On the areas of rational triangles

OR

How did Euler (and how can we) solve

$xyz(x+y+z) = a$?

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NCTS (Taiwan), August 2014

1. **Rational triangles, $xyz(x+y+z)=a$, and Euler**

2. **How Euler might have done it**

3. **How we might think about it**

4. **Further remarks**

1.1 The areas of rational triangles.

By a "rational triangle" we mean here a Euclidean triangle whose sides have rational lengths. Traditionally these lengths are called a, b, c (but after this page we'll have to re-use "a"). The area of a triangle is given by Hero(n)'s formula (*Metrica*, c.60 CE; possibly known much earlier):

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the triangle's semiperimeter

$$s = \frac{a+b+c}{2} = (s-a) + (s-b) + (s-c).$$

Thus

$$\text{Area}^2 = xyz(x+y+z)$$

where $(x, y, z) = (s-a, s-b, s-c)$.

1.2 $\boldsymbol{xyz(x+y+z)=a}$. Therefore the areas of rational triangles are precisely the square roots of positive numbers of the form $xyz(x+y+z)$ with $x,y,z \in \mathbf{Q}$. That leads us to ask: for which values of $a \in \mathbf{Q}$ [NB "a" is no longer the side of a triangle] does the Diophantine equation

$$xyz(x+y+z)=a$$

have a solution $(x,y,z) \in \mathbf{Q}^3$?

For areas of rational triangles, we care only about $a > 0$, but the equation is natural enough for arbitrary $a \in \mathbf{Q}^*$. NB the equation is even more symmetric than is obvious from the formula (or the Heron connection): setting $w = -(x+y+z)$, we may rewrite $xyz(x+y+z) = a$ as

$$w + x + y + z = wxyz + a = 0,$$

extending the S_3 symmetry to S_4. Better yet, $S_4 \times \{\pm 1\}$, using the involution $(w, x, y, z) \leftrightarrow (-w, -x, -y, -z)$.

1.3 Euler. Thanks to Franz Lemmermeyer for the following email (7.viii.09):

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In a letter to Goldbach dated April 15, 1749, Euler mentions
the equation xy(x+y) = a and conjectures that it does
not have rational solutions for a = 1 and a = 3.  This
can easily be confirmed [...]

Then he takes a look at xyz(x + y + z) = a and says
that he has found, with quite some effort, infinitely many
solutions given by some parametrization which I will not
copy here (I can send you the formulas if you want them).
Do you have any idea how to find such a parametrization [...]?
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In other words: Euler answered our question, but we don't know how he got his solution.

Now $xy(x+y)=a$ (or more symmetrically $w+x+y = wxy+a=0$) is an elliptic curve 3-isogenous with the "twisted Fermat cubic" $X^3+Y^3=aZ^3$. That is, each of these two curves admits a degree-3 map to the other over $\mathbf{Q}$; in one direction at least this is easy: let

$$(x,y,z)=(X^3,Y^3,-aZ^3)/XYZ.$$

Euler didn't have the "isogeny" or "twist" terminology, but he knew these maps: already in the letter to Goldbach he notes that if $a=pq(pm^3 \pm qn^3)$ then the equation $xy(x+y)=a$ has a solution [indeed $(x,y) = (pm^3, \pm qn^3)/mn$ works]; years later he would use both maps to (almost!) prove his conjecture for $a=1$ (and thus Fermat's "Last Theorem" for exponent 3) via an infinite descent that we now recognize as an example of "descent via a 3-isogeny".

As for $xyz(x + y + z) = a$, Euler writes:

Proposito numero a*, invenire tres numeros rationales* x*,* y*,* z*, ut sit* $xyz(x + y + z) = a$*, so ist das problema immer möglich und kann sogar in genere die Solution angegeben werden, welche ich endlich nach vieler angewandter Mühe herausgebracht. Nehmlich man setze (sumendo pro* s *et* t *numeros quoscunque pro lubitu)*

$$\begin{aligned} x &= \frac{6ast^3(at^4-2s^4)^2}{(4at^4+s^4)(2aat^8+10as^4t^4-s^8)}, \\ y &= \frac{3s^5(4at^4+s^4)^2}{2t(at^4-2s^4)(2aat^8+10as^4t^4-s^8)}, \\ z &= \frac{2(2aat^8+10as^4t^4-s^8)}{3s^3t(4at^4+s^4)} \end{aligned}$$

so wird

$$x+y+z=\frac{2aat^8+10as^4t^4-s^8}{6s^3t(at^4-2s^4)}$$

und hieraus bekommt man $xyz(x+y+z)=a$.

Als es sey $a=1$ *und man nehme* $t=2$, $s=1$, *so wird* [. . .]

followed by factorizations confirming the explicit solution

$$(x, y, z) = \left(\frac{48 \cdot 14^2}{65 \cdot 671}, \frac{3 \cdot 65^2}{56 \cdot 671}, \frac{2 \cdot 671}{6 \cdot 65} \right)$$

of $xyz(x+y+z) = 1$. But Euler gives no further explanation; nor have I found anything about this equation in Dickson's *History of the Theory of Numbers* except for the case $a = 1$ (which as we'll see is a bit easier).

