A Modeling Approach for Enhancing

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MODELING IS A FLEXIBLE, POWERFUL, AND engaging tool for middle school students to use to enhance their problem-solving skills. In this article, we describe one teacher's first efforts to develop her students' modeling expertise in a variety of arithmetic word problems. This modeling approach emanates, in part, from our observations of Singapore schools and curricula during the year 2001. We initially describe some of the basis for modeling that is grounded in current research and standards documents.

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Arithmetic word problems have long constituted an important component of the middle school curriculum. However, studies show that the traditional modes of introducing and teaching word problems often leave students searching for key words or phrases that suggest an operation or algorithm instead of drawing on previously acquired mathematical knowledge to use in solving the problems (Greer 1997). A recent study by Verschaffel, De Corte, and Vierstraete (1999) noted that "extensive experience with traditional arithmetic word problems induces in pupils a strong tendency to approach word problems in a mindless, superficial, routine-based way to identify the correct arithmetic operation needed to solve a word problem" (p. 265).

NCTM's *Principles and Standards for School Mathematics* (2000) points out that students in the lower grades should be able to use models to "make predictions, draw conclusions, or better understand quantitative situations" (p. 39). This document also suggests that upper elementary school students should have access to a repertoire of different mathematical representations and understand how to use these representations productively and how to move among them.

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The Modeling Approach in Singapore Schools

IN SINGAPORE, THE MINISTRY OF EDUCATION has established a modeling approach as a fundamental problem-solving framework that is taught to nearly every student in the country (Singapore Ministry of Education 1990). Essentially, this approach uses a pictorial representation of the quantities in a problem and the relationships among these quantities. The modeling approach is intended to help students visualize abstract mathematical relationships and accompanying problem structures (Fong 1994; Yeap and Kaur 2001).

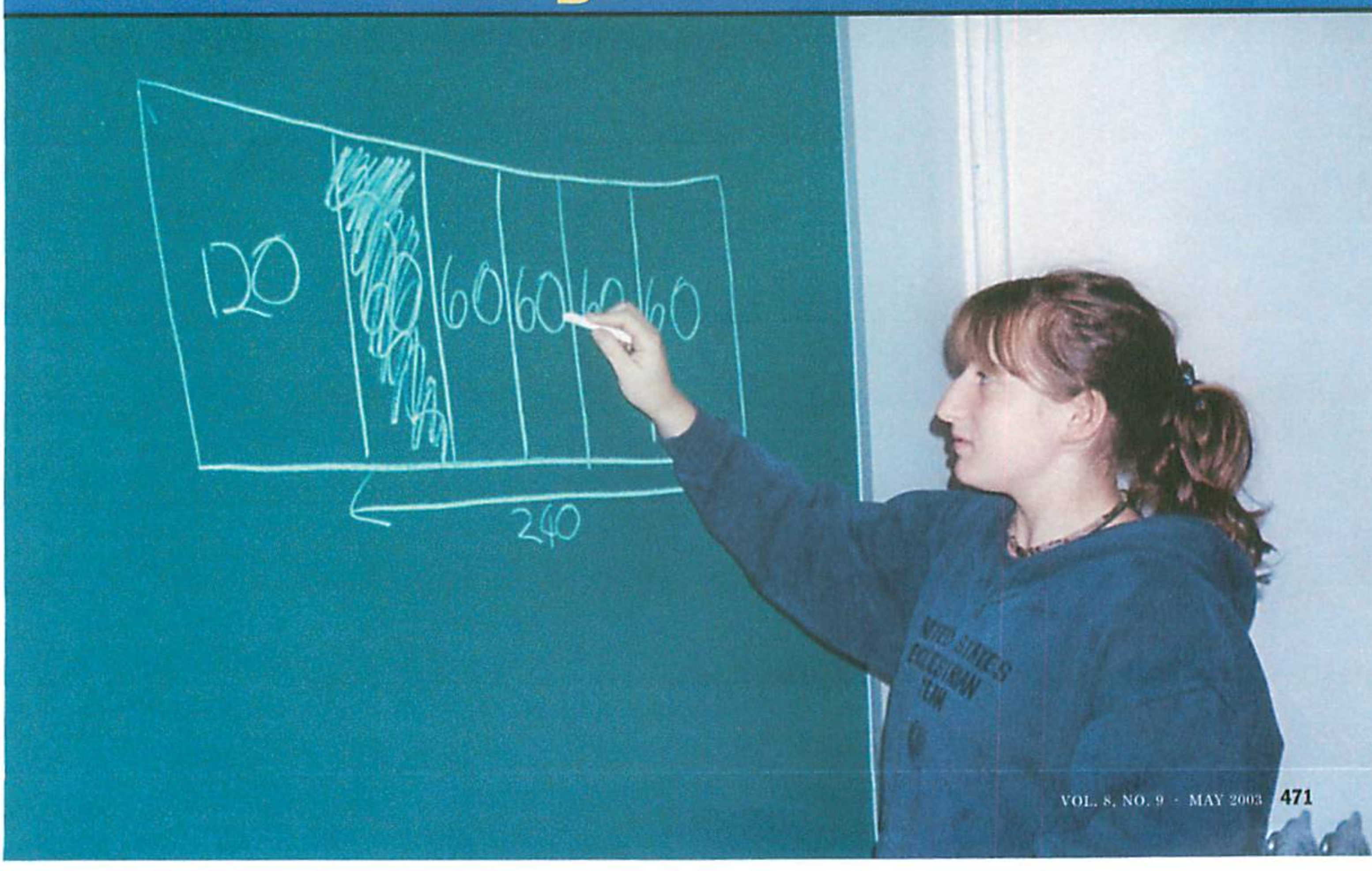
Today, a modeling approach permeates the school mathematics curriculum in Singapore. This approach may be a contributing factor to the country's success on international comparisons in mathematics. Students learn to use this approach through constant exposure in their textbooks and by frequently observing the use of diagrams for modeling in a wide variety of classroom examples. After observing this modeling approach in Singapore, we shared it with a group of American middle school teachers during a recent professional development class. Some initial teaching experiences of one of these middle school teachers, Ms. Y, are discussed in the following sections.

