, is just the grid of numbers made famous by French mathematician Blaise Pascal (1623-1662) for his work in probability theory.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

. . .

Each row of this triangle is a diagonal of the grid.

Regard the top row of the triangle (the single "1") as the zero-th row. (Then the sixth row of the triangle, for example, is 1 6 15 20 15 6 1).

In any row of the triangle call the left-most entry the zero-th entry. (Thus in the sixth row, "1" is the zero-th entry, "6" is the first entry, "15" is the second entry, and so on.)

Then the formula for the entry in the n-th row k places in from the left (and $n-k$ places from the right) is:

$$\frac{n!}{k!(n-k)!}$$

(Any honeycomb path to this cell requires k southeast right steps and $n-k$ southwest left steps.)

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

For the grid of numbers we saw that every entry was the sum of the entry just above it and just to the left of it. In Pascal's triangle this translates to:

EACH INTERIOR ENTRY IS THE SUM OF THE TWO ENTRIES ABOVE IT

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

...

EXERCISE: Notice:

$$1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 + 1 = 4$$
$$1 + 3 + 3 + 1 = 8$$
$$1 + 4 + 6 + 4 + 1 = 16$$

a) Without doing the computation, explain why the sum of entries in any row shown above will turn out to be double the sum of the entries in the row just above it.

HINT: Each entry in any one row is the sum of two entries in the row above it.

b) Explain why the sum of entries in the n-th row is sure to be 2^n. (Remember to call the top row of Pascal's triangle - the one with a single "1" - the zero-th row.)

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

EXERCISE: Explain why each alternating sum in Pascal's triangle, beyond the zero-th row, is zero:

$$1-1=0$$
$$1-2+1=0$$
$$1-3+3-1=0$$
$$1-4+6-4+1=0$$
$$1-5+10-10+5-1=0$$
$$\vdots$$

The following property is strange. Look at the powers of 11:

$$11^0=1$$
$$11^1=11$$
$$11^2=121$$
$$11^3=1331$$
$$11^4=14641$$
$$11^5=161051=1\,|\,5\,|10\,|10\,|\,5\,|\,1$$

Any guesses as to why these powers appear as rows of Pascal's triangle?

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

THE BINOMIAL THEOREM

Recall the act of expanding brackets: "Select one term from each set of parentheses and make sure to collect all possible combinations."

For example, $(a+b+c)(x+y)(p+q+r)(s+t+u+v) = axps + ayqu + cxps + \cdots$

Imagine expanding the quantity:

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

The term x^5 will appear once by choosing the term "*x*" from each set of parentheses.

The term x^4y will appear five times:

once by choosing *x, x, x ,x* and then *y*.
once by choosing *x, x, x, y*, and then *x*.
once by choosing *x, x, y, x*, and then *x*.
once by choosing *x, y, x, x*, and then *x*.
once by choosing *y, x, x, x*, and then *x*.

That is, x^4y will appear the same number of times as it is possible to arrange four *x*s and one *y*. This can be done $\frac{5!}{4!1!} = 5$ ways, which is also the number of paths in the honeycomb to the fifth row using 4 right steps and 1 left step. This is an entry of Pascal's triangle.

The term x^3y^2 will appear as many times as it is possible to arrange three *x*s and two *ys*, that is, $\frac{5!}{3!2!} = 10$ times.

The term x^2y^3 ten times, the term xy^4 five times, and the term x^5 once. We have:

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

The numbers 1, 5, 10, 10, 5, 1 are the entries of the fifth row of Pascal's triangle.

We have:

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

and so on.

SOME FUN ...

1. Put $x = 10$ and $y = 1$. Notice that, for instance:

$$11^4 = (10+1)^4 = 10^4 + 4\cdot 10^3 + 6\cdot 10^2 + 4\cdot 10 + 1 = 10000 + 4000 + 600 + 40 + 1 = 14641$$

.

This explains the connection of the powers of 11.

2. Put $x = 1$ and $y = 1$. Notice that, for instance:

$$2^4 = (1+1)^4 = 1^4 + 4\cdot 1^3 + 6\cdot 1^2 + 4\cdot 1 + 1 = 1+4+6+4+1$$

This explains - again - why the sum of entries in a row of entries of Pascal's triangle is a power of two.