[Note that Euler does not mention the connection with areas of rational triangles; this makes sense given the $xy(x+y) = a$ context. Dickson does treat "Heron triangles", i.e. rational triangles of rational area, but that's the easier problem $xyz(x+y+z) = \alpha^2$ with α as well as x, y, z variable; likewise $xyz(x+y+z) = a_0\alpha^2$ is easy for any $a_0 \in \mathbf{Q}^*$, but that doesn't seem to help to find a solution with $\alpha = 1$.]

2. How Euler might have done it

We ask (with F. Lemmermeyer): how might Euler have found his family of solutions? We also ask: why was it so hard (remember Euler admitted to “vieler angewandte Mühe”), and does such a solution have to be so complicated (deg. 20 in the homog. variables $(s : t)$)?

2.1 Euler’s tools. $xyz(x + y + z) = a$ is a quartic surface. To us this suggests various tools involving the geometry and arithmetic of K3 surfaces. Euler did not have this “technology”, but he did have other tools that (as already seen for $xy(x + y) = a$) closely correspond to parts of the modern theory of elliptic curves. He was also a consummate and fearless calculator with numbers and functions much more complicated than most of us ever manipulate without computer assistance!

2.2 Example: Euler on $A^4 + B^4 = C^4 + D^4$.

We illustrate with his work on another quartic surface (yes, only 2-dim., as $(A : B : C : D)$ are homogeneous variables whereas the (x, y, z) of $xyz(x + y + z) = a$ are affine — more on this later). Even this might be anachronistic, as Euler's solutions of the Diophantine equation $A^4 + B^4 = C^4 + D^4$ were published only in 1772; but it's not much of a stretch because the trickiest ingredient was already known to Fermat.

To find some nontrivial rational solutions of $A^4 + B^4 = C^4 + D^4$, Euler writes the equation as $A^4 - D^4 = C^4 - B^4$, and factors both sides:

$$(A - D)(A + D)(A^2 + D^2) = (C - B)(C + B)(C^2 + B^2).$$

Ignoring trivial solutions where $B = C$, we may let $A - D = x(C - B)$, and conversely given x we may remove a factor $B - C$ and write

$$(A + D)(A^2 + D^2) = x(B + C)(B^2 + C^2)$$

with $D = A + x(C - B)$. This is a plane cubic, which contains a rational point corresponding to another trivial solution $(A : B : C : D) = (x-1 : x+1 : x-1 : -(x+1))$. But Euler (and Fermat, maybe even Diophantus) knew how to start from a rational solution and (usually!) get an infinite sequence of solutions. Here this works, and yields a family of nontrivial solutions. One simple description of the method: intersect the curve with the tangent at the known point. This yields the original point with multiplicity 2 plus a new point, which in our case turns out to yield a nontrivial solution.

It was later noticed (Gérardin 1917) that the change of variable $x = (x_1 + x_0)/(x_1 - x_0)$ puts this new solution $(A : B : C : D)$ in the nice form

$$(P(x_1, x_0) : P(-x_0, x_1) : P(-x_1, x_0) : P(-x_0, -x_1)),$$

where $P(X_1, X_0)$ is the homogeneous polynomial

$$X_0 X_1^6 - 3X_0^2 X_1^5 - 2X_0^3 X_1^4 + X_0^5 X_1^2 + X_0^7.$$

So $\exists$ infinitely many solutions, parametrized by polynomials of degree 7 in one variable. E.g., taking $x = 3$ and removing a common factor 64 (that is, taking $x_1 = 2x_0$), Euler found the solution $59^4 + 158^4 = 133^4 + 134^4$, which turns out to be the smallest non-trivial integer solution of $A^4 + B^4 = C^4 + D^4$. To find further parametrized families one may apply the procedure again to the above solutions, either on the same cubic curve or one of the others with a constant value of $(A \pm D)/(B \pm C)$ or $(A \pm C)/(B \pm D)$.

Why the caveat "(usually!)" in the description of the method? Because it might fail by producing only trivial solutions. For example, we might have preferred to start with the simpler point $(A : B : C : D) = (x : 1 : -1 : -x)$ instead of $(x - 1 : x + 1 : x - 1 : -(x + 1))$, but $(x : 1 : -1 : -x)$ turns out to be an inflection point: the tangent line, here

$$2x^3 A + (B + C) = x^4 (B - C),$$

meets the cubic at $(x : 1 : -1 : -x)$ with multiplicity 3, so we get the same trivial solution again, not a new nontrivial one.

Still we expect that with enough "trivial" points to start from, **some** choice must work. . .

2.3 The quartic $Q_a : xyz(x + y + z) = a$.

. . . and so it does, until we try to find rational solutions of $xyz(x + y + z) = a$.

Call this surface Q_a. It is easy to find rational functions t on Q_a whose fibers are cubic curves. For instance, x itself works: given x, the equation $xyz(x+y+z) = a$ is cubic in y and z. There are even three obvious solutions ("at infinity", but Euler was OK with that), namely $(y : z : 1) = (0 : 1 : 0)$, $(1 : 0 : 0)$, and $(1 : -1 : 0)$. But alas all three are points of inflection. Also they're collinear, so using secants instead of tangents doesn't help either.