Some Examples to Illustrate the Modeling Approach

ARITHMETIC WORD PROBLEMS ARE A COMMON means for illustrating the power of modeling. Ms. Y used representative word problems involving fractions, ratio, and some initial algebraic notions from Singapore's school mathematics textbooks to present the approach to her students. Although she would normally introduce such problems with multiple manipulatives, such as fraction bars, Cuisenaire rods, or connecting cubes, Ms. Y decided to introduce her students to this modeling approach as another heuristic aid for them to learn and use.

The first example she selected was the multistep problem on the next page that included multiplication with fractions.

Problem Solving in the Middle Grades



Jim had 360 stamps. He sold 1/3 of them on Monday and 1/4 of the remainder on Tuesday. How many stamps did he sell on Tuesday?

Ms. Y began a demonstration of the modeling approach by asking her students how to represent the 360 stamps with a rectangle and how to represent 1/3 of the stamps in the same rectangle. Because these students had previously worked on numerous ways of displaying fractions with geometric figures, such as drawing a circle and dividing it into three sections, this step was easy for the class. The result was the representation in **figure 1a.**

Ms. Y next asked the students how many stamps were in the shaded region. Most students responded that the total number of stamps was 360, then calculated that 1/3 of that amount is 120 stamps. One student went to the overhead projector and wrote 120 in the appropriate section on the diagram (see **fig. 1b**).

Ms. Y then asked for ways to find the number of stamps in the unshaded regions. This question appeared to be another relatively easy one for the class. Many students noted that 120 could be subtracted from 360 to get the remaining amount of 240 stamps.

Ms. Y then read the second part of the problem and asked her students how they could represent 1/4 of the remaining stamps. After some discussion, the students decided to draw a second diagram with four partitions, each representing 1/4 of the remaining stamps. Although this strategy was a good first step, not many students were able to make the connection between the two parts of the original problem. In an attempt to help guide the learning of her students, Ms. Y encouraged them to put the two diagrams together. The resulting rectangle was drawn as shown in **figure 1c.**

Finally, in response to the teacher's question about what should be done next, a number of students suggested dividing by 4, and the remaining quantity (240) was divided by 4 to get a final result of 60. After another student wrote 60 in each of the remaining boxes, as shown in **figure 1d,** the students expressed the answer in the context of the problem statement. That is, Jim sold 60 stamps on Tuesday.