3. Put $x = 1$ and $y = -1$. Notice that, for instance:

$$0 = (1-1)^4 = 1^4 + 4\cdot 1^3\cdot(-1) + 6\cdot 1^2\cdot(-1)^2 + 4\cdot 1\cdot(-1)^3 + 1\cdot(-1)^4 = 1-4+6-4+1$$

This explains - again - why the alternating sum of entries in a row of Pascal's triangle is always zero.

EXERCISE: Take any row of Pascal's triangle and multiply its entries by consecutive powers of 2 taken in reverse order. Prove that their sum is sure to be a power of 3. (For example: $1\times 16 + 4\times 8 + 6\times 4 + 4\times 2 + 1\times 1 = 81 = 3^4$.)

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

Stated formally, we have ...

Binomial Theorem:

$$(x+y)^n = x^n + \frac{n!}{(n-1)!1!}x^{n-1}y + \frac{n!}{(n-2)!2!}x^{n-2}y^2 + \cdots + \frac{n!}{a!b!}x^a y^b + \cdots + y^n$$

The coefficients are the entries of the n-th row of Pascal's triangle.

<u>COMMENT</u>: We have used the notation $\begin{pmatrix} & n & \\ a & & b \end{pmatrix}$ for the expression $\frac{n!}{a!b!}$. Thus the binomial theorem can be written:

$$(x+y)^n = \begin{pmatrix} & n & \\ n & & 0 \end{pmatrix} x^n + \begin{pmatrix} & n & \\ n-1 & & 1 \end{pmatrix} x^{n-1}y + \begin{pmatrix} & n & \\ n-2 & & 2 \end{pmatrix} x^{n-2}y^2 + \cdots + \begin{pmatrix} & n & \\ a & & b \end{pmatrix} x^a y^b + \cdots + \begin{pmatrix} & n & \\ 0 & & n \end{pmatrix} y^n$$

Often mathematicians suppress one of the terms in the notation and write just $\binom{n}{a}$ for $\begin{pmatrix} & n & \\ a & & b \end{pmatrix}$. (We must have $b = n - a$.)

For example, $\binom{7}{5} = \begin{pmatrix} & 7 & \\ 5 & & 2 \end{pmatrix} = \frac{7!}{5!2!}$. Thus the binomial theorem might be written:

$$(x+y)^n = \binom{n}{n} x^n + \binom{n}{n-1} x^{n-1}y + \binom{n}{n-2} x^{n-2}y^2 + \cdots + \binom{n}{a} x^a y^b + \cdots + \binom{n}{0} y^n$$

<u>COMMENT</u>: The entries of Pascal's triangles - $\binom{n}{a}$ - are also called *binomial coefficients.*

EXERCISES

Question 1: Compute the powers: $101^0, 101^1, 101^2, 101^3, 101^4$ and 101^5. What do you notice? Explain what you notice.

Question 2: Without a calculator compute 201^6.

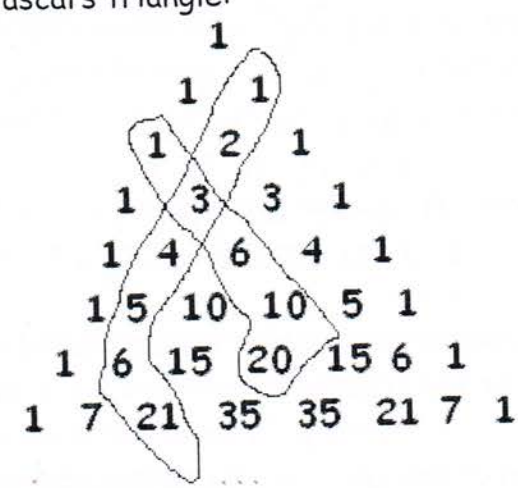
Question 3: a) Without doing the arithmetic, explain why

$$64 - 6 \cdot 32 + 15 \cdot 16 - 20 \cdot 8 + 15 \cdot 4 - 6 \cdot 2 + 1$$

must equal 1.

