On the toroidal crossing numbers of large complete graphs

MOVES conference II

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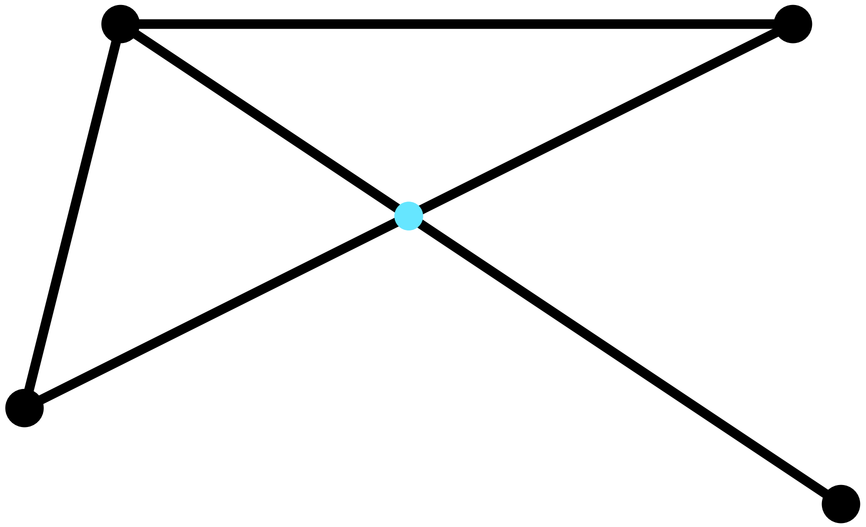
Review: Graphs and drawings

Our graphs will be finite, undirected, with no loops or multiple edges. Formally, we can say graph is a combinatorial structure $G = (V, E)$, with V (the set of **vertices**) finite, and E (the set of **edges**) a subset of $\binom{V}{2}$. [The notation "$\binom{V}{2}$" means the collection of all 2-element subsets of V, so called because there are $\binom{v}{2}$ of them if there are v vertices.] Less formally, each pair of vertices is either connected or not, and the edges keep track of the connections.

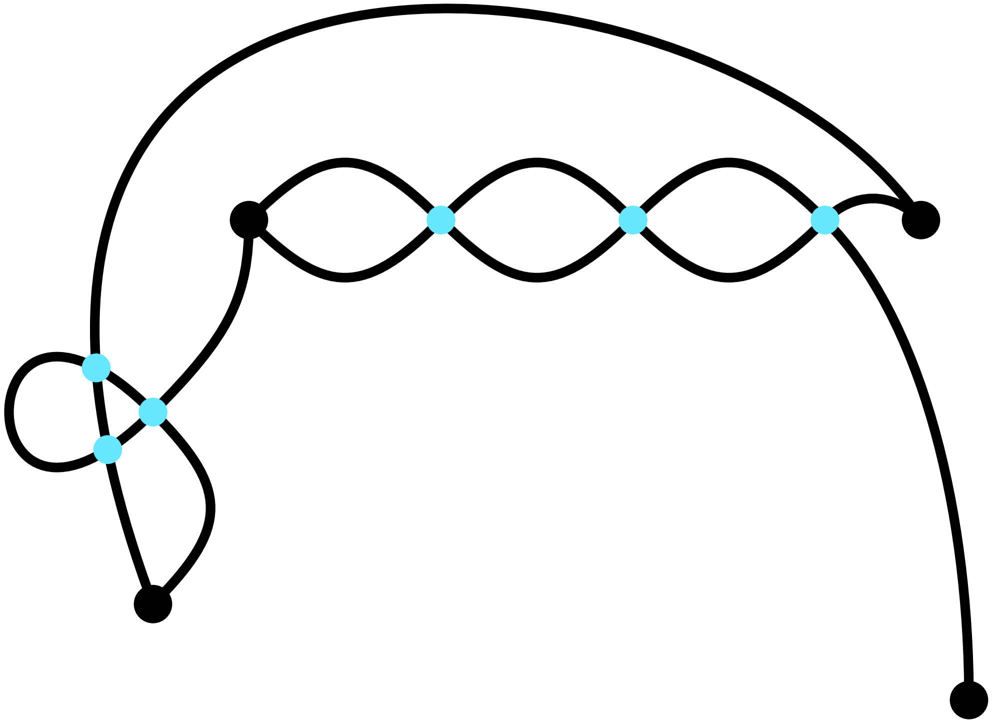
Again, connections go both ways (the graph is undirected); a vertex may not be "connected to itself" by an edge (no loops); nor can there be more than one edge between the same pair of vertices (no multiple edges).

We usually draw (small) graphs in the plane as diagrams where vertices are dots and edges are paths between them. For example, here's a drawing of a graph G with 4 vertices and 4 edges:

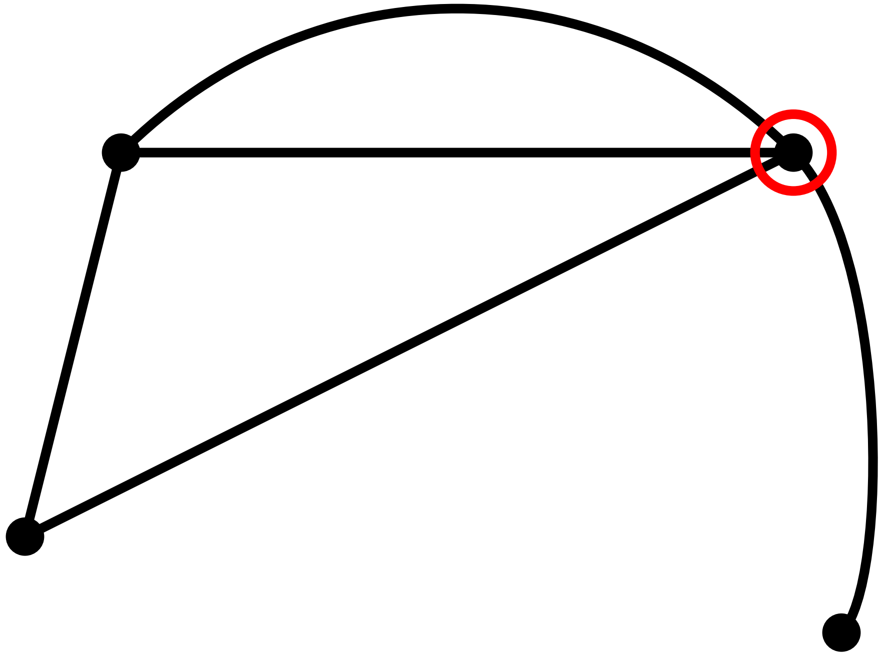


Note the "accidental" crossing between two of the edges, which does not count as a fifth vertex.

There are many ways to draw a given graph G. Here's another, more complicated one for the same G:

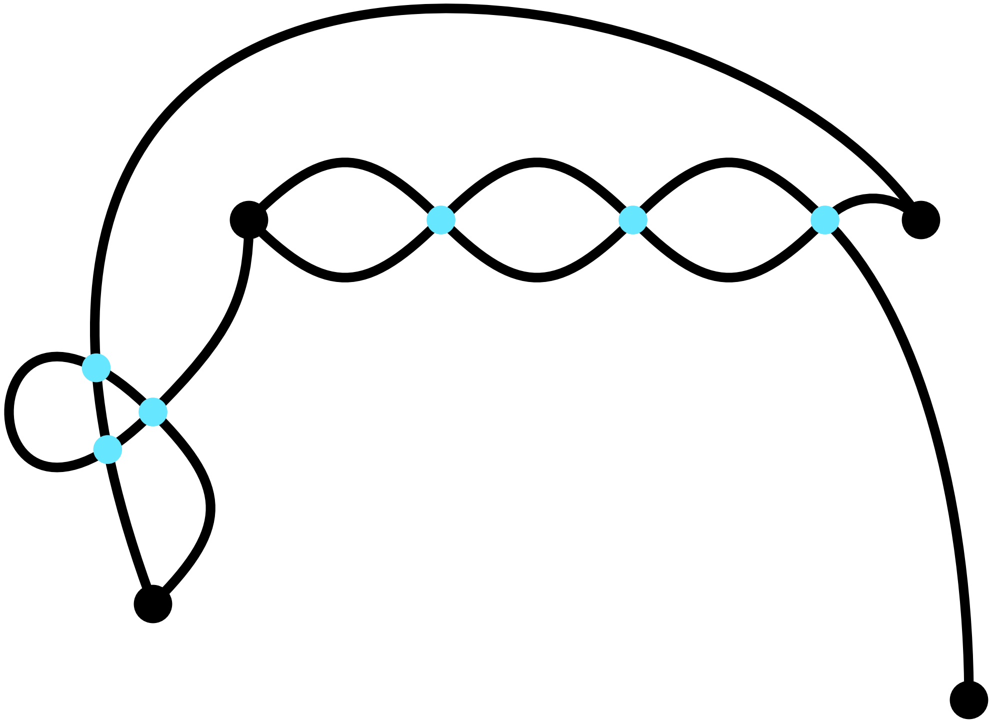


The only rule is that an edge is not allowed to go through a vertex except the edge's start and end. For example, this drawing of G is forbidden:



The crossing number of a drawing

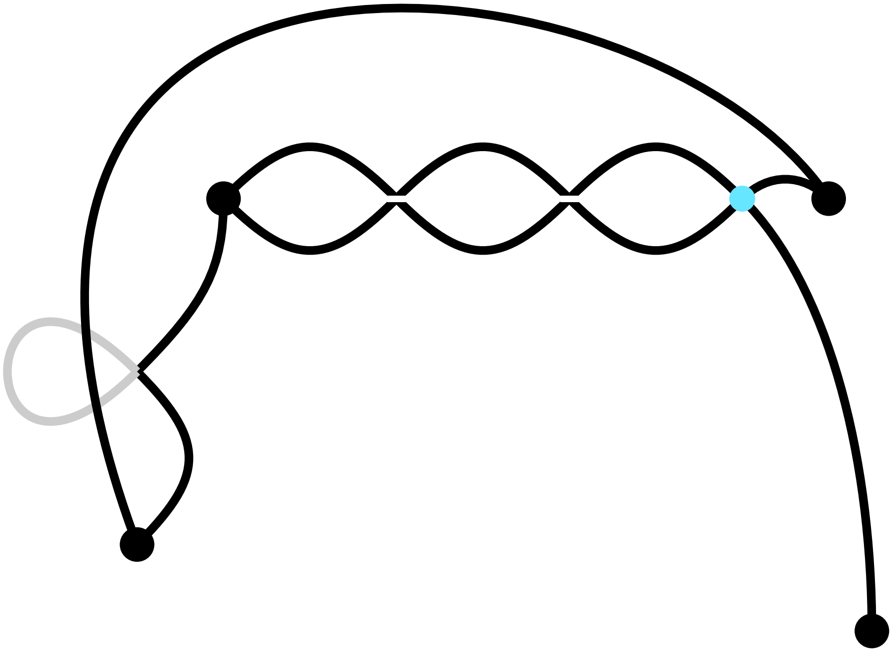
One way to quantify the complexity of a drawing of G: the drawing's **crossing number**, which counts those "accidental intersections" where two edges meet but not at a vertex. Our first drawing had just one crossing, the second rather more:



The (planar) crossing number of a graph

Given G, the task is to draw G with the fewest crossings possible. That smallest number is the **crossing number of** G.

[Simplifications: (i) an edge need never cross itself: erase the detour to get another drawing with fewer crossings. (ii) two edges need never cross more than once: multiple crossings cancel in pairs without affecting other intersections. (iii) two edges sharing a vertex need not cross at all.]



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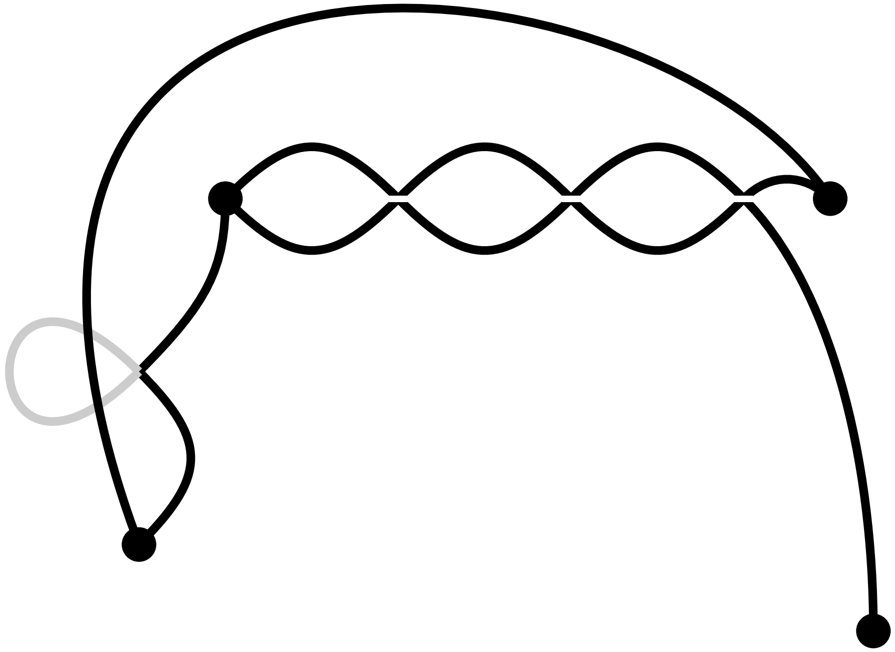
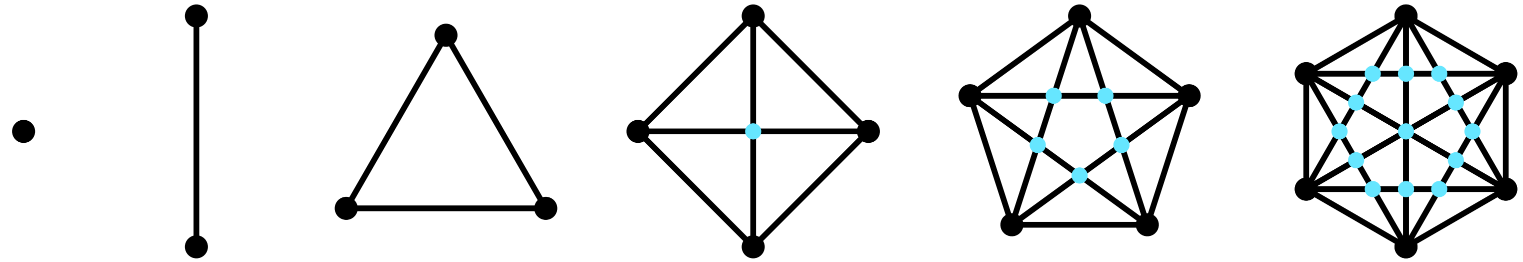


The complete graph K_v

Given the number v of vertices, adding edges to G can only increase the crossing number; so the largest crossing number is for the **complete graph** where every pair of vertices is connected, for a total of $\binom{v}{2}$ edges. This graph is called K_v. Here are drawings of K_v for $v \leq 6$:

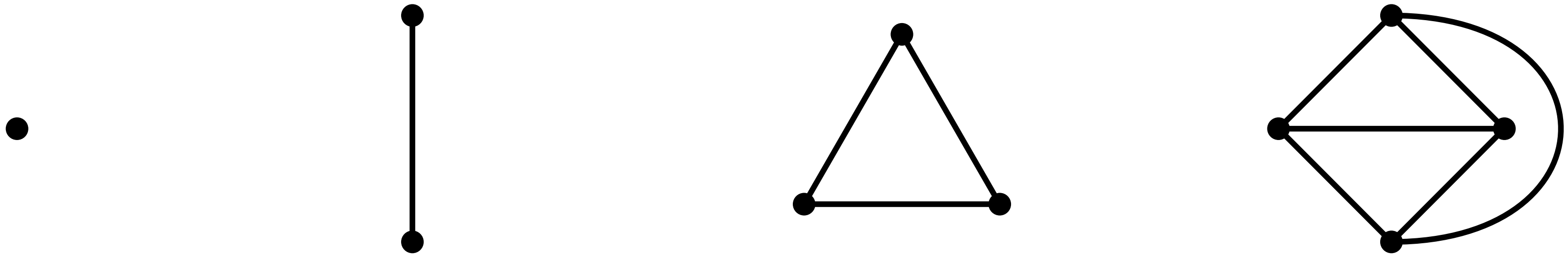


So: **What is the crossing number of K_v?**

[repeated:] So: **What is the crossing number of K_v?**

That's the topic of our next few slides, so let's give it a name: denote the crossing number of K_v by $N(v)$.