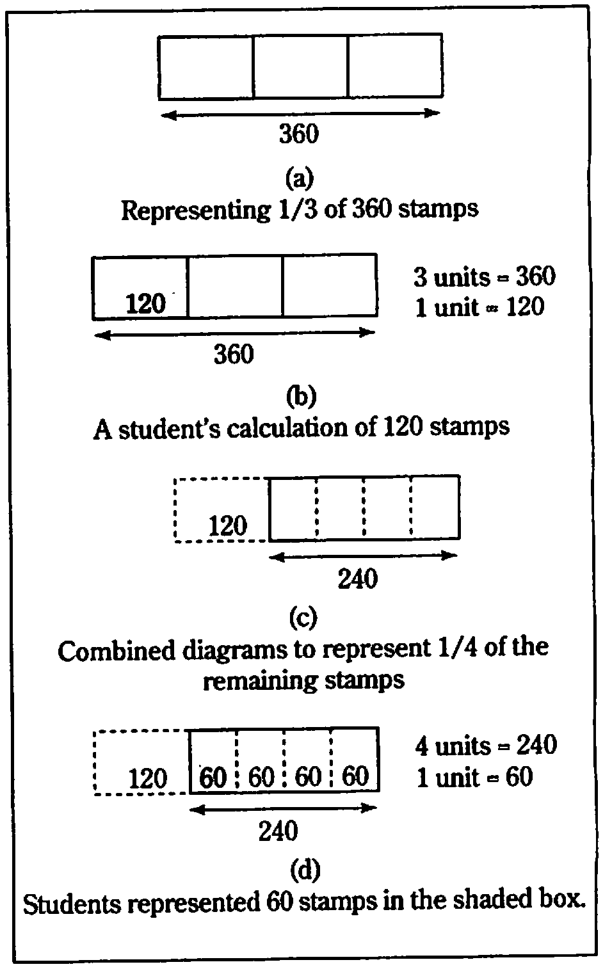


Fig. 1 Demonstration of the modeling approach

The Modeling Approach and Early Algebra

A MODELING APPROACH IS ALSO APPLICABLE IN problem-solving situations that deal with algebraic reasoning. One advantage of introducing students to a modeling approach is that they develop the ability to use representations to solve problems that involve algebraic thinking before they study algebra. Pictorial representations in the models help students to better comprehend both symbols and manipulations of algebraic equations, because they have already gained experience using icons, such as rectangles, and portions of these icons to represent quantities in a modeling approach (Kho 1987).

Another example that Ms. Y used in her class was an age problem. She said that she chose this example because she remembered the difficulties she had trying to solve such problems with algebra when she was a student. Moreover, she reported that this problem intrigued her because she believed that she would be able to help her students, who had not yet formally studied algebra, to use the modeling approach to solve the problem. The age problem was stated as follows:

Eiko is 5 years older than Juan. The sum of their ages is 101. Find Juan's age.

In class, Ms. Y began by asking her students how they could represent Eiko's and Juan's ages with rectangles. After some students suggested that two different rectangles could be drawn, Ms. Y asked a student to draw the two rectangles on a transparency on the overhead projector (see **fig. 2a**).

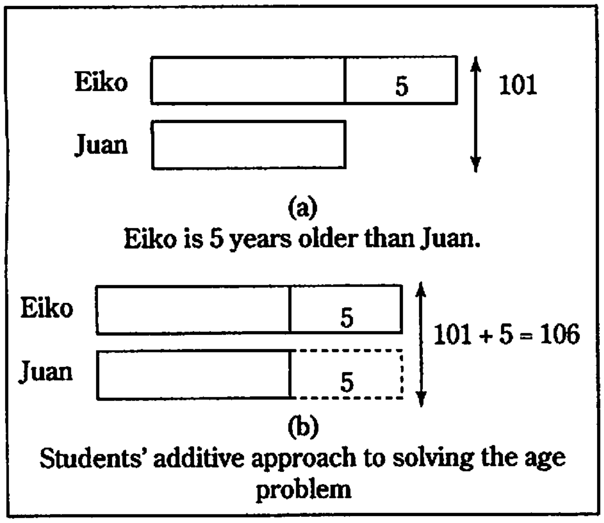


Fig. 2 Students represent the age problem.

