Math 55b: Honors Real and Complex Analysis

Homework Assignment #8 (26 March 2018):
Multivariate differential calculus; preview of complex contour integrals

> **Similarly**, *adv.*: At least one line of the proof of this case is the same as before.[1]

Some problems introducing multivariate differential calculus and its interaction with mathematics that we developed earlier this year:

1. [Rudin, Problem 7 in Chapter 9] Suppose $E \subset \mathbf{R}^n$ is open and $f : E \to \mathbf{R}$ has partial derivatives $D_1 f(x), \ldots, D_n f(x)$ that are bounded as x varies over E. Prove that f is continuous.

2. [Rudin, Problem 10 in Chapter 9] Suppose the open set $E \subset \mathbf{R}^n$ is *convex*, i.e., E contains the line segment joining any two of its points (so $x, y \in E$ and $0 \leq t \leq 1 \implies tx + (1-t)y \in E$). Prove that if $f : E \to \mathbf{R}$ has the property that $D_1 f(x)$ exists and is zero for all $x \in E$ then $f(x)$ depends only on $x_2, \ldots, x_n$. Show further that convexity can be replaced by a weaker hypothesis on the open set E, but that some condition is required; for instance if $n = 2$ and E is Ω-shaped the statement may be false.

3. The *Laplacian* Δf of a $\mathscr{C}^2$ function f on an open set $E \subseteq \mathbf{R}^n$ is defined by

$$\Delta f := \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = \sum_{j=1}^{n} D_{jj} f.$$

   Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Show that $\Delta(f \circ A) = (\Delta f) \circ A$ for all $f \in \mathscr{C}^2$ if and only if A is orthogonal. In particular, the "harmonic functions" (those in the kernel of Δ, i.e. $\mathscr{C}^2$ functions f with $\Delta f = 0$) are preserved by orthogonal changes of variable.

4. [Cauchy-Riemann equations] Let E be a nonempty open subset of $\mathbf{C}$. The usual identification of $\mathbf{C}$ with $\mathbf{R}^2$ (identify the complex number $z = x + iy$ with the vector (x, y)) lets us regard any map $w : E \to \mathbf{C}$ as a map from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^2$, or equivalently as a pair of real-valued functions $u(x, y) = \operatorname{Re} f(x + iy)$, $v(x, y) = \operatorname{Im} f(x + iy)$ on that subset. Prove the following criterion for a $\mathscr{C}^1$ function w to be differentiable *as a map from a subset of* $\mathbf{C}$ *to* $\mathbf{C}$, i.e., for there to exist a function $w' : E \to \mathbf{C}$ such that $w(z + h) = w(z) + hw'(z) + o(|h|)$ as $h \to 0$: the functions u, v must satisfy the *Cauchy-Riemann equations*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   Show that this is the case if w is any polynomial in z with complex coefficients, or the exponential function $w(z) = e^z = e^x(\cos y + i \sin y)$. Prove that such u, v, w are necessarily harmonic functions on $\mathbf{C}$. [We shall see that conversely every harmonic function on $\mathbf{C}$ is the real part of a differentiable function of a complex variable.]

The last batch of problems introduces contour integration over $\mathbf{C}$ in the special case of rectangular contours, which lets us illustrate the key properties even though we have

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al. See Problem 6. Note that this does not define an equivalence relation.

not yet defined line integrals and Green's theorem. This also starts one path to the key theorem that a differentiable function of a complex variable is automatically analytic.

We consider only rectangles with sides parallel to the real and imaginary axes, i.e. of the form $R = \{x + iy : a \le x \le b,\ c \le y \le d\}$ for some a, b, c, d with $a < b$ and $c < d$. For a continuous function $f : R \to \mathbf{C}$, we define $\oint_{\partial R} f(z)\, dz$ by

$$\oint_{\partial R} f(z)\, dz = \int_a^b f(x + ic)\, dx + i \int_c^d f(b + iy)\, dy - \int_a^b f(x + id)\, dx - i \int_c^d f(a + iy)\, dy.$$

(Each of the four terms, one for each side γ of R, is what we'll call the contour integral $\int_\gamma f(z)\, dz$ with γ oriented in the counterclockwise direction around the boundary ∂R of R.)

