Yet another proof of the uniqueness of the E_8 lattice

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Theorem (Mordell 1938 [2]). *Let L be an even unimodular positive-definite lattice of rank 8. Then $L \cong E_8$.*

Yet another proof: We construct an odd "2-neighbor" L' of L, identify L' with $\mathbf{Z}^8$ using our characterization of $\mathbf{Z}^n$ by its minimal characteristic norm [1], and use this to identify L with E_8.

A "2-neighbor" of L is a self-dual lattice $L' \subset L \otimes \mathbf{Q}$ such that $L_0 = L \cap L'$ is a sublattice of index 2 in each of L and L'. To obtain L_0 and L', first fix a lattice vector $v_0 \notin 2L$ such that $4|(v_0, v_0)$. (Such a vector exists in every even lattice of rank at least 3.) We may assume that $(v_0, v_0) \equiv 4 \bmod 8$, since if $8|(v_0, v_0)$ we may replace v_0 by $v_0 + 2w$ with (v_0, w) odd. Set

$$L_0 = \{v \in L : 2|(v, v_0)\}, \qquad L' = L_0 \cup (L_0 + \frac{v_0}{2}).$$

We claim that L' is an odd self-dual lattice. That it is a lattice is clear since certainly $v_0 \in L_0$. It is contained in its dual by construction. Since $[L' : L_0] = [L : L_0] = 2$, we have $\operatorname{disc}(L') = \operatorname{disc}(L) = 1$, so $L' \subseteq L'^*$ implies $L' = L'^*$. Finally L' is odd because it contains the vector $v_0/2$ of odd norm.

Recall (see for instance [1]) that every unimodular lattice M has characteristic vectors: lattice vectors $c \in M$ such that $(c, v) \equiv (v, v) \bmod 2$ for all $v \in M$. These constitute a coset of $2M$ in M, and if M is positive definite of rank n then $(c, c) \equiv n \bmod 8$ for all characteristic $c \in M$. Since L' is odd, 0 is not a characteristic vector of L'. Hence for every characteristic vector c of L' we have $(c, c) \geq 8 = \operatorname{rank}(L')$. By [1] it follows that $L' \cong \mathbf{Z}^8$.

Therefore L_0, being an even lattice contained with index 2 in L', is a D_8 lattice. Thus L is one of the three lattices intermediate between D_8 and D_8^*. One of these lattices is $L' = \mathbf{Z}^8$. The other two are isomorphic to E_8, Q.E.D.

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References

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