So, try some other t. For example, $t_4 = y/x$:

Taking $y = t_4 x$ in $xyz(x + y + z) = a$ makes

$$t_4(xz)^2 + (t_4^2 + t_4)x^2 \cdot xz = a,$$

a quadratic in xz whose discriminant is

$$(t_4^2 + t_4)^2 x^4 + 4at_4.$$

Setting this equal to a square produces a curve of genus 1 in another familiar kind of model, $Y^2 = \text{quartic}(x)$. Again, we have a couple of rational points at infinity (because the leading term is a square), but they do not lead us to a nontrivial solution.

There are other natural choices of t to try, such as xy or $x+y$. Less obvious choices arise from other models of Q_a. For example, Euler knew how to put $Y^2 = (t_4^2+t_4)^2x^4+4at_4$ in Weierstrass form (though of course he didn't call it that): set $Y = (t_4^2+t_4)x^2+X$, solve the resulting quadratic in x, etc., eventually obtaining

$$y_4^2 = x_4^3 - at_4^3(t_4+1)^2x_4$$

where

$$(x_4, y_4) = \left(\frac{-w(x+y)y^2}{x^2},\ \frac{w(x+y)^2y^3}{x^3}\right)$$

and $w = -(x+y+z)$ as above. [The points with $x=\infty$ go to $(x_4,y_4) = (\infty,\infty)$ and $(0,0)$.] This reveals the further possibility

$$t_7 = x_4/t_4 = -w(x+y)y/x,$$

which was not so easy to see from $xyz(x+y+z) = a$.

But there's only one rational point to be found over $\mathbf{Q}(t_7)\ldots$

It turns out there 9 choices for t in all (up to the $S_4 \times \{\pm 1\}$ symmetry and projective linear changes of co-ordinate), some even more baroque than $-w(x+y)y/x$; but the usual tricks fail on every one of them. Indeed in every case there are only finitely many $\mathbf{Q}(t)$-rational points on the elliptic curve, and they all come from points at infinity on Q_a!

One can imagine Euler's *vieler angewandter Mühe* transforming Q_a among various elliptic models, every once in a while finding a new one — perhaps reaching all 9 possibilities — and never finding a nontrivial solution on any of them.

2.4 Reverse-engineering Euler? Euler might still have used some of these elliptic models to find rational points on Q_a (= rat. solutions of $xyz(x+y+z)=a$) even if they do not come from a curve such as $(x_4, y_4) = (\xi_4(t_4), \eta_4(t_4))$ with rational functions ξ_4, η_4. If so, we might reconstruct Euler's approach by computing the coordinates of his solution in each of the 9 elliptic models (actually many more than 9, remember $S_4 \times \{\pm 1\}$), and see if it's particularly simple in any of them.

Doing this yields lots of very complicated formulas but also a few that look nice enough to pursue further. One of these is on our (t_4, x_4, y_4) model. Euler wrote his solution as a rational function of s/t, so we dehomogenize by setting his t equal 1, and find that one permutation of his solution yields (t_4, x_4, y_4) equal to . . .

$$\left(\frac{4(2s^4-a)}{s^4+4a}, \frac{36s^6}{s^4+4a}, \frac{216s^7(s^8-10as^4+2a^2)}{(s^4+4a)^3}\right).$$

So x_4 is not quite a rational function of t_4, but we do have

$$x_4 = \frac{8}{a}(8-t_4)\left(\frac{a(1+t_4)}{8-t_4}\right)^{3/2}$$

which makes $y_4^2 = x_4^3 - at_4^3(t_4+1)^2x_4$ equal to

$$-8(1+t_4)^2(8+4t_4-t_4^2)^2\left(\frac{a(1+t_4)}{8-t_4}\right)^{3/2}.$$

This yields a rational solution provided some projective coordinate on the t_4-line, namely $4a(1+t_4)/(8-t_4)$, is a 4th power. Making it s^4 recovers Euler's solution.

Of course we cannot be sure that this is how Euler did it, but it seems at least somewhat plausible. Since our formula $x_4^3 - at_4^3(t_4+1)^2 x_4$ for y_4^2 involves only odd powers of x_4, it is natural to try making x_4^2 a rational function of t_4, which is in some way the simplest possibility if x_4 cannot be such a function itself; choosing x_4^2 of the form $(1+t_4)^3/(bt+c)$ makes everything go through as long as the factor

$$1 + t_4 - act_4^3 - abt_4^4$$

of $x_4^2 - at_4^3(t_4+1)^2$ is the square of some quadratic polynomial in t_4^2, which quickly yields $(b, c) = (1/8a, -1/64a)$.

3. How we might think about it

3.1 Q_a as a K3 surface. It turns out that Q_a is birational to a "K3 surface". Such surfaces have an extensive theory, and we can use parts of it to describe the models of Q_a as an elliptic curve over $\mathbf{Q}(t)$ and to explain why it's hard to find rational curves on Q_a.

A K3 surface is a simply-connected projective algebraic surface with trivial canonical class. We projectivize Q_a by dividing each of w, x, y, z by a new variable v and clearing fractions:

$$Q'_a : w + x + y + z = wxyz + av^4 = 0,$$

a quartic surface in $\mathbf{P}^3$. If this were smooth it would automatically be K3 (adjunction formula, etc.). It turns out that Q'_a is not quite smooth, but the singularities are mild enough (A_n, D_n, or E_n) that we can resolve them to get a K3.