Easy for the first few v . . .



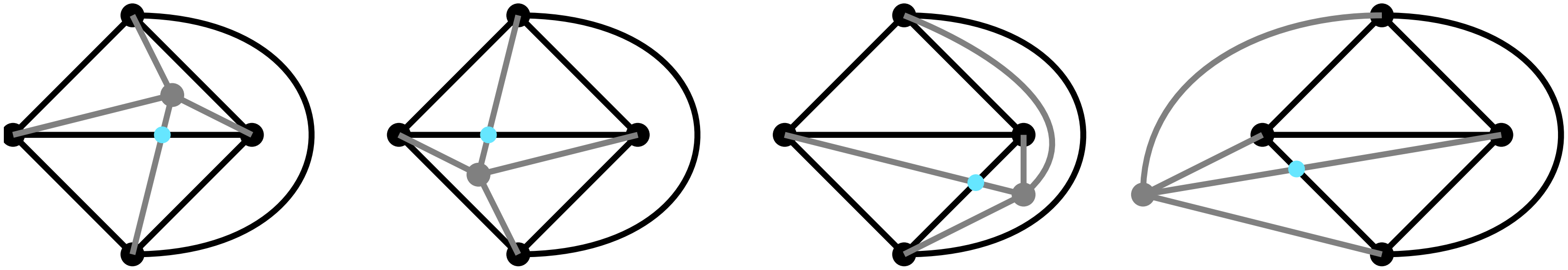
So $N(1) = N(2) = N(3) = N(4) = 0$.

What about $v = 5$?

Claim: $N(5) > 0$.

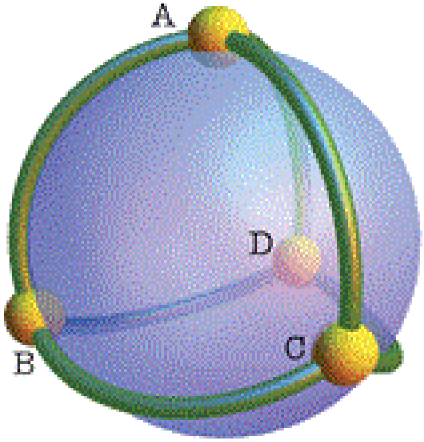
Equivalently: K_5 has no crossing-free drawing.

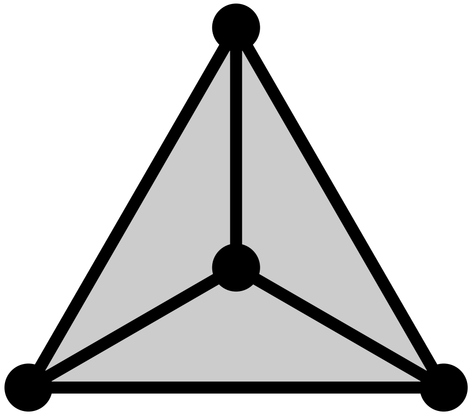
For example, starting from our crossing-free K_4, anywhere we put a 5th vertex v_5 forces at least one crossing to reach another vertex separated from v_5 by a closed circuit:



Remember v_5 is not allowed to be <u>on</u> an edge of the K_4!

Easier to see this by putting our K_4 drawing on the **sphere**, with tetrahedral symmetry. Here it is on the sphere Σ, and on the plane after removing a small disc from Σ opposite a vertex to flatten out K_4:





Anywhere we put v_5, it's in one of the four faces (possibly the "bottom"/"outside" face), and its edge to the opposite vertex must cross an edge of the tetrahedral K_4.

Maybe some other way works, though? No, because of **Euler's formula** $v-e+f=2$. Here $v=5$ and $e=\binom{5}{2}=10$, so $f=7$. But then we get a contradiction by counting in two ways the number of {face, edge} pairs with the edge bounding the face: 10 edges means 20 pairs; but 7 faces means at least $3\cdot 7=21$ pairs (remember no loops or multiple edges, so no "monogon" or "digon" faces with only one or two edges). **QED**

[Note that K_4 barely squeaks through: $v=4$, $e=\binom{4}{2}=6$, so $f=6$, and $2e=3f=12$.]

Conclusion: **The crossing number $N(5)$ of K_5 is 1.**

So what happens for $v > 5$? The answers are known for only a few further v:

v	6	7	8	9	10	11	12	13	14	15	...
$N(v)$	3	9	18	36	60	100	150	≤ 225	≤ 315	≤ 441	...

Known values proved by Richard Guy (1972) for $v \leq 10$ and by S. Pan and R. B. Richter (2007) for $v = 11$ and $v = 12$.

For each v it's a finite calculation, but it gets infeasible quickly as v grows.

Already in 1960 Guy gave a plausible guess for a formula for $N(v)$ for all v (including $v \leq 5$):

Guy's conjecture: a formula for the crossing number of K_v?

v	1–4	5	6	7	8	9	10	11	12	13	any v
$N(v)$	0	1	3	9	18	36	60	100	150	225?	$\frac{1}{4}\lfloor\frac{v}{2}\rfloor\lfloor\frac{v-1}{2}\rfloor\lfloor\frac{v-2}{2}\rfloor\lfloor\frac{v-3}{2}\rfloor$?

In other words, it should be

$$N(v) = \frac{v(v-2)^2(v-4)}{64} \qquad \text{or} \qquad \frac{(v-1)^2(v-3)^2}{64}$$

according as v is even or odd. In particular, Guy conjectures that **the crossing number grows as** $v^4/64$ for large v.

Why should $N(v)$ grow as fast as some multiple of v^4, and why is the multiple 1/64?

Total number of candidate crossings = number of pairs of non-overlapping pairs, which is $(1/2)\binom{v}{2}\binom{v-2}{2}$, that is

$$\frac{1}{2}\,\frac{v(v-1)}{2}\,\frac{(v-2)(v-3)}{2} = \frac{v(v-1)(v-2)(v-3)}{8} \sim \frac{1}{8}v^4.$$

So, according to Guy's conjecture, for large v the number of crossings can be about 1/8 the number of candidates [since $1/64 = (1/8)(1/8)$], but no smaller fraction.

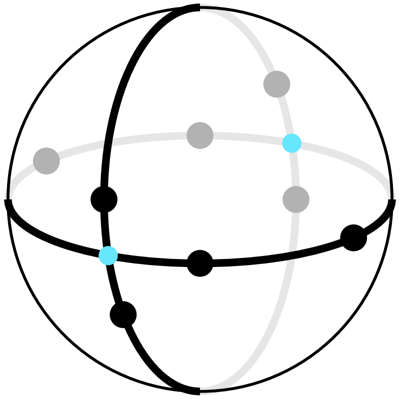
Certainly the fraction must be positive; e.g. use $N(5) = 1$ (out of $\frac{1}{2}10 \cdot 3 = 15$ candidates), and average over all 5-vertex subsets to get at least 1/15 once $v \geq 5$. Likewise $N(11) = 100$ gives $\geq 10/99 = 0.1010+$ for $v \geq 11$. But why 1/8 [= 0.125]?

Remarkably one can draw K_v with $\sim v^4/64$ crossings by just picking v **random** points on the sphere $\mathbf{\Sigma}$ and joining each pair by the shortest arc (great-circle path). Then puncture and project to the plane to get a plane drawing.

On the average, each candidate crossing happens exactly 1/8 of the time, so there are $v(v-1)(v-2)(v-3)/64$ crossings, with exactly the same leading term as the conjectured $N(v)$.

Why 1/8? Looks like a fearsome octuple integral (four points on 2-dim. S), or quintuple (since S also has 3 dimensions of symmetries). With some thought, can bring it down to the single integral $\int_0^\pi x \sin x \, dx$. But there's a neat trick that lets us compute this probability without any calculus!

We want the probability that for randomly chosen A, B, C, D the shorter great-circle arcs AB and CD intersect. The trick is that each of the 2^4 quadruples $(\pm A, \pm B, \pm C, \pm D)$ has the same distribution, and of these 16 exactly 2 have intersecting arcs. So probability is $2/16 = 1/8$ as claimed. [Arcs of the same two great circles, which are distinct with prob. 100%.]