5. Suppose F is a complex-valued function on a neighborhood of R that is differentiable *as a function of a complex variable*, with derivative F'. Prove that $\oint_{\partial R} F'(z)\, dz = 0$. On the other hand, evaluate $\oint_{\partial R} \mathrm{Re}(z)\, dz$ and $\oint_{\partial R} \mathrm{Im}(z)\, dz$, and show that they are *not* zero.

6. Divide R into rectangles R_1, R_2 by choosing some $x_1 \in (a, b)$ and setting

$$R_1 = \{x + iy : a \le x \le x_1,\ c \le y \le d\}, \quad R_2 = \{x + iy : x_1 \le x \le b,\ c \le y \le d\}.$$

Prove that $\oint_{\partial R} f(z)\, dz = \oint_{\partial R_1} f(z)\, dz + \oint_{\partial R_2} f(z)\, dz$. Do likewise for $y_1 \in (c, d)$, and generalize to any partitions of $[a, b]$ and $[c, d]$ into M and N intervals respectively (giving rise to what we may call a product partition of R into MN rectangles). Check that this agrees with your results from the previous problem.

7. [Goursat] Suppose f is a complex-valued function on a neighborhood of R that is differentiable *as a function of a complex variable*. Prove that $\oint_R f(z)\, dz = 0$ as follows.[2] Assume the integral is nonzero, and let C be its absolute value, with $C > 0$. Repeatedly applying the result of the previous problem, obtain a sequence of rectangles R_n (with $R_0 = R$), with each R_n ($n > 0$) being one quarter of R_{n-1} and satisfying $\left|\oint_{\partial R_n} f(z)\, dz\right| \ge C/4^n$. Then there exists some $z^* \in R$ contained in each R_n. Use differentiability of f at z^* to obtain a contradiction.

8. With f as in the previous problem, we can now define $F : R \to \mathbf{C}$ by

$$F(u+iv) = \int_a^u f(x+ic)\, dx + i \int_c^v f(u+iy)\, dy = \int_a^u f(x+iv)\, dx + i \int_c^v f(a+iy)\, dy$$

(why are these two expressions equal?). Prove that F is differentiable *as a function of a complex variable* on the interior of R, with $F'(z) = f(z)$ for all z in that interior. Show that F extends to some neighborhood R' of R and satisfies the same properties in that neighborhood. (In general we can't use for R' the full neighborhood on which f was defined, but (as suggested by the notation) we can use for R' the interior of some larger rectangle.)

9. Now suppose $0 < r < R$, and let f be a complex-valued function on a neighborhood N of the annulus $\{z \in \mathbf{C} : r \le |z| \le R\}$. Assume again that f is differentiable on N as a function of a complex variable. Prove that $\int_0^{2\pi} f(re^{i\theta})\, d\theta = \int_0^{2\pi} f(Re^{i\theta})\, d\theta$. (Apply $\oint_R f(z)\, dz = 0$ to a suitable function f on a neighborhood of the rectangle with $[a, b] = [\log r, \log R]$ and $[c, d] = [0, 2\pi]$.) Deduce that if N contains the full disc $\{z \in \mathbf{C} : |z| \le R\}$ then $f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta})\, d\theta$. [It is known in general that if f is harmonic on a closed ball then its value at the center equals its average on the boundary. But we cannot prove this yet because we haven't even defined the integrals needed to make sense of such an average.]

This problem set is due Monday, 2 April, at the beginning of class.

[2] Note that this proof manages to avoid any continuity hypothesis on f', so you cannot obtain the same result by appealing to Green's theorem even if you already know that approach.