Math 55b: Honors Advanced Calculus and Linear Algebra

Metric topology IV: Sequences and convergence;
the spaces $\mathcal{B}(X,Y)$ and $\mathcal{C}(X,Y)$, and uniform convergence

Sequences and convergence in metric spaces. [See Rudin, 3.1, 47–51.] The notion of convergence in the metric space $\mathbf{R}$ is implicit in such familiar contexts as infinite series and even nonterminating decimals, but was not made explicit until centuries after Euler spent considerable effort trying to evaluate such series as $1! - 2! + 3! - 4! + \cdots$ [sic]. As with many such notions, while our initial interest is in sequences in $\mathbf{R}$ or perhaps $\mathbf{R}^n$, the basics are just as easy to formulate in the context of an arbitrary metric space, and we shall have occasion to use this notion in that generality later in the course.

Let $\{p_n\} = \{p_1, p_2, p_3, \ldots\}$[1] be a sequence in a metric space X (i.e., with each $p_n \in X$, $n = 1, 2, 3, \ldots$). We say that $\{p_n\}$ *converges* if there is a point $p \in X$ such that: for every $\epsilon > 0$ there is an integer N such that $d(p_n, p) < \epsilon$ for each $n > N$. Equivalently, for every $\epsilon > 0$ we have $d(p_n, p) < \epsilon$ for all but finitely many n. [Why are the two definitions equivalent?] Such p is called the *limit* of $\{p_n\}$. The definite article must be justified: we must show that if p, p' are both limits of $\{p_n\}$ then $p = p'$. This is easily shown as follows [see Rudin, Thm. 3.2b, p.48]. For any $\epsilon > 0$, there are integers N, N' such that $d(p_n, p) < \epsilon$ for each $n > N$ and $d(p_n, p') < \epsilon$ for each $n > N'$. Let n be any integer that exceeds both N and N'. Then by the triangle inequality $d(p, p') < 2\epsilon$. Since ϵ is an arbitrary positive number, it follows that $p = p'$. We also use the following notations for "p is the limit of $\{p_n\}$": "$\{p_n\}$ converges to p", "p_n approaches p", "$p = \lim_{n\to\infty} p_n$", and "$p_n \to p$" (or "$p_n \to p$ as $n \to \infty$").[2]

A sequence that does not converge is said to *diverge*. Note that this notion can depend on X as well as $\{p_n\}$; e.g. $\{2^{-n}\}$ converges as a sequence in $\mathbf{R}$, but not as a sequence in $(0, 1)$. We can say "$\{p_n\}$ converges in X" if the ambient space may otherwise be ambiguous.

Like continuity of functions, the limit of a sequence may be defined topologically: $p_n \to p$ if every open set containing p also contains p_n for all but finitely many n.[3] This lets us define $\lim_{n\to\infty} p_n$ in an arbitrary topological space, but with the proviso that the limit of a sequence may not be unique without some assumption on the topology. (Challenge: formulate such an assumption that makes limits unique but still allows a non-metric topology!)

It should be clear that $p_n \to p$ if and only if the sequence $\{d(p_n, p)\}$ converges

[1] We use the braces to distinguish the sequence $\{p_n\}$ from its typical element p_n. Note that this contains more information than the *set* $\{p_n | n = 1, 2, 3, \ldots\}$, namely which element of that set occurs as which p_n.

[2] NB In such contexts ∞ is no more than a suggestive symbol; e.g., there is no point "∞" in $\mathbf{R}$ for $\{n\}$ to approach.

[3] NB this is not quite the same as "contains all but finitely many p_n"! Cf. the end of footnote 1.

to 0 in $\mathbf{R}$. It is easy to see that a sequence $\{(x_n, y_n)\}$ in a product metric space $X \times Y$ converges to (x, y) if and only if $x_n \to x$ and $y_n \to y$.

Convergence of sequences is also related to several notions already introduced, as follows.

Theorem. [Cf. Rudin, 3.2d on p.48] *If p is a limit point of $E \subseteq X$ then there exists a sequence $\{p_n\}$ in E with $p_n \to p$. The closure of any $E \subseteq X$ is the set of limits in X of sequences in E. In particular E is closed if and only if every sequence in E that converges in X also converges in E.*

Proof: We saw in the second handout that if p is a limit point then there are $q_n \in E$ $(n = 1, 2, 3, \ldots)$, with each $q_n \neq p$, such that for each $r > 0$ we have $d(p, q_n) < r$ for all but finitely many n. Then $q_n \to p$. If $p \in \overline{E}$ but p is not a limit point then $p \in E$, and we obtain $\{p_n\}$ by setting each $p_n = p$. Conversely, if $p_n \to p$ for some sequence $\{p_n\}$ in E, then every neighborhood of p contains a point of E, whence $p \in \overline{E}$. The last statement follows from Rudin 2.27b (also done in the second handout): $E = \overline{E}$ if and only if E is closed. $\square$

Theorem. [Cf. Rudin, Thm. 4.2 on p.84] *Let $f : X \to Y$ be a function between metric spaces, and $p \in X$. Then f is continuous at p if and only if $f(p_n) \to f(p)$ for every sequence $\{p_n\}$ in X such that $p_n \to p$.*