So, v *random* points attain $N(v) \sim v^4/64$. After 50+ years, we still can't do better. Nor prove that this is best possible, even in the limit; only that $> .1074$ of the edge-pairs must meet (de Klerk–Pasechnik–Schrijver 2007; but $.1074 < .125$).

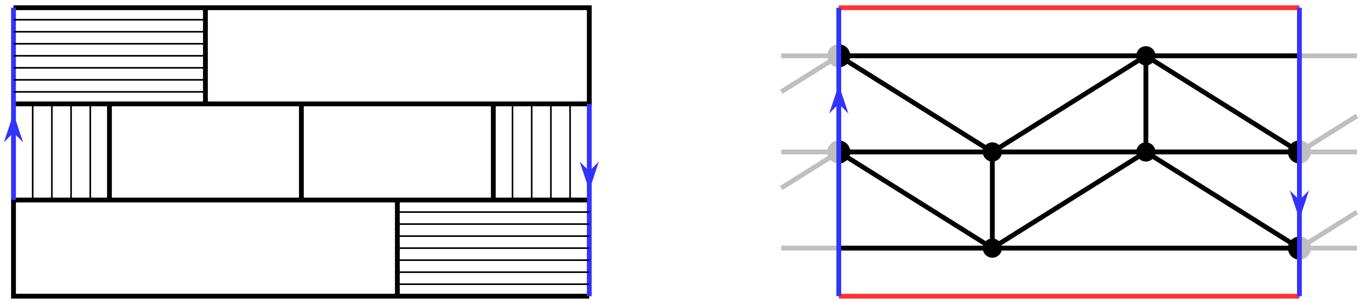
Crossing numbers on other surfaces

We can ask the same questions for drawing K_v (or any graph G) on any other surface $\mathbf{S}$, not just the plane. Denote by $N_{\mathbf{S}}(v)$ the crossing number of the complete graph K_v on $\mathbf{S}$. For us $\mathbf{S}$ will be the sphere $\mathbf{\Sigma}$, the Möbius band $\mathbf{M}$, the projective plane $\mathbf{P}$, and (of course, given our title) the torus $\mathbf{T}$.

Any planar drawing of G can be put on $\mathbf{S}$, so $N_{\mathbf{S}}(v) \leq N(v)$. If $\mathbf{S} = \mathbf{\Sigma}$ then in fact $N_{\mathbf{S}}(v) = N(v)$ because from any spherical drawing we can remove a small disc and flatten the rest to plane. But otherwise $N_{\mathbf{S}}(v)$ may be strictly smaller $\leq N(v)$.

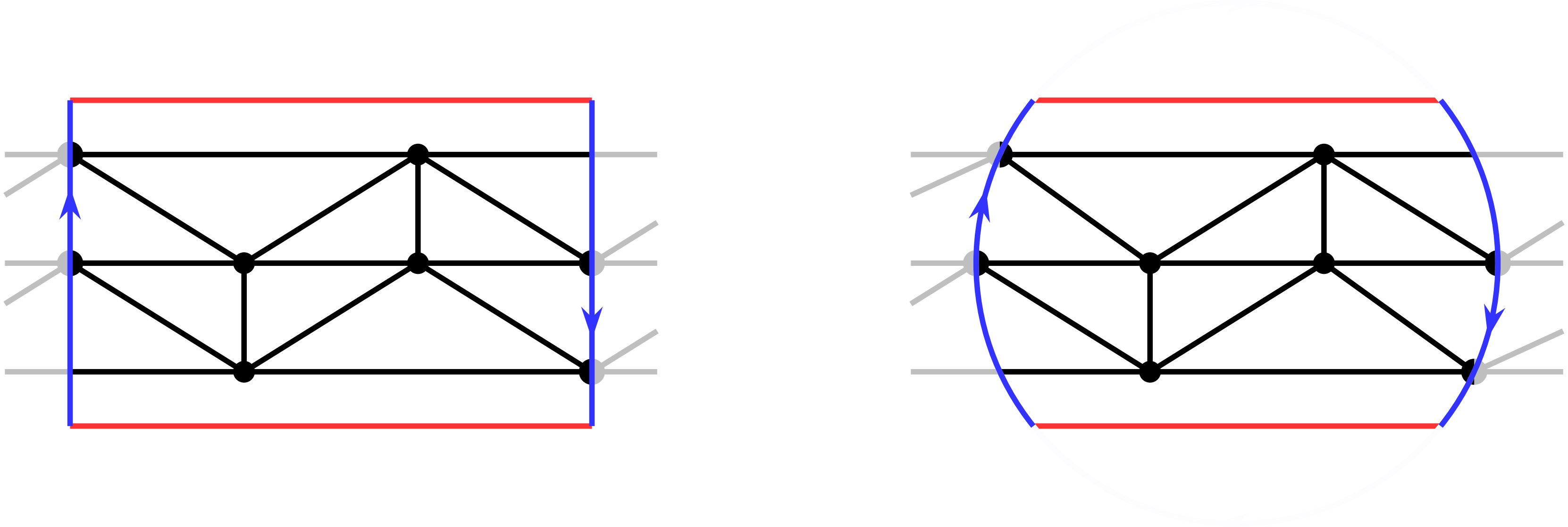
For instance, we saw $N(v) > 0$ once $v \geq 5$. But $N_{\mathbf{M}}(5) = N_{\mathbf{T}}(5) = 0$; indeed even $N_{\mathbf{M}}(6)$ and $N_{\mathbf{T}}(7)$ are zero: the graphs K_6 and K_7 can be drawn without crossings on the Möbius band and torus respectively!

We get $N_{\mathbf{M}}(6) = 0$ from Tietze's six-region map on the Möbius band $\mathbf{M}$ in which each region abuts every other:



This map can be constructed even more symmetrically on the **projective plane** $\mathbf{P}$, which is obtained from the circle C by identifying pairs of opposite points on the boundary, or from the sphere by identifying antipodal pairs (flatten the N or S hemisphere to C). Removing a disc from $\mathbf{P}$ recovers $\mathbf{M}$, so — as with the plane and sphere — we have $N_{\mathbf{M}}(v) = N_{\mathbf{P}}(v)$ for each v:

Möbius band as projective plane minus a disc, and drawings for $N_{\mathbf{M}}(6) = N_{\mathbf{P}}(6) = 0$ [note that the identifying arrows include a graph edge along the boundary]:



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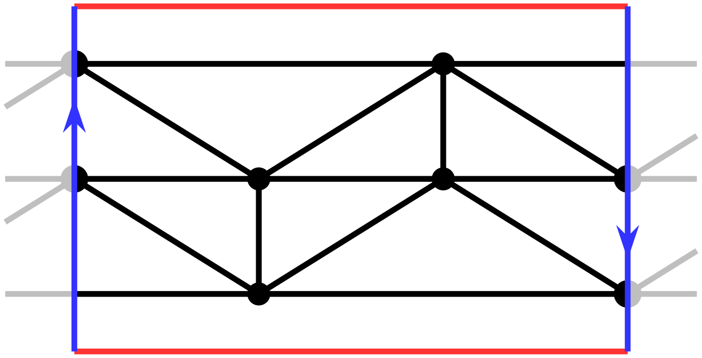
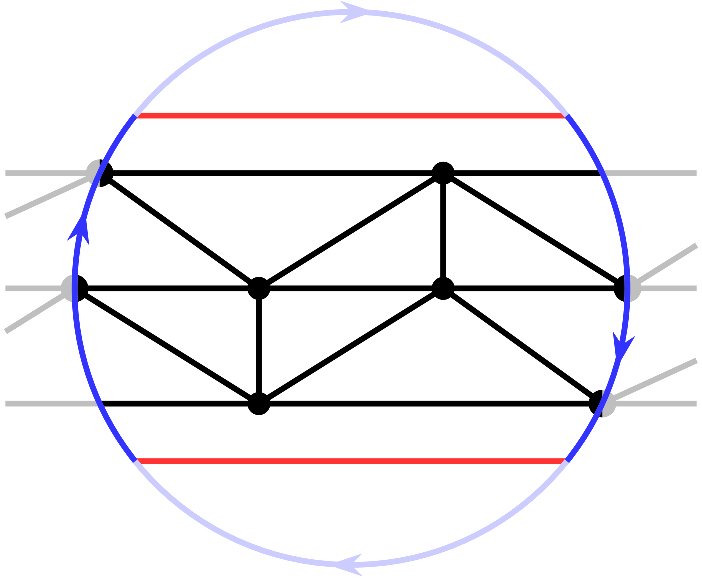
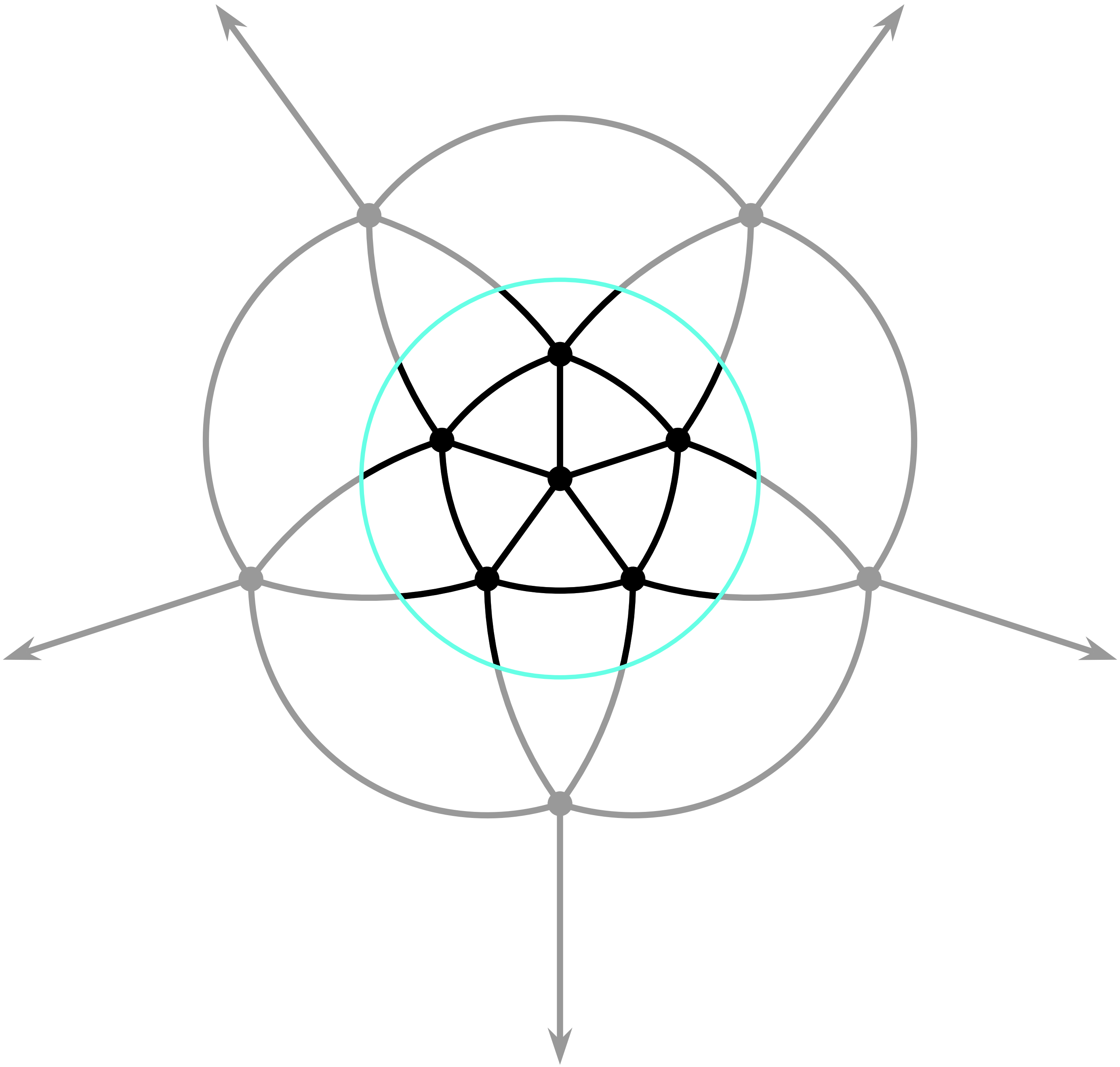




Since the projective plane $\mathbf{P}$ is “$\mathbf{\Sigma}/\{\pm 1\}$” (sphere modulo antipodal identification), it inherits the rotational symmetries of $\mathbf{\Sigma}$. This suggests a much more symmetrical crossing-free drawing of K_6 on $\mathbf{P}$, as follows.

Start from regular icosahedron, which has 12 vertices, each connected to 5 others. The antipode of a vertex is a vertex, so we get $12/2 = 6$ points of $\mathbf{P}$, each connected to 5 others — all different because they came from the same hemisphere of $\mathbf{\Sigma}$. So, 6 points, all pairs connected, no crossings, and lots (60) of symmetries:



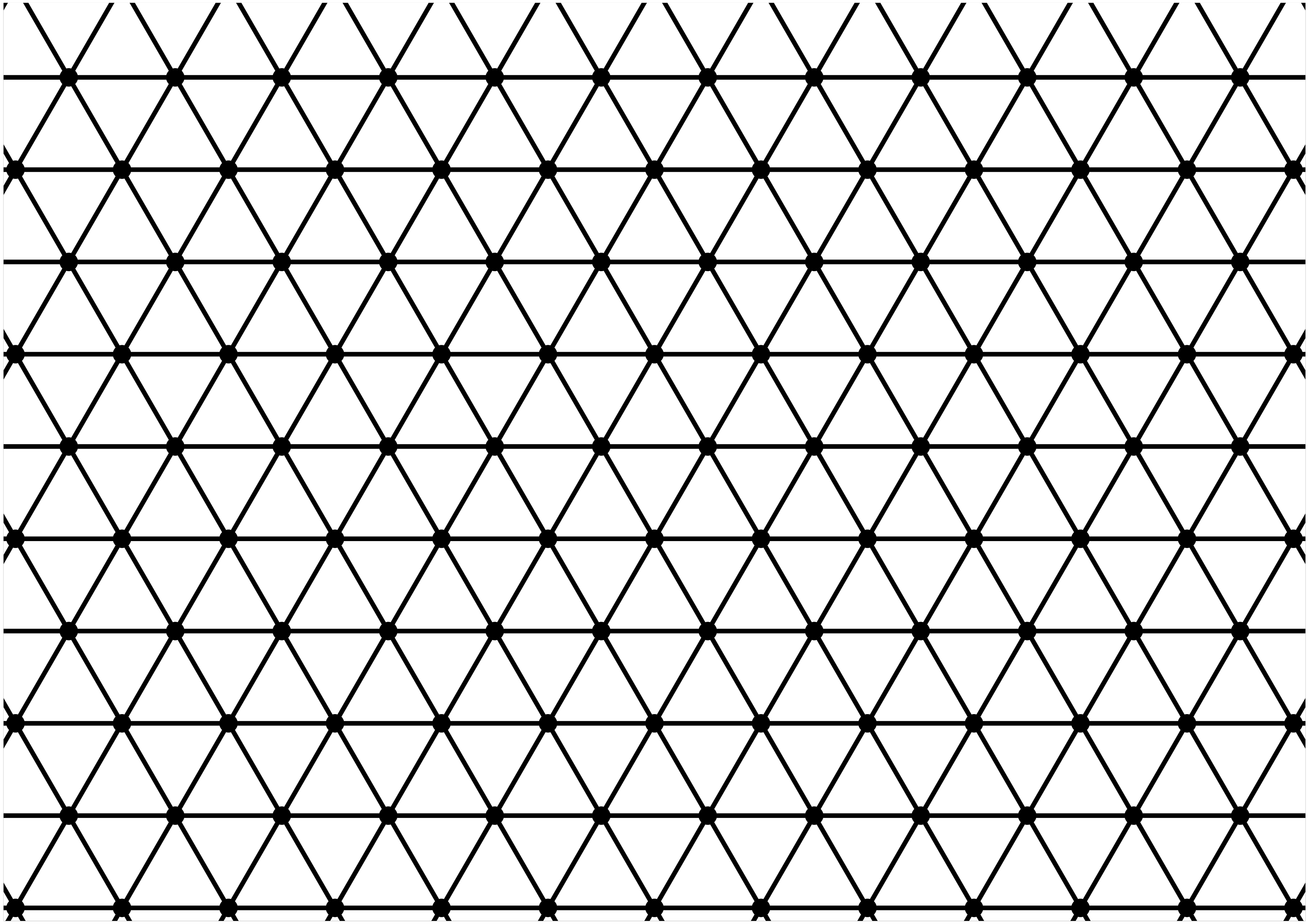
For large v, it is still true that $N_{\mathbf{M}}(v)$ (a.k.a. $N_{\mathbf{P}}(v)$) grows as $C_{\mathbf{M}}v^4$, because some positive fraction of the $\sim v^4/8$ pairs of edges must meet. But we don't even have a robust conjecture for that limiting value $C_{\mathbf{M}}$ (a.k.a. $C_{\mathbf{P}}$), let alone for exact crossing numbers.

