Math 55b: Honors Real and Complex Analysis

Homework Assignment #9 (2 April 2018):
More contour integrals and multivariate calculus

The shortest path between two real truths may pass through the complex plane.[1]

We continue the thread started by the second part of the previous problem set. We first illustrate the use of contour integration to calculate the definite integrals

$$\int_0^\infty e^{-x^2} \cos cx \, dx = \frac{1}{2} e^{-c^2/4} \sqrt{\pi}, \qquad \int_0^\infty \sin cx \, \frac{dx}{x} = \frac{\pi}{2}$$

for arbitrary $c > 0$. (The second integral is easily seen to be independent of c, but the fact that its constant value is $\pi/2$ is far from obvious.)

1. Fix $b > 0$. Apply $\oint_{R_n} f(z)\, dz = 0$ to $f(z) = \exp(-z^2)$ and $R = \{x + iy : -M \le x \le M,\ 0 \le y \le b\}$. Letting $M \to \infty$, deduce that

$$\int_{-\infty}^\infty e^{-(x+ib)^2} \, dx = \int_{-\infty}^\infty e^{-x^2} \, dx,$$

which we have already shown is equal to $\sqrt{\pi}$. Deduce the value of $\int_{-\infty}^\infty e^{-x^2} e^{icx}\, dx$ and thus of $\int_0^\infty e^{-x^2} \cos cx \, dx$. Prove that your formula for $\int_{-\infty}^\infty e^{-x^2} e^{icx}\, dx$ holds for all $c \in \mathbf{C}$.

2. In the last problem of the previous problem set, use $[c,d] = [0,\pi]$ instead of $[0, 2\pi]$ to find that

$$\int_{-r}^{-r_0} f(z) \frac{dz}{z} + \int_{r_0}^{r} f(z) \frac{dz}{z} = i \left(\int_0^\pi f(r_0 e^{i\theta})\, d\theta - \int_0^\pi f(re^{i\theta})\, d\theta \right).$$

Now take $f(z) = e^{icz}$ for some $c > 0$, and let $r_0 \to 0$ and $r \to \infty$. Deduce the formula for $\int_0^\infty \sin cz \, dz/z$.

The next problem concerns the geometry of the Riemann sphere, which is the complex projective line $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. Recall that the projective linear group $\mathrm{PGL}_2(\mathbf{C}) = \mathrm{GL}_2(\mathbf{C})/\mathbf{C}^*$ acts on $\mathbf{P}^1(\mathbf{C})$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = (az+b)/(cz+d)$, with the natural interpretation that $\infty \mapsto a/c$ and $-d/c \mapsto \infty$. A "circle" in $\mathbf{P}^1(\mathbf{C})$ is either an ordinary circle in $\mathbf{C}$ (i.e. a subset of the form $\{z \in \mathbf{C} : |z - z_0| = r\}$ for some $z_0 \in \mathbf{C}$ and $r > 0$), or the union of a line with $\{\infty\}$.

3. [Inversive geometry in $\mathbf{C}$] Prove that $\mathrm{PGL}_2(\mathbf{C})$ takes circles to circles, and acts transitively on the circles in $\mathbf{P}^1(\mathbf{C})$. Prove also that the stabilizer $\{g \in \mathrm{PGL}_2(\mathbf{C}) : g(C_0) = C_0\}$ of any circle C_0 acts transitively on $\mathbf{P}^1(\mathbf{C}) - C_0$. What is the subgroup of the stabilizer of $\{z \in \mathbf{C} : |z| = 1\}$ that does take 0 to 0?

4. In particular, there are fractional linear transformations that take C_0 to C_0 and take an arbitrary point not on C_0 to the center of C_0. For $C_0 = \{z \in \mathbf{C} : |z| = 1\}$, Find such a transformation w with $w(z_0) = 0$, and check that if $|z_0| < 1$ then $|w(z)| < 1$ for all z such that $|z| < 1$. Now if f is any complex-calued function on some neighborhood of C_0 that has a complex derivative, then we know from the last problem set that $f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta$. Applying this to $f \circ w$, generalize that integral to integral formula for $f(z_0)$ of the form $\int_0^{2\pi} p_{z_0}(\theta) f(e^{i\theta})\, d\theta$. Varying z_0, deduce that f is given by a power series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$, converging absolutely for all z such that $|z| < 1$.

[1]Usually attributed in some form to Jacques Hadamard (1865-1963), whose proof of the Prime Number Theorem (1896, also obtained independently by de la Vallée-Poussin) is, well, a prime example. The use of contour integration to evaluate real definite integrals is a more down-to-earth family of examples. According to http://homepage.math.uiowa.edu/~jorgen/hadamardquotesource.html it's adapted from a remark by Paul Painlevé (1863–1933), who besides his mathematical work served twice as Prime Minister of France.