Proof: $\Rightarrow$ is clear from the definition. For $\Leftarrow$, suppose on the contrary that f is not continuous at p. Then there exists $\epsilon > 0$ such that for each $\delta > 0$ some $p' \in X$ satisfies $d(p, p') < \delta$ but $d(f(p), f(p')) \geq \epsilon$. Let p_n be a p' that works for $\delta = 1/n$. Then $p_n \to p$ but $f(p_n)$ does not converge to $f(p)$. $\square$

We deduce the following for sequences of real or complex numbers:

Theorem. [Rudin, Thm. 3.3 on p.49] *Let $\{s_n\}, \{t_n\}$ be sequences of real or complex numbers. Assume that $s_n \to s$, $t_n \to t$. Then: the sequence $\{s_n + t_n\}$ converges to $s + t$; the sequence $\{s_n - t_n\}$ converges to $s - t$; the sequence $\{s_n t_n\}$ converges to st; and, if $s \neq 0$ and $s_n \neq 0$ for each n, the sequence $\{1/s_n\}$ converges to $1/s$.*

Proof: This follows immediately from the previous theorem together with the continuity of the functions $(s, t) \mapsto s \pm t$, $(s, t) \mapsto st$, $s \mapsto 1/s$ shown in the previous (third) handout. $\square$

[Note that this looks nothing like the proof in Rudin, which nevertheless should be familiar; what's going on here?]

Sequences in function spaces; uniform convergence. Let Y be a metric space, and X an arbitrary set. We noted that the set Y^X of functions from X to Y becomes a metric space under the sup metric if Y is bounded, or if X is finite, but that there is a problem if Y is unbounded and X is infinite: the purported distance between two functions might be infinite! The usual way around this problem is to restrict attention to *bounded* functions from X to Y, i.e. functions $f : X \to Y$ such that $f(X)$ is a bounded set in Y. The set of such functions is denoted $\mathcal{B}(X, Y)$, and it becomes a metric space under the sup metric $d(f, g) := \sup_{x \in X} d(f(x), g(x))$. Note that $\mathcal{B}(X, Y)$ is simply Y^X in the known cases of bounded Y or finite X.

Now let $\{f_n\}$ be a sequence in $\mathcal{B}(X, Y)$. What does it mean for $\{f_n\}$ to converge to f? Unwinding the definition, we find that the condition is:

> For each $\epsilon > 0$ there exists an integer N such that $d(f_n(x), f(x)) < \epsilon$ for all $x \in X$ and $n > N$.

If f_n, f are any functions (not necessarily bounded) from X to Y, we say that f_n *converges uniformly* to f if the above condition is satisfied. [Cf. Rudin, Def. 7.7 on p.147, which is the special case $Y = \mathbf{R}$ or $\mathbf{C}$.] Note that if each f_n is bounded then so is f (if $f_n(X)$ is contained in a neighborhood of radius M, and $d(f, f_n) < \epsilon$, then $f(X)$ is contained in a neighborhood of radius $M + \epsilon$ with the same center). If $f_n \to f$ uniformly, then certainly $f_n(x) \to f(x)$ for each $x \in X$, i.e. f_n also *converges pointwise* to f. Note that pointwise convergence of f_n to f, defined by

> For each $x \in X$ and each $\epsilon > 0$ there exists an integer N such that $d(f_n(x), f(x)) < \epsilon$ for all $n > N$,

is a strictly weaker notion than uniform convergence, since N is allowed to depend on x as well as ϵ. (Cf. our earlier distinction between continuity and uniform continuity.) For instance, let $X = Y = [0, 1]$, and $f_n(x) = x^n$. Then f_n converges pointwise, but not uniformly, to the function

$$f(x) := \begin{cases} 1, & \text{if } x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If X is itself a metric space[4] then we are often interested in the subspace $\mathcal{C}(X, Y)$ of $\mathcal{B}(X, Y)$ consisting of *continuous* bounded functions. The key fact here is:

Theorem. *$\mathcal{C}(X, Y)$ is closed in $\mathcal{B}(X, Y)$ for any metric spaces X, Y.*

That is, the uniform limit of bounded continuous functions is again bounded and continuous. We have already shown that the uniform limit of bounded functions

[4] More generally X, but not Y, can be a general topological space.

is bounded. We next show that the uniform limit of continuous functions is again continuous:

Theorem. [Cf. Rudin, Thms. 7.11 and 7.12 on p.149–150] *Suppose X, Y are metric spaces, and $\{f_n\}$ a sequence of continuous functions from X to Y. If $\{f_n\}$ converges uniformly to a function $f : X \to Y$, then f is continuous.*

Proof: Fix $x \in X$. We show more generally that if f_n is continuous at x and $f_n \to f$ uniformly then f is continuous at x. Indeed, let ϵ be any positive real number. Since $f_n \to f$ uniformly, there exists N such that $d(f_n, f) < \epsilon/3$ for all $n > N$. Fix one such n. Since f_n is continuous at x, there exists $\delta > 0$ such that $d(f_n(x), f_n(x')) < \epsilon/3$ provided $d(x, x') < \delta$. Then for all such x we have

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(x')) + d(f_n(x'), f(x')) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus f is continuous at x as claimed. $\square$

This also completes the proof that $\mathcal{C}(X, Y)$ is closed in $\mathcal{B}(X, Y)$.

Note that the example of $f_n(x) = x^n$ shows that the pointwise limit of continuous functions may fail to be continuous. Must the pointwise limit of bounded functions be bounded?