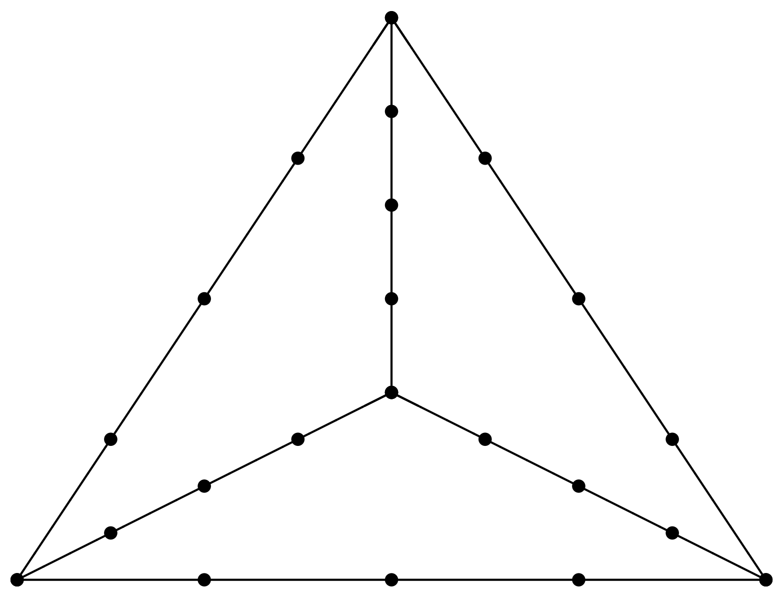
The affine part $v \neq 0$ of Q'_a (which is just Q_a) is smooth. The infinite part is the intersection with the plane $v = w + x + y + z = 0$, which gives $wxyz = 0$. This is the union of 4 lines that meet in pairs in $\binom{4}{2} = 6$ points. Each of these points is an A_3 singularity of Q'_a. Resolving it yields a K3 surface, which we'll call $\widetilde{Q}_a$.

This already shows that the function field of Q_a is not rational. Therefore there is no complete parametrization of $xyz(x + y + z) = a$ (unlike the situation for $xyz(x+y+z) = a_0\alpha^2$).

[Recall: $\tilde{Q}_a$ is birationally $w+x+y+z = wxyz+av^4 = 0$]

The preimage of $v = 0$ on $\tilde{Q}_a$ is then the union of $22 = 4 + 3 \cdot 6$ curves: the "proper transforms" of the 4 lines $wxyz = 0$, and 3 rational curves above each A_3 singularity. These 22 are all "-2 curves": smooth curves l of genus 0, which thus (again by adjunction) have self-intersection $l \cdot l = -2$. Any two distinct lines l, l' among them are either disjoint or meet transversely at one point; that is, $l \cdot l' = 0$ or 1.

We describe this configuration by a graph G_{22} with a vertex for each of the 22 lines and an edge for each intersection. This is a complete graph on 4 vertices with each edge replaced by a path of length 4: the paths come from the A_3 singularities, and each joins two of the four vertices representing the curves $w = 0$, $x = 0$, $y = 0$, $z = 0$.



3.2 The Néron–Severi lattice NS($\widetilde{Q}_a$). A fundamental invariant of a K3 surface (or indeed any smooth projective surface) S is its Néron–Severi lattice, which is a free finitely-generated abelian group NS(S) equipped with a (possibly indefinite!) symmetric pairing

$$\mathrm{NS}(S) \times \mathrm{NS}(S) \to \mathbf{Z}, \quad (D, D') \to D \cdot D'.$$

The group NS(S) consists of divisors on S *defined over an algebraic closure*, modulo algebraic equivalence; the pairing is the intersection number (this is where we use dimension 2). By the index theorem, the pairing on NS(S) $\otimes_{\mathbf{Z}} \mathbf{R}$ has signature $(1, \rho - 1)$ where $\rho = \rho(S)$ is the rank of NS(S), also known as the "Picard number" of S. When S is a K3 surface, algebraic and rational equivalence coincide, and the pairing is even: the homomorphism NS(S) $\to \mathbf{Z}/2\mathbf{Z}$ defined by $D \mapsto D \cdot D \bmod 2$ is trivial; i.e., the self-intersection $D \cdot D$ is always even. If D is the class of a curve of arithmetic genus g then $D \cdot D = 2g - 2$; in particular $D \cdot D \geq 0$ unless D is a smooth curve of genus 0, when $D \cdot D = -2$ as already noted.

In characteristic zero we have $\rho(S) \leq h^{1,1}(S)$ using the cycle class map. If S is K3 then $h^{1,1}(S) = 20$. K3 surfaces with $\rho = 20$ are an important special case, analogous to CM elliptic curves, and like them classically called "singular" (though of necessity they're geometrically smooth). As with CM curves, there are only countably many $\mathbf{C}$-isomorphism classes of singular K3's; and of these, only a finite (though still unknown) number have representatives defined over $\mathbf{Q}$ [Šafarevič].

The Fermat surface $A^4 + B^4 = C^4 + D^4$ is a famous example of a singular K3 surface; we shall see that the $\widetilde{Q}_a$, too, have $\rho = 20$.