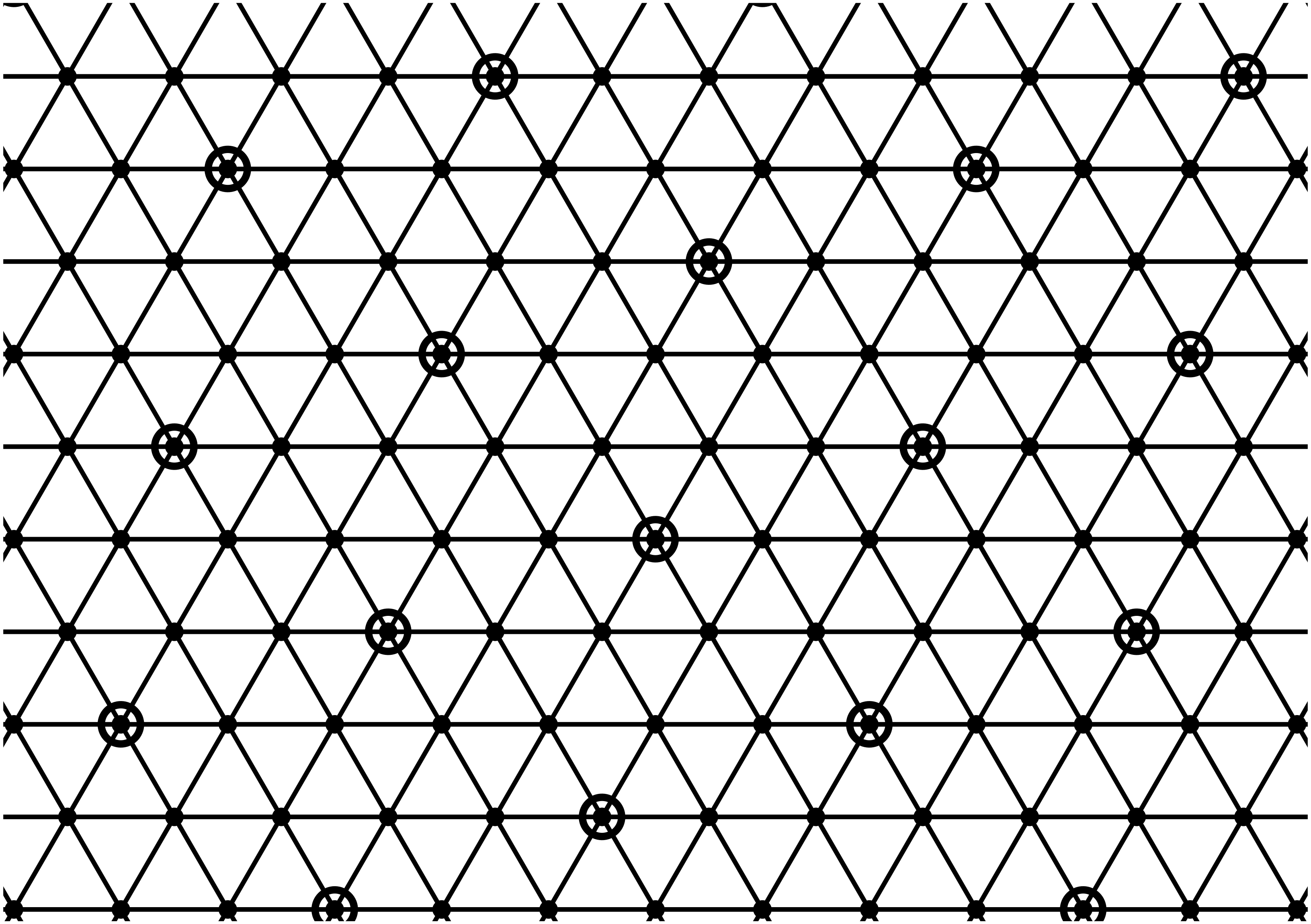
We get $C_{\mathbf{P}} \leq 1/(8\pi^2) = 0.012655148-$ from the strategy of sprinkling points at random on $\mathbf{P}$ and joining each pair by the shortest path (two such paths meet with probability $1/\pi^2$, as against $1/8$ on $\mathbf{\Sigma}$). Is this best possible? It seems the best explicit construction known [M. Koman 1969] gives $13/2^{10} = 0.0126953125$, a tiny bit worse (just under $\frac{1}{4}\%$).

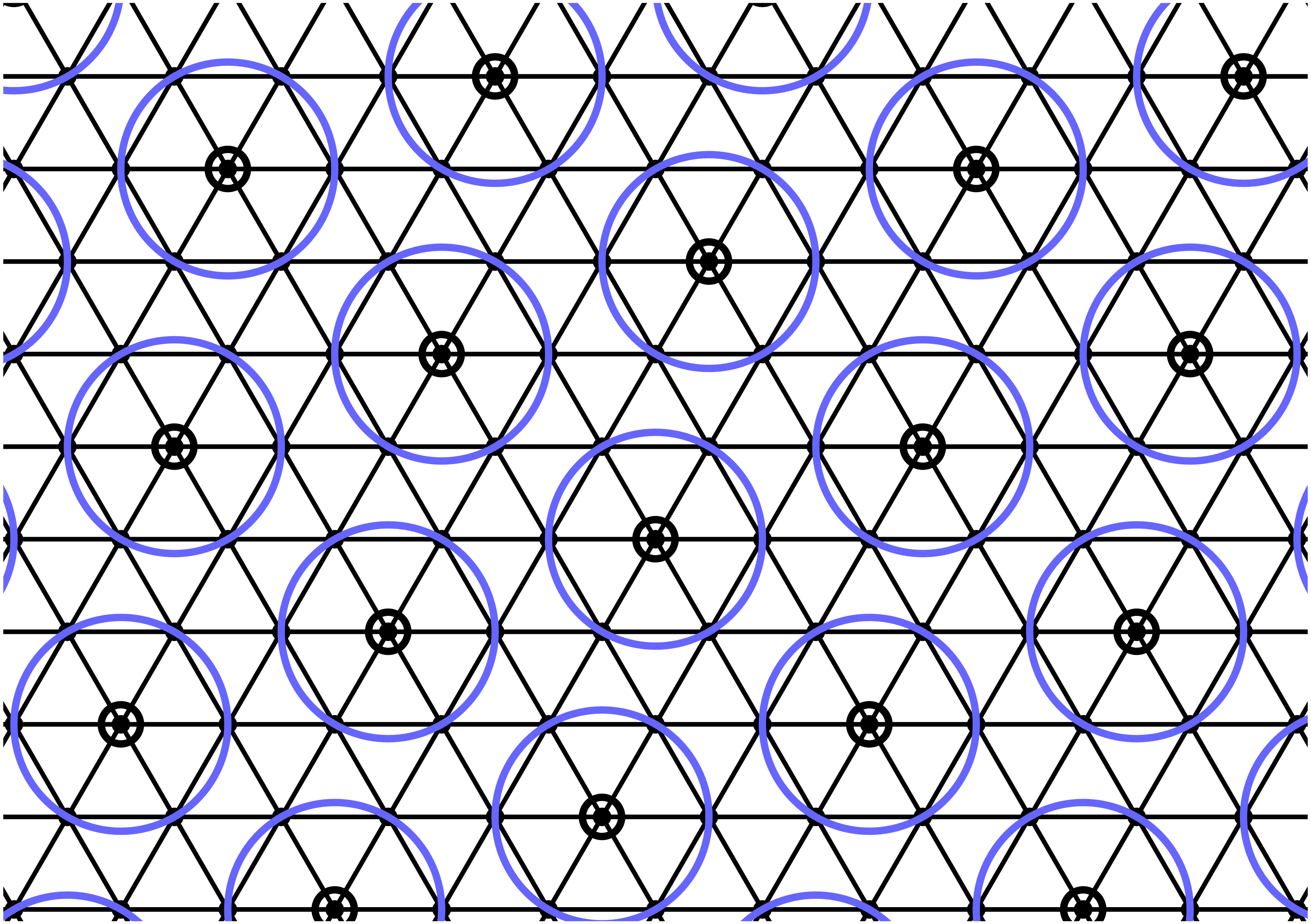
Crossing numbers on the torus (finally)