The first rectangle represented Eiko's age and was 5 units longer than the second rectangle, which represented Juan's age.

The students used different strategies to complete the problem. Some students remarked that because Eiko was 5 years older than Juan and the sum of the ages was 101, the difference in the ages, 5, could be subtracted from 101 to get 96. The students now saw rectangles of equal size and concluded that they needed only to divide the result by 2 to get the solution. These students wrote—

$$101 - 5 = 96$$
$$96 \div 2 = 48$$
Juan is 48 years old.

Other students also noted that the sum of the ages was 101, but they added 5 to the rectangle representing Juan's age to obtain the results in **figure 2b.**

In the next step, they added 5 to 101 to get 106, saw that they had two rectangles of equal size, then divided by 2. Because the problem stated that Eiko was 5 years older, these students subtracted 5 from her age to obtain Juan's age. They wrote—

$$5 + 101 = 106$$
$$106 \div 2 = 53$$
Eiko is 53 years old.
Juan is $53 - 5 = 48$ years old.

As a conclusion to this problem, Ms. Y contrasted the two modeling approaches with a traditional algebraic approach. That is, she noted that students could begin by representing the two ages with a single variable, then manipulate algebraic expressions to solve for the variable, as shown at the top of the next column:

Juan's age $= x$
Eiko's age $= x + 5$
Total ages $= x + (x + 5) = 2x + 5$
$$2x + 5 = 101$$
$$2x + 5 - 5 = 101 - 5$$
$$2x = 96$$
$$x = 48$$
Juan's age = 48 years old

The next word problem was challenging for the middle school students to solve. This example illustrated another kind of situation that also encouraged the development of algebraic thinking. The problem was stated as follows:

> Ali had 3 times as much money as John. After Ali spent \$60 and John spent \$10, they each had an equal amount of money. How much did Ali have at first?

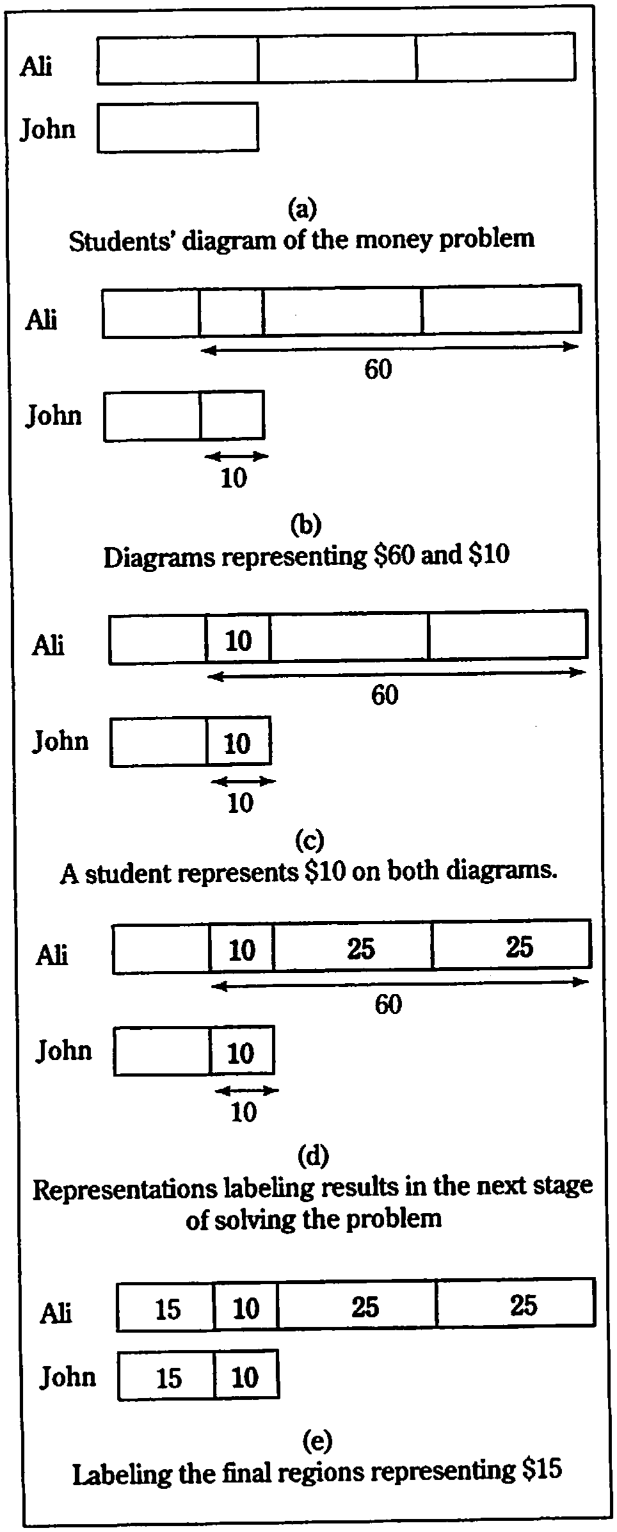
As she did with most of these problems, Ms. Y initially asked the class how they could begin to solve this problem by drawing rectangles. After discussing the problem in groups, a consensus emerged to draw two rectangles, with one partitioned into thirds to represent the fact that Ali had 3 times as much money as John. At this point, many students remarked that these diagrams made sense to them because they could easily see that Ali's amount was three times as much as John's amount. The students' diagram is shown in **figure 3a.**