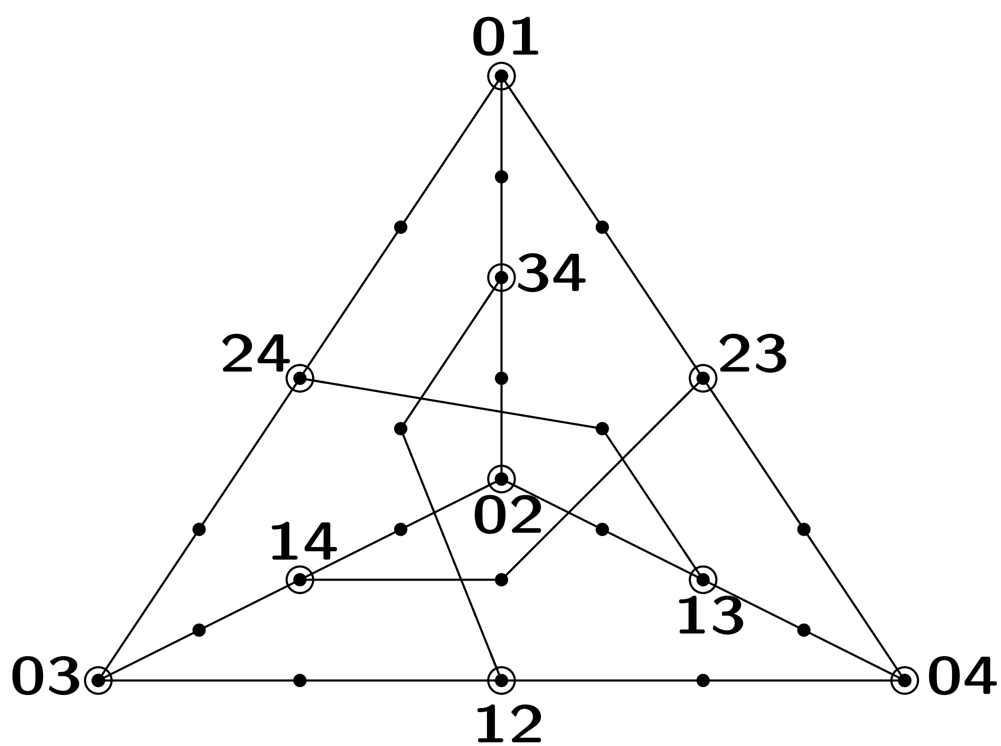
Start with the subgroup $L_\infty \subset \mathrm{NS}(\widetilde{Q}_a)$ generated by (the classes of) our 22 rational curves at infinity. We claim that this subgroup has rank 19.

Construct nondegenerate lattice $L_{22} \cong L'_{22}/K$ as follows. L'_{22} is free on 22 generators identified with vertices of G_{22}; for generators l, l', we have $l \cdot l' = -2$, $+1$, or 0 according as the vertices are the same, adjacent, or disjoint [as with root systems, but multiplied by -1]; and K is the kernel of this pairing. We calculate $\mathrm{rk}(K) = 3$, whence L_{22} has rank $22 - 3 = 19$, and L_{22} has signature $(1, 18)$ (for instance the sum H of all 22 vertices has $H \cdot H = 4$). By construction the composite map $L_\infty \to L'_{22} \to L_{22}$ is consistent with the pairing, and by the index theorem this map is an isomorphism.

Now it turns out that the hyperbolic lattice L_∞ and its G_{22} configuration are known. L_∞ is the "even sublattice of $\mathrm{I}_{1,18}$"; here $\mathrm{I}_{1,18}$ is the unimodular lattice with an orthogonal basis $e_1, \ldots, e_{19}$ where $e_1^2 = 1$ and $e_i^2 = -1$ for $i > 1$, and L_∞ consists of $\sum_{i=1}^{19} c_i e_i$ with $2 \mid \sum_{i=1}^{19} c_i$. Our 22 vectors l with $l \cdot l = -2$ are Vinberg's first batch of roots of L_∞.

As promised, $\mathrm{NS}(\widetilde{Q}_a)$ is larger than L_∞. In fact $\mathrm{NS}(\widetilde{Q}_a)$ is the even sublattice of $\mathrm{I}_{1,19}$. We find a 20th generator by intersecting $\widetilde{Q}_a$ with the plane $w+x = y+z = 0$. This produces two conics, $xz = \pm\sqrt{-a} \cdot v^2$, which sum to H in $\mathrm{NS}(\widetilde{Q}_a)$. Either of them together with L_∞ generates a lattice of rank 20 and discriminant -4, and that must be all of $\mathrm{NS}(\widetilde{Q}_a)$ because there is no unimodular even lattice of signature $(1, 19)$.

Instead of $w + x = y + z = 0$ we could have used $w + y = z + x = 0$ or $w + z = x + y = 0$. The respective conics $xz = \sqrt{-a} \cdot v^2$, $yx = \sqrt{-a} \cdot v^2$, $zy = \sqrt{-a} \cdot v^2$ (with the same choice of $\sqrt{-a}$) are pairwise disjoint, and each meets just two of the G_{22} curves, with multiplicity 1. Extending the G_{22} configuration by these three (-2) curves yields a graph G_{25}, labeled to show it's a Petersen graph with edges replaced by length-2 paths:



As with G_{22}, the 25-curve configuration was already obtained by Vinberg as the first batch of roots of the even sublattice of $\mathrm{I}_{1,19}$. He also noted the connection with -2 curves on a singular K3 surface (though he did not write the surface in the form $xyz(x+y+z) = a$ or $xyz(x+y+z) = av^4$).