Replacing z by $az+b$ for any $a \in \mathbf{C}^*$ and $b \in \mathbf{C}$, we recover the theorem that if f is a complex-valued function on any open disc $\{z \in \mathbf{C} : |z - z_0| < r\}$, and f has a complex derivative on this disc, then it has a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ convergent inside the disc. That is, such a function is automatically analytic! We can thus henceforth drop cumbersome phrases such as "differentiable as a function of a complex variable" and simply write "analytic". Using the integral formula we further see that such functions, unlike analytic functions on $\mathbf{R}$, are closed under uniform limits:[2]

5. Suppose now that $f_m = \sum_{n=0}^{\infty} a_n(f_m) z^n$ are analytic functions on the open disc $D = \{z \in \mathbf{C} : |z| < 1\}$, and $f : D \to \mathbf{C}$ is a function such that $f_m \to f$ uniformly in $|z| \leq r$ for each $r < 1$. Prove that for each n the sequence $\{a_n(f_m)\}$ converges, and that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in D$; in particular, f is analytic on D.

This will be an important tool for constructing and studying analytic functions.

Meanwhile we still have some multivariate real calculus to pursue... The next few problems concern multivariate differentiation of functions coming from linear algebra:

6. Let E be the open subset of the $(n+1)$-dimensional real vector space $\mathcal{P}_n$ consisting of the polynomials of degree n, i.e. $E = \{\sum_{j=0}^{n} a_j T^j : a_n \neq 0\}$. Fix $P_0 \in E$ and a real root t_0 of P_0. Give necessary and sufficient conditions on P_0, t_0 for there to exist a $\mathscr{C}^1$ real-valued function t on a neighborhood of P_0 such that $t(P_0) = t_0$ and $t(P)$ is a root of P for each P in the neighborhood. What is the derivative $t'(P_0)$?

7. Let E be the set $\mathrm{GL}_n(\mathbf{R})$ of invertible matrices in the n^2-dimensional vector space $\mathcal{M}_n$ of $n \times n$ real matrices. Then E is open (why?). Let $f : E \to \mathcal{M}_n$ be the map taking any matrix $A \in E$ to A^{-1}. Equivalently, f takes A to the solution of $AX = I$. Use the Implicit Function Theorem to show that f is differentiable, and compute its derivative, i.e. give a formula for $f'(A)B$ for any $A \in E$ and $B \in \mathcal{M}_n$. Check that your formula is consistent with the identities $f(TA) = f(A)T^{-1}$, $f(AT) = T^{-1}f(A)$ for all $T \in \mathrm{GL}_n(\mathbf{R})$. [NB the maps $A \mapsto TA$ and $A \mapsto AT$ are linear.]

8. Recall that we proved the Implicit Function Theorem via the Inverse Function Theorem, so the proof came down to constructing $f(A)$ for A near A_0 as the fixed point of some contraction mapping ϕ_A on a neighborhood of A_0. Having constructed $f(A) = A^{-1}$ this way as the solution of $AX = I$, take $A_0 = I$, determine ϕ_A, and check directly that its iterates converge to A^{-1} in some neighborhood of I.

And finally a warning about Fubini (changing the order of integration):

9. [Rudin p.288 #2, extended] For $i = 1, 2, 3, \ldots$, let ϕ_i be a continuous function on $\mathbf{R}$ supported[3] on $(1/2^i, 1/2^{i-1})$, such that $\int_0^1 \phi_i(x)\,dx = 1$. For $(x, y) \in \mathbf{R}^2$ define

$$f(x,y) = \sum_{i=1}^{\infty} [\phi_i(x) - \phi_{i+1}(x)]\phi_i(y).$$

Show that $f : \mathbf{R}^2 \to \mathbf{R}$ has compact support and is continuous except at $(0,0)$, and moreover that f must be unbounded in every neighborhood of $(0,0)$; $\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = 0$ but $\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = 1$. Likewise, construct a *bounded* function $f : [0,1] \times [0,1] \to \mathbf{R}$ such that: for each x, the function $y \mapsto f(x,y)$ is Riemann integrable with $\int_0^1 f(x,y)\,dy = 0$ (from which it follows that $\int_0^1 \left[\int_0^1 f(x,y)\,dy \right] dx = 0$); but there exist y such that the function $y \mapsto f(x,y)$ is not Riemann integrable (whence $\int_0^1 \left[\int_0^1 f(x,y)\,dx \right] dy$ doesn't even make sense).

This problem set is due Monday, 9 April, at the beginning of class.

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[2] Recall that by Weierstrass *any* continuous function on a real interval is uniformly approximated by polynomials; certainly polynomials are analytic and "most" continuous functions aren't.

[3] Recall that "f is supported on S" means "$f(x) = 0$ if $x \notin S$."