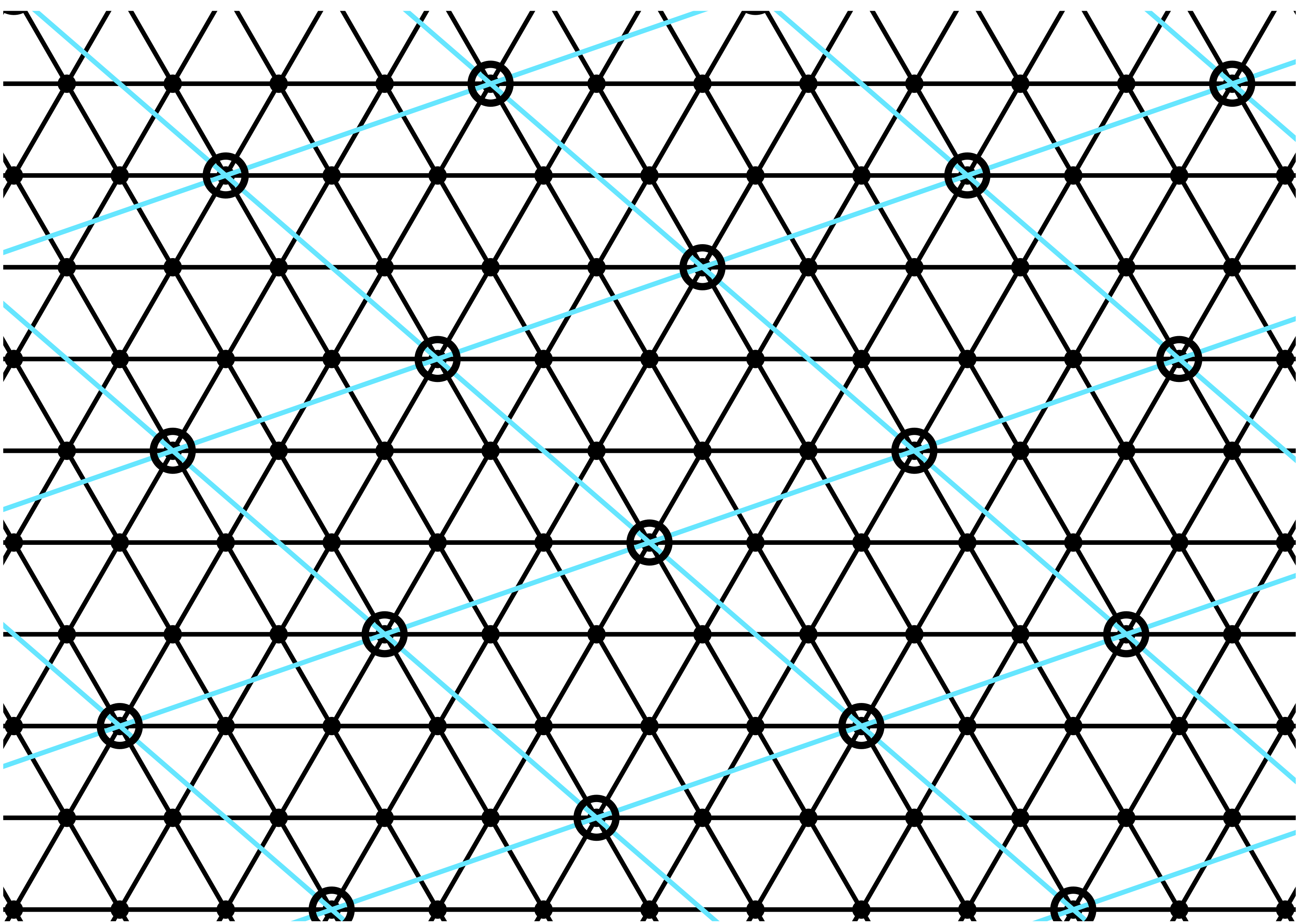
What of the torus? Here, as promised, $N_{\mathbf{T}}(v) = 0$ for $v \leq 7$. As with $N(4) = 0$ and $N_{\mathbf{P}}(6) = 0$, this is barely consistent with Euler's formula, and only with all faces triangular so that f is as small as $2e/3$: since $e = \binom{7}{2} = 21$, we then have $\chi = 7 - 21 + \frac{2}{3}21 = 0$, which indeed is the "Euler characteristic" of $\mathbf{T}$ (while $\chi = 1$ for $\mathbf{P}$, and "of course" $\chi = 2$ for $\mathbf{S}$).

Recall that our crossing-free drawings of K_4 on $\mathbf{\Sigma}$ and of K_6 on $\mathbf{P}$ were tilings by triangles meeting 3 (resp. 5) at a vertex. On $\mathbf{T}$, we need a tiling by triangles meeting 6 at a vertex (in each case the count is $v - 1$). Here's how to do it. As $\mathbf{P} = \mathbf{\Sigma}/\{\pm 1\}$, the torus $\mathbf{T}$ is $\mathbf{R}^2/L$, the plane modulo lattice translations, i.e. a parallelogram with opposite sides identified:









StIll, K_8 and all further K_v can't be drawn on $\mathbf{T}$ without some crossings, and the crossing number $N_{\mathbf{T}}(v)$ must grow as $C_{\mathbf{T}}(v^4)$ for some $C_{\mathbf{T}} > 0$, and again we'd like to find $C_{\mathbf{T}}$, or at least to get upper and lower bounds.

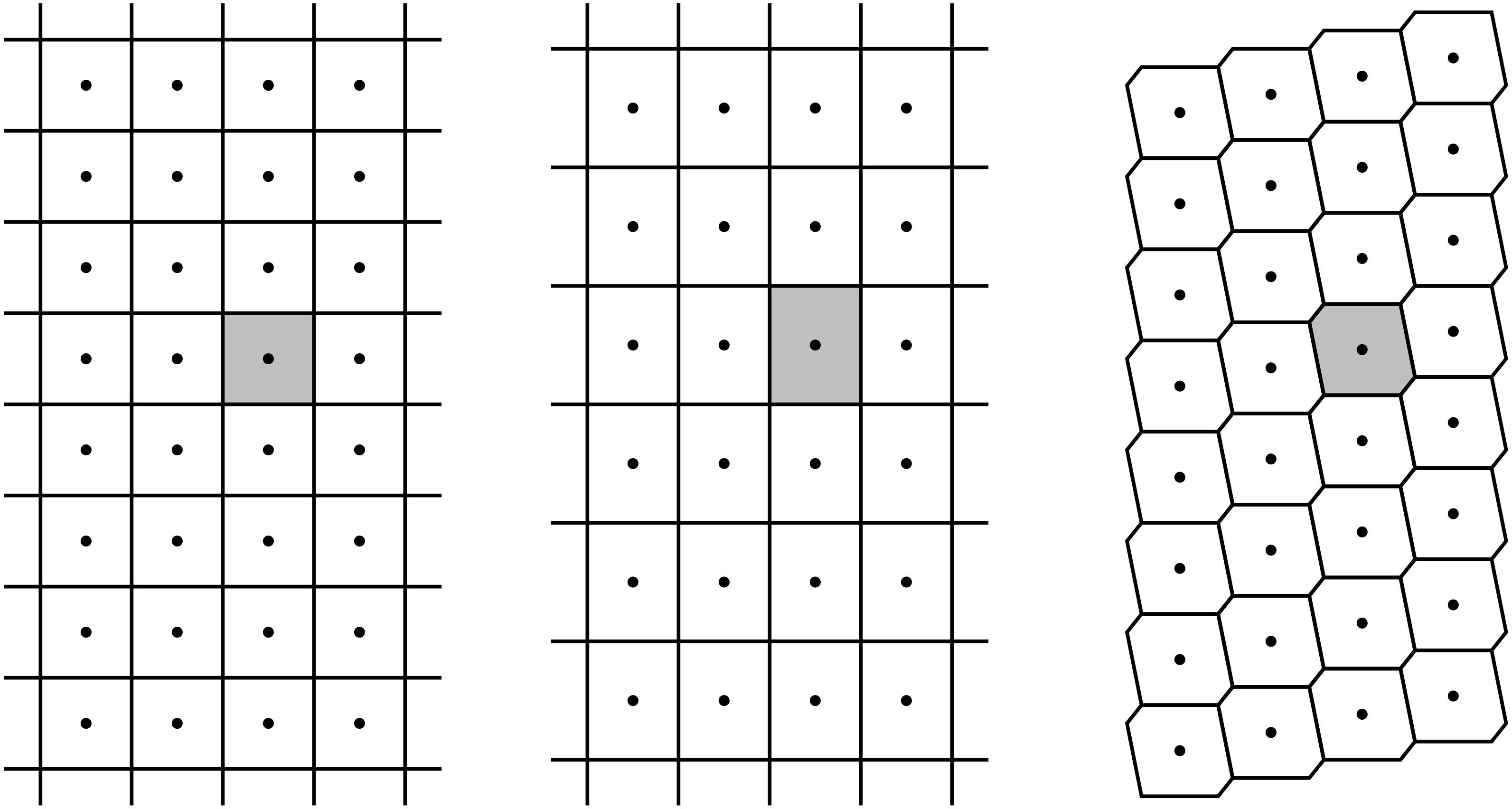
Upper bound means showing that K_v can be drawn on $\mathbf{T}$ with at most $\sim cv^4$ crossings, and again we try random strategy: place v points randomly on the torus, and connect each pair with the shortest path. So, as before with $C_{\mathbf{\Sigma}}$ and $C_{\mathbf{P}}$, we see that $C_{\mathbf{T}}$ is at most 1/8 of the probability that two such paths intersect. What is this probability?