Going from G_{22} to G_{25} preserves the S_4 symmetry and indeed extends it to the S_5 symmetry of the Petersen graph. This too reflects a symmetry of $\widetilde{Q}_a$, though we must choose a square root of $-a$ to see it, so it is defined only over $\mathbf{Q}(\sqrt{-a})$.

3.3 Elliptic fibrations of $\widetilde{Q}_a$. The Néron–Severi lattice of a K3 surface S lets us describe the elliptic fibrations $t : S \to \mathbf{P}^1$, i.e., the rational functions t on S whose generic fiber is an elliptic curve $E/\mathbf{Q}(t)$.

Rational points of E are sections of the fibration, which are (-2) curves on S. Let s_0 be the zero-section and $f \in \mathrm{NS}(S)$ the class of the fiber. Then $s_0 \cdot f = 1$ and $f \cdot f = 0$, so s_0 and f generate a "hyperbolic plane" $U \subset \mathrm{NS}(S)$ (indeed f and $f + s_0$ are isotropic vectors with $f \cdot (f + s_0) = 1$; "x is isotropic" means $x \cdot x = 0$). Since $\mathrm{disc}(U) = -1$ is a unit, U is a direct summand of $\mathrm{NS}(S)$. The orthogonal complement is negative definite, so $\mathrm{NS}(S) = U \oplus L\langle -1 \rangle$ for some positive-definite even lattice L of rank $\rho - 2$, called the <u>essential lattice</u> of the fibration.

The "roots" of a pos.-def. even lattice Λ are its vectors of norm 2; the "root lattice" $R(\Lambda)$ is the lattice generated by the roots. Such a lattice is a direct sum of simple root lattices A_n, D_n, E_n. For $\Lambda = L$, these simple summands of $R(L)$ correpond to the reducible fibers of t, and can be determined by Tate's algorithm.

By a theorem of Shioda and Tate, the group $E(\bar{\mathbf{Q}}(t))$ of sections is isomorphic with $L/R(L)$. It inherits an action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ whose fixed sublattice is $E(\mathbf{Q}(t))$.

Conversely, given an imbedding $\iota : U \hookrightarrow \mathrm{NS}(S)$ we can find an effective isotropic $f_1 \in \iota(U)$ with a 2-dim. space of sections whose ratio t is an elliptic fibration with $f \leq f_1$.

Example: a 12-cycle in G_{22} yields $f = f_1$, a reducible fiber of type I_{12} giving an A_{11} component of $R(L)$. This is our first elliptic fibration, with $t_1 = x$. We find a Weierstrass form

$$y_1^2 = x_1^3 + t_1^4(x_1 + 4a)^2$$

where $(x_1, y_1) = (4ax/y, 4a(w - z)x^2/y)$. The A_{11} fiber is at $t_1 = \infty$, and there's an E_6 at $t_1 = 0$ and 3-torsion point $0, 4at_1^2$, all visible on G_{22}. There is also a point of infinite order $(-4a, (-4a)^{3/2})$ visible on G_{25}, but it is not fixed by Galois unless $-a$ is a square.

Our t_4 fibration $y_4^2 = x_4^3 - at_4^3(t_4 + 1)^2 x_4$ has E_7 fibers at $t_4 = 0$ and $t_4 = \infty$, which are visible on G_{22}, as are both torsion points; and a D_4 fiber at $t_4 = -a$, only part of which is visible: two components are defined only over $\mathbf{Q}(\sqrt{-a})$ and appear one in each of the two G_{25}'s containing G_{22}.

For any elliptic fibration of $\widetilde{Q}_a$, the essential lattice L is an even lattice of rank $20-2=18$ and discriminant 4, and unless $-a$ is a square L inherits a Galois involution σ whose fixed sublattice L_0 has rank $19-2=17$ and also discriminant 4. Conversely, for such L we can identify $U \oplus L$ and $U \oplus L_0$ with $\mathrm{NS}(\widetilde{Q}_a)$ and $\mathrm{NS}(\widetilde{Q}_a)^\sigma = L_\infty$ respectively, and recover an elliptic fibration.

We find all such (L, L_0) by Kneser's gluing method (rediscovered in this K3 context by Nishiyama). It can be shown that the lattice $D_7 \oplus L_0$ of rank $7+17=24$ is contained with index 4, uniquely up to automorphism of the root lattice D_7, in some even unimodular lattice N, from which we can recover L_0 as the orthogonal complement of D_7, and L as the orthogonal complement of $D_6 \subset D_7$. So we need to classify pairs $(N, D_7 \hookrightarrow N)$ up to isomorphism.

Now even unimodular lattices in rank 24 were classified by Niemeier: there are 24, each characterized by its root lattice $R(N)$. Any map $D_7 \hookrightarrow N$ must send D_7 into a simple summand of $R(N)$, which must be D_n (some $n \geq 7$) or E_8. This lets us list all possible L_0. We tabulate them in the next page. As promised, there are 9, none of which has a section of infinite order defined over $\mathbf{Q}$ unless $-a$ is square. Indeed one can check from Vinberg's analysis of the hyperbolic roots of L_∞ and $\mathrm{NS}(\widetilde{Q}_a)$ that (again unless $-a \in \mathbf{Q}^{*2}$) the only -2 curves on $\widetilde{Q}_a$ are the 22 curves at infinity!