b) Say something interesting that can be deduced from examining $(3-1)^6$

Question 4: Here's Pascal's triangle:



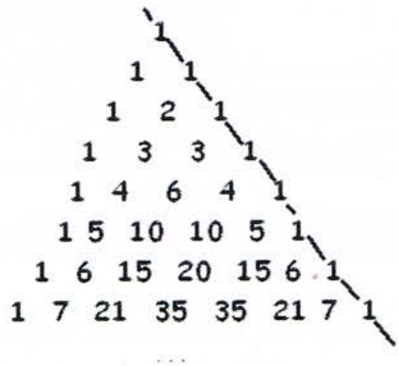
The stocking property can be interpreted as follows:

Choose any entry "1" on the side of the triangle and follow the diagonal from that entry into the interior of the triangle. Turn downwards 90^0 *to form the toe of the stocking. Then the number in the toe equals the sum of the numbers in the leg of the stocking.*

Convince yourself - again - that this property is valid.

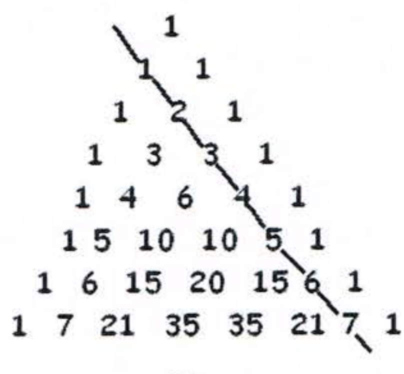
CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques

a) The entries on the right-most diagonal of Pascal's triangle are: 1 1 1 1 1 1...



Explain why the stocking property tells us that the entries of the next diagonal of Pascal's triangle must be: 1 2 3 4 5 ....

b) Knowing that the second diagonal of Pascal's triangle has entries 1 2 3 4 5 6 ...



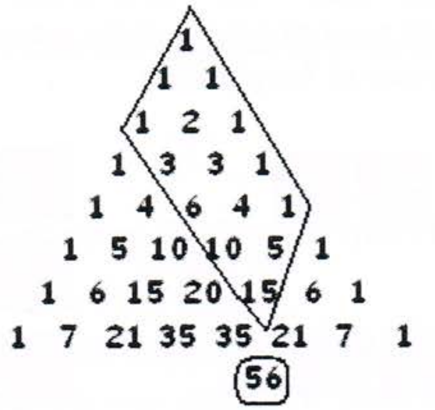
explain, using the stocking property, why the entries of next diagonal of Pascal's triangle must be the triangular numbers: 1 3 6 10 15 21 ....

c) Use the stocking property to quickly determine the sum of the first six triangle numbers.

d) The sum of the first six triangle numbers is called the sixth *tetrahedral number*. What is a tetrahedron? Why is this name appropriate?

e) Use Pascal's triangle to find the sum of the first six tetrahedral numbers.

Question 5: Choose any entry in Pascal's triangle - say the "15" circled in the diagram below. Form the parallelogram of entries above it just as shown.



Sum all the entries in the parallelogram. (In this case we have: 15 + 10 + 5 + 6 + 4 + 1 + 3 + 3 + 1 + 1 + 2 + 1 + 1 + 1 + 1.) The answer is one less than the value of the entry directly below the circled entry two rows down. (In this case, one less than 56.)

a) Play with this. Can you see why 56 is one more than the sum of entries in the parallelogram? (A yes/no answer will suffice.)

HINT: Think of the entries in the parallelogram as a series of parallel stockings.

b) Can you see that this parallelogram property always works? (Again an honest yes or no will suffice.)

Question 6: Suppose we wished to expand the product $(x+y+z)^7$

a) How many times will the term y^7 appear?
b) How many times will the term xz^6 appear?
c) How many times will the term $x^2y^3z^2$ appear?
d) Write a formula for the number of times a general term $x^ay^bz^c$ will appear.

e) State the general "trinomial theorem:"

$$(x+y+z)^n =$$

f) State a general "quadronomial" theorem:

$$(x+y+z+w)^n =$$

CURRICULUM CONNECTIONS
Pascal's triangle; Binomial theorem; Counting techniques