This time we likely need <u>some</u> calculus to compute the answer, and the computation is somewhat delicate [average over $\vec{x}, \vec{y}$ in Voronoi region of $|\vec{x} \times \vec{y}|$, and integrating the absolute value of a cross product is annoying].

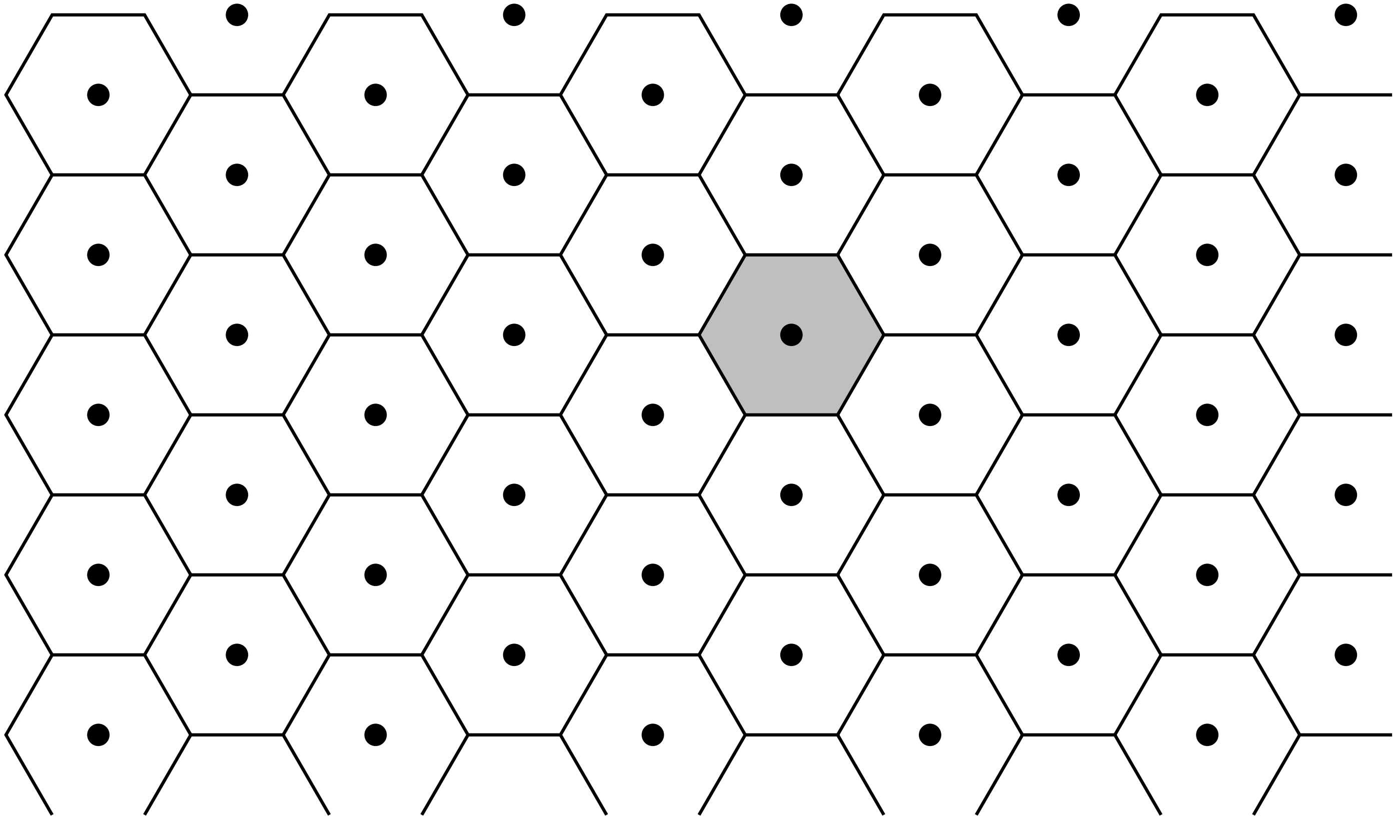
Still, R.K. Guy, T. Jenkyns, and J. Schaer did it in 1968, finding an average of $5/54 = 0.09259+$. They then improved this by a factor of 59/60 by rerouting some paths around corners, getting $59/648 = 0.09105-$, so $C_{\mathbf{T}} < 0.01138+$. According to several sources, including Guy himself, that was still the best upper bound known in 2011.

But it turns out that one can do even better, shaving off a further factor of 176/177 (a bit more than $\frac{1}{2}$%) to get $(44/45)(5/54) = 22/243 = 0.090535-$, so $C_{\mathbf{T}} < 0.011317-$.

G-J-S construct $\mathbf{T}$ as $\mathbf{R}^2/L$ where L is the **square** lattice. Now topologically any lattice L yields the same $\mathbf{T}$. But geometrically different L can make for different "shortest paths" mod L, and thus different $C_{\mathbf{T}}$ estimates, via different Voronoi regions:



After some more annoying integrations, it turns out that square (and rectangular) L's are the *worst* for this purpose, while the best — as elsewhere [see SPLAG] — is the triangular one, with regular hexagons as Voronoi regions, improving by 44/45 over the square $\mathbf{T}$.



As usual in math, answering one question (or even just finding better partial answer) raises a slew of new questions.

- For starters, it's natural to apply the random strategy with a *uniform* distribution of points on $\mathbf{\Sigma}$, $\mathbf{P}$, or $\mathbf{T}$; but is this best? Seems hard to prove; at any rate I don't know how.

- What about other surfaces? The Klein bottle is a 2:1 quotient of $\mathbf{T}$, so the same general approach should give an answer, though the details will differ. But for surfaces of genus > 1 it's not even clear that the integrals can be done "in closed form", and already for genus 2 surfaces there's a 6-dimensional space of constant-curvature structures to try.

- Meanwhile, the $\int \cdots \int |\vec{x} \times \vec{y}|$ formula suggests a new isoperimetric question: regular hexagon better than square; in general is regular N-gon better than irregular, and circle best of all?

ACKNOWLEDGEMENTS

Thanks to John Mackey for introducing me to this family of questions, to Richard Guy for correspondence on the status of the $C_{\mathbf{T}}$ problem, and (again) to MOVES 2 for welcoming this presentation.

THE END

THANK YOU!

Any (more) questions?

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