Thus for $a \notin \mathbf{Q}^{*2}$ any rational curve on $\tilde{Q}_a$ must be singular. The general K3 theory does not account for singular rational curves nearly as well as it does for (-2) curves. But it does supply various elliptic fibrations on which to seek singular rational curves or explain ones previously obtained.

In the following table, D_7 or D_6 always goes into the component of $R(N)$ listed first. $R^+(\Lambda)$ denotes the saturation $(R(\Lambda) \otimes \mathbf{Q}) \cap \Lambda$, and the superscripts $+k$ give the torsion group R^+/R:

ESSENTIAL LATTICES OF $\mathbf{Q}(a)$-RATIONAL ELLIPTIC FIBRATIONS OF $\tilde{Q}_a$

#	$R(N)$	$R^+(L_0)$	$R^+(L)$
1	$D_7A_{11}E_6$	$(A_{11}E_6)^{+3}$	$(A_{11}E_6)^{+3}$
2	D_8^3	$(D_8^2)^{+2}$	$(A_1^2D_8^2)^{+2+2}$
3	D_9A_{15}	$(A_1^2A_{15})^{+4}$	$(A_3A_{15})^{+4}$
4	$D_{10}E_7^2$	$(A_3E_7^2)^{+2}$	$(D_4E_7^2)^{+2}$
5	D_{12}^2	$(D_5D_{12})^{+2}$	$(D_6D_{12})^{+2}$
6	E_8^3	E_8^2	$A_1^2E_8^2$
7	$D_{16}E_8$	D_9E_8	$D_{10}E_8$
8	E_8D_{16}	D_{16}^{+2}	$A_1^2D_{16}^{+2}$
9	D_{24}	D_{17}	D_{18}

[For #1, the essential lattice L has an extra generator not fixed by $\mathrm{Gal}(\mathbf{Q}(\sqrt{-a})/\mathbf{Q}(a))$; for #2, #6, and #8, L_0 has an extra generator that ends up in $R(L)$.]

4. Further remarks

4.1 The other four elliptic fibrations of $\widetilde{Q}_a$. If we drop the condition that the fibration be defined over $\mathbf{Q}(a)$, there's no L_0 and we need only require $D_6 \hookrightarrow N$. This lets us use components D_6 and E_7 of $R(N)$, and gives four new possibilities, with $R(N) = D_6^4$, $D_6 A_9^2$, $E_7 A_{17}$, and $E_7^2 D_{10}$ (with $D_6 \hookrightarrow E_7$ in the last case). We have obtained explicit equations for all $9+4$ elliptic fibrations. Once we work over $\mathbf{Q}(\sqrt{-a}\,)$ the surface has infinitely many automorphisms, so we cannot list all possible functions t, only one representative from each class.

4.2 Petersen-graph trivia. It so happens that in none of these fibrations does $R(L)$ have a component A_{13} (ultimately because no $R(N)$ has such a component). Hence G_{25} has no 14-cycle, and we recover the tidbit that there is no heptagon in the Petersen graph. The more familiar fact that the Petersen graph is non-Hamiltonian is seen even more easily in this way: a Hamiltonian circuit would give a 20-cycle in G_{25}, which is impossible because then $\mathrm{NS}(\widetilde{Q}_a)$ would contain a negative-semidefinite lattice of rank 20.

4.3 Another rational curve on Q_a. Before locating online the transcription of Euler's letter, I found another singular curve on Q_a, which I thought might be Euler's, but turns out to be different and a bit simpler (degree 16, not 20): w, x, y, z are some permutation of

$$(s^4 - 4a)^2/(2s^3(s^4 + 12a)),$$
$$2a(3s^4 + 4a)^2/(s^3(s^4 - 4a)(s^4 + 12a)),$$
$$(s^5 + 12as)/(2(3s^4 + 4a)),$$
$$-2s^5(s^4 + 12a)/((s^4 - 4a)(3s^4 + 4a)).$$

This first turned up as a "quadratic section" of #9 (with $N = D_{24}^{+2}$, $L_0 = D_{17}$, $L = D_{18}$), but could also have been found on #4, where (t_4, x_4, y_4) is

$$\left(\frac{4a - s^4}{4s^4}, \frac{a(3s^4 + 4a)^2}{16s^{10}}, \frac{a(3s^4 + 4a)^2(s^4 + 12a)}{128s^{13}}\right).$$

4.4 The $\tilde{Q}_a$ as a family of quartic twists. Any two Q_a are isomorphic over $\mathbf{C}$: to get from Q_a to $Q_{a'}$, multiply x, y, z by $(a'/a)^{1/4}$. Under this identification, all the elliptic fibrations and curves that we found over $\mathbf{Q}(a)$ become independent of a.

Over $\mathbf{Q}$ we have $Q_a \cong Q_{a'}$ if $a'/a \in \mathbf{Q}^{*4}$. We shall see that this sufficient condition is necessary, even for the weaker conclusion $\tilde{Q}_a \cong \tilde{Q}_{a'}$.

In general Q_a and $Q_{a'}$ are quartic twists, possible thanks to an automorphism of Q_a defined over $\mathbf{Q}(i)$ but not $\mathbf{Q}$: multiply x, y, z by i. This automorphism extends to our fibrations and curves; e.g. on both Euler's curve and the new one, w, x, y, z as well as t_4, x_4, y_4 are all of the form $s^m f(s^4)$. Imposing this condition can simplify the search for rational curves on Q_a.