After this diagram was drawn, Ms. Y asked the students to shade the rectangular regions to indicate the amount of money Ali and John had spent. The students returned to their group discussions and eventually agreed that they should shade a portion of Ali's diagram to show \$60 and a smaller portion of John's diagram to show \$10. Knowing in advance that the placement of these shaded regions was important in solving the problem, Ms. Y directed her students to draw their shading from the right side of the diagram. One student drew the shaded regions on the overhead projector (see **fig. 3b**).

To enhance their understanding, Ms. Y next asked students to display a representation on both rectangles of the \$10 John had spent. **Figure 3c** shows how another student wrote \$10 in the rectangles on the diagram.

This representation seemed to help students see the solution to the teacher's question of how to determine the amounts in Ali's shaded regions. Many students replied that they could subtract 10 from 60 to determine the remaining amount of money that Ali had spent. When the students subtracted 10 from 60 and divided the remainder in half, they could then display the result of 25 in each of the two rectangles that together showed two-thirds of Ali's original amount. These calculations resulted in the representation shown in **figure 3d.**

From this perspective, most students were able to see that Ali's three portions each equaled $25.



(a) Students' diagram of the money problem

(b) Diagrams representing $60 and $10

(c) A student represents $10 on both diagrams.

(d) Representations labeling results in the next stage of solving the problem

(e) Labeling the final regions representing $15

Fig. 3 Students use representations to solve a more difficult problem.

More subtraction would reveal the values needed to label the last unidentified regions, that is, $25 – $10 = $15, as shown in **figure 3e.** The students added the values in the rectangular regions to conclude that Ali initially had $75.

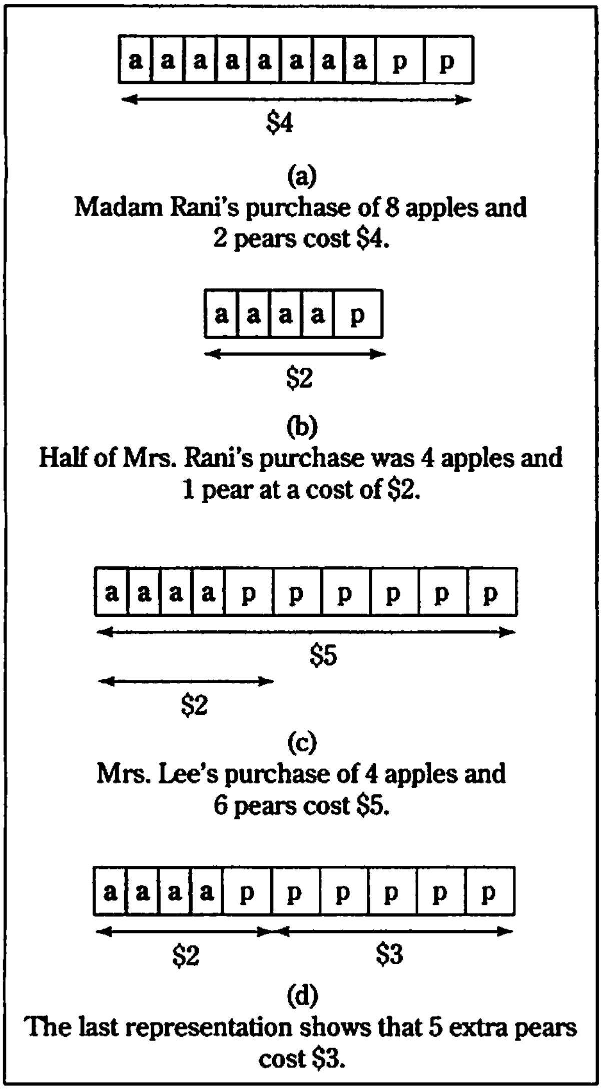
In the final example from Ms. Y's classroom, the modeling approach served as a precursor for the formal algebraic techniques involved in solving simultaneous linear equations in two unknowns. Ms. Y especially liked this problem because it accentuated the importance of engaging students in making connections between logical reasoning and visual models. The problem concerns a pair of purchases at a produce market, as follows:

> Madam Rani bought 8 apples and 2 pears for $4. Mrs. Lee bought 4 apples and 6 pears for $5. Find the price of an apple.

A few students began by guessing numbers arbitrarily; however, through collaboration with their group members, they realized the value of searching for a more mathematical way to solve the problem. After several minutes of discussing the problem in their groups and recalling how they had used rectangles to solve previous problems, several students started their models by drawing a rectangle with 10 sections that showed the quantities and cost of Madam Rani's purchase. To make the diagrams easier to understand, the teacher encouraged students to use contrasting colors to represent the different types of fruit or to use symbols, such as an *a* for apple and a *p* for pear, to clarify the models. Students then represented the first part of the problem as in **figure 4a.**

Ms. Y next instructed her students to return to the original problem and look for a relationship among the quantities. For those students who needed help at this point, the teacher suggested that they think about how they could represent or model the cost of 4 apples and 1 pear. Ms. Y kept drawing the students' attention back to the given statement that Madam Rani bought 8 apples and 2 pears. She continued to ask her students to look at the relationships between the initial purchase, 8 apples and 2 pears, and the new model of 4 apples and 1 pear. Soon, almost all students realized that they could divide the amounts in half and draw another rectangle with these halved amounts (see **fig. 4b**).

After this step, Ms. Y asked the students how they could represent Mrs. Lee's purchase. At this point, the students seemed to be more comfortable in working with the models. Many of the students remembered from the previous stamp problem that rectangles could be put together; for this reason, they decided to copy the halved representation of



(a)
Madam Rani's purchase of 8 apples and 2 pears cost $4.

(b)
Half of Mrs. Rani's purchase was 4 apples and 1 pear at a cost of $2.

(c)
Mrs. Lee's purchase of 4 apples and 6 pears cost $5.

(d)
The last representation shows that 5 extra pears cost $3.

Fig. 4 Modeling simultaneous equations

Mrs. Rani's purchase and add 5 more pears to it. Because these students already knew the cost of 4 apples and 1 pear, they included the previously modeled $2 cost as part of the new model, as shown in **figure 4c. Figure 4d** shows how students completed the problem, subtracting $2 from $5 to get $3 as the cost of the 5 extra pears.

The final step required dividing the $3 by the 5 pears to get the cost of $0.60 for each pear. Students commonly *noted* that $2.00 – $0.60 (the cost of a pear) = $1.40, which could be divided by 4 to get $0.35, the cost of an apple.

Research on the modeling approach continues in Asia and often includes word problems similar to those in this section. The effectiveness of the modeling approach was recently confirmed in a series of case studies in which more than 80 percent of fifth graders in Singapore were able to solve problems with two simultaneous unknowns (Yeap and Kaur 2001).

Conclusion

SINGAPORE TEACHERS' USE OF THE MODELING approach to teach mathematics skills and concepts in a problem-solving environment is likely to continue to enhance their students' learning of mathematics. Anecdotal evidence from Singapore's secondary school teachers has shown that students tend to return to the modeling approach to solve problems in the upper grades (Fong 1994). Further, once students become familiar with the modeling approach, they often omit the detailed diagrams in their solutions. Armed with this information, even teachers of the upper grades have begun to incorporate modeling methods into their instruction. Once students master a modeling approach, making the transition to algorithmic methods for solving advanced problems is often easier and more apt to be successful.

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