Existence and uniqueness of the Hoffman-Singleton Graph (outline)

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0. Let G be a Moore graph of degree 7, i.e., a strongly regular graph with parameters $(50, 7, 0, 1)$, or equivalently a graph of degree 7 on 50 vertices with diameter 2 and girth 5. Such G contains $50 \cdot 7 \cdot 6 \cdot 6/10 = 1260$ pentagons.

1. Let A be the neighborhood of one of those pentagons, and let B be its complement in $V(G)$. Then $V(G) = A \cup B$ is a partition of $V(G)$ into two 25-vertex parts such that every vertex of G is adjacent to 2 in the same part and 5 in the other part. This is all we'll use about A, B.

2. Of the 1260 pentagons, 1000 are of type $ABxAB$ and 250 of type $AAxBB$. This leaves only 10 of type $AAAAx$ and $BBBBx$. It soon follows that each of A, B consists of five pentagons.

3. We now know what the AA and BB edges look like; the condition that G have no 3- or 4-cycles will force the AB edges uniquely up to automorphism of A, B. First, each A vertex has its one of its five B neighbors in each of the five B pentagons and vice versa. Thus the restriction of G to the union of an A and a B pentagon is a Petersen graph.

4. We can orient the A and B pentagons compatibly, i.e., label the 50 vertices $A_i(x), B_j(y)$ $(i, j, x, y \in \mathbf{F}_5)$ so that the pentagon edges are $\{A_i(x), A_i(x+1)\}$ and $\{B_j(y), B_j(y+1)\}$ and the AB edges are $\{A_i(x), B_j(2x + c_{ij})\}$ for some c_{ij}. [A priori it might have been necessary to use $-2x + c_{ij}$ instead of $2x + c_{ij}$ for some i, j, but thanks to the girth condition we can always flip some pentagons, i.e., relabel $A_i(x), B_j(x)$ as $A_i(-x), B_j(-x)$ for certain i, j, to eliminate all the minus signs.]

5. To avoid 4-cycles it is now only necessary that $c_{ij} + c_{i'j'} \neq c_{i'j} + c_{ij'}$ $(i \neq i', j \neq j')$. This determines the c_{ij} up to rotating the pentagons [i.e., relabeling $A_i(x)$ as $A_i(x - \delta_i)$, which translates the c_{ij} by δ_i, and likewise for $B_j(y)$] and permuting the i's and j's among themselves. For instance we may take $c_{ij} = i \cdot j$.