4.5 The L-function of $\tilde{Q}_a$, and an isogeny. Since a K3 surface S has trivial H^1 and H^3, its only interesting L-function is for $H^2(S)$. For singular S, the L-function is particularly simple: a product of shifted Artin L-functions from $\mathrm{NS}(S)$, multiplied by the Hecke L-function associated to a Grossencharacter for the imaginary quadratic field $\mathbf{Q}(\sqrt{\operatorname{disc}\mathrm{NS}(S)}\,)$, associated to the "transcendental lattice" of S.

For our surfaces $S = \tilde{Q}_a$, the contribution of $\mathrm{NS}(S)$ is just $\zeta(s-1)^{19}L(\chi, s-1)$ where χ is the quadratic character $(-a/\cdot)$. The remaining factor depends on the quartic character $(-a/\cdot)_4$ in $\mathbf{Q}(i)$. We can use this to show that if a'/a is <u>not</u> a 4th power then $\tilde{Q}_a$ and $\tilde{Q}_{a'}$ are not isomorphic over $\mathbf{Q}$, and thus neither are their open subsets Q_a and $Q_{a'}$.

But in one case it's closer than expected. . .

While -4 is not a 4th power in $\mathbf{Q}$, it is one in $\mathbf{Q}(i)$, namely $(1+i)^4$. Hence the L-functions of $\widetilde{Q}_a$ and $\widetilde{Q}_{-4a}$ differ only in the factor $L(\chi, s-1)$: the quadratic character χ gets multiplied by $(-1/\cdot)$.

This is explained by the existence of an isogeny between $\widetilde{Q}_a$ and $\widetilde{Q}_{-4a}$. In general an "isogeny" between K3 surfaces is a dominant rational map. It is known that the L-functions of isogenous K3's might not be the same but always have the same transcendental factors. The easiest nontrivial kind of K3 isogeny is obtained from an isogeny of elliptic curves over $\mathbf{Q}(t)$, and indeed the elliptic models of $\widetilde{Q}_a$ and $\widetilde{Q}_{-4a}$ over $\mathbf{Q}(t_4)$ are related by such an isogeny.

Recall that this model of $\widetilde{Q}_a$ is

$$y_4^2 = x_4^3 - a t_4^3 (t_4 + 1)^2 x_4.$$

The isogeny between $\widetilde{Q}_a$ and $\widetilde{Q}_{-4a}$ is a special case of the 2-isogeny

$$(X', Y') = (X - AX^{-1}, (1 + AX^{-2})Y)$$

that always relates the CM elliptic curves $Y^2 = X^3 - AX$ and $Y'^2 = X'^3 + 4AX'$. (This too goes back to Fermat, since the cases $A = \pm 1$ are key to his "descent" proofs that the only rational solutions of $Y^2 = X^3 \pm X$ have $Y = 0$.)

Using this isogeny we can start from a rational curve on $\widetilde{Q}_{-a/4}$ and get one on $\widetilde{Q}_a$ of about twice the degree. This explains why the special case $a \in \mathbf{Q}^{*2}$ is a bit easier: then $-a/4$ is -1 times a square, so $\widetilde{Q}_{-a/4}$ has (-2) curves not in G_{22}, which map to rational albeit singular curves on $\widetilde{Q}_a$.

We can also apply this to Euler's degree-20 curve on $\widetilde{Q}_{-a/4}$, or our new degree-16 curve, to get curves on $\widetilde{Q}_a$ of degree 36 and 28 respectively. The latter curve has w, x, y, z a permutation of

$$\frac{s^5(s^4-3a)^3}{(2(s^4+a)(s^{12}+12as^8-3a^2s^4+2a^3))}, \quad \frac{(s^{12}+12as^8-3a^2s^4+2a^3)}{(2s^3(s^4-3a)(3s^4-a))},$$
$$\frac{2a(s^4+a)^2(3s^4-a)^2}{(s^3(s^4-3a)(s^{12}+12as^8-3a^2s^4+2a^3))}, \quad \frac{-2s(s^{12}+12as^8-3a^2s^4+2a^3)}{((s^4-3a)(s^4+a)(3s^4-a))};$$

and as for the curve of degree 36 . . .

. . . I'll conclude by thanking you for your patience, and will for once refrain from imposing on it further by exhibiting yet another explicit rational parametrization solution of the equation $xyz(x+y+z) = a$ from that 1749 letter of Euler.

`http://math.dartmouth.edu/∼euler/correspondence/letters/OO0853.pdf`

Well, if you <u>`insist`</u>:

$$\frac{(8s^8+a^2)(8s^8-88as^4-a^2)}{(12s^3(s^4-a)(8s^8+20as^4-a^2))}\ ,$$

$$\frac{(8s^8+a^2)(8s^8-88as^4-a^2)}{(12s^3(8s^4+a)(8s^8+20as^4-a^2))}\ ,$$

$$\frac{192as^5(s^4-a)^2(8s^4+a)^2}{((8s^8+a^2)(8s^8-88as^4-a^2)(8s^8+20as^4-a^2))}\ ,$$

$$\frac{-3s(8s^8+20as^4-a^2)^3}{(4(s^4-a)(8s^4+a)(8s^8+a^2)(8s^8-88as^4-a^2))}\ .$$